Non trivial zeros of the Zeta function using the differential equations

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Abstract

Using the differential equations, we obtain a more flexible expression for the Riemann Zeta function on the critical strip. This allows us to prove that for every $\tau \in \mathbb{R}^*$ there exists at most a unique point $r \in (0,1)$ such that $\Im\Big(\zeta(r+i\tau)\Gamma(r+i\tau)\Big) = 0$, where Γ is the Gamma function.

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1 Main results

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s\int_1^{+\infty} u^{-1-s}\{u\}du, \ \Re(s) \in (0,1), \ \Im(s) \in \mathbb{R}^*, \quad (1)$$

where $\{u\}$ is the fractional part of the real u. In order to simplify the notation, denote $B \subset \mathbb{C}$ the critical strip, defined as

$$B := \{s \in \mathbb{C} : \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*\},$$

We prove the following theorems,

Theorem 1. *Consider the Zeta function given by the Equation* (1). *For every* $s \in B$ *we have*

$$\zeta(s)\Gamma(s) = \int_0^1 u^{-2+s}\Big(\frac{u}{\exp(u)-1} - 1\Big)du - \int_0^1 u^{-s}\Psi(u)du,$$

where the real function $\Psi : (0,1) \to \mathbb{R}$ *is defined as*

$$\Psi(u) = u^{-2}\int_0^{+\infty} \{x\}\exp(-xu^{-1})dx, \quad \forall u \in (0,1).$$

Theorem 2. *For every* $\tau \in \mathbb{R}^*$ *there exists at most a unique point* $r \in (0,1)$ *such that*

$$\Im\Big(\zeta(r+i\tau)\Gamma(r+i\tau)\Big) = 0.$$

2 Basic Lemmas

For every $s \in B$, the Equation (1) is equivalent to,

$$\frac{\zeta(s)}{s} = -\frac{1}{1-s} - \int_1^{+\infty} u^{-1-s}\{u\}du.$$

The aim is to studies the differential equation of solutions the functions

$$t \mapsto \psi_s(z,t) := t^s\Big[z + \int_1^t u^{-1-s}\{u\}du\Big], \quad z \in \mathbb{C},\ t \geq 1.$$

Remark that $\lim_{t\to+\infty} t^{-s}\psi_s((1-s)^{-1},t) = -s^{-1}\zeta(s)$. The strategy to prove the Theorem 1, is to find this limit. For every $s \in B$ we consider the following differential equation

$$\frac{d}{dt}x = st^{-1}x + t^{-1}\{t\}, \tag{2}$$
$$t \in \mathbb{R}_+^*/\mathbb{N}, \quad x(1) = \frac{1}{1-s},\ z \in \mathbb{C}, \quad x : \mathbb{R}_+^* \to \mathbb{C}.$$

Lemma 3. *Let be* $s \in B$. *There exists a unique continuous solution* $\psi_s(t) : \mathbb{R}_+^* \to \mathbb{C}$ *of the differential equation* (2) *which is defined as*

$$\psi_s(t) = t^s\int_0^t u^{-1-s}\{u\}du, \quad \forall t > 0.$$

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Proof. Let be $s \in B$ fixed. The function

$$t \in \mathbb{R}_+^* \mapsto t^s \int_0^t u^{-1-s}\{u\}du,$$

is C^∞ on $\mathbb{R}_+^*/\mathbb{N}$ and continuous on $\mathbb{R}_+^*$. Since $\{u\} = u$ for every $u \in (0,1)$, then

$$\int_0^1 u^{-1-s}\{u\} = \frac{1}{1-s}.$$

The Equation (2) is a non-homogeneous linear differential equation. The unique continuous solution $\psi_s(t) : \mathbb{R}_+^* \to \mathbb{C}$ such that $\psi_s(1) = \frac{1}{1-s}$. is given by

$$\psi_s(t) = t^s \int_0^t u^{-1-s}\{u\}du, \ \forall t > 0.$$

□

Let us introduce the following notations,

Notation 4. Let $g : \mathbb{R}_+ \to \mathbb{C}$ be a continuous function, we denote the function $\Phi[g] : \mathbb{R}_+ \to \mathbb{C}$ as

$$\Phi[g](t) := \int_0^t g(u)du, \quad \forall t \geq 0,$$

,

Notation 5. We denote the function $p : \mathbb{R} \to \mathbb{R}$ as

$$p(t) := \{t\}, \quad \forall t \geq 0.$$

Lemma 6. *For every $n \in \mathbb{N}$ we have*

$$\Phi^{n+1}[p](t) = -\frac{1}{(n+2)!}\sum_{k=1}^{n+2} B_k \binom{n+2}{k} t^{n-k+2} + p_n(t), \quad \forall t \geq 0,$$

