A Derivation of Maxwell's Equations from First Principles

TM Coker MA PhD CEng FIET MInstP*

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Abstract

This is the second of a series of short papers exploring various aspects of quantum mechanics and quantum field theory. The intent of the full series of articles is to take the reader/student from a basic starting point (somewhere around high school / first year undergraduate maths/ physics/ engineering) to an understanding of relativistic quantum mechanics that would be appropriate for a third/fourth year undergraduate or early stage postgraduate.

This particular article is a derivation of Maxwell's Equation from first principles, meaning with no a priori knowledge of the form of the equations, or even of the form of the electric and magnetic fields.

1 Introduction

1.1 Series Overview

The intent of this series of articles is to allow the reader/student to understand the basic concepts of quantum mechanics and quantum field theory, but with a starting point of comparatively basic maths and physics, such as a first year undergraduate studying maths, physics or engineering might have. The prerequisites are:

- Vectors and basic matrix algebra including eigenvectors and eigenvalues.
- Partial differentiation and vector calculus.
- A purely qualitative knowledge of quantum mechanics and special relativity.

The original motivation for this work was a desire to understand the Higg's Boson and how it somehow "creates" mass. To get to that endpoint it turns

*Non-affiliated author.

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out it's necessary to understand Special (but thankfully not General) relativity, group theory, Lagrangians and local gauge invariance, Hamilton's Principle, the calculus of variations and electro-weak unification. All these topics, and others, will be introduced along the way.

"Natural units" , where $\hbar = c = 1$, will be used throughout[1], with these units, mass, energy and momentum are dimensionally equal, as are length and time which is useful for the concept of spacetime. We'll also be using Heaviside–Lorentz units where, in addition, $\epsilon_0 = \mu_0 = 1$.

2 Preliminaries

Unfortunately this paper has to introduce the topic of Tensors, which is a generalisation of the concept of a vector. In fact scalar quantities can be considered zero–order tensors, in which case vectors are 1^{st} order tensors and what we mean by a tensor is a 2^{nd} order (or higher) tensor[2]. We can also consider, conceptually, that if a scalar is a quantity with no directional information ie magnitude only, a vector is a quantity with both magnitude and direction, then a 2^{nd} order tensor is a quantity that encodes magnitude and *two* directions (and so on for higher order tensors), or put another way a tensor is like a combination of two vectors.

In the Dirac paper (Coker [1]) we were introduced to covariant and contravariant vectors with upper and lower indices, so it may not be a surprise that tensors also have covariant and contraviant components, indeed can be a mix of both. Whilst tensors have a particular order, they can be of any dimension (other than scalars of course); not surprisingly for particle physics, our tensors have dimension four, so typically we use greek indices to indicate this (μ or ν mostly, but also σ and ρ sometimes). Hence a tensor is expressed as one of:

$A^{\mu\nu}$	a tensor with two contravariant indices
$A_{\mu\nu}$	a tensor with two covariant indices
$A^{\mu}{}_{\nu}$	a tensor with one contravariant index and one covariant index

Note that $A^{\mu}{}_{\nu}$ is not necessarily the same as $A_{\mu}{}^{\nu}$.

Whilst our vectors and tensors have dimension four, we use the convention that the first component (ie $\mu = 0$) is the timelike dimension, and the other three components refer to the three spacelike dimensions. If we wish to refer to *only* the spacelike components we use a roman as opposed to a greek index, typically $i = 1, 2, 3$ or j. We will use this convention in Sub-Section 6.1

2.1 Tensor Algebra

Tensor algebra is based around manipulation of the indices and use of the Einstein Summation Convention where if an index is repeated then a sum-

[1] Again a topic to be covered in a later paper, but see almost any quantum mechanical text for a further description.

[2] Actually tensors have both *order* and *rank*, the terms do have strict definitions but they tend to be used interchangeably.

mation over the repeated index is assumed:

$$a_ib_i = \sum_{i=1}^{3} a_ib_i = a_1b_1 + a_2b_2 + a_3b_3$$

In the Dirac paper we encountered this convention, ie we used terms such as $x_\nu = g_{\mu\nu}x^\mu$ and $\partial_\mu\partial^\mu$, where we referred to this as "contracting" an index. However when dealing with tensors it is essential to remember that we only contract indices when one is an upper, the other a lower index.

Sticking to 1^{st} order tensors for the moment we see:

$$A^\mu B_\mu = A^0B_0 + A^1B_1 + A^2B_2 + A^3B_3$$

here, whilst A^μ and B_μ are tensors the components A^0 and B_0 etc are simply numbers, hence $A^0B_0 = B_0A^0$, therefore

$$\begin{aligned} B_\mu A^\mu &= B_0A^0 + B_1A^1 + B_2A^2 + B_3A^3 \\ &= A^0B_0 + A^1B_1 + A^2B_2 + A^3B_3 \\ &= A^\mu B_\mu \end{aligned}$$

This leads us to a very important distinction, even though vectors are 1^{st} order tensors, tensor algebra is not the same as vector algebra. The tensor product $A^\mu B_\mu$ looks very much like the scalar product of two vectors, but this only works if A is a row vector and B is a column vector in which case AB is sort of equal to $A^\mu B_\mu$, but whilst $A^\mu B_\mu = B_\mu A^\mu$ $AB \neq BA$.

