

# Note for the Beal's Conjecture

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**Abstract:** This work explores two famous conjectures in number theory: Fermat's Last Theorem and Beal's Conjecture. Fermat's Last Theorem, posed by Pierre de Fermat in the 17th century, states that there are no positive integer solutions for the equation  $a^n + b^n = c^n$ , where  $n$  is greater than 2. This theorem remained unproven for centuries until Andrew Wiles published a proof in 1994. Beal's Conjecture, formulated in 1997 by Andrew Beal, generalizes Fermat's Last Theorem. It states that for positive integers  $A$ ,  $B$ ,  $C$ ,  $x$ ,  $y$ , and  $z$ , if  $A^x + B^y = C^z$  (where  $x$ ,  $y$ , and  $z$  are all greater than 2), then  $A$ ,  $B$ , and  $C$  must share a common prime factor. Beal's Conjecture remains unproven, and a significant prize is offered for a solution. This paper provides a concise introduction to both conjectures, highlighting their connection and the ongoing challenge that a short proof for the Beal's Conjecture presents to mathematicians.

**Keywords:** Generalized Fermat Equation; prime numbers; binomial theorem; coprime numbers

**MSC:** 11D41

## 1. Introduction

Around 1637, Pierre de Fermat, a French mathematician, scribbled a now-famous remark in the margin of a book he was reading. He claimed to have discovered a proof for the equation  $a^n + b^n = c^n$  having no solutions in positive integers for  $n$  greater than 2 [1]. However, he didn't provide the details of his proof [1]. Mathematicians like Leonhard Euler and Sophie Germain made significant contributions years later [2] [3]. In the 20th century, mathematicians like Ernst Kummer proved the theorem for a specific class of numbers [4]. However, a complete solution remained out of reach. Finally, in 1994, Andrew Wiles, a British mathematician, announced a proof for Fermat's Last Theorem. The proof was incredibly complex, drawing on advanced areas of mathematics like elliptic curves. After some initial errors were addressed, Wiles' work was accepted as the long-awaited solution to the theorem [5].

In 1993, Andrew Beal, an amateur mathematician and banker, formulates the Beal's Conjecture while investigating generalizations of Fermat's Last Theorem. The conjecture is stated publicly for the first time where Beal offers a prize of \$5,000 for a proven solution or disproof of the conjecture [6]. This prize was later increased several times, reaching its current value of \$1 million held by the American Mathematical Society (AMS). The Beal's Conjecture says that if  $A^x + B^y = C^z$ , where  $A$ ,  $B$ ,  $C$ ,  $x$ ,  $y$ , and  $z$  are all positive integers, and  $x$ ,  $y$ , and  $z$  are greater than 2, then  $A$ ,  $B$ , and  $C$  must share a common prime factor [6]. In other words, there are no solutions where  $A$ ,  $B$ , and  $C$  are completely coprime numbers [6]. This conjecture has occasionally been referred to as a generalized Fermat equation. Indeed, Fermat's Last Theorem can be seen as a special case of the Beal's Conjecture restricted to  $x = y = z$ . New important advances for this problem have emerged in the last years [7], [8], [9].

The proof of Fermat's Last Theorem was described as a "stunning advance" in the citation for Wiles's Abel Prize award in 2016. It also proved much of the Taniyama-Shimura conjecture, subsequently known as the modularity theorem, and opened up entire new approaches to numerous other problems and mathematically powerful modularity lifting techniques [10]. However, the Beal's Conjecture remains unsolved. In this note, using only simple arguments, we show that the Beal's Conjecture is true. Wiles' proof is very far from being close to Fermat's claimed theorem due to its long extension, complexity and tools

that were only available during the 20th century. A trustworthy and short proof for Beal's Conjecture could considerably impact pure mathematics and spur new advances in number theory. Besides, this work unveils the long known mystery about the possible existence of Fermat's claimed theorem. Certainly, this work could be closer to Fermat's claimed proof.

## 2. Materials and methods

According to the binomial theorem, the expansion of any nonnegative integer power  $n$  of the binomial  $x + y$  is a sum of the form

$$(x + y)^n = \binom{n}{0} \cdot x^n \cdot y^0 + \binom{n}{1} \cdot x^{n-1} \cdot y^1 + \dots + \binom{n}{n} \cdot x^0 \cdot y^n,$$

where each  $\binom{n}{k}$  is a positive integer known as a binomial coefficient, defined as

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdot \dots \cdot 2 \cdot 1}.$$

This formula is also referred to as the binomial formula or the binomial identity [11]. The expression  $d \mid n$  means the integer  $d$  divides  $n$  while  $d \nmid n$  means the integer  $d$  does not divide  $n$ .

**Proposition 1.** A natural number  $p$  is prime if and only if  $p \nmid \binom{p}{k}$  for all integers  $0 < k < p$  [12].

Putting all together yields the proof of the Beal's Conjecture.

## 3. Results

The following is a key Lemma.

