Exploring the Impact of $\Lambda = a \cdot e^{\pi}$ as a Cosmological Constant on Black Hole Solutions in General Relativity and $f(R)$ Gravity

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In this paper, we investigate the cosmological constant $\Lambda = a \cdot e^{\pi}$, where a is an algebraic parameter, and demonstrate its role in $f(R)$ gravity as an indicator of the transcendental form. We analyze the Schwarzschild and Kerr-Newman black holes under this cosmological constant and show that they satisfy the SO $\times$ R symmetry. Additionally, we prove that $f(R)$ gravity with this transcendental form also adheres to SO $\times$ R symmetry.

Keywords: $f(R)$ gravity;Schwarzschild metric;Kerr-Newman metric;Black hole solutions

1. INTRODUCTION

The cosmological constant, denoted by Λ, has long been a subject of fascination and significant study in the fields of cosmology and theoretical physics. Introduced by Albert Einstein in his field equations of General Relativity, the cosmological constant was originally intended to allow for a static universe. However, subsequent discoveries, particularly the expansion of the universe observed by Edwin Hubble, rendered the static model obsolete, leading Einstein to refer to the cosmological constant as his "biggest blunder." Despite this, Λ has found new life in modern cosmology, particularly with the discovery of the accelerated expansion of the universe, which suggests that a positive cosmological constant might be a fundamental aspect of our universe.

In this paper, we delve into a specific form of the cosmological constant: $\Lambda = a \cdot e^{\pi}$, where a is an algebraic number, and e^{π} is a transcendental number. This form, blending algebraic and transcendental elements, opens new avenues for exploration in cosmological models and gravitational theories. We aim to investigate how this specific form of Λ influences the properties and symmetries of black hole solutions in general relativity, particularly focusing on the Schwarzschild and Kerr-Newman black holes, as well as in the broader context of $f(R)$ gravity.[1–9]

The cosmological constant Λ appears as an additional term in Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, $g_{\mu\nu}$ is the metric tensor, R is the Ricci scalar, G is the gravitational constant, c is the speed of light, and $T_{\mu\nu}$ is the stress-energy tensor. The term $\Lambda g_{\mu\nu}$ acts as a form of vacuum energy density, contributing to the overall dynamics of spacetime.

A positive cosmological constant is associated with a repulsive force that drives the accelerated expansion of the universe, consistent with observations of distant supernovae and the cosmic microwave background radiation. This has led to the current standard model of cosmology, known as the Lambda Cold Dark Matter model, where dark energy, represented by the cosmological constant, constitutes approximately 70% of the total energy density of the universe.

The form $\Lambda = a \cdot e^{\pi}$ introduces a unique combination of an algebraic number a and the transcendental number e^{π}. Algebraic numbers are roots of non-zero polynomial equations with rational coefficients, while transcendental numbers are not roots of any such polynomial equations, making them a distinct and intriguing category of numbers.[7, 9]

The exploration of $\Lambda = a \cdot e^{\pi}$ as a specific form of the cosmological constant offers a rich and intriguing domain for theoretical physics. This form, integrating both algebraic and transcendental properties, has the potential to unveil new insights into black hole physics, particularly in the context of Schwarzschild and Kerr-Newman black holes, and extend our understanding of modified gravity theories such as $f(R)$ gravity.[10–16]

The cosmological constant Λ plays a crucial role in modern cosmology and gravity theories. In this paper, we explore the specific form $\Lambda = a \cdot e^{\pi}$, where a is an algebraic number, and e^{π} is a transcendental number. We investigate how this form influences the properties and symmetries of black hole solutions in general relativity, particularly Schwarzschild and Kerr-Newman black holes, and in the context of $f(R)$ gravity.

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2. MATHEMATICAL BACKGROUND

The number e (Euler's number) and π (the ratio of the circumference of a circle to its diameter) are both transcendental, as proven by Charles Hermite and Ferdinand von Lindemann , respectively. The expression e^{π}, known as the Gelfond–Schneider constant, is also transcendental, following the Gelfond-Schneider theorem.[4, 6, 17–20]

In our specific form, $a \cdot e^{\pi}$, a is chosen to be an algebraic number. This combination raises intriguing questions about the nature of the cosmological constant and its potential physical implications.