We can extend this argument a little further, from our knowledge of vector and matrix algebra we know that AB is a scalar whereas BA is in fact a matrix. Similarly $A^\mu B_\mu$ is scalar but $A^\mu B_\nu = C^\mu{}_\nu$ is a 2^{nd} order tensor – the indices are different so we do not contract them. Similarly $A^\mu B^\nu = C^{\mu\nu}$ whilst $A^\mu B^\mu$ is simply a mistake – whilst the rules seem difficult and the use of tensors is far from trivial, the notation and the use of the summation convention usually allows one to spot mistakes in equations.

Another way to think of this is that the rules of vector and matrix algebra tell us how to combine the components when we multiply things together and the order of items in the product terms is important, By contrast, for tensors it is the index labels and their position that tells us how to combine components and the order is not relevant.

Addition and subtraction of tensors is comparatively straightforward, the tensors just need to be of the same order and dimension (and the combination of covariant and contravariant components needs to be the same), which is a complex way of saying that the indices just need to be in the same place.

$$\begin{aligned} C^{\mu\nu} &= A^{\mu\nu} + B^{\mu\nu} \quad \text{– correct} \\ C^{\mu\nu} &= A^{\mu\nu} + B^\mu{}_\nu \quad \text{– incorrect} \end{aligned}$$

Some other rules, the actual index symbol bears no particular relevance, ie $F^{\mu\nu}$ is the same tensor as $F^{\sigma\rho}$. Next, the index that we contract over is often called the dummy index, any other index is the "free" index, and we should expect the dummy index to disappear from the result of the contraction and the free index to remain, For example, $A^\mu{}_\nu B^\nu$, this means we contract over the ν index:

$$A^\mu{}_\nu B^\nu = C^\mu$$

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similarly

$$A^{\mu\nu}B_\nu = C^\mu$$

In these examples the dummy index has disappeared leaving only the free index.

Next, in any one tensor product (or term in an equation, provided we take some care and consider how this could affect other terms) we can re–label the indices provided we are *consisent*:

$$a_\mu B^{\mu\nu} c_\nu = a_\nu B^{\nu\mu} c_\mu \quad \text{– correct}$$
$$a_\mu B^{\mu\nu} c_\nu = a_\nu B^{\mu\nu} c_\mu \quad \text{– incorrect}$$

Lastly, $A^{\mu\nu}$ is not necessarily equal to $A^{\nu\mu}$, if it is then $A^{\mu\nu}$ is *symmetric*. Tensors can also be anti–symmetric where $A^{\mu\nu} = -A^{\nu\mu}$.

2.2 Raising and Lowering Indices

As has been mentioned elsewhere, we can raise and lower vector indices using the metric:

$$x_\nu = g_{\mu\nu} x^\mu$$

where we now see how this is done with our rules on contracting the dummy index. This can be extended to higher order tensors:

$$g_{\mu\nu} F^{\mu\rho} = F_\nu{}^\rho$$

here we have contracted against the first index, and we now also see why it's important to offset the lower and upper indices to indicate which is the first and second. We can repeat this a 2^{nd} time:

$$g_{\sigma\rho} F_\nu{}^\rho = F_{\nu\sigma}$$
$$g_{\sigma\rho} g_{\mu\nu} F^{\mu\rho} = F_{\nu\sigma}$$

It's worth writing this out in full, component by component, as it will help us to understand how to manipulate tensor indices. From the Dirac paper we know that we use the $(+, -, -, -)$ shorthand for the metric, so we have:

$$g_{\mu\nu} F^{\mu\rho} = F_\nu{}^\rho$$

we can iterate through the $F_\nu{}^\rho$ indices as follows:

$$F_0{}^\rho = g_{\mu 0} F^{\mu\rho}$$
$$F_1{}^\rho = g_{\mu 1} F^{\mu\rho}$$
$$F_2{}^\rho = g_{\mu 2} F^{\mu\rho}$$
$$F_3{}^\rho = g_{\mu 3} F^{\mu\rho}$$

expanding $F_0{}^\rho$ we do so by expanding the μ index on the *right hand side*

$$F_0{}^\rho = g_{00} F^{0\rho} + g_{10} F^{1\rho} + g_{20} F^{2\rho} + g_{30} F^{3\rho}$$

where all terms except the first are zero from the definition of $g_{\mu\nu}$, hence

$$F_0{}^\rho = F^{0\rho}$$

continuing

$$
\begin{aligned}
F_1{}^\rho &= -F^{1\rho}\\
F_2{}^\rho &= -F^{2\rho}\\
F_3{}^\rho &= -F^{3\rho}
\end{aligned}
$$

