(Yet another) way to prove the Pythagoras' theorem

Ajay Kumar K S[1]

1 Introduction

This article attempts a new way of proving the Pythagoras' theorem. For centuries, people have used diverse tools such as combinatorics, calculus, geometry, algebra and trigonometry to come up with hundreds of different ways to prove the theorem [1][2][3]. The contributors who have shown these new ways have been equally varied and diverse - spanning from ancient mathematicians to contemporary academics, from recreational mathematicians (such as Miss E.A. Coolidge, a remarkable blind girl [1]) to philosophers, from high-school students [4] to a former chief economist of the World Bank [5], and even extending to a former president of the United States [6]. Here, I use some geometry, trigonometry and algebra to prove the theorem. There are few proofs using trigonometry [4][7][8]. The reason why I even tried looking for one more way to prove is probably best paraphrased by the character Lisa Simpson (from *The Simpsons*) - I guess the hunt was more fun than the catch. [9]

2 Some basic definitions

The three internal angles of any triangle add up to $180°$.

In the figure, ΔABC depicts a right-angled triangle, where $\angle BCA = 90°$. Let $\angle CAB = \theta°$ and $|AB| = c$ (hypotenuse), $|BC| = a$ and $|AC| = b$. $\angle ABC = (90 - \theta)°$ $(180° - (\angle BCA + \angle CAB))$. (The figure appears as though $\angle CAB < \angle ABC$, but it should not matter; we can have $\angle CAB \geq \angle ABC$ and that would look like the figure flipped horizontally.)

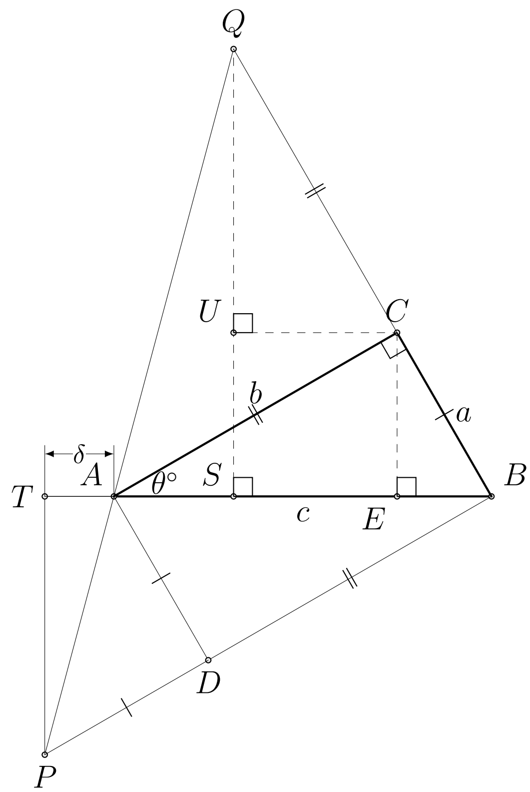
From the point C, we construct a triangle ΔACQ such that $\angle ACQ = 90°$ and $|CQ| = |AC|$. This is an isosceles right-angled triangle. $\angle QAC = \angle CQA = 45°$ and $|CQ| = b$. $\angle QAB = \angle QAC + \angle CAB = (45 + \theta)°$.

We construct another triangle ΔADB such that $\angle ADB = 90°$, and $|AD| = a$ and $|DB| = b$ ($|AB|$ is equal to c). We can see that $\angle BAD = (90 - \theta)°$ and $\angle DBA = \theta°$.

From the point D, we construct a triangle ΔDAP such that $\angle PDA = 90°$ and $|AD| = |PD|$. This is an isosceles right-angled triangle. $\angle DAP = \angle APD = 45°$ and

[1]Ajay Kumar K S is currently director of pharma solutions at *Strand Life Sciences*. To contact him regarding this article, please send email to p11ajayks@iima.ac.in

https://doi.org/10.33774/coe-2024-knw1p-v3 **ORCID:** https://orcid.org/0009-0009-0253-6208 Content not peer-reviewed by Cambridge University Press. **License:** All Rights Reserved



$|PD| = a$.