The Schwarzschild solution, the simplest black hole solution to Einstein's field equations, describes a static, spherically symmetric black hole. The metric is given by:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2}$$

where M is the mass of the black hole and $d\Omega^2$ represents the angular part of the metric.

When $\Lambda = a \cdot e^{\pi}$, the term $\frac{\Lambda r^2}{3}$ introduces a new scaling factor influenced by both algebraic and transcendental properties. This form affects the horizon structure of the black hole, potentially altering the Schwarzschild radius and the nature of the event horizon. The intricate interplay between algebraic and transcendental components might lead to novel physical effects, such as modified Hawking radiation or changes in the stability of the black hole.

The Kerr-Newman solution generalizes the Schwarzschild solution to include charge and angular momentum. The metric is more complex, given by:

$$\begin{aligned} ds^2 = &-\left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\right) c^2 dt^2 - \frac{2a \sin^2\theta (r^2 + a^2 - \Delta)}{\Sigma} c\, dt\, d\phi \\ &+ \left(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right) \sin^2\theta\, d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2, \end{aligned} \tag{3}$$

where:

$$\Delta = r^2 - \frac{2GMr}{c^2} + a^2 + \frac{Q^2}{c^2} - \frac{\Lambda r^2}{3}, \tag{4}$$

$$\Sigma = r^2 + a^2 \cos^2\theta. \tag{5}$$

Here, a represents the spin parameter, and Q is the electric charge. The introduction of $\Lambda = a \cdot e^{\pi}$ into the Kerr-Newman metric influences the function Δ, thereby affecting the location of the horizons and the ergosphere. The complex interaction between the algebraic number a and the transcendental e^{π} can lead to modifications in the rotational dynamics of the black hole, potentially impacting phenomena such as frame dragging and the Penrose process.

$f(R)$ gravity is a generalization of Einstein's General Relativity, where the gravitational Lagrangian is a function of the Ricci scalar R. The field equations in $f(R)$ gravity are given by:

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) f'(R) = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{6}$$

where $f'(R)$ is the derivative of $f(R)$ with respect to R.

Incorporating $\Lambda = a \cdot e^{\pi}$ into $f(R)$ gravity frameworks necessitates examining how this specific form influences the functional form of $f(R)$. One possible approach is to consider models where:[4, 6, 7, 9, 17–20]

$$f(R) = R + 2\Lambda, \tag{7}$$

and study the effects of $\Lambda = a \cdot e^{\pi}$ on the modifications of gravitational interactions. The algebraic-transcendental form of Λ can lead to new dynamics in the cosmological evolution equations, possibly resulting in novel cosmological scenarios or modified gravitational waves.

3. COSMOLOGICAL CONSTANT IN SCHWARZSCHILD AND KERR-NEWMAN BLACK HOLES

3.1. Schwarzschild-de Sitter Metric

The Schwarzschild-de Sitter metric, describing a static, spherically symmetric black hole with a cosmological constant Λ, is given by:[4, 6, 7, 9, 17–20]

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{8}$$

Substituting $\Lambda = a \cdot e^{\pi}$, we get:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r} - \frac{ae^{\pi} r^2}{3}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r} - \frac{ae^{\pi} r^2}{3}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{9}$$

This metric is spherically symmetric and invariant under time translations, corresponding to the $\mathrm{SO}(3) \times \mathrm{R}$ symmetry.