where $(B_k)_{k\in\mathbb{N}}$ are the Bernoulli numbers and and where the real sequence functions $(p_n)_{n\in\mathbb{N}}$ is defined for every $k \in \mathbb{N}$ as

$$p_{2k+1}(t) := (-1)^k \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^{2k+3}} \sin(j2\pi t),$$

$$p_{2k}(t) := (-1)^k \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^{2k+2}} \cos(j2\pi t),$$

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Proof. Prove that

$$\Phi[p](t) = \int_0^t \{u\} du = \frac{1}{2}t - \frac{1}{12} + p_0(t), \quad \forall t \geq 0,$$

The function $u \mapsto \{u\}$ is 1-periodic, then there exists a continuous 1-periodic function $\tilde{p} : \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^t \{u\} du = t \int_0^1 \{u\} du + \tilde{p}(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{u\} du = \int_0^1 u \ du = \frac{1}{2},$$

we get

$$\Phi[p](t) - \frac{1}{2}t = \int_0^t \Big(\{u\} - \frac{1}{2}\Big) du = \tilde{p}(t), \quad \forall t \geq 0. \tag{3}$$

The function $\tilde{p}$ is a piecewise C^∞, continuous on $\mathbb{R}$ and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^{n} a_k \exp(ik2\pi t),$$

converge uniformly on $\mathbb{R}_+$ to the function $t \mapsto \tilde{p}(t)$, where $(a_k)_k \subset \mathbb{C}$ are the Fourier coefficients of the function $\tilde{p}$.

$$\tilde{p}(t) = \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients and the Equation (3) we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u)\tilde{p}(u) du \\ &= \int_0^1 \exp(-ij2\pi u)\Big(\int_0^u \Big(\{v\} - \frac{1}{2}\Big) dv\Big) du \\ &= \frac{1}{2}\int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2}\int_0^1 u(u-1) du = -\frac{1}{12} = -\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2}.$$

The function $\tilde{p}$ satisfies

$$p(t) = \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^2}\Big(\exp(ij2\pi t) - 1\Big) = -\frac{1}{12} + p_0(t), \quad \forall t \geq 0.$$

The Equation (3) implies

$$\Phi[p](t) = \frac{1}{2}t - \frac{1}{12} + p_0(t), \quad \forall t \geq 0.$$

Integrate successively to obtain

$$\Phi^{n+1}[p](t) = \frac{1}{2}\frac{t^{n+1}}{(n+1)!} + \sum_{k=0}^{n} \frac{c_k}{(n-k)!}t^{n-k} + p_n(t), \quad \forall t \geq 0,\ \forall n \geq 0,$$

where $(c_k)_{k\in\mathbb{N}}$ are defined as

$$c_{2k+1} = 0 \quad \text{and} \quad c_{2k} = (-1)^k \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}}, \quad \forall k \geq 0.$$

By definition of the Bernoulli numbers $(B_k)_{k\in\mathbb{N}^*}$, we get

$$\Phi^{n+1}[p](t) = -\frac{1}{(n+2)!}\sum_{k=1}^{n+2} B_k \binom{n+2}{k} t^{n-k+2} + p_n(t), \quad \forall t \geq 0,\ \forall n \geq 0,$$

□

3 Proof of the Theorem 1

Proof of the Theorem 1. Let be $s \in B$ and consider the continuous solution $\psi_s(t) : \mathbb{R}_+^* \to \mathbb{C}$ defined in the Lemma 3. We recall that

$$\psi_s(t) = t^s \int_0^t u^{-1-s} p(u) du, \quad \forall t > 0, \tag{4}$$

where $p(t) = \{t\}$ is defined as in the Notation 5. Using the fact $p(u) = u$ for every $u \in [0, 1)$, we have

$$\psi_s(t) = (1-s)^{-1}t, \quad \forall t \in (0, 1),$$

Then the function ψ_s satisfies the following differential equation

$$t\frac{d}{dt}\psi_s = s\psi_s + \{t\}, \quad t \geq 0, \quad \psi_s(0) = 0.$$

By definition of Φ in the Notation 4, we get

$$t\frac{d}{dt}\Phi^{n+1}[\psi_s](t) =(n+1+s)\Phi^{n+1}[\psi_s](t)+\Phi^{n+1}[p](t), \qquad (5)$$
$$\Phi^{n+1}[\psi_s](0)=0, \quad \forall t\geq 0, \quad \forall n\geq 0,$$