QP is a straight line passing through A (points Q, A and P are collinear) because

$$\begin{aligned} \angle QAP &= \angle QAC + \angle CAB + \angle BAD + \angle DAP \\ &= 45^\circ + \theta^\circ + (90 - \theta)^\circ + 45^\circ \\ &= 180^\circ \end{aligned}$$

$\angle ABQ + \angle PBA = (90 - \theta)^\circ + \theta^\circ = 90^\circ$. This means that ΔPBQ is a right-angled triangle. $|PB| = |PD| + |DB|$ which is $a + b$, and $|BQ| = |BC| + |CQ|$ which is also $a + b$.

From the point P, we draw a line PT that is perpendicular to AB such that we get a triangle ΔATP (by extending AB to meet T) such that $\angle ATP = 90^\circ$ and let $|AT| = \delta$. $\angle PAT = \angle QAB = (45 + \theta)^\circ$ (vertically opposite angles).

In a right-angled triangle, we define the sine of an angle as the ratio of the length of the opposite side to the length of the hypotenuse, and the cosine of an angle as the ratio of the length of the adjacent side to the length of the hypotenuse. By this definition, in the right-angled triangle ΔABC, $\sin\theta^\circ = \frac{|BC|}{|AB|} = \frac{a}{c}$, and $\cos\theta^\circ = \frac{|AC|}{|AB|} = \frac{b}{c}$. $\sin(\angle ABC) = \sin(90 - \theta)^\circ = \frac{|AC|}{|AB|}$, which is the same as $\cos\theta^\circ$. So $\cos\theta^\circ = \sin(90 - \theta)^\circ$.

It follows that for $45°$, $\sin 45° = \cos 45°$.

Deriving $\cos(45+\theta)°$

In this section, we derive the formula for $\cos(45+\theta)°$ without using the Pythagoras' theorem. This is well known [10] but is being presented again for use in the next section.

We draw a line from the point Q which is perpendicular to AB and meets AB at the point S. From C, we draw a line that is perpendicular to QS and meets QS at the point U. From C, we draw another line that is perpendicular to AB and meets AB at the point E.

Referring to ΔQAS, we can see that $\angle QAS = (45+\theta)°$, and that $|QS|$ is equal to $|QU| + |US|$ and $|AS|$ is equal to $|AE| - |ES|$. Also, $|US|$ is equal to $|CE|$ and $|UC|$ is equal to $|ES|$.

UC is parallel to AB and QS is parallel to CE. $\angle ACU = \angle CAB = \theta°$ and $\angle UCQ = (90-\theta)°$ (because $\angle ACU + \angle UCQ = \angle ACQ = 90°$). It follows that $\angle CQU = \theta°$.

We now look at $\cos(\angle QAS)$ which is $\cos(45+\theta)°$.

$$\begin{aligned}
\cos(45+\theta)° &= \frac{|AS|}{|AQ|} \\
&= \frac{|AE| - |ES|}{|AQ|} \\
&= \frac{|AE|}{|AQ|} - \frac{|UC|}{|AQ|} \\
&= \frac{|AE|}{|AC|}.\frac{|AC|}{|AQ|} - \frac{|UC|}{|QC|}.\frac{|QC|}{|AQ|}
\end{aligned}$$

From ΔCAE, we know that $\cos(\angle CAE) = \cos\theta° = \frac{|AE|}{|AC|}$, from ΔQUC, we know that $\sin(\angle CQU) = \sin\theta° = \frac{|UC|}{|QC|}$, and from ΔACQ, we know that $\cos(\angle QAC) = \cos 45° = \frac{|AC|}{|AQ|}$ and $\sin(\angle QAC) = \sin 45° = \frac{|QC|}{|AQ|}$. So

$$\cos(45+\theta)° = \cos\theta°.\cos 45° - \sin\theta°.\sin 45°$$

Because $\sin 45° = \cos 45°$,

$$\boldsymbol{\cos(45+\theta)° = \sin 45°(\cos\theta° - \sin\theta°)}$$

https://doi.org/10.33774/coe-2024-knw1p-v3 **ORCID:** https://orcid.org/0009-0009-0253-6208