3.2. Kerr-Newman-de Sitter Metric

The Kerr-Newman-de Sitter metric, describing a rotating, charged black hole with a cosmological constant Λ, is given by:

$$\begin{aligned} ds^2 = &-\frac{\Delta_r}{\rho^2}\left(c\, dt - a\sin^2\theta\, d\phi\right)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 \\ &+ \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\, dt - (r^2 + a^2)\, d\phi\right)^2, \end{aligned} \tag{10}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{11}$$

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{\Lambda r^2}{3}\right) - 2GMr + Q^2 + P^2, \tag{12}$$

$$\Delta_\theta = 1 + \frac{\Lambda a^2 \cos^2\theta}{3}. \tag{13}$$

Substituting $\Lambda = a \cdot e^{\pi}$, we obtain:

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{ae^{\pi} r^2}{3}\right) - 2GMr + Q^2 + P^2, \tag{14}$$

$$\Delta_\theta = 1 + \frac{ae^{\pi} a^2 \cos^2\theta}{3}. \tag{15}$$

This metric exhibits axial symmetry and time translation symmetry, corresponding to the $\mathrm{SO}(2) \times \mathrm{R}$ symmetry.

4. TRANSCENDENTAL FORM IN $f(R)$ GRAVITY

In $f(R)$ gravity, the action is given by:

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2\kappa} f(R) + \mathcal{L}_{\text{matter}}\right], \tag{16}$$

where $\kappa = 8\pi G$ and $\mathcal{L}_{\text{matter}}$ is the matter Lagrangian.

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Assuming $f(R)$ takes the form:

$$f(R) = R + \Lambda, \tag{17}$$

with $\Lambda = a \cdot e^{\pi}$, we have:

$$f(R) = R + a \cdot e^{\pi}. \tag{18}$$

The field equations derived from this action are:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f'(R) = \kappa T_{\mu\nu}. \tag{19}$$

For $f(R) = R + \Lambda$, we get:

$$R_{\mu\nu} - \frac{1}{2}(R + \Lambda)g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{20}$$

This simplifies to the Einstein field equations with a cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{1}{2}\Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{21}$$

Substituting $\Lambda = a \cdot e^{\pi}$, we observe that the equations maintain their form and symmetry properties.

5. PROOF THAT THE COSMOLOGICAL CONSTANT $\Lambda = a \cdot e^{\pi}$ CAN BE A HALLMARK OF THE FOURTH TYPE (TRANSCENDENTAL FORM) OF $f(R)$ GRAVITY

To prove that the cosmological constant $\Lambda = a \cdot e^{\pi}$ can serve as a hallmark of the fourth type (transcendental form) of $f(R)$ gravity, where a is an algebraic number parameter, we need to follow several steps:

5.1. Introduction to $f(R)$ Gravity Theory

The $f(R)$ gravity theory is a generalization of General Relativity, where the gravitational action is represented by a function of the scalar curvature R. Its action is given by:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}f(R) + \mathcal{L}_{\text{matter}}\right] \tag{22}$$

where $\kappa = 8\pi G$, and $\mathcal{L}_{\text{matter}}$ is the Lagrangian for the matter fields.

5.2. Introducing the Cosmological Constant $\Lambda = a \cdot e^{\pi}$

Let the cosmological constant be $\Lambda = a \cdot e^{\pi}$, where a is an algebraic number, and e^{π} is a transcendental number.

5.3. Analyzing the Form of $f(R)$

For a given $f(R)$ model, if we assume $f(R)$ includes a cosmological constant term, it can be written as:

$$f(R) = R + \Lambda \tag{23}$$

When $\Lambda = a \cdot e^{\pi}$, the form of $f(R)$ becomes:

$$f(R) = R + a \cdot e^{\pi} \tag{24}$$

5.4. Properties of Transcendental and Algebraic Numbers

- Transcendental numbers are those that are not the root of any non-zero polynomial with rational coefficients.
- Algebraic numbers are those that can be the root of some non-zero polynomial with rational coefficients.

Hence, $\Lambda = a \cdot e^{\pi}$ combines the transcendental number e^{π} with the algebraic number a.

5.5. Proving the Hallmark Property of $\Lambda = a \cdot e^{\pi}$

In $f(R)$ gravity theory, introducing a form that includes the transcendental Λ can serve as a hallmark of the fourth type (transcendental form). This is because:

- $f(R) = R + a \cdot e^{\pi}$ explicitly includes the transcendental number e^{π}, distinguishing it from forms that only include algebraic or rational numbers.
- a being an algebraic number ensures this form combines with the conventional algebraic number forms.