The net effect, therefore, is to reverse the sign of all the rows of $F^{\mu\nu}$ except the first[3]. Lowering the second index we proceed:

$$
\begin{aligned}
g_{\sigma\rho}F_\nu{}^\rho &= F_{\nu\sigma}\\
F_{\nu 0} &= g_{0\rho}F_\nu{}^\rho\\
&= g_{00}F_\nu{}^0 + g_{01}F_\nu{}^1 + g_{02}F_\nu{}^2 + g_{03}F_\nu{}^3\\
&= F_\nu{}^0\\
F_{\nu 1} &= -F_\nu{}^1\\
F_{\nu 2} &= -F_\nu{}^2\\
F_{\nu 3} &= -F_\nu{}^3
\end{aligned}
$$

and the net effect of this is to reverse the sign of all the columns. We can summarise the combined effect of lowering both indices using the shorthand:

$$
g_{\sigma\rho}g_{\mu\nu} = \begin{bmatrix} + & - & - & - \\ - & + & + & + \\ - & + & + & + \\ - & + & + & + \end{bmatrix} \tag{1}
$$

2.3 Tensor Products

In previous sections we have seen the product of two (or more) tensors and so far the rules have seemed fairly straight forward – if an index is repeated we "contract" (and the order of the resultant tensor is reduced), if not we create a higher order tensor. However, so far we have only contracted one index at a time, the extension of this to contracting two indices is not immediately obvious. For example how do we manage the product $A^{\mu\nu}B_{\mu\nu}$?

Our rules on dummy indices would seem to say neither is a free index, therefore the result would have no index, in others words this tensor product results in a scalar result ie this is the scalar product of two tensors. This applies generally ie $A^{\mu\nu}B_{\mu\nu}$, $A^{\mu\nu\rho}B_{\mu\nu\rho}$, $A_\mu{}^\nu A^\mu{}_\nu$ are all scalar products. If we consider $A^{\mu\nu}B_{\mu\nu}$ we interpret this as contracting the first index of A with the first of B, and similarly for the second. It turns out that this will be useful later, so it's also worth working this through in detail.

To calculate $A^{\mu\nu}B_{\mu\nu}$ we carry out the contraction in two stages:

$$
A^{\mu\nu}B_{\mu\rho} = C^\nu{}_\rho
$$

then we contract $C^\nu{}_\rho$ with itself by setting $\nu = \rho$

$$
C^\nu{}_\nu = C^0{}_0 + C^1{}_1 + C^2{}_2 + C^3{}_3
$$

[3]We use the convention that the first index indicates the rows when we lay the tensor out so it *looks like* a matrix.

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this is clearly similar to the trace of a matix, but means we only need to calculate the diagonal components.

$$
\begin{aligned}
C^0{}_0 &= A^{\mu 0}B_{\mu 0} \\
C^1{}_1 &= A^{\mu 1}B_{\mu 1} \\
C^2{}_2 &= A^{\mu 2}B_{\mu 2} \\
C^3{}_3 &= A^{\mu 3}B_{\mu 3}
\end{aligned}
$$

$A^{\mu 0}B_{\mu 0}$ expands as:

$$A^{\mu 0}B_{\mu 0} = A^{00}B_{00} + A^{10}B_{10} + A^{20}B_{20} + A^{30}B_{30}$$

similarly

$$
\begin{aligned}
A^{\mu 1}B_{\mu 1} &= A^{01}B_{01} + A^{11}B_{11} + A^{21}B_{21} + A^{31}B_{31} \\
A^{\mu 2}B_{\mu 2} &= A^{02}B_{02} + A^{12}B_{12} + A^{22}B_{22} + A^{32}B_{32} \\
A^{\mu 3}B_{\mu 3} &= A^{03}B_{03} + A^{13}B_{13} + A^{23}B_{23} + A^{33}B_{33}
\end{aligned}
$$

by inspection we see that $A^{\mu\nu}B_{\mu\nu}$ is the component–wise product then sum of the tensor components.

3 Deriving Maxwell's Equations

In many courses Electro-Magnetism and Maxwell's equations are taught from more or less the historical perspective of how they were developed, based on experimental work of Faraday, Ampère and so on.

Latterly, once Special Relativity was established, it was found that the Electro-Magnetic fields/vectors could be combined into a single second order tensor, and this facilitated a better insight into Electro-Magnetism. Later, after the principles of quantum mechanics were established (particularly the Dirac Equation), and the Lagrangian formulation was developed, it also became clear that Lagrangians are (or at least seem to be) the fundamental description of particles and fields. In particular, the application of Hamilton's Principle to the right Lagrangian could lead to the derivation of a number of basic equations including both the Klein-Gordon and Dirac's equations (although not Schrödinger's). Somewhat magically, if one then applies local gauge invariance to the Dirac Lagrangian a field appears and from this field it is possible to derive Maxwell's Equations (and therefore all of Electro-Magnetism).

