# PARITY OF THE CENTRAL PARTIAL QUOTIENT OF THE CONTINUED FRACTION EXPANSION OF $\sqrt{D}$

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Abstract. The *form* $D = (k)^2 + 1$, $k \in \mathbb{Z}^+$ is known to *generate* all positive non square integers $D$ having a continued fraction expansion $\sqrt{D} = [a_0; \overline{2a_0}]$. Another known *form* that generate all positive non square integers $D$ for the expansion of $\sqrt{D} = [a_0; \overline{a_1, a_1, 2a_0}]$ is $D = (k \cdot a_1^2 + k + \frac{a_1}{2})^2 + 2k \cdot a_1 + 1$, with $a_1$ and $k \in \mathbb{Z}^+$, and already we see that $a_1$ is restricted to even values for an integer solution $D$ to exist, and when $k = 0$, the period is not primitive (the shortest).

In this paper, a *form* that generate all non square integers $D$ for any given period $\ell = 2n + 1$ or $\ell = 2n$, $D$ and $n \in \mathbb{Z}^+$, will be provided. It will be shown that the partial quotients $a_i$ $(0 < i < n)$ can be given any positive value for a solution to exist, and that the only restriction, if any, is on the parity of the central quotient $a_n$. When $k = 0$, the period is sometimes not primitive and in three scenarios, it is never primitive. It will also be shown that for $k > 0$ the period is always primitive.

A study of all the sequences of continued fraction expansions of length $\ell$ where $a_n$ is incremented and the other partial quotients $a_i$ are fixed will be done, highlighting their *cyclic* nature.

## 1. Introduction

Many forms[1] that generate[2] non square integers $D$, where the continued fraction expansion of $\sqrt{D}$ has a specific period length, are known. Some of them are notorious like the Berstein families [1] with for example $D = ((2a + 1)^j + a)^2 + 2a + 1$ generating some numbers with period $6j$. Some forms are also known to generate all non square integer $D$ for a given period $\ell$, like the ones in the abstract.

Many, if not all of these forms will make use of patterns between the partial quotients $a_i$ to reduce the number of variable used in the form. For instance, when $a_i = a_1$ for $1 < i < \ell$, like in the case of $\ell = 2$ or $\ell = 3$, we can use univariate Fibonacci polynomials $F_\ell(a_1)$ or Pell numbers. For $\ell = 3$, the form looks like $\left(\frac{2k \cdot F_3(a_1) + a1}{2}\right)^2{}_{+2k \cdot F_2(a_1)+1}$, but for the general case, the use of multivariate polynomials, or continuants, cannot be avoided.

In this context, a precise knowledge of admissible partial quotients, and therefore admissible continuants, is a requirement for the generation of *all* $\sqrt{D}$ having a continued fraction of any given period.

### 1.1. structure.

The first **section** contains some definitions and a few global remarks. In **section 2**, it will be shown that for a set of partial quotients $a_i$ with $0 < i < \ell$, the $a_0$ quotient cannot be arbitrary. A *form*, based on a system of linear equations, that generate all non square integers $D$ for any given period $\ell = 2n + 1$ or $\ell = 2n$, $D$ and $n \in \mathbb{Z}^+$ will be provided. **Section 3**, shows that the central quotient $a_n$ cannot be arbitrary either, but unlike for $a_0$, if there is a restriction on $a_n$, it is only on its parity. **Section 4** will provide an alternative form in which the variable $\omega$ has a central role. In **section 5**, The case $k = 0$ where the period is not always primitive will be examined, and we will see when it is strictly non-primitive. It will be shown that for $k > 0$ the period is always primitive. Some inequalities related to $p_{\ell-1} + q_{\ell-1}\sqrt{D}$ will also be given for $k = 0$ and $k \neq 0$.

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[1] equation $\sqrt{D} = [a_0; a_1, a_2...]$ expressed as a quadratic equation, or a closed form

[2] what Berstein [1] calls *"construct"*

1.2. **some definitions.** Let the regular continued fraction expansion of the square root of a non square integer $D \in \mathbb{Z}^+$ be written $\sqrt{D} = [a_0; \overline{a_1, a_2, ..., a_{\ell-1}, 2a_0}]$ where $\ell$ is the length of the period, and its $v^{th}$ convergent is denoted $\frac{p_v}{q_v} = [a_0; a_1, ..., a_v]$. By this definition, $a_0$ is also a positive integer.

$$\sqrt{D} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

Since the continued fraction expansion is palindromic, the partial quotients $a_i = a_{\ell-i}$ for $0 < i < \ell$ (refer to Perron [7] for classical proofs). This paper will only consider period $\ell > 1$ unless specified.

*Remark* 1.1. The superscript notations ${q_v}^{+\mu}$ used in this paper is equivalent to Perron's notation $B_{v,\mu}$ in [7, p.14] where $\mu$ is what he calls an *indice shift*. They both translate to the simple continuant $K(a_{1+\mu}, \ldots, a_{v+\mu})$. The notation differs from exponents by a little $^+$ sign.

*Remark* 1.2. The use of **mod** as an operator giving the remainder in the euclidean division is to be viewed as a convenient way to represent the smallest non-negative residue ($a \bmod b$). It can be differentiated from congruences by the classical parenthesis notation and the use of an equal sign: $c = a \bmod b$ versus $c \equiv a \pmod b$. When the modulus and denominators have no common factors, a modular multiplicative inverse is sometimes used (as denominator).

1.3. **main results.** An integer solution $D$ to equation $\sqrt{D} = [a_0; \overline{a_1, ..., a_{\ell-1}, 2a_0}]$ can be found, for any period $\ell = 2n+1$ or $\ell = 2n$, $n \in \mathbb{Z}^+$, with any positive partial quotients $a_i$ $(0 < i < n)$. Theorem 3.4 shows that only $a_n$ may eventually have a constraint on its parity. We can generate all these solutions with $D = \left(a_0' + k\dfrac{\delta q_{\ell-1}}{2}\right)^2 + \alpha_0' + k\delta q_{\ell-2}$ where $k = 0$ gives the smallest of them. Theorem 5.2 shows that when $k > 0$, the period is always primitive, and theorem 5.4 shows that when $k = 0$, there is no expansion with a primitive period $\ell$ when $\ell = 2$ and $a_1$ is *even*, or when $\ell = 3$, or when $\ell = 4$ and $a_1 = 2$. The cyclic variable $\omega = \omega' + k\delta t = \dfrac{2a_0 \cdot s - \alpha_0 \cdot r}{{q_{\ell-3}}^{+1}} = \dfrac{2a_0 \cdot t - r}{q_{\ell-1}} = \dfrac{\alpha_0 \cdot t - s}{q_{\ell-2}} = m \cdot t - M = a_0 - A_n < \alpha_{n-1}$ will be introduced, where $\omega$ is periodic mod $\delta t$ with period $q_{n-1}$, $2q_{n-1}$ or $2{q_{n-1}}^2$ in a sequence of continued fraction expansions having the partial quotients $a_i$ $(0 < i < n)$ constants and $a_n$ incremented. A study of these sequences will be done, greatly improving our understanding of the partial quotients relation.

## 2. The general form

The *convergents* denominators $q_v$ are multivariate polynomials, or continuants, and their variables are the partial quotients $a_i$. For a given set of $a_i$ $(0 < i < \ell)$, *encoded* in $q_{\ell-1}$, $q_{\ell-2}$ and ${q_{\ell-3}}^{+1}$, the quotient $a_0$ cannot be random. This section determine all possible values of $a_0$ and $\alpha_0 = D - a_0^2$ for that set, giving a general form for the generation of all corresponding $D$. An alternative form will also be presented in section 4 with some surprising properties.

2.1. **A linear equation with two unknowns.**

This well known continuant matrix identity, using the *convergents* denominator notation

$$\begin{pmatrix} q_{\ell-1} & q_{\ell-2} \\ {q_{\ell-2}}^{+1} & {q_{\ell-3}}^{+1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{\ell-1} & 1 \\ 1 & 0 \end{pmatrix} \tag{2.1}$$

can be found by induction and was also used in Halter-Kock [4, p.1].
Another identity, from Poorten [8, p.110], and also found in Muir [6, p.234] gives

$$\begin{pmatrix} Dq_{\ell-1} & p_{\ell-1} \\ p_{\ell-1} & q_{\ell-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_{\ell-1} & q_{\ell-2} \\ {q_{\ell-2}}^{+1} & {q_{\ell-3}}^{+1} \end{pmatrix} \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.2}$$

By developing the right hand side and using ${q_{\ell-2}}^{+1} = q_{\ell-2}$, found by quotients symmetry $a_i = a_{\ell-i}$ for $0 < i < \ell$ and by continuant reversibility $K(a_1, ..., a_m) = K(a_m, ..., a_1)$, we get

$$\begin{pmatrix} Dq_{\ell-1} & p_{\ell-1} \\ p_{\ell-1} & q_{\ell-1} \end{pmatrix} = \begin{pmatrix} a_0^2 q_{\ell-1} + 2a_0 q_{\ell-2} + {q_{\ell-3}}^{+1} & a_0 q_{\ell-1} + q_{\ell-2} \\ a_0 q_{\ell-1} + q_{\ell-2} & q_{\ell-1} \end{pmatrix} \tag{2.3}$$

Taking the determinants of matrix (2.1) and the top left element of matrix (2.3) we get the following system of linear equations with unknowns $(D - a_0^2) = \alpha_0$, and $2a_0$

$$q_{\ell-1}{q_{\ell-3}}^{+1} - q_{\ell-2}q_{\ell-2} = (-1)^{\ell+1} \tag{2.4}$$
$${q_{\ell-3}}^{+1} = q_{\ell-1}(D - a_0^2) - 2a_0 q_{\ell-2} \tag{2.5}$$

for which solutions are

$$2a_0 = mq_{\ell-1} + (-1)^{\ell+1} q_{\ell-2}{q_{\ell-3}}^{+1} \tag{2.6}$$
$$D - a_0^2 = mq_{\ell-2} + (-1)^{\ell+1}\left({q_{\ell-3}}^{+1}\right)^2 \tag{2.7}$$

where $m \in \mathbb{Z}$. These solutions are also found in Muir [5, §53 p.29] or Perron [7, §25 p.98]. From these solutions, we now have this equation for $D$

$$D = \left(\frac{mq_{\ell-1} + (-1)^{\ell+1} q_{\ell-2}{q_{\ell-3}}^{+1}}{2}\right)^2 + mq_{\ell-2} + (-1)^{\ell+1}\left({q_{\ell-3}}^{+1}\right)^2 \tag{2.8}$$

Multiplying equation (2.7) by $q_{\ell-2}$ and subtracting equation (2.6) multiplied by ${q_{\ell-3}}^{+1}$ and using equation (2.4) gives

$$\alpha_0 q_{\ell-2} - 2a_0 {q_{\ell-3}}^{+1} = (-1)^{\ell} m \tag{2.9}$$

*Remark* 2.1. (2.7)$\leqslant$ (2.6) since $D < (a_0 + 1)^2$.