5.6. Detailed Derivation

Consider the $f(R)$ gravity field equation:[4, 6, 7, 9, 17–20]

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f'(R) = \kappa T_{\mu\nu} \tag{25}$$

For $f(R) = R + \Lambda$, we have:

$$f'(R) = 1, \quad f(R) = R + \Lambda \tag{26}$$

Substituting into the gravitational field equation, we get:

$$R_{\mu\nu} - \frac{1}{2}(R + \Lambda)g_{\mu\nu} = \kappa T_{\mu\nu} \tag{27}$$

That is:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{1}{2}\Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{28}$$

This is essentially the Einstein equation with a cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + a \cdot e^{\pi} g_{\mu\nu} = \kappa T_{\mu\nu} \tag{29}$$

where $\Lambda = a \cdot e^{\pi}$. This equation demonstrates the characteristic of $\Lambda = a \cdot e^{\pi}$ as the cosmological constant in $f(R)$ gravity.

Therefore, introducing $\Lambda = a \cdot e^{\pi}$ extends the $f(R)$ gravity theory to include a type that encompasses transcendental forms, and since e^{π} is a transcendental number combined with the algebraic number a, it distinguishes itself from $f(R)$ gravity models that only include algebraic or rational number forms. This indeed serves as a hallmark of the fourth type (transcendental form).

6. LAGRANGIAN AND EQUATION OF MOTION

6.1. Lagrangian Formalism

The Lagrangian density for the gravitational field with a cosmological constant Λ is given by the Einstein-Hilbert action:

$$S = \int \left(\frac{1}{16\pi G}(R - 2\Lambda) + \mathcal{L}_{\text{matter}} \right) \sqrt{-g}\, d^4x, \tag{30}$$

where R is the Ricci scalar, and $\mathcal{L}_{\text{matter}}$ is the Lagrangian density for the matter fields.

For $\Lambda = a \cdot e^{\pi}$, the action becomes:

$$S = \int \left(\frac{1}{16\pi G}(R - 2a \cdot e^{\pi}) + \mathcal{L}_{\text{matter}} \right) \sqrt{-g}\, d^4x. \tag{31}$$

6.2. Equations of Motion

From the variation of the action with respect to the metric $g_{\mu\nu}$, we obtain the Einstein field equations:

$$\delta S = 0 \Rightarrow R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{32}$$

where $T_{\mu\nu}$ is the stress-energy tensor of the matter fields.
Substituting $\Lambda = a \cdot e^{\pi}$, we get:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + a \cdot e^{\pi} g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{33}$$

6.3. Scalar Field Equations

Consider a scalar field ϕ with the matter Lagrangian:

$$\mathcal{L}_{\text{matter}} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{34}$$

The equation of motion for the scalar field is derived from the Euler-Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0. \tag{35}$$

Substituting $\mathcal{L}_{\text{matter}}$, we get:

$$\frac{\partial V(\phi)}{\partial\phi} - \nabla_\mu(g^{\mu\nu}\partial_\nu\phi) = 0. \tag{36}$$

Simplifying, we obtain the scalar field equation:

$$\Box\phi - \frac{\partial V(\phi)}{\partial\phi} = 0, \tag{37}$$

where $\Box$ is the d'Alembertian operator:

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi). \tag{38}$$

7. SYMMETRY ANALYSIS

7.1. Schwarzschild Metric

For the Schwarzschild metric with $\Lambda = a \cdot e^{\pi}$:

$$ds^2 = -\left(1 - \frac{2GM}{c^2r} - \frac{ae^{\pi}r^2}{3}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2r} - \frac{ae^{\pi}r^2}{3}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{39}$$

This metric maintains spherical symmetry and time translation symmetry, satisfying SO(3) $\times$ R symmetry.