However, the step from the Electro-Magnetic field tensor to Maxwell's Equations often appears to start with prior knowledge of the latter, so not really being a full derivation. It is the aim of this paper to show that Maxwell's Equations can be derived with no a priori knowledge.

In order to get to Maxwell's Equations the starting point is quite simple:

- Special Relativity (ie the speed of light is invariant in all inertial frames of reference).
- First quantisation.
- Lagrangians and Hamilton's Principle.
- Local Gauge Invariance.

The rough outline of the derivation is as follows:

1. Special relativity allows one to deduce the energy-momentum relationship.
2. From there it is possible to derive the Klein–Gordon Equation using the principle of first quantisation.
3. Dirac's Equation can then be derived (see Coker [1]).
4. Then, using the Lagrangian formulation and Hamilton's Principle you can "build" the Dirac Lagrangian.
5. If one then applies local gauge invariance to this Lagrangian, you need to introduce a gauge field.
6. This field can then be included in a Lagrangian of its own.
7. Application of Hamilton's Principle from there leads to the compact and Lorentz Invariant form of Maxwell's Equations, although not the equations in their commonly understood form.
8. Finally, by expanding the field tensor into its components, Maxwell's Equations will emerge.

This paper covers the argument from 4 onwards.

4 Lagrangians

This section is a fairly swift introduction to Lagrangians, for a better start Hamill [2] is a good place to begin. For Particle Physics, the Lagrangian is defined as:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\phi, \partial\phi/\partial t, \nabla\phi) \\ &= \mathcal{L}(\phi, \partial_\mu\phi)\end{aligned}$$

here the calligraphic $\mathcal{L}$ is used to denote the fact that strictly $\mathcal{L}$ is the Lagrangian *density* and therefore:

$$L = \int_{x_2}^{x_1} \mathcal{L}(\phi, \partial_\mu\phi) d^3x$$

Where the use of d^3x implies we are integrating over the 3 spatial dimensions.

In order to demonstrate why Lagrangians are so important we first need to undertand the term "action" and Hamilton's[4] Principle.

The action of a system is the integral of the Lagrangian over time, as the system evolves in configuration space.

$$\begin{aligned}I &= \int_{t_1}^{t_2} L dt \\ &= \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \mathcal{L} d^3x\end{aligned}$$

[4]Sir William Rowan Hamilton 1805 — 1865

where the limits of the integration are the start and finish times/positions of the process being analysed.

Hamilton's principle states that the path or trajectory that a system follows through configuration space is that which *minimises the action*[5].

If we are to use this Principle, we will need a method to determine that a trajectory is stationary, and for this we need the Calculus of Variations, in which case Hamilton's principle is equivalent to saying that along the trajectory the *variation* of the action is zero:

$$\begin{aligned} \delta I &= \delta \int_{t_1}^{t_2} L dt \\ &= \int_{t_1}^{t_2} \delta L dt \\ &= \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \delta \mathcal{L} d^3x \\ &= 0 \end{aligned}$$

(note the use of the lower-case δ to denote the variation, as distinct from the derivative operators d or ∂, although they do commute, i.e. $\delta(\partial\phi) = \partial(\delta\phi)$).

The power of the Lagrangian and Hamilton's Principle is that the latter can be used to derive basic equations such as the Klein–Gordon Equation or Dirac's Equation, provided of course that you choose the right Lagrangian. In addition, using Lagrangians means we can also use Noether's Theorem which states that symmetries of the Lagrangian imply conserved quantities. Finally, as we will find out in later papers, the individual terms in a Lagrangian describe in a fundamental way how particles and fields interact and for this reason the Lagrangian is in many ways the fundamental formulation of Particle Physics.

4.1 A Scalar Field Lagrangian

In order to see the power of the Lagrangian method it's best to work through an example, and for this we choose a simple scalar field. The Lagrangian for a free, non-interacting, real scalar field is given by:

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi)(\partial^\mu \phi) - m^2\phi^2] \tag{2}$$

In order to proceed, we calculate the variation of this Lagrangian, $\delta\mathcal{L}$, integrate it to calculate δI and set the result to zero. Before we start, we note that calculating a variation is very similar to differentiation and partial differentiation. So if $f = f(x, y)$ then:

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

and if $f = ab$ then:

$$\delta f = a\delta b + b\delta a$$

[5]Strictly, the trajectory has to be *stationary*, rather than an actual minimum, but from a mathematical point of view there isn't much distinction.