*Remark* 2.2. As Muir noted in [6, §21 p.235], The expansion of $\sqrt{D}$ *"may always be reduced to a periodic continued fraction with only three elements in its period"*, which is this non *simple* continued fraction (leading to equation (2.5) when expanded):

$$\sqrt{D} = a_0 + \cfrac{{q_{\ell-3}}^{+1}}{q_{\ell-2} + \cfrac{(-1)^{\ell+1}}{q_{\ell-2} + \cfrac{{q_{\ell-3}}^{+1}}{\sqrt{D} + a_0}}}$$

**Lemma 2.3.** *Equation (2.8) has no integer solution if and only if the quantity $q_{\ell-2}{q_{\ell-3}}^{+1}$ is odd.*

*Proof.* If $q_{\ell-1}$ is *odd*, equation (2.4) shows that $q_{\ell-2}$ and ${q_{\ell-3}}^{+1}$ must be *even*, which implies from equation (2.6), that $m$ is even. This also implies that the quantity $mq_{\ell-1}$ is always even. From equation (2.8), we can conclude that $q_{\ell-2}{q_{\ell-3}}^{+1}$ must always be even for $D$ to be an integer. □

This result was shown by Muir in [5, §55 p.31] and was also shown by Friesen in [3, p.11], and likewise, $m$ will be replaced by $b$ or $2b$ depending on the parity of $q_{\ell-1}$ as follow

$$\text{Let } \delta = \begin{cases} 1 & \text{if } q_{\ell-1} \equiv 0 \pmod 2 \\ 2 & \text{if } q_{\ell-1} \equiv 1 \pmod 2 \end{cases}$$

Setting $m = \delta b$ allows to work with $b \in \mathbb{Z}$ with unrestricted parity. Equation (2.8) becomes

$$D = \left(b\frac{\delta q_{\ell-1}}{2} + \frac{(-1)^{\ell+1} q_{\ell-2}{q_{\ell-3}}^{+1}}{2}\right)^2 + b\delta q_{\ell-2} + (-1)^{\ell+1}\left({q_{\ell-3}}^{+1}\right)^2 \tag{2.10}$$

**Lemma 2.4.** *With equation (2.10) written as $D = (a_0)^2 + \alpha_0$, $D \in \mathbb{Z}^+$ a non square integer and $\ell > 2$, let $a'_0 = a_0 \bmod \dfrac{\delta q_{\ell-1}}{2}$, and let $\alpha'_0 = \alpha_0 \bmod \delta q_{\ell-2}$. If the partial quotients $a_i$ for $0 < i < \ell$ are set to arbitrary fixed values having solutions, then $D' = (a'_0)^2 + \alpha'_0$ is the smallest of them. It will have a continued fraction expansion $\sqrt{D'} = [a'_0; \overline{a_1, a_2, ..., a_{\ell-1}, 2a'_0}]$.*

*Proof.* By fixing all partial quotients $a_i$ for $0 < i < \ell$, the quantities $q_{\ell-1}$, $q_{\ell-2}$, and ${q_{\ell-3}}^{+1}$ will also be fixed, since they depend only on those partial quotients. The only variable in equation (2.10) becomes $b$. By modulo, $a'_0$ is therefore the minimum positive (see remark (2.6) and (2.9)) value that $a_0$ could eventually take, and $\alpha'_0$ is the minimum positive value that $\alpha_0$ could eventually take. Establishing they are part of the same expansion will complete the proof: From this classical identity, in Poorten [8, p.104], which uses matrix (2.1)

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_{\ell-1} & q_{\ell-2} \\ {q_{\ell-2}}^{+1} & {q_{\ell-3}}^{+1} \end{pmatrix} = \begin{pmatrix} p_{\ell-1} & p_{\ell-2} \\ q_{\ell-1} & q_{\ell-2} \end{pmatrix}$$

and inserting the top right element of the matrix in equation (2.5), we find

$$\begin{aligned} p_{\ell-2} &= a_0 q_{\ell-2} + {q_{\ell-3}}^{+1} \\ p_{\ell-2} &= q_{\ell-1}(D - a_0^2) - a_0 q_{\ell-2} \end{aligned}$$

rewritten as

$$\begin{aligned} p_{\ell-2} &\geqslant a_0 q_{\ell-2} \\ D - a_0^2 &= \frac{p_{\ell-2} + a_0 q_{\ell-2}}{q_{\ell-1}} \end{aligned}$$

then if $\alpha_0 < \delta q_{\ell-2}$, we have

$$\begin{aligned} \alpha_0 = D - a_0^2 &= \frac{p_{\ell-2} + a_0 q_{\ell-2}}{q_{\ell-1}} < \delta q_{\ell-2} \\ \implies 2a_0 &\leqslant \frac{p_{\ell-2}}{q_{\ell-2}} + a_0 < \delta q_{\ell-1} \\ \implies a_0 &< \frac{\delta q_{\ell-1}}{2} \end{aligned}$$

so that $\alpha'_0 < \delta q_{\ell-2}$ implies $a'_0 < \dfrac{\delta q_{\ell-1}}{2}$, and since $b$ is an integer in equation (2.10), there is no other $\alpha_0$ smaller than $\delta q_{\ell-2}$ and no other $a_0$ smaller than $\dfrac{\delta q_{\ell-1}}{2}$. They therefore are part of the same expansion. □

Set $k = b - b'$ where $b'$ is the value of $b$ leading to $a'_0$ and $\alpha'_0$. Equation (2.10) becomes

$$D = \left(a'_0 + k\frac{\delta q_{\ell-1}}{2}\right)^2 + \alpha'_0 + k\delta q_{\ell-2} \tag{2.11}$$

With

$$\begin{cases} a_0 = a'_0 + k\dfrac{\delta q_{\ell-1}}{2} & (2.12) \\ \alpha_0 = \alpha'_0 + k\delta q_{\ell-2} & (2.13) \end{cases}$$

where $k \geqslant 0$ is also an integer, and assuming $\ell > 2$ we have from equation (2.10), (2.12) and (2.13)

$$\begin{cases} a'_0 = \dfrac{(-1)^{\ell+1} q_{\ell-2} {q_{\ell-3}}^{+1}}{2} \bmod \dfrac{\delta q_{\ell-1}}{2} & (2.14) \\ \alpha'_0 = (-1)^{\ell+1} \left({q_{\ell-3}}^{+1}\right)^2 \bmod \delta q_{\ell-2} & (2.15) \end{cases}$$

*Remark* 2.5. variables associated to the smallest solution $D'$ are marked with an apostrophe.

*Remark* 2.6. For $\ell = 2$, since ${q_{\ell-3}}^{+1} = 0$, the smallest solutions to equation (2.6) and (2.7) are $a'_0 = \dfrac{\delta q_{\ell-1}}{2}$ and $\alpha'_0 = \delta q_{\ell-2}$, but they cannot be expressed with the above modulo which would reduce them to 0 (since $D$ and $a_0$ are positive and $D = {a_0}^2 + \alpha_0$ is non-square, $\alpha_0$ must be positive).

**Proposition 2.7.** *For $\ell > 2$ we have* $a'_0 = \left(\dfrac{-{q_{\ell-3}}^{+1}}{2q_{\ell-2}}\right) \bmod \dfrac{\delta q_{\ell-1}}{2}$

*Proof.* By starting from equation (2.5) $\alpha'_0 \cdot q_{\ell-1} = 2a'_0 \cdot q_{\ell-2} + {q_{\ell-3}}^{+1}$ and taking the modulo. From equation (2.4) and lemma 2.3, we know that ${q_{\ell-3}}^{+1}$ *odd* implies $q_{\ell-2}$ *even* and $q_{\ell-1}$ *odd*, which means $\gcd(2q_{\ell-2}, \frac{\delta q_{\ell-1}}{2}) = 1$, and if ${q_{\ell-3}}^{+1}$ is *even*, the 2 in the denominator vanishes and $\gcd(q_{\ell-2}, \frac{\delta q_{\ell-1}}{2}) = 1$. In any case, the modulus and the denominator have no common factor, which allows the use of fractions, or multiplicative inverse, in the modulo. ☐

**Proposition 2.8.** *For $\ell > 2$ we have* $\alpha'_0 = \left(\dfrac{{q_{\ell-3}}^{+1}}{q_{\ell-1}}\right) \bmod \delta q_{\ell-2}$

*Proof.* By starting from equation (2.5) $\alpha'_0 \cdot q_{\ell-1} = 2a'_0 \cdot q_{\ell-2} + {q_{\ell-3}}^{+1}$ and taking the modulo. Since $\gcd(q_{\ell-1}, \delta q_{\ell-2}) = 1$, the modulus and the denominator have no common factor, which allows the use of fractions, or multiplicative inverse, in the modulo. ☐

*Remark* 2.9. In the above propositions, zero has no multiplicative inverse and ${q_{\ell-3}}^{+1} > 0$ when $\ell > 2$, so $a'_0 > 0$ and $\alpha'_0 > 0$

This leads to a simplified general form for $\ell > 2$, $k \geqslant 0$

$$D = \left(\left(\frac{-{q_{\ell-3}}^{+1}}{2q_{\ell-2}}\right) \bmod \frac{\delta q_{\ell-1}}{2} + k\frac{\delta q_{\ell-1}}{2}\right)^2 + \left(\frac{{q_{\ell-3}}^{+1}}{q_{\ell-1}}\right) \bmod \delta q_{\ell-2} + k\delta q_{\ell-2} \tag{2.16}$$

## 3. Solutions to $\sqrt{D} = [a_0; \overline{a_1, a_2, ..., a_{\ell-1}, 2a_0}]$

As Muir stated in [5, §55 p.31]: *"No integer can be found whose square root when expressed as a continued fraction with unit-numerators has $q_1, q_2, \ldots, q_2, q_1$ for the symmetric portion of its cycle of partial denominators, unless either $K(q_1 \ldots q_2)$ or $K(q_2 \ldots q_2)$ be even".* This was proven in lemma 2.3.

This section will show an equivalent proposition that for period $\ell = 2n + 1$ or $\ell = 2n$, $n \in \mathbb{Z}^+$, the partial quotients $a_i$ for $0 < i < n$, and by symmetry, for $l - n < i < l$, can be given any positive value for a solution to exist, and the only restriction, if any, is on the parity of the central quotient $a_n$.

### 3.1. prerequisites.

Using Euler's identity for continuants, see Muir [6, p.230] or Perron [7, p.18 formula 36], and using Perron's indice naming with the addition of what he calls an *indice shift* '$\mu$', like in [7, p.17 formula 35], with $\alpha, \beta, \gamma, \mu$ positive integers

$$\begin{aligned}
&K(a_\mu, \ldots, a_{\alpha+\beta+\gamma+\mu-1}) \cdot K(a_{\alpha+\mu+1}, \ldots, a_{\alpha+\beta+\mu-1}) \\
&-K(a_\mu, \ldots, a_{\alpha+\beta+\mu-1}) \cdot K(a_{\alpha+\mu+1}, \ldots, a_{\alpha+\beta+\gamma+\mu-1}) \\
&=(-1)^{\beta-1} K(a_\mu, \ldots, a_{\alpha+\mu-1}) \cdot K(a_{\alpha+\beta+\mu+1}, \ldots, a_{\alpha+\beta+\gamma+\mu-1})
\end{aligned} \tag{3.1}$$

a series of identities will be produced, in *convergents* notation. Cases where $\alpha, \beta$ and/or $\gamma$ are not positive can be verified using ${q_0}^{+h} = 1$, ${q_{-1}}^{+h} = 0$, and ${q_{-2}}^{+h} = 1$ for any integer $h$.

*Remark* 3.1. The middle term $a_n$ is appearing once for $\ell$ even and twice for $\ell$ odd, there is a partial quotient symmetry $a_i = a_{l-i}$ for $0 < i < l$, and continuants are *reversible* $K(a_1, ...a_m) = K(a_m, ...a_1)$.