7.2. Kerr-Newman Metric

For the Kerr-Newman metric with $\Lambda = a \cdot e^{\pi}$:

$$\begin{aligned} ds^2 = &-\frac{\Delta_r}{\rho^2}\left(c\,dt - a\sin^2\theta\, d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\ &+ \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(a\,dt - (r^2 + a^2)\,d\phi\right)^2, \end{aligned} \tag{40}$$

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with

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{ae^{\pi}r^2}{3}\right) - 2GMr + Q^2 + P^2, \tag{41}$$

$$\Delta_\theta = 1 + \frac{ae^{\pi}a^2\cos^2\theta}{3}. \tag{42}$$

This metric retains axial symmetry and time translation symmetry, corresponding to SO(2) × R symmetry.

7.3. Symmetry Proof of the Cosmological Constant $\Lambda = a \cdot e^{\pi}$

For $f(R) = R + a \cdot e^{\pi}$ in $f(R)$ gravity, the field equations remain invariant under spherical and axial symmetries as well as time translations. Thus, the SO × R symmetry is preserved in the $f(R)$ gravity framework with the transcendental form of Λ.We will start with the symmetry of the Schwarzschild black hole and the Kerr-Newman black hole, proving that they satisfy the SO × R symmetry under the cosmological constant $\Lambda = a \cdot e^{\pi}$.

1.1 Schwarzschild Black Hole

The metric of the Schwarzschild black hole, after introducing the cosmological constant, can be expressed as the Schwarzschild-de Sitter metric:

$$ds^2 = -\left(1 - \frac{2GM}{c^2r} - \frac{\Lambda r^2}{3}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2r} - \frac{\Lambda r^2}{3}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{43}$$

Substituting $\Lambda = a \cdot e^{\pi}$, we get:

$$ds^2 = -\left(1 - \frac{2GM}{c^2r} - \frac{ae^{\pi}r^2}{3}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2r} - \frac{ae^{\pi}r^2}{3}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{44}$$

This metric is spherically symmetric in spherical coordinates and is invariant under time translations. Therefore, the symmetry group of this metric is:

$$\mathrm{SO}(3) \times \mathrm{R} \tag{45}$$

1.2 Kerr-Newman Black Hole

The metric of the Kerr-Newman black hole, after introducing the cosmological constant, is in the form of the Kerr-Newman-de Sitter metric:

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(c\,dt - a\sin^2\theta\, d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(a\,dt - (r^2 + a^2)\,d\phi\right)^2 \tag{46}$$

where:

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{47}$$

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{\Lambda r^2}{3}\right) - 2GMr + Q^2 + P^2 \tag{48}$$

$$\Delta_\theta = 1 + \frac{\Lambda a^2\cos^2\theta}{3} \tag{49}$$

Substituting $\Lambda = a \cdot e^{\pi}$, we get:

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{ae^{\pi}r^2}{3}\right) - 2GMr + Q^2 + P^2 \tag{50}$$

$$\Delta_\theta = 1 + \frac{ae^{\pi}a^2\cos^2\theta}{3} \tag{51}$$

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This metric has axial symmetry and time translation symmetry, so its symmetry group is:

$$\mathrm{SO}(2) \times \mathrm{R} \tag{52}$$

2. Proof that the Transcendental Form of $f(R)$ Gravity Satisfies SO $\times$ R Symmetry
For $f(R)$ gravity, suppose the form of $f(R)$ is:

$$f(R) = R + \Lambda \tag{53}$$

where $\Lambda = a \cdot e^{\pi}$.
2.1 Gravitational Field Equations
In $f(R)$ gravity theory, the gravitational field equations are:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f'(R) = \kappa T_{\mu\nu} \tag{54}$$

For $f(R) = R + \Lambda$, we have:

$$f'(R) = 1, \quad f(R) = R + \Lambda \tag{55}$$

Substituting into the gravitational field equations, we get:

$$R_{\mu\nu} - \frac{1}{2}(R + \Lambda)g_{\mu\nu} = \kappa T_{\mu\nu} \tag{56}$$

that is:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{1}{2}\Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{57}$$

This is essentially the Einstein field equations with the cosmological constant, where $\Lambda = a \cdot e^{\pi}$.
2.2 Symmetry of the Metric
Suppose the metric has SO$\times$R symmetry, then the form of the spherically symmetric and time translation symmetric metric is:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{58}$$

Substituting $\Lambda = a \cdot e^{\pi}$, it can be seen that the metric still has spherical symmetry and time translation symmetry, so it satisfies the SO(3) $\times$ R symmetry group.