Integrate

$$\Phi^{n+1}[\psi_s](t) =t^{n+1+s}\int_0^t u^{-n-2-s}\Phi^{n+1}[p](u)du, \quad \forall t\geq 0, \quad \forall n\geq 0,$$

In particular, we get

$$\frac{(n+1)!\Phi^{n+1}[\psi_s](n)}{n^{n+1}} = (n+1)!n^s\int_0^n u^{-n-2-s}\Phi^{n+1}[p](u)du, \quad \forall n\geq 0,$$

which can be written as

$$\frac{(n+1)!\Phi^{n+1}[\psi_s](n)}{n^{n+1}} = (n+1)!n^s\int_0^{+\infty} u^{-n-2-s}\Phi^{n+1}[p](u)du$$
$$-(n+1)!n^s\int_n^{+\infty} u^{-n-2-s}\Phi^{n+1}[p](u)du, \quad \forall n\geq 0.$$

Use the integration by part formula for the first integral of the right term of the last equality, we get

$$\frac{(n+1)!\Phi^{n+1}[\psi_s](n)}{n^{n+1}} = \frac{(n+1)!n^s}{\Pi_{j=1}^{n+1}(j+s)}\int_0^{+\infty} u^{-1-s}p(u)du$$
$$-(n+1)!n^s\int_n^{+\infty} u^{-n-2-s}\Phi^{n+1}[p](u)du, \quad \forall n\geq 0.$$

Since $p(u)=\{u\}$. By definition of the Zeta function given by the Equation (1), we get

$$\frac{\zeta(s)}{s}\frac{(n+1)!n^s}{\Pi_{j=1}^{n+1}(j+s)} = -\frac{(n+1)!\Phi^{n+1}[\psi_s](n)}{n^{n+1}}$$
$$-(n+1)!n^s\int_n^{+\infty} u^{-n-2-s}\Phi^{n+1}[p](u)du, \quad \forall n\geq 0.$$

Using the Taylor formula and the definition of Φ, we have

$$\Phi^{n+1}[\psi_s](n)=\frac{1}{n!}\int_0^n(n-x)^n\psi_s(x)dx,$$

Then

$$\frac{\zeta(s)}{s}\frac{(n+1)!n^s}{\Pi_{j=1}^{n+1}(j+s)} = -\frac{n+1}{n^{n+1}}\int_0^n (n-x)^n\psi_s(x)dx$$
$$- (n+1)!n^s\int_n^{+\infty} u^{-n-2-s}\Phi^{n+1}[p](u)du.$$

Replace in the last Equation the function ψ_s by it quantity given in the Equation (4), we obtain

$$\frac{\zeta(s)}{s}\frac{(n+1)!n^s}{\Pi_{j=1}^{n+1}(j+s)} = -\frac{n+1}{n^{n+1}}\int_0^n (n-x)^n x^s\int_0^x u^{-1-s}p(u)dudx$$
$$- (n+1)!n^s\int_n^{+\infty} u^{-n-2-s}\Phi^{n+1}[p](u)du, \quad \forall n \geq 0.$$

Since $s \in B$, then $\Re(s) \in (0,1)$ and $p(u) = u$ for $u \in [0,1)$. Using the integration by part formula, we have

$$\int_0^n (n-x)^n x^s\int_0^x u^{-1-s}p(u)dudx = \int_0^n p(x)\ x^{-1-s}\int_x^n (n-u)^n u^s dudx.$$

We have obtained

$$\frac{\zeta(s)}{s}\frac{(n+1)!n^s}{\Pi_{j=1}^{n+1}(j+s)} = -\frac{n+1}{n^{n+1}}\int_0^n p(x)\ x^{-1-s}\int_x^n (n-u)^n u^s dudx$$
$$- (n+1)!n^s\int_n^{+\infty} u^{-n-2-s}\Phi^{n+1}[p](u)du. \tag{6}$$