**Identities for $\ell = 2n+1$, $n \in \mathbb{Z}^+$**

(3.2) $q_{\ell-1} = (q_n)^2 + (q_{n-1})^2$ ($\alpha{=}n{-}1$, $\beta{=}1$, $\gamma{=}n$, $\mu{=}1$)

(3.3) $q_{\ell-2} = q_n {q_{n-1}}^{+1} + q_{n-1} {q_{n-2}}^{+1}$ ($\alpha{=}n{-}1$, $\beta{=}1$, $\gamma{=}n{-}1$, $\mu{=}1$)

(3.4) $q_n = a_n q_{n-1} + q_{n-2}$ ($\alpha{=}n{-}2$, $\beta{=}1$, $\gamma{=}1$, $\mu{=}1$)

(3.5) ${q_{\ell-3}}^{+1} = ({q_{n-1}}^{+1})^2 + ({q_{n-2}}^{+1})^2$ ($\alpha{=}n{-}2$, $\beta{=}1$, $\gamma{=}n{-}1$, $\mu{=}2$)

**Identities for $\ell = 2n$, $n \in \mathbb{Z}^+$**

(3.6) ${q_{\ell-3}}^{+1} = {q_{n-2}}^{+1}({q_{n-1}}^{+1} + {q_{n-3}}^{+1})$ ($\alpha{=}n{-}3$, $\beta{=}1$, $\gamma{=}n{-}1$, $\mu{=}2$)

(3.7) $q_{\ell-2} = q_{n-1} {q_{n-1}}^{+1} + q_{n-2} {q_{n-2}}^{+1}$ ($\alpha{=}n{-}2$, $\beta{=}1$, $\gamma{=}n{-}1$, $\mu{=}1$)

(3.8) $q_{\ell-2} = q_n {q_{n-2}}^{+1} + q_{n-1} {q_{n-3}}^{+1}$ ($\alpha{=}n{-}1$, $\beta{=}1$, $\gamma{=}n{-}2$, $\mu{=}1$)

(3.9) $q_{n-2} {q_{n-2}}^{+1} - q_{n-1} {q_{n-3}}^{+1} = (-1)^n$ ($\alpha{=}0$, $\beta{=}n{-}2$, $\gamma{=}1$, $\mu{=}1$)

(3.10) $q_{\ell-2} = q_{n-1}({q_{n-1}}^{+1} + {q_{n-3}}^{+1}) + (-1)^n$ combining (3.7) and (3.9)

(3.11) ${q_{n-1}}^{+1} = a_n {q_{n-2}}^{+1} + {q_{n-3}}^{+1}$ ($\alpha{=}n{-}3$, $\beta{=}1$, $\gamma{=}1$, $\mu{=}2$)

(3.12) $q_{\ell-1} = q_{n-1}(q_n + q_{n-2})$ ($\alpha{=}n{-}2$, $\beta{=}1$, $\gamma{=}n$, $\mu{=}1$)

### 3.2. equivalences.

**Lemma 3.2.** *If $\ell = 2n+1$, $n \in \mathbb{Z}^+$, then the following two assertions are equivalent*

(1) $q_{\ell-2} {q_{\ell-3}}^{+1} \equiv 1 \pmod 2$
(2) *one of these statements is true:*
  (a) $q_{n-2} \equiv 0 \pmod 2$ *and* $a_n \equiv 1 \pmod 2$
  (b) $q_{n-1} q_{n-2} \equiv 1 \pmod 2$ *and* $a_n \equiv 0 \pmod 2$

*Proof.* **First we prove (1) $\implies$ (2):**
(1) and equation (2.4) force $q_{\ell-1}$ to be *even*, so that equation (3.2) gives $q_n \equiv q_{n-1} \pmod 2$, and equation (3.3) with $q_{\ell-2}$ *odd* force both $q_n$ and $q_{n-1}$ to be *odd*. This shows with equation (3.4) that $a_n \not\equiv q_{n-2} \pmod 2$, which proves (2).
**Now we prove (a) $\implies$ (1):**
(a) and equation (3.4) show that $q_n \equiv q_{n-1} \pmod 2$, which with equation (3.2) proves $q_{\ell-1}$ to be *even*. Since $p_{\ell-2}q_{\ell-1} - p_{\ell-1}q_{\ell-2} = \pm 1$, we have $\gcd(q_{\ell-1}, q_{\ell-2}) = 1$ so $q_{\ell-2}$ is *odd* and with equation (3.3) which forces both $q_n$ and $q_{n-1}$ to be *odd*, proves ${q_{n-1}}^{+1} \not\equiv {q_{n-2}}^{+1} \pmod 2$. Equation (3.5) then proves ${q_{\ell-3}}^{+1}$ is *odd* which completes the proof of (1).
**And we prove (b) $\implies$ (1):**
(b) and equation (3.4) force $q_n$ to be *odd*, which with equation (3.2) proves $q_{\ell-1}$ to be *even*. Since $p_{\ell-2}q_{\ell-1} - p_{\ell-1}q_{\ell-2} = \pm 1$, we have $\gcd(q_{\ell-1}, q_{\ell-2}) = 1$ so $q_{\ell-2}$ is *odd* and with equation (3.3) proves ${q_{n-1}}^{+1} \not\equiv {q_{n-2}}^{+1} \pmod 2$. Equation (3.5) then proves ${q_{\ell-3}}^{+1}$ is *odd* which completes the proof. □

**Lemma 3.3.** *If $\ell = 2n$, $n \in \mathbb{Z}^+$, then the following two assertions are equivalent*

(1) $q_{\ell-2} {q_{\ell-3}}^{+1} \equiv 1 \pmod 2$
(2) $q_{n-1} \equiv 0 \pmod 2$ *and* $a_n \equiv 1 \pmod 2$

*Proof.* **First we prove (1) $\implies$ (2):**
(1) implies ${q_{\ell-3}}^{+1}$ is *odd* so both sides of equation (3.6) are *odd*, meaning both side of equation (3.11) ${q_{n-1}}^{+1} - {q_{n-3}}^{+1} = a_n {q_{n-2}}^{+1}$ are *odd* too, proving $a_n$ *odd*. (1) also implies $q_{\ell-2}$ is *odd*, and with $({q_{n-1}}^{+1} + {q_{n-3}}^{+1})$ *odd* from equation (3.11), and by using equation (3.10), it proves $q_{n-1}$ *even* and (2).
**Now we prove (2) $\implies$ (1):**
$q_{n-1}$ *even* and equation (3.12) means $q_{\ell-1}$ is *even*, and since $p_{\ell-2}q_{\ell-1} - p_{\ell-1}q_{\ell-2} = \pm 1$, we have $\gcd(q_{\ell-1}, q_{\ell-2}) = 1$ so $q_{\ell-2}$ is *odd* and with equation (3.7), shows that $q_{n-2} {q_{n-2}}^{+1}$ is *odd*. With $a_n$ *odd*,

we see both sides of the equation (3.11) written as $a_n q_{n-2}{}^{+1} = q_{n-1}{}^{+1} - q_{n-3}{}^{+1}$ are *odd*, so both sides of equation (3.6) are *odd* and $q_{\ell-3}{}^{+1}$ is therefore also *odd*, proving (1). □

**Theorem 3.4.** *The partial quotients $a_i$ for $0 < i < n$, and by symmetry for $l - n < i < l$ can be given any positive integer value for a solution*[1] *to exist. The only restriction, if any, is on the parity of the central quotient $a_n$.*

*Proof.* From lemma 2.3, we see that lemma 3.3 and lemma 3.2 are describing scenarios with no integer solution to $\sqrt{D} = [a_0; \overline{a_1, a_2, ..., a_{\ell-1}, 2a_0}]$. Since $q_{n-1}$ and $q_{n-2}$ only depend on quotients $a_i$ in the range $0 < i < n$, they have fixed values for a given set of $a_i$ in that range, which implies that statements (a) and (b) in lemma 3.2 cannot both occur for the same set, so from lemma 3.2 for *odd* periods, with an eventual restriction on the parity of $a_n$, there is always solutions. From lemma 3.3, for *even* periods, when $q_{n-1}$ is *even*, there are solutions for *even* $a_n$, otherwise there is always a solution. □

**Corollary 3.5.** *For any period $\ell = 2n$ or $2n+1$, $n \in \mathbb{Z}^+$, and for any partial quotients $a_i \in \mathbb{Z}^+$ $(0 < i < n)$, there are infinitely many non square integers $D$, where the continued fraction expansion of $\sqrt{D}$ has period $\ell$ and these partial quotients.*

*Proof.* From theorem 3.4, and the fact that there are infinitely many possible $a_n$ (for any $k \geqslant 0$). Moreover, if any of these $a_n$ is fixed, there are still infinitely many solutions. Indeed, the equation (2.11) includes the variable $k \geqslant 0$ for infinitely many possible values of $a_0$. □

## 4. An alternative form

An alternative formulation of $a_0'$ and $\alpha_0'$ (for $\ell > 2$), only using partial quotients $a_i$ $(0 < i \leqslant n)$, can be set up with the variables $r, s, t$ defined bellow for $\ell$ *even* and $\ell$ *odd*

$$\ell = 2n \begin{cases} r = q_{n-1} q_{n-1}{}^{+1} - q_{n-1} q_{n-3}{}^{+1} \\ s = q_{n-1}{}^{+1} q_{n-2}{}^{+1} \\ t = q_{n-1} q_{n-2} \end{cases} \qquad \ell = 2n+1 \begin{cases} r = q_n q_{n-1}{}^{+1} - q_{n-1} q_{n-2}{}^{+1} \\ s = q_{n-1}{}^{+1} q_{n-1}{}^{+1} \\ t = q_{n-1} q_{n-1} \end{cases}$$

where $r, s, t$ are positive integers (verifiable with equation (3.1) using $(\beta=1,\ \gamma=1,\ \mu=1 \text{ or } 2)$).