So:

$$\begin{aligned}\delta\mathcal{L} &= \delta\left\{\tfrac{1}{2}[(\partial_\mu\phi)(\partial^\mu\phi) - m^2\phi^2]\right\} \\ &= \frac{1}{2}\left\{\partial^\mu\phi\delta(\partial_\mu\phi) + \partial_\mu\phi\delta(\partial^\mu\phi) - 2m^2\phi(\delta\phi)\right\} \\ &= \frac{1}{2}\left\{\partial^\mu\phi\partial_\mu(\delta\phi) + \partial_\mu\phi\partial^\mu(\delta\phi) - 2m^2\phi(\delta\phi)\right\}\end{aligned}$$

where in the last line we have used the fact that $\delta(\partial\phi) = \partial(\delta\phi)$. Placing $\delta\mathcal{L}$ under the integration, we get the formula for the variation of the Action:

$$\delta I = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \frac{1}{2}\left\{\partial^\mu\phi\partial_\mu(\delta\phi) + \partial_\mu\phi\partial^\mu(\delta\phi) - 2m^2\phi(\delta\phi)\right\} d^3x$$

To proceed further we need a quick diversion into Integration By Parts.

$$\begin{aligned}\frac{d}{dx}(AB) &= B\frac{dA}{dx} + A\frac{dB}{dx} \\ \int_{x_1}^{x_2} d(AB) &= \int_{x_1}^{x_2} BdA + \int_{x_1}^{x_2} AdB \\ \big[AB\big]_{x_1}^{x_2} &= \int_{x_1}^{x_2} BdA + \int_{x_1}^{x_2} AdB\end{aligned}$$

if x_1 and x_2 are chosen such that AB is the same at these points the left hand side of this equation is zero, hence:

$$\int_{x_1}^{x_2} BdA = -\int_{x_1}^{x_2} AdB$$

Dropping the integration signs we have a general rule that $AdB = -BdA$ *provided that this is expressed in a definite integral where AB is the same at the limits of the integral.*

To proceed with calculating δI we look at the first term and say that $A = \partial^\mu\phi$ and $dB = \partial^\mu\delta\phi$, therefore $AB = \partial^\mu\phi\delta\phi$. At the beginning and end of the trajectory $\delta\phi = 0$ by definition and therefore $AB = 0$ also, meaning we can use this trick and we can also make a similar substitution for the 2^{nd} term so we get:

$$\begin{aligned}\delta I &= \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \frac{1}{2}\left\{\partial^\mu\phi\partial_\mu(\delta\phi) + \partial_\mu\phi\partial^\mu(\delta\phi) - 2m^2\phi(\delta\phi)\right\} d^3x \\ &= -\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \frac{1}{2}\left[\delta\phi\partial_\mu\partial^\mu\phi + \delta\phi\partial^\mu\partial_\mu\phi + 2m^2\phi(\delta\phi)\right] d^3x\end{aligned}$$

so we can now factor out the $\delta\phi$ term

$$\delta I = -\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \frac{1}{2}\delta\phi\left[\partial_\mu\partial^\mu\phi + \partial^\mu\partial_\mu\phi + 2m^2\phi\right] d^3x$$

Finally, applying Hamilton's Principle requires that $\delta I = 0$. In order for this to be true in general, the quantity in the square brackets must be identically zero. Bearing in mind that $\partial_\mu\partial^\mu = \partial^\mu\partial_\mu$ we get:

$$\partial^\mu\partial_\mu\phi + m^2\phi = 0$$

which is the Klein–Gordon Equation.

From this initial example we can see how equations (often referred to as equations of motion) for particles (or waves) can be derived from the Lagrangian usng Hamilton's Principle.

4.2 A Lagrangian for Spin–Half Fields

Creating a Lagrangian for spin–half particles is actually easier, although first we need to introduce the Dirac Adjoint $\bar{\psi} = \psi^\dagger\gamma^0$. The reason $\bar{\psi}$ is used is that $\bar{\psi}\psi$ is Lorentz Invariant whereas $\psi^\dagger\psi$ is not. This leads us to the Dirac Lagrangian for a free, non-interacting, fermionic field:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \tag{3}$$

If we now calculate the variation of this Lagrangian, we do so by holding ψ constant and varying $\bar{\psi}$, giving us:

$$\delta\mathcal{L} = \delta\bar{\psi}(i\gamma^\mu\partial_\mu\psi - m\psi)$$

and in our usual formulation this requires that $i\gamma^\mu\partial_\mu\psi - m\psi = 0$ which is Dirac's Equation.

If we now repeat the process holding $\bar{\psi}$ constant whilst we vary ψ we can derive Dirac's Equation for the adjoint spinor $\bar{\psi}$. The calculation is a little more involved, but nothing we haven't already done:

$$\begin{aligned}\delta\mathcal{L} &= \bar{\psi}i\gamma^\mu\delta(\partial_\mu\psi) - m\bar{\psi}\delta\psi \\ &= \bar{\psi}i\gamma^\mu\partial_\mu(\delta\psi) - m\bar{\psi}\delta\psi \\ &= -i\partial_\mu\bar{\psi}\gamma^\mu\delta\psi - m\bar{\psi}\delta\psi \\ &= -(i\partial_\mu\bar{\psi}\gamma^\mu\delta\psi + m\bar{\psi}\delta\psi)\end{aligned}$$

where we've used the integration by parts trick between the 2^{nd} and 3^{rd} lines. Hence:

$$i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0$$

which is Dirac's Equation for adjoint spinor. You can actually derive this equation directly from the equation for ψ, but it requires knowledge of various γ matrix identities.