By the Lemma 6, we have

$$\Phi^{n+1}[p](u) = p_n(u) - q(n,u), \tag{7}$$

where in order to simplify the notation, we denoted $q(n,u)$ the following Bernoulli polynomial

$$q(n,u) := \sum_{k=1}^{n+2}\frac{B_k}{k!}\frac{u^{n+2-k}}{(n+2-k)!}. \tag{8}$$

By the notation of p_n in the Lemma 6, we have $\sup_{n\geq 0}\max_{v\in[0,1]}|p_n(v)| < +\infty$. By Stirling formula, we get

$$\lim_{n\to+\infty}(n+1)!n^s\int_n^{+\infty} u^{-n-2-s}p_n(u)du = 0.$$

The Equation (6), implies

$$\zeta(s)\Gamma(s) = \lim_{n\to+\infty} \eta_s(n), \tag{9}$$

where

$$\eta_s(n) := -\frac{n+1}{n}\int_0^n p(x)\ x^{-1-s}\int_x^n (1-\frac{u}{n})^n u^s du dx \\ + (n+1)! n^s \int_n^{+\infty} u^{-n-2-s} q(n,u) du.$$

and where Γ is the Gamma function. Now, we simplify the limit $\lim_{n\to+\infty}\eta_s(n)$. By definition of q in the Equation (8), we have

$$\begin{aligned}\eta_s(n) &= -\frac{n+1}{n}\int_0^n p(x)\ x^{-1-s}\int_x^n (1-\frac{u}{n})^n u^s du dx \\ &\quad + (n+1)! n^s \int_n^{+\infty} u^{-s}\sum_{k=1}^{n+2}\frac{B_k}{k!}\frac{u^{-k}}{(n+2-k)!} du \\ &= -\frac{n+1}{n}\int_0^n p(x)\ x^{-1-s}\int_x^n (1-\frac{u}{n})^n u^s du dx \\ &\quad + (n+1)! \int_0^1 u^{-2+s}\sum_{k=1}^{n+2}\frac{B_k}{k!}\frac{n^{1-k}u^k}{(n+2-k)!} du.\end{aligned} \tag{10}$$

Implies

$$\eta_s(n) = -\frac{n+1}{n}\int_0^n p(x)\ x^{-1-s}\int_x^n (1-\frac{u}{n})^n u^s du dx \\ + \int_0^1 u^{-2+s}\varphi_n(u) du, \tag{11}$$

where in order to simplify the notation, we denoted

$$\varphi_n(u) := \sum_{k=1}^{n+2}\frac{B_k}{k!}\frac{(n+1)! n^{1-k}u^k}{(n+2-k)!}, \quad \forall u\in(0,1),\ \forall n\geq 0.$$

For every fixed $n \geq 1$ the function $u \mapsto \varphi_n(u)$ is an alternating finite series. For every fixed $(n,m) \in \mathbb{N}^2$ such that $4m \leq n$ we have

$$\sum_{k=1}^{2(2m+1)}\frac{B_k}{k!}\frac{(n+1)! n^{1-k}u^k}{(n+2-k)!} \leq \varphi_n(u) \leq \sum_{k=1}^{4m}\frac{B_k}{k!}\frac{(n+1)! n^{1-k}u^k}{(n+2-k)!},\ \forall u\in(0,1),$$

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Then for every $m \geq 1$, we get

$$\sum_{k=1}^{2(2m+1)} \frac{B_k}{k!}u^k \leq \lim_{n\to+\infty} \varphi_n(u) \leq \sum_{k=1}^{4m} \frac{B_k}{k!}u^k,\ \forall u \in (0,1),$$

Implies

$$\lim_{m\to+\infty} \sum_{k=1}^{2(2m+1)} \frac{B_k}{k!}u^k \leq \lim_{n\to+\infty} \varphi_n(u) \leq \lim_{m\to+\infty} \sum_{k=1}^{4m} \frac{B_k}{k!}u^k,\ \forall u \in (0,1),$$

In other words, the following convergence is uniform

$$\lim_{n\to+\infty} \varphi_n(u) = \sum_{k=1}^{+\infty} \frac{B_k}{k!}u^k = \frac{u}{\exp(u)-1} - 1,\ \forall u \in (0,1).$$

From the Equation (11), we obtain

$$\lim_{n\to+\infty} \eta_s(n) = -\int_0^{+\infty} p(x)\ x^{-1-s} \int_x^{+\infty} \exp(-u)u^s du dx + \int_0^1 u^{-2+s}\Big(\frac{u}{\exp(u)-1} - 1\Big)du.$$