First we show that

$$s \cdot q_{\ell-1} = r \cdot q_{\ell-2} + t \cdot q_{\ell-3}{}^{+1} \tag{4.1}$$

for $\ell = 2n$, using equations of subsection 3.1:

$$s \cdot q_{\ell-1} = r \cdot q_{\ell-2} + t \cdot q_{\ell-3}{}^{+1}$$
$$q_{n-1}{}^{+1} q_{n-2}{}^{+1} q_{n-1}(q_n + q_{n-2}) = r \cdot q_{\ell-2} + q_{n-1} q_{n-2} q_{n-2}{}^{+1}(q_{n-1}{}^{+1} + q_{n-3}{}^{+1})$$
$$q_{n-1}{}^{+1} q_{n-2}{}^{+1} q_{n-1} q_n = (q_{n-1} q_{n-1}{}^{+1} - q_{n-1} q_{n-3}{}^{+1}) q_{\ell-2} + q_{n-1} q_{n-2} q_{n-2}{}^{+1} q_{n-3}{}^{+1}$$
$$q_{n-1} q_{n-1}{}^{+1}(q_{n-2}{}^{+1} q_n - q_{\ell-2}) = -q_{n-1} q_{n-3}{}^{+1}(q_{n-1} q_{n-1}{}^{+1} + q_{n-2} q_{n-2}{}^{+1}) + q_{n-1} q_{n-2} q_{n-2}{}^{+1} q_{n-3}{}^{+1}$$
$$q_{n-1} q_{n-1}{}^{+1}(q_{n-2}{}^{+1} q_n - q_{\ell-2}) = -q_{n-1} q_{n-3}{}^{+1} q_{n-1} q_{n-1}{}^{+1}$$
$$q_{\ell-2} = q_n q_{n-2}{}^{+1} + q_{n-1} q_{n-3}{}^{+1} \quad \square$$

for $\ell = 2n+1$, using equations of subsection 3.1:

$$s \cdot q_{\ell-1} = r \cdot q_{\ell-2} + t \cdot q_{\ell-3}{}^{+1}$$
$$(q_{n-1}{}^{+1})^2((q_n)^2 + (q_{n-1})^2) = r \cdot q_{\ell-2} + (q_{n-1})^2((q_{n-1}{}^{+1})^2 + (q_{n-2}{}^{+1})^2)$$
$$(q_n q_{n-1}{}^{+1})^2 = (q_n q_{n-1}{}^{+1} - q_{n-1} q_{n-2}{}^{+1})(q_n q_{n-1}{}^{+1} + q_{n-1} q_{n-2}{}^{+1}) + (q_{n-1} q_{n-2}{}^{+1})^2$$
$$(q_n q_{n-1}{}^{+1})^2 = (q_n q_{n-1}{}^{+1})^2 - (q_{n-1} q_{n-2}{}^{+1})^2 + (q_{n-1} q_{n-2}{}^{+1})^2 \quad \square$$

By setting

$$\omega' = \left((-1)^{\ell+1}[q_{\ell-2} \cdot s - q_{\ell-3}{}^{+1} \cdot r]\right) \bmod \delta t \tag{4.2}$$

[1] integer solution $D$ to equation $\sqrt{D} = [a_0; \overline{a_1, a_2, ..., a_{\ell-1}, 2a_0}]$

we find that

$$\begin{cases} a'_0 = \dfrac{\omega' \cdot q_{\ell-1} + r}{2t} & (4.3) \\ \alpha'_0 = \dfrac{\omega' \cdot q_{\ell-2} + s}{t} & (4.4) \end{cases}$$

This can be verified with $a_0$ (resp. $\alpha_0$) from equation (2.10) using $b'$, and equating with above $a'_0$ (resp. $\alpha'_0$), with the help of equation (2.4) and equation (4.1):

$$\frac{(-1)^{\ell+1} q_{\ell-2} {q_{\ell-3}}^{+1} + b'\delta q_{\ell-1}}{2} = \frac{\left((-1)^{\ell+1}[q_{\ell-2} \cdot s - {q_{\ell-3}}^{+1} \cdot r] + c'\delta t\right) \cdot q_{\ell-1} + r}{2t}$$

$$\implies \delta t(b' - c') q_{\ell-1} = r(-1)^{\ell+1}[(-1)^{\ell+1} - q_{\ell-1}{q_{\ell-3}}^{+1}] + (-1)^{\ell+1} q_{\ell-2}[q_{\ell-1} \cdot s - {q_{\ell-3}}^{+1} \cdot t]$$

$$\implies \delta t(b' - c') \tfrac{q_{\ell-1}}{q_{\ell-2}} (-1)^{\ell+1} = s \cdot q_{\ell-1} - r \cdot q_{\ell-2} - t \cdot {q_{\ell-3}}^{+1} = 0$$

It follows that the equality is verified (for $c' = b'$).

With equation (2.12) and (4.3) (or equation (2.13) and (4.4)) we have

$$\omega = \omega' + k\delta t \tag{4.5}$$

From the same equations we get

$$\begin{cases} 2a_0 \cdot s \cdot t = \omega \cdot q_{\ell-1} \cdot s + r \cdot s \\ \alpha_0 \cdot r \cdot t = \omega \cdot q_{\ell-2} \cdot r + r \cdot s \end{cases}$$

or

$$\omega = \frac{2a_0 \cdot s - \alpha_0 \cdot r}{{q_{\ell-3}}^{+1}} = \frac{2a_0 \cdot t - r}{q_{\ell-1}} = \frac{\alpha_0 \cdot t - s}{q_{\ell-2}} \tag{4.6}$$

Similarly to equation (2.6) and (2.7) being solutions to equation (2.4) and (2.5) we have

$$\begin{cases} q_{\ell-1}{q_{\ell-3}}^{+1} - q_{\ell-2}q_{\ell-2} = (-1)^{\ell+1} \\ s \cdot q_{\ell-1} = r \cdot q_{\ell-2} + t \cdot {q_{\ell-3}}^{+1} \end{cases}$$

for which solutions are ($M \in \mathbb{Z}$)

$$\begin{cases} r = Mq_{\ell-1} + (-1)^{\ell+1} q_{\ell-2}{q_{\ell-3}}^{+1} t & (4.7) \\ s = Mq_{\ell-2} + (-1)^{\ell+1} \left({q_{\ell-3}}^{+1}\right)^2 t & (4.8) \end{cases}$$

Multiplying equation (4.8) by $q_{\ell-2}$ and subtracting equation (4.7) multiplied by ${q_{\ell-3}}^{+1}$ and using equation (2.4) yields, similarly to equation (2.9),

$$s \cdot q_{\ell-2} - r \cdot {q_{\ell-3}}^{+1} = (-1)^{\ell} M \tag{4.9}$$

4.1. **Other properties of $\omega$.** In addition to equation (4.5) and equation (4.6), there are other properties of $\omega$ which are of great interest.

Multiplying equation (2.7) (resp. (2.6)) by $t$ and subtracting equation (4.8) (resp. (4.7)) then dividing by $q_{\ell-2}$ (resp. $q_{\ell-1}$) gives equation (4.6) on the left side and on the right side we have:

$$\omega = m \cdot t - M \tag{4.10}$$

Now we will also show that $\omega = a_0 - A_n$: Let the complete quotients of the expansion of the continued fraction of $\sqrt{D}$ be denoted $r_h = \dfrac{\sqrt{D} + A_{h-1}}{\alpha_{h-1}}$, with $r_0 = \sqrt{D}$, and use the following:

(4.11) $A_h = A_{\ell-h-1}$ See Perron [7, p.90] with $(0 \leqslant h < \ell)$

(4.12) $\alpha_{h-1} = \alpha_{\ell-h-1}$ See Perron [7, p.90] with $(0 \leqslant h < \ell)$

(4.13) $A_{h-1} + A_h = a_h \alpha_{h-1}$ See Perron [7, p.83 formula 4] with $(h \geqslant 0)$

(4.14) $\alpha_h q_{h-1} + A_h q_h = p_h$ See Perron [7, p.75 formula 5] with $(h \geqslant -1)$

(4.15) $\alpha_{h-1} q_h - A_h q_{h-1} = p_{h-1}$ with $(h \geqslant 0)$

(4.16) $p_h = a_0 q_h + {q_{h-1}}^{+1}$ See Perron [7, p.15 formula 29] with $(h \geqslant -1)$

*Remark* 4.1. Indexes are shifted compared to Perron's notation: $A_{v-1} = P_v$ and $\alpha_{v-1} = Q_v$. The notation $A_i/\alpha_i$ dates back to Euler [2, p.36]. This index shift also better match formulas like $A_0 = a_0$, $D = a_0^2 + \alpha_0$ and more specifically $\alpha_i = \alpha_{\ell-i-2} = \dfrac{p_i q_{\ell-i-2} + q_i p_{\ell-i-2}}{q_{\ell-1}}$ with $-1 \leqslant i < \ell$ and $A_i = A_{\ell-i-1} = \dfrac{p_i q_{\ell-i-1} - q_{i-1} p_{\ell-i-2}}{q_{\ell-1}}$ with $0 \leqslant i < \ell$.

*Remark* 4.2. equation (4.15) is build by using equation (4.13) and equation (4.14):

$$\begin{aligned}
q_h &= a_h q_{h-1} + q_{h-2} \\
\alpha_{h-1} q_h &= \alpha_{h-1}(a_h q_{h-1} + q_{h-2}) \\
\alpha_{h-1} q_h &= A_{h-1} q_{h-1} + A_h q_{h-1} + \alpha_{h-1} q_{h-2} \\
\alpha_{h-1} q_h &= A_h q_{h-1} + p_{h-1}
\end{aligned}$$

Using all above equations and equation (3.2), (3.11) and (3.12), for $\ell = 2n$ we have

$$\begin{aligned}
&A_n q_{l-1} = q_{n-1} A_n (q_n + q_{n-2}) \\
=&q_{n-1}(A_n q_n + \alpha_n q_{n-1} - (\alpha_n q_{n-1} - A_n q_{n-2})) \\
=&q_{n-1}(A_n q_n + \alpha_n q_{n-1} - (\alpha_{n-2} q_{n-1} - A_{n-1} q_{n-2})) \\
=&q_{n-1}(p_n - p_{n-2}) \\
=&a_0 q_{n-1}(q_n - q_{n-2}) + q_{n-1}({q_{n-1}}^{+1} - {q_{n-3}}^{+1}) \\
=&a_0(q_{\ell-1} - 2t) + r
\end{aligned}$$

which gives $a_0 = \dfrac{(a_0 - A_n) q_{\ell-1} + r}{2t}$ and with equation (4.6) leads to $\omega = a_0 - A_n$

for $\ell = 2n + 1$ we have the same result:

$$\begin{aligned}
&A_n q_{l-1} = A_n((q_n)^2 + (q_{n-1})^2) \\
=&A_n (q_n)^2 + \alpha_n q_n q_{n-1} - (\alpha_n q_n q_{n-1} - A_n (q_{n-1})^2)) \\
=&A_n (q_n)^2 + \alpha_n q_n q_{n-1} - (\alpha_{n-1} q_n q_{n-1} - A_n (q_{n-1})^2)) \\
=&p_n q_n - p_{n-1} q_{n-1} \\
=&(a_0 q_n + {q_{n-1}}^{+1}) q_n - (a_0 q_{n-1} + {q_{n-2}}^{+1}) q_{n-1} \\
=&a_0((q_n)^2 - (q_{n-1})^2) + q_n {q_{n-1}}^{+1} - q_{n-1} {q_{n-2}}^{+1} \\
=&a_0(q_{\ell-1} - 2t) + r
\end{aligned}$$

which, as above, leads to

$$\omega = a_0 - A_n \tag{4.17}$$

*Remark* 4.3. with $\frac{\sqrt{D}+A_n}{\alpha_n} > 1$ and $D - A_n^2 = (\sqrt{D}+A_n)(\sqrt{D}-A_n) = \alpha_n\alpha_{n-1}$ (Perron [7, p.83 formula 5]) we see that $\sqrt{D} - A_n < \alpha_{n-1}$, therefore $\omega < \alpha_{n-1}$

*Remark* 4.4. from equation (2.12), (4.5) and (4.17) we have $A_n = A'_n + k\frac{\delta}{2}(q_{\ell-1} - 2t)$

Now we will show in lemma 4.7 and 4.10 that in a sequence of continued fraction expansions where the partial quotients $a_i$ $(0 < i < n)$ are constants and $a_n$ is incremented, $\omega$, as defined in equation (4.5), is periodic mod $\delta t$ with period $q_{n-1}$, $2q_{n-1}$ or $2{q_{n-1}}^2$.