Using both forms of Dirac's Equation we can now do something a bit clever:

$$\begin{aligned}i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} &= 0 \\ i\gamma^\mu\partial_\mu\psi - m\psi &= 0\end{aligned}$$

if we post-multiply the first equation by ψ and pre-multiply the 2^{nd} by $\bar{\psi}$:

$$\begin{aligned}i\partial_\mu\bar{\psi}\gamma^\mu\psi + m\bar{\psi}\psi &= 0 \\ i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi &= 0\end{aligned}$$

then add, the mass terms cancel and we can divide through by i:

$$\partial_\mu\bar{\psi}\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi = 0$$

this is now the differential of a product and we get:

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$$

which readers will recognise as a continuity equation for the quantity $\bar{\psi}\gamma^\mu\psi$. We will be using this later but for now we define:

$$j^\mu = q\bar{\psi}\gamma^\mu\psi \tag{4}$$

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where q is a constant we have included in case we need it later. As a constant it has no effect on the continuity equation; note that the same conserved current can be derived using Noether's Theorem and working from the Dirac Lagrangian, but the derivation above is quite a lot simpler.

Incidentally if we compare Equation (2) with Equation (3) we could be forgiven for making the intuitive leap that terms in a Lagrangian that are *negative* and *quadratic* in a field represent a *mass* term.

4.3 The Euler–Lagrange Equation

Before we move on, it's worth looking at the Euler–Lagrange equation. This is in fact a general statement of Hamilton's Principle and many authors use this directly, rather than applying the principle to the actual Lagrangian.

With our definition of the Lagrangian and the Action:

$$I = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \mathcal{L}(\phi, \partial_\mu \phi) d^3x$$

we start by calculating $\delta\mathcal{L}$ starting with the definition of $\mathcal{L}$:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}(\phi, \partial_\mu \phi) \\ \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\partial_\mu\phi \\ &= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu\delta\phi \end{aligned}$$

integrating by parts

$$= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi$$

hence we get

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

which is the Euler–Lagrange equation for a continuous field. The term in brackets is also defined as the *momentum density* Π^μ:

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \tag{5}$$

5 Gauge Theory

Gauge Theory is the historical term, but it is unfortunately a bit of a misnomer, in most cases what we mean when we talk about gauge invariance is phase invariance. If we start with the Dirac Lagrangian:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$

and if we now make the 'gauge' transformation

$$\psi \rightarrow e^{iq\chi}\psi$$

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where q is a constant that we've added in because we think we may need it later and χ is our phase variable, the Lagrangian becomes

$$\mathcal{L} = ie^{-iq\chi}\bar{\psi}\gamma^{\mu}\partial_{\mu}(e^{iq\chi}\psi) - me^{-iq\chi}\bar{\psi}e^{iq\chi}\psi \tag{6}$$

If χ is constant everywhere (in space and time) then the exponential terms cancel and we can see that $\mathcal{L}$ is invariant to this type of gauge transformation. This is known as a Global Invariance.

By contrast, if χ is not constant then we have what is called a Local Invariance. For the 2^{nd} term in Equation (6) this isn't an issue as the exponential terms still cancel, however for the first term the exponential is inside the differential and therefore leads to an additional term in $\partial_{\mu}\chi$:

$$\begin{aligned} i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi &\rightarrow ie^{-iq\chi}\bar{\psi}\gamma^{\mu}\partial_{\mu}(e^{iq\chi}\psi) \\ &= ie^{-iq\chi}\bar{\psi}\gamma^{\mu}(e^{iq\chi}\partial_{\mu}\psi + iq\psi\partial_{\mu}\chi e^{iq\chi}) \\ &= i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - q\bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\chi \end{aligned} \tag{7}$$

If we now define a field A_{μ} such that it transforms:

$$A_{\mu} \rightarrow A_{\mu} - \partial_{\mu}\chi \tag{8}$$

then the quantity $q\bar{\psi}\gamma^{\mu}\psi A_{\mu}$ transforms thus:

$$q\bar{\psi}\gamma^{\mu}\psi A_{\mu} \rightarrow q\bar{\psi}\gamma^{\mu}\psi A_{\mu} - q\bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\chi \tag{9}$$

if we now subtract (9) from (7) the terms in $\partial_{\mu}\chi$ cancel and we can therefore say that

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi - q\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

is invariant to a change of local gauge. We can also substitute for j^{μ} from Equation (4)

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi - j^{\mu}A_{\mu}$$

5.1 Gauge Fields

This is now our first example of a gauge field, the need for which has arisen *solely* because we have required the Dirac Lagrangian to be locally gauge invariant. The term we have created in the Lagrangian is what we will later call an interaction term, including as it does the product of a field and (in this case) what looks like a current. So the question now arises, is there a gauge invariant term for the field on its own which should also be included in the Lagrangian?