3 The proof

In ΔATP, $\angle ATP = 90°$, and $\angle PAT = (45 + \theta)°$ (because $\angle PAT = \angle QAB$). Let $|AT| = \delta$. We know that $\sin(\angle APD) = \sin 45° = \frac{|AD|}{|AP|} = \frac{a}{|AP|}$. This means $|AP| = \frac{a}{\sin 45°}$.

$$\cos(\angle PAT) = \frac{|AT|}{|AP|}$$

$$\cos(45 + \theta)° = \frac{\delta}{\frac{a}{\sin 45°}} = \frac{\delta \sin 45°}{a}$$

We know that $\cos(45 + \theta)° = \sin 45°(\cos\theta° - \sin\theta°)$. This means that

$$\sin 45°(\cos\theta° - \sin\theta°) = \frac{\delta \sin 45°}{a}$$

We know that $\cos\theta° = \frac{b}{c}$ and $\sin\theta° = \frac{a}{c}$.

$$\left(\frac{b}{c} - \frac{a}{c}\right) = \frac{\delta}{a}$$

$$\boldsymbol{\delta = \left(\frac{ab - a^2}{c}\right)}$$

In ΔBTP, $\angle BTP = 90°$.

$$\cos(\angle PBT) = \cos\theta° = \frac{|BT|}{|BP|}$$

$$|BT| = |AB| + |AT| = (c + \delta)$$

$$|BP| = |PD| + |BD| = (a + b)$$

$$\cos\theta° = \frac{(c + \delta)}{(a + b)}$$

We know that $\cos\theta° = \frac{b}{c}$. This means that

$$\frac{(c + \delta)}{(a + b)} = \frac{b}{c}$$

$$c^2 + \delta c = ab + b^2$$

Replacing δ by $\left(\frac{ab-a^2}{c}\right)$, we get

$$c^2 + \left(\frac{ab - a^2}{c}\right)c = ab + b^2$$

$$c^2 + ab - a^2 = ab + b^2$$

$$\boldsymbol{c^2 = a^2 + b^2}$$

So, the hypotenuse $|\boldsymbol{AB}|^2$ is equal to $|\boldsymbol{BC}|^2$ + $|\boldsymbol{AC}|^2$. ■

https://doi.org/10.33774/coe-2024-knw1p-v3 **ORCID:** https://orcid.org/0009-0009-0253-6208

References

[1] E S. Loomis, The Pythagorean Proposition, *National Council of Teachers of Mathematics* (1968), https://files.eric.ed.gov/fulltext/ED037335.pdf

[2] A. Bogomolny, Pythagorean Theorem, https://www.cut-the-knot.org/pythagoras/index.shtml

[3] S J. Kolpas, The Pythagorean Theorem: Eight Classic Proofs, *Dale Seymour Publications* (1992)

[4] N D. Jackson and C R. Johnson, An Impossible Proof Of Pythagoras, *AMS Special Session on Undergraduate Mathematics and Statistics Research* (2023), https://meetings.ams.org/math/spring2023se/meetingapp.cgi/Paper/23621

[5] K. Basu, A New and Very Long Proof of the Pythagoras Theorem By Way of a Proposition on Isosceles Triangles, *Munich Personal RePEc Archive* (2015), https://mpra.ub.uni-muenchen.de/61125/1/MPRA_paper_61125.pdf

[6] S J. Kolpas, Mathematical Treasure: James A. Garfield's Proof of the Pythagorean Theorem, *Convergence* (2016), https://maa.org/press/periodicals/convergence/mathematical-treasure-james-a-garfields-proof-of-the-pythagorean-theorem

[7] J. Zimba, On the Possibility of Trigonometric Proofs of the Pythagorean Theorem, *Forum Geometricorum (Volume 9)* (2009), https://forumgeom.fau.edu/FG2009volume9/FG200925.pdf

[8] N. Luzia, Other trigonometric proofs of Pythagoras theorem, *arXiv* (2004), https://arxiv.org/abs/1502.06628

[9] I. Maxtone-Graham, Catch 'em If You Can, *The Simpsons: Season 15, Episode 18* (2004)

[10] I M. Gelfand and M. Saul, Trigonometry, *Birkhäuser* (2001)