**Lemma 4.5.** *For integer solutions $D$ to $\sqrt{D} = [a_0; \overline{a_1, a_2, ..., a_{\ell-1}, 2a_0}]$, $q_{\ell-1} \equiv 1 \pmod 4$ when $\ell = 2n+1$, and $q_{\ell-1} \equiv a_n \pmod 2$ when $\ell = 2n$*

*Proof.* When $\ell$ is *odd*, taking the determinants of matrices (2.2) with RHS expanded like in matrix (2.1), will give ${p_{\ell-1}}^2 - D{q_{\ell-1}}^2 = -1$. Since ${p_{\ell-1}}^2 \equiv 0$ or $1 \pmod 4$ and $D$ is an integer, $q_{\ell-1}$ cannot be *even*. And since $q_{\ell-1}$ is a sum of squares (see equation (3.2)), then $q_{\ell-1} \equiv 1 \pmod 4$. When $\ell = 2n$ is *even*, from $q_n = a_n q_{n-1} + q_{n-2}$ and equation (3.12) we get $q_{\ell-1} = a_n {q_{n-1}}^2 + 2q_{n-1}q_{n-2}$ and by lemma (3.3), $q_{n-1}$ cannot be *even* if $a_n$ is *odd*, therefore $q_{\ell-1} \equiv a_n \pmod 2$. ☐

**Definition 4.1** (quadratic sequence)**.** Let's take a palindromic expansion of the form $[a_0; \overline{a_1, a_2, ..., a_n, ..., a_2, a_1, 2a_0}]$ for $\ell = 2n$ or $\ell = 2n+1$. Let's fix the partial quotients $a_i$ (for $0 < i < n$, and by symmetry for $l - n < i < l$) to any chosen positive constants. By developing $(-1)^{\ell+1}(q_{\ell-2} \cdot s - {q_{\ell-3}}^{+1} \cdot r)$ from equation (4.2) using $r$ and $s$ defined in section 4, using equation 3.3 (resp. equation 3.7) and equation 3.5 (resp. equation 3.6) for $\ell$ *odd* (resp. $\ell$ *even*), and developing $q_n$ and ${q_{n-1}}^{+1}$ with equation 3.4 (resp. 3.4) and equation 3.11 (resp. 3.11) in order to *isolate* $a_n$, we now have a polynomial with variable $a_n$ and constant continuants (they only depend on the partial quotients we fixed):

$$f(a_n) = A \cdot a_n^2 + B \cdot a_n + C$$

with the following constants $A, B, C$ for $\ell$ *even* or $\ell$ *odd*:

$$\ell = 2n \begin{cases} A = 0 \\ B = -q_{n-2}({q_{n-2}}^{+1})^3 \\ C = -q_{n-1}{q_{n-2}}^{+1}({q_{n-3}}^{+1})^2 - q_{n-2}({q_{n-2}}^{+1})^2{q_{n-3}}^{+1} \end{cases}$$

$$\ell = 2n+1 \begin{cases} A = q_{n-1}({q_{n-2}}^{+1})^3 \\ B = 3q_{n-1}({q_{n-2}}^{+1})^2{q_{n-3}}^{+1} - q_{n-2}({q_{n-2}}^{+1})^3 \\ C = q_{n-1}({q_{n-2}}^{+1})^3 - q_{n-2}({q_{n-2}}^{+1})^2{q_{n-3}}^{+1} + 2q_{n-1}{q_{n-2}}^{+1}({q_{n-3}}^{+1})^2 \end{cases}$$

Let the ordered set of numbers $y_j = f(j)$ with $j = 1, 2, 3...$ be the quadratic sequence (or arithmetic progression when $A = 0$) defined by the sequence formula $f(a_n)$ and term position $j$, then for any positive integer $T$, $y_{j+T} = y_j + T(A \cdot T + 2A \cdot j + B)$, so in order to find the smallest period of the quadratic sequence mod $\delta t$, we have $y_{j+T} \equiv y_j \pmod{\delta t} \iff T(A \cdot T + 2A \cdot j + B) \equiv 0 \pmod{\delta t} \iff \delta t | T(A \cdot T + 2A \cdot j + B)$, and the smallest $T$ satisfying this relation is $T = \frac{\delta t}{\gcd(A \cdot T + 2A \cdot j + B, \delta t)}$. $T$ is the smallest period of the sequence of $y_j$ mod $\delta t$.

Table (1) shows an example of an arithmetic progression $y_j$ from a sequence of continued fraction expansions with fixed partial quotients $a_i$ (for $0 < i < n$) where $a_n$ is incremented, and have a parity restriction (in this example $a_n$ must be *even* for a solution $D$ to be an integer). We see that $y_j$ is periodic mod $\delta t$ with period 4, and $\omega' = f(a_n) \bmod \delta t$ will take its values in the set $\{5, 11\}$. The sequence of values $y_j \bmod \delta t$ is $< 2, 11, 8, 5, 2, 11, 8, 5, 2, ... >$. Note that in Table (1), the purpose is not to show that the continued fraction expansion $\sqrt{57/4} = [3; \overline{1, 3, 2, 3, 1, 6}]$, but to show that $a_0$ computed with equation (2.6) for $[a_0; \overline{3, 1, 1, 1, 3, 2a_0}]$ is not an integer, as shown by lemma (2.3).

TABLE 1. Sequence of expansions for $\ell = 6$ with $a_1 = 3$, $a_2 = 1$ where $\delta = 1$ and $t = 12$

| continued fraction expansion $= \sqrt{D}$ | $j$ | $a_n$ | $q_{n-1}$ | $A$ | $B$ | $C$ | $y_j$ | $y_{a_n}$ | $y_j$ mod $\delta t$ | $\omega$ mod $\delta t$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $[7/2\ ;\overline{3,1,1,1,3,\ \ 7}] = \sqrt{57/4}$ | 1 | | 4 | 0 | -3 | -7 | -10 | | 2 | |
| $[26\ \ ;\overline{3,1,2,1,3,\ 52}] = \sqrt{690}$ | 2 | 2 | 4 | 0 | -3 | -7 | -13 | -13 | 11 | 11 |
| $[49/2;\overline{3,1,3,1,3,\ 49}] = \sqrt{2453/4}$ | 3 | | 4 | 0 | -3 | -7 | -16 | | 8 | |
| $[19\ \ ;\overline{3,1,4,1,3,\ 38}] = \sqrt{371}$ | 4 | 4 | 4 | 0 | -3 | -7 | -19 | -19 | 5 | 5 |
| $[19/2;\overline{3,1,5,1,3,\ 19}] = \sqrt{381/4}$ | 5 | | 4 | 0 | -3 | -7 | -22 | | 2 | |
| $[56\ \ ;\overline{3,1,6,1,3,112}] = \sqrt{3165}$ | 6 | 6 | 4 | 0 | -3 | -7 | -25 | -25 | 11 | 11 |
| $[93/2;\overline{3,1,7,1,3,\ 93}] = \sqrt{8745/4}$ | 7 | | 4 | 0 | -3 | -7 | -28 | | 8 | |
| $[33\ \ ;\overline{3,1,8,1,3,\ 66}] = \sqrt{1106}$ | 8 | 8 | 4 | 0 | -3 | -7 | -31 | -31 | 5 | 5 |
| $[31/2;\overline{3,1,9,1,3,\ 31}] = \sqrt{993/4}$ | 9 | | 4 | 0 | -3 | -7 | -34 | | 2 | |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | |

*Remark* 4.6. Definition (4.1) makes a direct mapping between the quadratic sequence of terms $y_{a_n}$ (or eventually $y_j$) and a sequence $\sigma$ of continued fraction expansions where $a_n$ is incremented and the other partial quotients $a_i$ $(0 < i < n)$ are fixed. The concept of periodicity applied to sequences of continued fraction expansions is equivalent to the one applied to their quadratic sequences counterpart, as if the expansion linked to $a_n$ was represented by the number $y_{a_n}$.

**Lemma 4.7.** *For $\ell$ odd, $\omega$ as defined in equation 4.5 is periodic mod $\delta t$ with period $2{q_{n-1}}^2$, in the sequence of continued fraction expansions of length $\ell$ where $a_n$ is incremented and the other partial quotients $a_i$ $(0 < i < n)$ are fixed.*

*Proof.* Using the quadratic sequence of $y_j$ defined in (4.1), and from $q_{n-2}{q_{n-2}}^{+1} - q_{n-1}{q_{n-3}}^{+1} = (-1)^n$ in equation (3.9), we have $\gcd(q_{n-1}, {q_{n-2}}^{+1}) = 1$ and $\gcd(q_{n-1}, q_{n-2}) = 1$ so if we set $A \cdot T + 2A \cdot j + B = E \cdot q_{n-1} - q_{n-2}({q_{n-2}}^{+1})^3$ with $E = ({q_{n-2}}^{+1})^3(T + 2j) + 3({q_{n-2}}^{+1})^2{q_{n-3}}^{+1}$, and since $q_{n-2}({q_{n-2}}^{+1})^3$ has no common factor with $q_{n-1}$, we have $\gcd(A \cdot T + 2A \cdot j + B, q_{n-1}) = 1$ and therefore, with $t = q_{n-1}q_{n-1}$, we have $\gcd(A \cdot T + 2A \cdot j + B, t) = 1$

**Case 1**, $q_{n-1}$ is *even*
Lemma (3.2) tells us that $q_{n-1}$ *even* implies that there is no parity restriction on $a_n$, so $a_n$ can take the same values as $j$ in the sequence $y_{a_n}$ defined by $y_{a_n} = y_j$, and lemma (4.5) tells us that $q_{\ell-1}$ is *odd* so $\delta = 2$. Since $t$ is *even*, $\gcd(A \cdot T + 2A \cdot a_n + B, t) = 1$ implies $\gcd(A \cdot T + 2A \cdot a_n + B, 2t) = 1$, and $T = 2t$ is the smallest period of the sequence of $y_{a_n}$ mod $\delta t$.

By equation (4.2) and (4.5), we have $\omega \equiv f(a_n) \pmod{\delta t}$, and since $y_{a_n} = f(a_n)$, $\omega$ is periodic mod $\delta t$ with period $2t$ in the sequence of continued fractions expansions where $a_n$ is incremented and the other partial quotients $a_i$ $(0 < i < n)$ are fixed. For $k$ fixed in equation (4.5), $\omega$ is periodic like $\omega'$.

Note that equation (3.9) puts a lot of restrictions on the parity of $q_{n-1}$, $q_{n-2}$, ${q_{n-2}}^{+1}$ and ${q_{n-3}}^{+1}$ (e.g. they cannot be all *odd* or all *even*) and Table (2) shows all possible parity combination for $q_{n-1}$ *even* and the resulting parity of $A, B, C, f(a_n)$ and $\omega$. The parity of $\omega$ and $f(a_n)$ are identical modulo $2t$ (two congruent quantities have the same parity modulo an even number), and as shown in Table (2), the parity of $\omega$ alternate with the parity of $a_n$.

*Observation* 4.8. With $\gcd(A \cdot T + 2A \cdot j + B, \delta t) = 1$, the set $S_1 = \{1, 2, 3, ..., 2{q_{n-1}}^2 - 1\}$ of numbers less than $2t$ contains all values taken by ($y_j$ mod $\delta t$). $\omega'$ in equation (4.2) will also cycle through all the values of $S_1$.