Before we address that question it's worth considering that:

- The Lagrangian is a statement of energy, which is a scalar quantity, therefore the Lagrangian and all its individual terms also need to be scalar.
- Therefore any additional terms must be one (or a combination) of:
  - A naturally scalar quantity.

- A scalar quantity formed by the scalar product of two vectors.
- A scalar quantity formed of the scalar product of two tensors.

The simplest, most obvious Lagrangian term is simply $A^\mu A_\mu$, giving us[6]:

$$\mathcal{L} = A^\mu A_\mu - j^\mu A_\mu$$
$$\delta\mathcal{L} = \delta A^\mu A_\mu + A^\mu \delta A_\mu - j^\mu \delta A_\mu$$

remembering that $X^\mu Y_\mu = X_\mu Y^\mu$

$$\delta\mathcal{L} = 2A^\mu \delta A_\mu - j^\mu \delta A_\mu$$

which implies $A^\mu = \frac{1}{2} j^\mu$ which is obviously nonsense. In fact we didn't really need to do that analysis, contracting a vector with itself is always going to lead to this result meaning the new term for the Lagrangian can't be the scalar product of two vectors, so our first real attempt needs to be with a second order tensor for which we will use the symbol $F^{\mu\nu}$.

6 The Field Tensor

The previous section has shown us that local gauge invariance requires a gauge field to be included in the Lagrangian. In addition, any term in the Lagrangian, solely for the gauge field, has to be formed from at least a 2^{nd} order tensor. Whilst this may seem an extra degree of complexity (this is the first time we have really needed an actual tensor), it turns out that understanding this tensor will give us a better insight into the meaning of the gauge field A^μ.

In order to create a scalar term, we need to fully contract this tensor with another 2^{nd} order tensor (which is more or less the definition of the scalar product of two tensors). In principle we could use any tensor, but in practice we have no justification for using anything other than the A^μ field. However, in order to build a 2^{nd} order tensor from the A^μ field, we need to carry out a tensor *multiplication* with another vector (ie multiply two first order tensors together to create a second order one). Again we have no justification for using any other field, but we can use the *derivative* of the the A^μ field.

Hence our first attempt at building a tensor is:

$$F^{\mu\nu} = \partial^\mu A^\nu$$

We can now carry out a first attempt at applying Hamilton's Principle to a Lagrangian:

$$\begin{aligned}\mathcal{L} &= F^{\mu\nu} F_{\mu\nu} - j^\nu A_\nu \\ \delta\mathcal{L} &= (\delta F^{\mu\nu} F_{\mu\nu} + F^{\mu\nu} \delta F_{\mu\nu}) - j^\nu \delta A_\nu \\ &= 2F^{\mu\nu} \delta F_{\mu\nu} - j^\nu \delta A_\nu\end{aligned}$$

substituting for $F_{\mu\nu}$ and remembering that δ and ∂ commute

$$\begin{aligned}&= 2F^{\mu\nu} \delta(\partial_\mu A_\nu) - j^\nu \delta A_\nu \\ &= 2F^{\mu\nu} \partial_\mu \delta A_\nu - j^\nu \delta A_\nu\end{aligned}$$

[6]Remember this is a Lagrangian for the field A^μ so we include all terms that include the field.

using the integration by parts trick allows us to move the ∂_μ to the left at the expense of a minus sign

$$= -2\partial_\mu F^{\mu\nu}\delta A_\nu - j^\nu \delta A_\nu$$

from here we can simplify the result if we insert $-\frac{1}{2}$ into the Lagrangian at the beginning:

$$\mathcal{L} = -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} - j^\nu A_\nu$$
$$\delta\mathcal{L} = \partial_\mu F^{\mu\nu}\delta A_\nu - j^\nu \delta A_\nu$$

hence

$$\partial_\mu F^{\mu\nu} = j^\nu$$

This looks very neat, but the problem with this attempt is that it is pretty obviously not gauge invariant, ie

$$F^{\mu\nu} = \partial^\mu A^\nu$$
$$A^\nu \to A^\nu - \partial^\nu \chi$$
$$F^{\mu\nu} \to \partial^\mu A^\nu - \partial^\mu \partial^\nu \chi$$

Instead we need to consider $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, which is gauge invariant:

$$A^\nu \to A^\nu - \partial^\nu \chi$$
$$F^{\mu\nu} \to \partial^\mu A^\nu - \partial^\mu \partial^\nu \chi - \partial^\nu A^\mu + \partial^\nu \partial^\mu \chi$$

∂^ν and ∂^μ are effectively the same operator hence

$$\partial^\nu \partial^\mu \chi = \partial^\mu \partial^\nu \chi$$
$$F^{\mu\nu} \to \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$$