**Lemma 1.** Let  $a$ ,  $b$  and  $c$  be natural numbers greater than 1 and  $p$ ,  $q$  and  $r$  be different prime numbers. If we have  $p \mid (a + b)$ ,  $q \mid (c - b)$ ,  $r \mid (c - a)$ ,  $p \mid c$ ,  $q \mid a$ ,  $r \mid b$ ,  $c \neq (a + b)$ ,  $a \neq (c - b)$  and  $b \neq (c - a)$ , then this implies that  $\text{GCD}(a, b, c) > 1$  where the function  $\text{GCD}(\dots)$  is the greatest common divisor.

**Proof.** Suppose that  $a$ ,  $b$  and  $c$  are coprime numbers. We can rewrite the same statement as

$$\begin{aligned} a + b &= p \cdot u, \\ c - b &= q \cdot v, \\ c - a &= r \cdot w, \\ c &= p \cdot U, \\ a &= q \cdot V, \\ b &= r \cdot W, \end{aligned}$$

such that  $u$ ,  $v$ ,  $w$ ,  $U$ ,  $V$  and  $W$  are natural numbers. We claim that the Lemma 1 is true by solving this simple system of linear equations. Indeed, we would have

1. First, substituting  $a$  and  $c$  into

$$\begin{aligned} q \cdot V + b &= p \cdot u, \\ p \cdot U - b &= q \cdot v, \end{aligned}$$

which is

$$p \cdot U + q \cdot V = p \cdot u + q \cdot v.$$

2. Next, substituting  $b$  and  $c$  again into

$$\begin{aligned}a + r \cdot W &= p \cdot u, \\ p \cdot U - a &= r \cdot w,\end{aligned}$$

which is

$$p \cdot U + r \cdot W = p \cdot u + r \cdot w.$$

3. Finally, mixing previous equations into

$$\begin{aligned}p \cdot U - p \cdot u &= q \cdot v - q \cdot V, \\ p \cdot U - p \cdot u &= r \cdot w - r \cdot W,\end{aligned}$$

which is

$$r \cdot W - q \cdot V = r \cdot w - q \cdot v.$$

Solving the following equations:

$$\begin{aligned}p \cdot U + q \cdot V &= p \cdot u + q \cdot v \\ p \cdot U + r \cdot W &= p \cdot u + r \cdot w \\ r \cdot W - q \cdot V &= r \cdot w - q \cdot v,\end{aligned}$$

then we find that necessarily  $u = U$ ,  $v = V$  and  $w = W$ , which means that

$$a = q \cdot V = q \cdot v = c - b.$$

This contradicts our initial precondition that  $a \neq (c - b)$ . Since this implies that the natural numbers  $a$ ,  $b$ , and  $c$  cannot be coprimes, we reach a final contradiction. Consequently, by reductio ad absurdum, we prove that  $\text{GCD}(a, b, c) > 1$ .  $\square$

This is the main theorem.

**Theorem 1.** *The Beal's Conjecture is true.*

**Proof.** Suppose that the Beal's Conjecture is false. Hence, there would exist an equation  $A^x + B^y = C^z$ , where  $A, B, C, x, y$ , and  $z$  are all positive integers, and  $x, y$ , and  $z$  are greater than 2, and  $A, B$ , and  $C$  are coprime numbers. We can confirm that  $A, B, C > 1$  according to the Catalan solution [13]. Let's take three different prime numbers  $p, q$  and  $r$  such that  $p \mid C^z$ ,  $q \mid A^x$  and  $r \mid B^y$ . Putting together the binomial theorem and Proposition 1, we can rewrite the equation  $A^x + B^y = C^z$  as

$$\begin{aligned}a + b + p \cdot k &= c \\ a &= c - b + p \cdot n \\ b &= c - a + p \cdot m\end{aligned}$$

such that  $a = A^{x \cdot p}$ ,  $b = B^{y \cdot p}$  and  $c = C^{z \cdot p}$  where  $k, m$  and  $n$  are natural numbers. After that, we substitute the previous values of  $a, b, c, p, q$  and  $r$  inside of the Lemma 1. Certainly, we deduce that  $p \mid (a + b)$ ,  $q \mid (c - b)$ ,  $r \mid (c - a)$ ,  $p \mid c$ ,  $q \mid a$ ,  $r \mid b$ ,  $c \neq (a + b)$ ,  $a \neq (c - b)$  and  $b \neq (c - a)$  because of  $q \mid n$ ,  $r \mid m$  and  $p \mid (p \cdot k)$  over the coefficients expansion of the binomial theorem. In general, we can show that there is a contradiction under the assumption that  $a, b$ , and  $c$  are coprimes according to the Lemma 1. Since this implies that the natural numbers  $A, B$ , and  $C$  cannot be coprimes, then we prove that the Beal's Conjecture is true.  $\square$

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## Short Biography of Authors



**Frank Vega** is essentially a Back-End Programmer and Mathematical Hobbyist who graduated in Computer Science in 2007. In May 2022, The Ramanujan Journal accepted his mathematical article about the Riemann hypothesis. The article “Robin’s criterion on divisibility” makes several significant contributions to the field of number theory. It provides a proof of the Robin inequality for a large class of integers, and it suggests new directions for research in the area of analytic number theory.

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