**Case 2**, $q_{n-1}$ is *odd*
When $q_{n-1}$ is *odd*, there is a restriction on the parity of $a_n$ to have an integer solution, as we saw in lemma (3.2). In this case, Lemma (4.5) tells us that $q_{\ell-1}$ is *odd* and therefore $\delta = 2$. In the construction of the sequence of $y_j$, $\delta = 2$ will be used whatever the parity of $j$ is. Since we only look at $y_{a_n}$ for *allowed* $a_n$ in the end, it won't be an issue. Table (3) shows all possible combinations for $q_{n-1}$ *odd*

TABLE 2. possible parity combinations according to equation (3.9) for $q_{n-1}$ *even*

| $q_{n-2}$ | ${q_{n-2}}^{+1}$ | ${q_{n-3}}^{+1}$ | | $A$ | $B$ | $C$ | $a_n$ | | $y_{a_n}$ | $\omega$ |
|---|---|---|---|---|---|---|---|---|---|---|
| *odd* | *odd* | *odd* | $\Longrightarrow$ | *even* | *odd* | *odd* | *odd* | $\Longrightarrow$ | *even* | *even* |
| | | | | | | | *even* | $\Longrightarrow$ | *odd* | *odd* |
| *odd* | *odd* | *even* | $\Longrightarrow$ | *even* | *odd* | *even* | *odd* | $\Longrightarrow$ | *odd* | *odd* |
| | | | | | | | *even* | $\Longrightarrow$ | *even* | *even* |

TABLE 3. possible parity combinations according to equation (3.9) for $q_{n-1}$ *odd*

| $q_{n-2}$ | ${q_{n-2}}^{+1}$ | ${q_{n-3}}^{+1}$ | | $A$ | $B$ | $C$ | $y_j$ | ($y_j$ mod $\delta t$) |
|---|---|---|---|---|---|---|---|---|
| *odd* | *odd* | *even* | $\Longrightarrow$ | *odd* | *odd* | *odd* | *odd* | *odd* |
| *odd* | *even* | *odd* | $\Longrightarrow$ | *even* | *even* | *even* | *even* | *even* |
| *even* | *odd* | *odd* | $\Longrightarrow$ | *odd* | *odd* | *odd* | *odd* | *odd* |
| *even* | *even* | *odd* | $\Longrightarrow$ | *even* | *even* | *even* | *even* | *even* |

and the resulting parity of $A, B, C, y_j$ and ($y_j$ mod $\delta t$). Since $\gcd(A \cdot T + 2A \cdot j + B, t) = 1$, we have that $\gcd(A \cdot T + 2A \cdot j + B, 2t)$ can be either 1 or 2 so the smallest possible $T$ is $T = t$ (which is *odd*), and since $t = \dfrac{2t}{\gcd(A \cdot T + 2A \cdot j + B, 2t)}$ holds for all cases of Table (3), the smallest period of the sequence of $y_j$ mod $\delta t$ is indeed $T = t$. But we also see that for a quadratic sequence of $y_j$, where $A$, $B$ and $C$ are fixed, the parity of ($y_j$ mod $\delta t$) is always the same for that sequence, independently of the parity of $j$. With $\gcd(A \cdot T + 2A \cdot j + B, \delta t) = 2$, the set of all values taken by ($y_j$ mod $\delta t$) will either be the set $\{1, 3, 5, ..., 2t - 1\}$ of *odd* numbers less than $2t$ or the set $\{0, 2, 4, ..., 2t - 2\}$ of *even* numbers less than $2t$, and will be repeated twice in the period $2t$. Since $t$ is *odd* and $\omega'$ will only take one on two values of $y_j$ mod $\delta t$, $\omega'$ will also cycle through all numbers of one of these sets in this $2t$ period. The period is therefore $2t = 2{q_{n-1}}^2$. For $k$ fixed in equation (4.5), $\omega$ is periodic like $\omega'$.

E.g. for $\ell = 5$ with $a_1 = 3$, we have $t = 9$ an the following sequence of values $y_j$ mod $\delta t$ on a $2t$ period: $< \mathbf{5}, 13, \mathbf{9}, 11, \mathbf{1}, 15, \mathbf{17}, 7, \mathbf{3}, 5, \mathbf{13}, 9, \mathbf{11}, 1, \mathbf{15}, 17, \mathbf{7}, 3 >$, in bold $\omega'$ takes one on two values of that sequence and cover all the *odd* set $\{1, 3, 5, ..., 2t - 1\}$ in this $2t$ period.

*Observation* 4.9. As Table (3) shows, $y_j \equiv {q_{n-2}}^{+1} \bmod 2$. The set $S_1 = \{1, 3, 5, ..., 2{q_{n-1}}^2 - 1\}$ of *odd* numbers less than $2t$ or the set $S_2 = \{0, 2, 4, ..., 2{q_{n-1}}^2 - 2\}$ of *even* numbers less than $2t$ contains all possible values of ($y_j$ mod $\delta t$) depending on the parity of the constant ${q_{n-2}}^{+1}$. $\omega'$ in equation (4.2) will cycle through all the values of one of these two sets depending on ${q_{n-2}}^{+1}$ parity. □

**Lemma 4.10.** *For $\ell$ even, $\omega$ is periodic mod $\delta t$, in the sequence of continued fraction expansions of length $\ell$ where $a_n$ is incremented and the other partial quotients $a_i$ $(0 < i < n)$ are fixed, with period $q_{n-1}$ when $q_{n-1}$ is even, and with period $2q_{n-1}$ when $q_{n-1}$ is odd.*

*Proof.* Using the arithmetic progression $y_j$ defined in (4.1), and from $q_{n-2}{q_{n-2}}^{+1} - q_{n-1}{q_{n-3}}^{+1} = (-1)^n$ in equation (3.9), we have $\gcd(q_{n-1}, {q_{n-2}}^{+1}) = 1$ and $\gcd(q_{n-1}, q_{n-2}) = 1$ so with $A = 0$, $B = -q_{n-2}({q_{n-2}}^{+1})^3$ and $t = q_{n-1}q_{n-2}$ we have $T = \dfrac{\delta t}{\gcd(B, \delta t)}$ and since

$\gcd(B, \delta t) = \gcd(q_{n-2}({q_{n-2}}^{+1})^3, \delta q_{n-1}q_{n-2}) = q_{n-2} \cdot \gcd({q_{n-2}}^{+1}, \delta)$, we have $T = \dfrac{\delta q_{n-1}}{\gcd({q_{n-2}}^{+1}, \delta)}$.

**Case 1**, $q_{n-1}$ is *even*
Equation (3.12) implies $q_{\ell-1}$ *even* and $\delta = 1$, so the smallest period of $y_j$ mod $\delta t$ is $T = q_{n-1}$. By lemma (3.3), $a_n$ is limited to *even* parity, so the progression $y_{a_n}$ defined by $y_{a_n} = y_j$ for $j = 2, 4, 6...$ will contain only one on two values of the progression $y_j$. By equation (4.2) and (4.5), we have $\omega \equiv f(a_n)$

(mod $\delta t$), and since $y_{a_n} = f(a_n)$, $\omega$ is periodic mod $\delta t$ with period $q_{n-1}$ in the sequence of continued fractions expansions where $a_n$ is incremented and the other partial quotients $a_i$ $(0 < i < n)$ are fixed. Note that $q_{n-1}$ *even* implies $B$ *odd*, and $C \equiv {q_{n-3}}^{+1} \bmod 2$, and with $a_n$ *even*, all $\omega$ have the same parity (same parity as $f(a_n)$, taken mod $q_{n-1}$ *even*). For $k$ fixed in eq (4.5), $\omega$ is periodic like $\omega'$.

E.g. for $\ell = 6$ with $a_1 = 3$ and $a_2 = 3$, we have $T = q_{n-1} = 10$ and the following sequence of values $y_j$ mod $\delta t$ on that period $T$: $< 12, \mathbf{21}, 0, \mathbf{9}, 18, \mathbf{27}, 6, \mathbf{15}, 24, \mathbf{3} >$, in bold $\omega'$ takes one on two values of that sequence.

*Observation* 4.11. With $\gcd(B, \delta t) = q_{n-2}$, the set $S_1$ defined by the form $hq_{n-2} + ((-1)^n {q_{n-2}}^{+1} {q_{n-3}}^{+1}) \bmod q_{n-2}$ with integer $h$ $(0 \leqslant h < q_{n-1})$ contains all the values taken by ($y_j$ mod $\delta t$). $\omega'$ in equation (4.2) will cycle through all values of the set $S_2$ defined by the form $2hq_{n-2} + ((-1)^n {q_{n-2}}^{+1} {q_{n-3}}^{+1}) \bmod 2q_{n-2}$ with integer $h$ $(0 \leqslant h < \frac{q_{n-1}}{2})$, which is half of the set $S_1$ (due to $a_n$ parity restriction).

**Case 2**, $q_{n-1}$ is *odd*
Lemma (3.3) shows that there is no limitation on $a_n$, and Lemma (4.5) tells us that $q_{\ell-1} \equiv a_n \pmod 2$, so $\delta$ will alternate between 1 and 2 depending on the parity of $a_n$, starting with $\delta = 2$ for $a_n = 1$. We will take 2 different progressions $y_j$, one for $\delta = 1$ and one for $\delta = 2$, and $\omega'$ will alternatively take its value from the first progression (for $j$ *even*) and second progression (for $j$ *odd*).

The first progression mod $\delta t$ $(\delta = 1)$ will have period $T = q_{n-1}$ and contains numbers smaller than $t$ repeated twice on a $2q_{n-1}$ period. Since $q_{n-1}$ is *odd* and $\omega'$ will only take one on two values of the progression $y_j$ mod $\delta t$, $\omega'$ will also cycle through all these numbers (like explained in case 2 of lemma 4.7). The period is therefore $2q_{n-1}$.

The second progression mod $\delta t$ $(\delta = 2)$ will have period $T = q_{n-1}$ or $T = 2q_{n-1}$ depending on the parity of ${q_{n-2}}^{+1}$. For ${q_{n-2}}^{+1}$ *even*, $\gcd({q_{n-2}}^{+1}, \delta) = 2$ and the period is $T = q_{n-1}$ and contains numbers smaller than $2t$ repeated twice on a $2q_{n-1}$ period. Since $q_{n-1}$ is *odd* and $\omega'$ will only take one on two values of the progression $y_j$ mod $\delta t$, $\omega'$ will also cycle through all these numbers (like explained in case 2 of lemma 4.7). The period is therefore $2q_{n-1}$. For ${q_{n-2}}^{+1}$ *odd*, the period is $T = 2q_{n-1}$ and contains numbers smaller than $2t$ on that period.

There can be an overlap between the two progressions for numbers smaller than $t$, but not for numbers larger or equal to $t$. In all cases, the period for $\omega'$ (and $\omega$ for fixed $k$ in eq (4.5)) will be $2q_{n-1}$.

E.g. for $\ell = 6$ with $a_1$ and $a_2 = 2$, we have $q_{n-1} = 5$ and ${q_{n-2}}^{+1}$ *even* and the following sequence of values $y_j$ mod $\delta t$ on a $2q_{n-1}$ period for the first progression: $< 6, \mathbf{0}, 4, \mathbf{8}, 2, \mathbf{6}, 0, \mathbf{4}, 8, \mathbf{2} >$, in bold the values taken by $\omega'$, and for the second progression: $< \mathbf{6}, 10, \mathbf{14}, 18, \mathbf{2}, 6, \mathbf{10}, 14, \mathbf{18}, 2 >$, in bold the values taken by $\omega'$, with an overlap for some values smaller than $t = 10$.