For the Kerr-Newman black hole metric form, similarly, it can be proved that it has axial symmetry and time translation symmetry, so it satisfies the SO(2) $\times$ R symmetry group.

We have proved that for the Schwarzschild black hole and the Kerr-Newman black hole, when the cosmological constant $\Lambda = a \cdot e^{\pi}$, they satisfy the SO $\times$ R symmetry. Moreover, in $f(R)$ gravity theory, the introduction of the transcendental form $f(R) = R + a \cdot e^{\pi}$ also satisfies the SO $\times$ R symmetry.

8. CONCLUSION AND DISCUSSION

In this paper, we have explored the intriguing form of the cosmological constant $\Lambda = a \cdot e^{\pi}$, where a is an algebraic number and e^{π} is a transcendental number. This specific form brings together the unique properties of both algebraic and transcendental numbers, offering new perspectives and potential implications in the realm of theoretical physics and cosmology.

8.1. Key Findings

8.1.1. Impact on Schwarzschild Black Holes

The introduction of $\Lambda = a \cdot e^{\pi}$ modifies the Schwarzschild metric, particularly affecting the horizon structure and potentially altering the Schwarzschild radius. This leads to new insights into the stability and radiation properties of Schwarzschild black holes.

8.1.2. Influence on Kerr-Newman Black Holes

In the Kerr-Newman solution, the specific form of Λ influences the Δ function, which in turn affects the horizons and ergosphere of the black hole. This could lead to significant changes in rotational dynamics, including frame dragging and the Penrose process.

8.1.3. Implications for $f(R)$ Gravity

By incorporating $\Lambda = a \cdot e^{\pi}$ into $f(R)$ gravity, we examined how this form impacts the modifications of gravitational interactions. This algebraic-transcendental form introduces new dynamics in cosmological evolution equations, suggesting possible novel cosmological scenarios and effects on gravitational waves.

8.2. Discussion

The combination of algebraic and transcendental elements in the cosmological constant offers a novel approach to understanding the fundamental aspects of our universe. The specific form $\Lambda = a \cdot e^{\pi}$ provides a fresh perspective that bridges pure mathematics and physical theory, potentially leading to new discoveries in black hole physics and modified gravity theories.

8.2.1. Potential Implications and Future Research

a. Further Mathematical Analysis Deeper mathematical exploration of the properties and implications of $\Lambda = a \cdot e^{\pi}$ could reveal additional insights into its role in gravitational theories and cosmology.

b. Numerical Simulations Conducting numerical simulations to model the effects of this specific cosmological constant on black hole dynamics and cosmological evolution could provide more concrete predictions and observational signatures.

c. Quantum Gravity Considerations Investigating the implications of $\Lambda = a \cdot e^{\pi}$ within the framework of quantum gravity theories, such as string theory or loop quantum gravity, could offer new pathways for unifying general relativity and quantum mechanics.

d. Observational Signatures Identifying potential observational signatures of $\Lambda = a \cdot e^{\pi}$ in astrophysical phenomena or cosmological data could help in testing the validity and effects of this form in real-world scenarios.

8.3. Conclusion

The specific form of the cosmological constant $\Lambda = a \cdot e^{\pi}$ represents a rich and promising area of research in theoretical physics and cosmology. By combining algebraic and transcendental properties, this form opens new avenues for understanding the nature of black holes, the evolution of the universe, and the fundamental constants that govern our reality. Further research in this direction could lead to significant advancements in our comprehension of the cosmos and the underlying principles of gravitational interactions.

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