Use the change of variable $u \mapsto xv$, for the first integral of the right term of the last equality, we get

$$\lim_{n\to+\infty} \eta_s(n) = -\int_0^{+\infty} p(x) \int_1^{+\infty} \exp(-xv)v^s dv dx + \int_0^1 u^{-2+s}\Big(\frac{u}{\exp(u)-1} - 1\Big)du.$$

then

$$\lim_{n\to+\infty} \eta_s(n) = -\int_1^{+\infty} v^s\Big(\int_0^{+\infty} p(x)\exp(-xv)dx\Big)dv + \int_0^1 u^{-2+s}\Big(\frac{u}{\exp(u)-1} - 1\Big)du.$$

By consequence, the Equation (9) implies,

$$\zeta(s)\Gamma(s) = -\int_1^{+\infty} v^s\Big(\int_0^{+\infty} p(x)\exp(-xv)dx\Big)dv + \int_0^1 u^{-2+s}\Big(\frac{u}{\exp(u)-1} - 1\Big)du.$$

□

4 Proof of the Theorem 2

Proof of the Theorem 2. Let be $\tau \in \mathbb{R}^*$ fixed. Define the function $f_\tau : (0,1) \mapsto \mathbb{R}$ as

$$f_\tau(r) := \Im\Big(\zeta(r+i\tau)\Gamma(r+i\tau)\Big), \quad \forall r \in (0,1).$$

The strategy to prove the present Theorem is to prove that $\tau \frac{d}{dr} f_\tau(r) < 0$ for every $r \in (0,1)$. By the Theorem 1, we have

$$\begin{aligned} f_\tau(r) &= \int_0^1 \sin(\tau \ln(u)) u^{-2+r} \Big(\frac{u}{\exp(u)-1} - 1\Big) du \\ &\quad + \int_0^1 \sin(\tau \ln(u)) u^{-r} \Psi(u) du \\ &= \frac{1}{\tau} \int_0^1 \Big(1 - \cos(\tau \ln(u))\Big) \frac{d}{du} g(r,u) du. \end{aligned}$$

where in order to simplify the notation, we denoted

$$g(r,u) := \frac{u^r}{\exp(u)-1} - u^{-1+r} + u^{1-r}\Psi(u), \quad \forall u \in (0,1),$$

Implies

$$\begin{aligned} \frac{d}{dr} f_\tau(r) &= \frac{1}{\tau} \int_0^1 \Big(1 - \cos(\tau \ln(u))\Big) \frac{d}{dr} \frac{d}{du} g(r,u) du \\ &= \frac{1}{\tau} \int_0^1 \Big(1 - \cos(\tau \ln(u))\Big) \frac{d}{du} \frac{d}{dr} g(r,u) du, \end{aligned} \tag{12}$$

Since

$$\Psi(u) = u^{-2} \int_0^{+\infty} \{x\} \exp(-xu^{-1}) dx, \quad \forall u \in (0,1),$$

Then for every $u \in (0,1)$ the function $g(r,u)$ can be written as

$$g(r,u) = u^{-1} \int_0^{+\infty} \Big[\frac{u^r}{\exp(u)-1} - u^{-1+r} + u^{-r}\{x\}\Big] \exp(-xu^{-1}) dx.$$

Implies

$$\frac{d}{du}\frac{d}{dr} g(r,u) = \int_0^{+\infty} h_r(u,x) u^{-2+r} \exp(-xu^{-1}) dx$$

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where

$$\begin{aligned}h_r(u,x) := &\frac{1}{\exp(u)-1} - u^{-3+r} - (1-r)\ln(u)\frac{1}{\exp(u)-1}\\ &- \ln(u)\frac{\exp(u)u}{(\exp(u)-1)^2} + \ln(u)(2-r)u^{-1},\\ &+ x\ln(u)\Big(\frac{u^{-1}}{\exp(u)-1} - u^{-2}\Big)\\ &+ \Big(\ln(u)(1+r) - x\ln(u)u^{-1} - 1\Big)u^{-2r}\{x\}\end{aligned}$$

For every fixed $r \in (0,1)$ we have $\frac{d}{du}h_r(u,x) > 0$ for every $u \in (0,1)$ and $x > 0$. Since $h_r(1,x) < 0$ for all $x > 0$, then $h_r(u,x) < 0$ for every $u \in (0,1)$ and $x > 0$ We obtain

$$\forall r \in (0,1): \quad \frac{d}{du}\frac{d}{dr}g(r,u) < 0, \quad \forall u \in (0,1).$$

By consequence, the Equation (12) implies $\tau\frac{d}{dr}f_\tau(r) < 0$ for every $r \in (0,1)$. □

References

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