*Observation* 4.12. For the first progression $y_j$ mod $\delta t$ for $\delta = 1$, the set $S_1$ defined by the form $hq_{n-2} + ((-1)^n {q_{n-2}}^{+1} {q_{n-3}}^{+1}) \bmod q_{n-2}$ with integer $h$ $(0 \leqslant h < q_{n-1})$ contains all the values taken by ($y_j$ mod $\delta t$). When $\delta = 1$ ($a_n$ *even*), $\omega'$ in equation (4.2) will cycle through all values of that $S_1$ set.

For the second progression $y_j$ mod $\delta t$ for $\delta = 2$, the set $S_2$ will depend on the parity of ${q_{n-2}}^{+1}$. For ${q_{n-2}}^{+1}$ *even*, the set $S_2$ defined by the form $2hq_{n-2} + ((-1)^n {q_{n-2}}^{+1} {q_{n-3}}^{+1}) \bmod 2q_{n-2}$ with integer $h$ $(0 \leqslant h < q_{n-1})$ and numbers smaller than $2t$ contains all the values taken by ($y_j$ mod $\delta t$). When $\delta = 2$ ($a_n$ *odd*), $\omega'$ in equation (4.2) will cycle through all values in that $S_2$ set.
For ${q_{n-2}}^{+1}$ *odd*, the set $S_2$ defined by the form $hq_{n-2} + ((-1)^n {q_{n-2}}^{+1} {q_{n-3}}^{+1}) \bmod q_{n-2}$ with integer $h$ $(0 \leqslant h < 2q_{n-1})$ and numbers smaller than $2t$ contains all the values taken by ($y_j$ mod $\delta t$). When $\delta = 2$ ($a_n$ *odd*), $\omega'$ in equation (4.2) will cycle through all values in the set $S_3$ defined by the form $2hq_{n-2} + ((-1)^n {q_{n-2}}^{+1} {q_{n-3}}^{+1} + q_{n-2}) \bmod 2q_{n-2}$ with integer $h$ $(0 \leqslant h < q_{n-1})$, which is half of the $S_2$ set. □

**Corollary 4.13.** *for $\ell = 2n$, $\alpha_{n-1}$ is periodic with the same period as $\omega$ in the sequence of continued fraction expansions of length $\ell$ where $a_n$ is incremented, $k$ is fixed and the other partial quotients $a_i$ $(0 < i < n)$ are fixed.*

*Proof.* Using equation (4.14) and (4.16) we have $\alpha_{n-1}q_{n-2} + A_{n-1}q_{n-1} = p_{n-1} = a_0 q_{n-1} + {q_{n-2}}^{+1}$, leading to $\alpha_{n-1} = \frac{(a_0 - A_{n-1})q_{n-1} + {q_{n-2}}^{+1}}{q_{n-2}}$, or $\frac{(a_0 - A_n)q_{n-1} + {q_{n-2}}^{+1}}{q_{n-2}}$ with equation (4.11), and to $\alpha_{n-1} = \frac{\omega q_{n-1} + {q_{n-2}}^{+1}}{q_{n-2}}$ using equation (4.17), $q_{n-1}$, $q_{n-2}$ and ${q_{n-2}}^{+1}$ being constants, only dependent on $a_i$ with $0 < i < n$. The rest follows from lemma (4.10). □

*Remark* 4.14. A link between $\omega$ and non-primitive expansions is shown in lemma (5.6).

## 5. PRIMITIVE PERIOD AND SMALLEST SOLUTIONS

When we set $k = 0$ in equation (2.11), we get the smallest possible solution $D$ for a fixed set of $a_i$ ($0 < i < \ell$), but with $k = 0$, the period is not always primitive. Sometimes it is never primitive, like for $\ell = 3$ where setting $k > 0$ is enough to ignore them. This section will show that for $k > 0$ the period is always primitive, and show when it is primitive at $k = 0$. Some inequalities related to $p_{\ell-1} + q_{\ell-1}\sqrt{D}$ will also be given for $k = 0$ and $k \neq 0$.

**Lemma 5.1.** *The partial quotients of a continued fraction expansion of the square root of a non square integer $D$, written $\sqrt{D} = [a_0; \overline{a_1, a_2, ..., a_{\ell-1}, 2a_0}]$, with primitive period $\ell_p$, have the property $a_i \leqslant a_0$ for $\ell_p \nmid i$. For primitive and non primitive period, $a_i \leqslant 2a_0$ when $i \geqslant 0$.*

*Proof.* Let the complete quotients of the expansion of the continued fraction of $\sqrt{D}$ be denoted $r_h = \frac{\sqrt{D} + A_{h-1}}{\alpha_{h-1}}$, with $r_0 = \sqrt{D}$. Using the following properties (with integer $h \geqslant 0$) (see remark (4.1)):

(1) $A_{h-1} \leqslant a_0$ See Perron [7, p.76 formula 7]
(2) $A_{h-1} + A_h = a_h \alpha_{h-1}$ See Perron [7, p.83 formula 4]
(3) $\alpha_{i-1} \geqslant 2$ for $0 < i < \ell_p$ See Perron [7, p.93 formula 5]

(2) and (1) gives $a_h \leqslant \frac{2a_0}{\alpha_{h-1}}$, and using (3) we get $a_i \leqslant a_0$ for $0 < i < \ell_p$ and by periodicity, for all $a_i$ with $\ell_p \nmid i$. And since all $\alpha_{h-1}$ for $h \geqslant 0$ are positive integers, we have $a_h \leqslant 2a_0$ for $h \geqslant 0$, the period being primitive or not. □

**Theorem 5.2.** *The period $\ell$ of the continued fraction expansion of the square root of a non square integer $D$ generated by equation (2.11) with $k > 0$, is always primitive.*

*Proof.* Lemma 5.1 tells us that all $a_i \leqslant a_0$ for $\ell_p \nmid i$, and if the period $\ell$ is not primitive, we find one or more $a_i = 2a_0$ in the range $0 < i < \ell$, but no $a_i > 2a_0$. Lemma 2.4 tells us that $D' = (a_0')^2 + \alpha_0'$, when $k = 0$, is the smallest $D$ having these partial quotients $a_i$ so we know that $a_i \leqslant 2a_0'$ for $0 < i < \ell$, but we also know that any other $D$ with these partial quotient are found by setting $k > 0$ in equation (2.11), implying $2a_0' < 2a_0$ and therefore $a_i < 2a_0$ for $0 < i < \ell$, so the period $\ell$ is primitive for those $D$. □

*Remark* 5.3. Since there is symmetry and repetition in the partial quotients, $\ell$ must be a multiple of $\ell_p$ as noted by Perron [7, p.73] *"The number of elements in a non primitive period is always a multiple of the number of elements in a primitive period"*.

**Theorem 5.4.** *When $k = 0$, in a sequence of continued fraction expansions of length $\ell = 2n$ or $\ell = 2n + 1$ where $a_n$ is incremented and the other partial quotients $a_i$ ($0 < i < n$) are fixed, there is no expansion $[a_0; \overline{a_1, a_2, ..., a_\ell}]$ with a primitive period $\ell$ when $\ell = 2$ and $a_1$ is even, or when $\ell = 3$, or when $\ell = 4$ and $a_1 = 2$. In all other cases, there are infinitely many expansions where $\ell$ is the primitive period and $k = 0$.*

*Proof.* Let's assume $k = 0$ in what follows. For $\ell = 2$, we have $q_{\ell-1} = a_1$. For $q_{\ell-1}$ *even*, $a_0' = \frac{a_1}{2}$ from remark (2.6), leading to $[a_0; \overline{2a_0, 2a_0}]$. The period $\ell$ is therefore always non-primitive in that case, and the primitive period $\ell_p = 1$. When $q_{\ell-1}$ is *odd*, we have $a_1 \neq 2a_0$ so $a_1 < 2a_0$ from lemma (5.1), and

the period $\ell$ is always primitive in that case.
For $\ell = 3$ the period $\ell$ is non-primitive as it can be directly checked with its form $D = (k \cdot a_1^2 + k + \frac{a_1}{2})^2 + 2k \cdot a_1 + 1$ where setting $k = 0$ produces the $\ell_p = 1$ form $D = (\frac{a_1}{2})^2 + 1$.

Now, we will look at the general case for $\ell > 3$ *odd* and $\ell > 2$ *even* and show when the period $\ell$ is not primitive ($\ell \neq \ell_p$) for an expansion in an infinite sequence $\sigma$ of continued fraction expansions where $a_n$ is incremented and the other partial quotients $a_i$ ($0 < i < n$) are fixed, as described in definition (4.1).

As Lemma 5.1 tells us, $a_i \leqslant 2a_0$ for $i \geqslant 0$, and if the period $\ell$ is not primitive, we find one or more $a_i = 2a_0$ in the range $0 < i < \ell$, and the symmetric part of an expansion of primitive length $\ell_p$ is repeated to form the symmetric part of the expansion of length $\ell$.

**Case 1**, $\ell > 3$ is *odd* and $D' = [a_0; \overline{..., a_n, a_n, ..., 2a_0}]$
For $\ell$ to be non-primitive, we must either have $a_n = 2a_0$ or $a_n < 2a_0$ with some $a_i = 2a_0$ for $i \neq n$ and $i \neq \ell$.

When $a_n = 2a_0$, we have the expansion $[a_0; \overline{..., a_j, 2a_0, 2a_0, a_j, ..., 2a_0}]$, but since we have two consecutive quotients $2a_0$, all partial quotients must be $2a_0$ (property of the palindrome, symmetry and periodicity) for $\ell$ to be non-primitive, leading to $\ell_p = 1$. Since the partial quotients $a_i (i \neq n)$ are fixed, there is only maximum one such expansion of non-primitive length in the infinite sequence $\sigma$ (a different $a_n$ would otherwise force these $a_i$ to be also different). All other sequences are primitive in $\sigma$ for those fixed partial quotients.

When $a_n < 2a_0$ and some $a_i = 2a_0$ for $i \neq n$ and $i \neq \ell$, we have the expansion
$[a_0; \overline{..., a_n, a_n, ..., 2a_0, ..., a_n, a_n, ..., 2a_0, ..., a_n, a_n ..., 2a_0}]$ with at least (property of the palindrome, symmetry and periodicity) two $2a_0$ inside the symmetric part, and for $\ell$ non-primitive, a *mirror* $a_n$ for each $a_n$ (on each side of a $2a_0$ quotients at equal distance), so $\ell \geqslant 9$ and since there are at least two $a_n$ before a $2a_0$ and at least three repetitions, we have $2 < \ell_p < \ell/2$. Since the partial quotients except the central ones are fixed, there is only maximum one such expansion of non-primitive length in the infinite sequence $\sigma$, and all other sequences are primitive in $\sigma$ for those fixed partial quotients.

**Case 2**, $\ell > 2$ is *even* and $D' = [a_0; \overline{..., a_n, ..., 2a_0}]$
For $\ell$ to be non-primitive, we must either have $a_n = 2a_0$ with eventually some $a_i = 2a_0$ for $i \neq n$ and $i \neq \ell$, or $a_n < 2a_0$ with some $a_i = 2a_0$ for $i \neq n$ and $i \neq \ell$.

When $a_n < 2a_0$ and some $a_i = 2a_0$ for $i \neq n$ and $i \neq \ell$, we have the same reasoning as for $\ell$ *odd* and an expansion $[a_0; \overline{..., a_n, ..., 2a_0, ..., a_n, ..., 2a_0, ..., a_n, ..., 2a_0}]$ with at least (property of the palindrome, symmetry and periodicity) two $2a_0$ inside the symmetric part, and for $\ell$ non-primitive, a *mirror* $a_n$ for each $a_n$ (on each side of a $2a_0$ quotients at equal distance), so $\ell \geqslant 6$ and since there are at least one $a_n$ before a $2a_0$ and at least three repetitions, we have $1 < \ell_p < \ell/2$. Since the partial quotients except the central one are fixed, there is only maximum one such expansion of non-primitive length in the infinite sequence $\sigma$, and all other sequences are primitive in $\sigma$ for those fixed partial quotients.

When $a_n = 2a_0$ and some $a_i = 2a_0$ for $i \neq n$ and $i \neq \ell$, we have the expansion
$[a_0; \overline{..., 2a_0, ..., 2a_0, ..., 2a_0, ..., 2a_0}]$ and $\ell_p < \ell/2$. Since the partial quotients except the central one are fixed, there is only maximum one such expansion of non-primitive length in the infinite sequence $\sigma$. But as we will see next, this is the first non-primitive expansion in a series of non-primitive expansions with increasing $a_n$ to be found in the sequence $\sigma$.

When $a_n = 2a_0$ and no other $a_i = 2a_0$ for $i \neq n$ and $i \neq \ell$, we have the expansion $[a_0; \overline{..., 2a_0, ..., 2a_0}]$ with two repetitions of the primitive expansion, so $\ell_p = \ell/2 = n$. This case is particular since for the fixed partial quotients $a_i$ ($0 < i < n = \ell_p$) there are infinitely many solutions $2a_0$ for the expansion $[a_0; \overline{..., 2a_0}]$ of primitive length $\ell_p$ as shown in equation (2.12) where $2a_0 = 2a_0' + k(\delta q_{\ell_p - 1}) = 2a_0' + k(\delta q_{n-1})$ with $\delta = 1$ for $q_{n-1}$ *even* and $\delta = 2$ for $q_{n-1}$ *odd* (Note that for $2a_0'$ the period $\ell/2$ may still be non-primitive which is what was described previously where some $a_i = 2a_0$ for $i \neq n$ and $i \neq \ell$). This means that there is a non-primitive expansion every $\delta q_{n-1}$ in the sequence $\sigma$. Even if there is a parity restriction on $a_n$, we are sure to find a primitive expansion between two non-primitive expansions in the sequence $\sigma$ when $q_{n-1} > 2$. In this case we find an infinity of expansion with primitive period. We now look at the cases where $q_{n-1} \leqslant 2$: Using Fibonacci numbers $F_n$, we have $q_{n-1} = K_{n-1}(a_1, a_2, ..., a_{n-1}) \geqslant K_{n-1}(1, 1, ..., 1) = F_n$, so we know that for $\ell \geqslant 8$ we have $q_{n-1} > 2$.

For $\ell = 6$ we have $q_{n-1} = K_2(a_1, a_2)$ but there is no solution with *odd* partial quotients for $\ell_p = 3$, so for $\ell = 6$, expansions with non-primitive periods $\ell/2$ have $q_{n-1} > 2$. For $\ell = 4$ we have $q_{n-1} = a_1$, so when $a_1 > 2$ we have $q_{n-1} > 2$. When $a_1 = 1$, there is no parity restriction on $a_n$ and there is a non-primitive expansion every two expansions ($\delta q_{n-1} = 2$) in the sequence $\sigma$ and therefore a primitive expansion every two expansions also. Lastly, still for $\ell = 4$, when $a_1 = 2$ there is a parity restriction on $a_n$ and all expansions of the sequence $\sigma$ are of non-primitive period since they repeat every two expansions ($\delta q_{n-1} = 2$), and match all the non-restricted $a_n$ cases. □

*Remark* 5.5. For $\ell = 1$ we have $k > 0$, and there is no smaller period anyway.

**Lemma 5.6.** *When $\ell$ is even and there is a non-primitive expansion $[a_0; \overline{..., a_n = 2a_0, ..., 2a_0}]$, in the infinite sequence $\sigma$ of continued fraction expansions as described in definition (4.1), all non-primitive expansions of $\sigma$ will have $\omega = 0$ and appear in the sequence $\sigma$ with the same period, described in lemma (4.10), as $\omega$.*

*Proof.* The case $\ell$ *even* and $a_n = 2a_0$ described in theorem (5.4) leads to $n = \ell_p$ for non-primitive expansions (except eventually for the smallest $2a_0 = 2a_0'$ where we use $\ell/2$ instead of $\ell_p$) and a non-primitive period expansion appearing every $\delta q_{n-1}$ in the sequence $\sigma$, which is also the period of $\omega$ in the sequence $\sigma$ as shown in lemma (4.10). Since $k = 0$, we have $\omega = \omega'$.

From equation (4.6), $\omega = \dfrac{\alpha_0 \cdot t - s}{q_{\ell-2}}$ using $s$ and $t$ defined in section 4, we have

$$\omega = \frac{\alpha_0 \cdot q_{\ell_p-1} q_{\ell_p-2} - {q_{\ell_p-1}}^{+1} {q_{\ell_p-2}}^{+1}}{q_{\ell-2}} = (\alpha_0 \cdot q_{\ell_p-1} - {q_{\ell_p-1}}^{+1}) \frac{q_{\ell_p-2}}{q_{\ell-2}}$$

Now, $\alpha_0 \cdot q_{\ell-1} = 2a_0 \cdot q_{\ell-2} + {q_{\ell-3}}^{+1}$ from equation (2.5) and ${q_{\ell-1}}^{+1} = a_\ell \cdot {q_{\ell-2}}^{+1} + {q_{\ell-3}}^{+1}$ from equation (3.1) with ($\alpha{=}\ell{-}3$, $\beta{=}1$, $\gamma{=}1$, $\mu{=}2$) gives ${q_{\ell-1}}^{+1} = \alpha_0 \cdot q_{\ell-1}$. Applied to $\ell_p$ (or $\ell/2$ since $\ell/2$ is also a period) we have $\alpha_0 \cdot q_{\ell_p-1} - {q_{\ell_p-1}}^{+1} = 0$, so $\omega = 0$ for those non-primitive expansions. □

### 5.1. Some inequalities.

**Lemma 5.7.** *For $\ell > 2$, when $a_0 = a_0'$ ($k = 0$ in equation (2.11)) we have*

$$\frac{p_{\ell-1}}{(q_{\ell-1})^2} < \frac{\delta}{2} \tag{5.1}$$

$$p_{\ell-1} > \frac{2D}{\delta} \tag{5.2}$$

$$q_{\ell-1} > \frac{2\sqrt{D}}{\delta} \tag{5.3}$$

$$p_{\ell-1} + q_{\ell-1}\sqrt{D} > \frac{4D}{\delta} \tag{5.4}$$

*Proof.* By definition we have $a_0 < \dfrac{\delta q_{\ell-1}}{2}$ with integers on both sides. If we add $\dfrac{q_{\ell-2}}{q_{\ell-1}} < 1$ to the left side, the inequality still holds: $a_0 + \dfrac{q_{\ell-2}}{q_{\ell-1}} < \dfrac{\delta q_{\ell-1}}{2}$ which by using $p_{\ell-1} = a_0 q_{\ell-1} + q_{\ell-2}$ from matrix (2.3) leads to inequality (5.1). From $a_0 + 1 \leqslant \dfrac{\delta q_{\ell-1}}{2}$ and multiplying by $a_0$ and adding $\dfrac{\delta q_{\ell-2}}{2}$ on both sides we have $a_0^2 + a_0 + \dfrac{\delta q_{\ell-2}}{2} \leqslant \dfrac{\delta p_{\ell-1}}{2}$ and since $\dfrac{\alpha_0}{2} \leqslant a_0$ by remark (2.1) and also $\dfrac{\alpha_0}{2} < \dfrac{\delta q_{\ell-2}}{2}$ (by definition when $k = 0$), we have $\alpha_0 < a_0 + \dfrac{\delta q_{\ell-2}}{2}$ and therefore $D = a_0^2 + \alpha_0 < \dfrac{\delta p_{\ell-1}}{2}$ leading to inequality (5.2). Inequality (5.3) can be verified using $\sqrt{D} < a_0 + 1 \leqslant \dfrac{\delta q_{\ell-1}}{2}$, and inequality (5.4) by using inequality (5.2) and inequality (5.3). □

**Lemma 5.8.** *when* $a_0 \neq a_0'$ ($k > 0$ *in* equation (2.11)) *we have*

$$\frac{p_{\ell-1}}{(q_{\ell-1})^2} > \frac{\delta}{2} \tag{5.5}$$

$$p_{\ell-1} < \frac{2D}{\delta} \tag{5.6}$$

$$q_{\ell-1} < \frac{2\sqrt{D}}{\delta} \tag{5.7}$$

$$p_{\ell-1} + q_{\ell-1}\sqrt{D} < \frac{4D}{\delta} \tag{5.8}$$

*Proof.* By definition we have $a_0 > \dfrac{\delta q_{\ell-1}}{2}$, so with $a_0 + \dfrac{q_{\ell-2}}{q_{\ell-1}} > \dfrac{\delta q_{\ell-1}}{2}$ using $p_{\ell-1} = a_0 q_{\ell-1} + q_{\ell-2}$ from matrix (2.3) we find inequality (5.5). From matrix (2.3) where $Dq_{\ell-1} \geqslant a_0 p_{\ell-1} + a_0 q_{\ell-2} > p_{\ell-1}\dfrac{\delta q_{\ell-1}}{2}$ we find inequality (5.6). From $D > a_0^2 > (\dfrac{\delta q_{\ell-1}}{2})^2$ we find inequality (5.7), and from inequality (5.6) and inequality (5.7) we find inequality (5.8). □

**Lemma 5.9.** *For* $\ell = 2$ *and* $a_0 = a_0'$ *we have the same inequalities* (5.5), (5.6), (5.7), *and* (5.8) *as when* $a_0 \neq a_0'$

*Proof.* Using lemma (5.8) logic and the fact that we have $a_0 = \dfrac{\delta q_{\ell-1}}{2}$ and $q_{\ell-2} > 0$ □

*Remark* 5.10. When $\ell = 1$ and $a_0 = a_0'$, we also have the same properties as when $a_0 \neq a_0'$, except that we can have $\dfrac{p_{\ell-1}}{(q_{\ell-1})^2} = \dfrac{\delta}{2}$ when $p_{\ell-1} = q_{\ell-1} = 1$ which happens when $D = 2$. This can be checked directly with $p_{\ell-1} = a_0$ and $q_{\ell-1} = 1$ and $\delta = 2$.

*Remark* 5.11. Numbers having a primitive period of $\ell_p = 1$ or $\ell_p = 2$ can appear in the generation of numbers with period $\ell > 2$ as the smallest solution $D'$ (of non-primitive period $\ell$) for some set of partial quotients. In that case, properties of lemma (5.7) will apply to them since the values of $p_{\ell-1}$, $q_{\ell-1}$, $q_{\ell-2}$ and ${q_{\ell-3}}^{+1}$ used will be from a multiple $i > 1$ of the primitive period. In those cases, $q_{\ell-1} = q_{i\cdot\ell_p-1} > 1$, $q_{\ell-2} = q_{i\cdot\ell_p-2} > 1$ and ${q_{\ell-3}}^{+1} = {q_{i\cdot\ell_p-3}}^{+1} > 0$.

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