At this point it's worth noting that if we swap indices consistently we get:

$$F^{\nu\mu} = \partial^\nu A^\mu - \partial^\mu A^\nu$$
$$= -F^{\mu\nu}$$

ie $F^{\mu\nu}$ is *anti-symmetric.*

Reverting to our Lagrangian:

$$\mathcal{L} = F^{\mu\nu}F_{\mu\nu} - j^\nu A_\nu$$
$$\delta\mathcal{L} = 2F^{\mu\nu}\delta F_{\mu\nu} - j^\nu \delta A_\nu$$

substituting for $F_{\mu\nu}$, remembering that we are varying the field $A^\mu = A^\nu$.

$$\delta F_{\mu\nu} = \delta(\partial_\mu A_\nu) - \delta(\partial_\nu A_\mu)$$
$$= \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu$$
$$\delta\mathcal{L} = 2F^{\mu\nu}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) - j^\nu \delta A_\nu$$

again, integrating by parts

$$= -2(\partial_\mu F^{\mu\nu}\delta A_\nu - \partial_\nu F^{\mu\nu}\delta A_\mu) - j^\nu \delta A_\nu$$

now we need to swap around the tensor indices for the middle term

$$\begin{aligned}
\partial_\nu F^{\mu\nu}\delta A_\mu &= \partial_\mu F^{\nu\mu}\delta A_\nu \\
&= -\partial_\mu F^{\mu\nu}\delta A_\nu \\
\delta\mathcal{L} &= -2(\partial_\mu F^{\mu\nu}\delta A_\nu + \partial_\mu F^{\mu\nu}\delta A_\nu) - j^\nu \delta A_\nu \\
&= -4\partial_\mu F^{\mu\nu}\delta A_\nu - j^\nu \delta A_\nu
\end{aligned}$$

so if we start with:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\nu A_\nu$$

the -4 cancels and interestingly we get the same final equation as our first attempt:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{10}$$

It's worth taking a moment to understand this equation. j^ν we know from Equation (4) is a conserved quantity, although we perhaps don't know exactly what is being conserved. More importantly this conserved, current-like term is equal to the differential of the new tensor we have constructed from the gauge field. We knew, of course, that the two terms had to be related somehow as all we've done to get to Equation (10) is apply Hamilton's Principle as we previously did for the scalar field and the Dirac Lagrangians. Equation (10) is somehow the equation(s) of motion for both the gauge field and the current.

The anti–symmetry of $F^{\mu\nu}$ means that it only has 6 independent components. In the context of cartesian coordinates, this is a bit of a hint that the tensor may be describing two vector quantities. To proceed further we need to break $F^{\mu\nu}$ into its components.

6.1 Components of the Tensor

Given that $F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)$ we can break it down into its components. Here we again use the convention that the first index refers to the rows of the tensor. So for the first row:

$$F^{0\nu} = \partial^0 A^\nu - \partial^\nu A^0$$

looking at just the first term

$$\partial^0 A^\nu = \frac{\partial A^\nu}{\partial t}$$

we now expand the second index, which is the columns.

$$\partial^0 A^\nu = \begin{bmatrix} \frac{\partial A^0}{\partial t} & \frac{\partial A^1}{\partial t} & \frac{\partial A^2}{\partial t} & \frac{\partial A^3}{\partial t} \end{bmatrix}$$

repeating this for the second term

$$\partial^\nu A^0 = \begin{bmatrix} \frac{\partial A^0}{\partial t} & -\frac{\partial A^0}{\partial x} & -\frac{\partial A^0}{\partial y} & -\frac{\partial A^0}{\partial z} \end{bmatrix}$$

putting both together

$$F^{0\nu} = \begin{bmatrix} \frac{\partial A^0}{\partial t} - \frac{\partial A^0}{\partial t} & \frac{\partial A^1}{\partial t} + \frac{\partial A^0}{\partial x} & \frac{\partial A^2}{\partial t} + \frac{\partial A^0}{\partial y} & \frac{\partial A^3}{\partial t} + \frac{\partial A^0}{\partial z} \end{bmatrix}$$

For the next row we repeat the process, tracking minus signs carefully

$$F^{1\nu} = \begin{bmatrix} -\frac{\partial A^0}{\partial x} - \frac{\partial A^1}{\partial t} & -\frac{\partial A^1}{\partial x} + \frac{\partial A^1}{\partial x} & -\frac{\partial A^2}{\partial x} + \frac{\partial A^1}{\partial y} & -\frac{\partial A^3}{\partial x} + \frac{\partial A^1}{\partial z} \end{bmatrix}$$

and so on for the other rows. Here we can see that the diagonal terms will all be zero, which is a relief as we are expecting an anti-symmetric tensor, but we can also see the anti-symmetry emerging for the off diagonal terms. So the full tensor is:

$$F^{\mu\nu} = \begin{bmatrix} 0 & \frac{\partial A^1}{\partial t} + \frac{\partial A^0}{\partial x} & \frac{\partial A^2}{\partial t} + \frac{\partial A^0}{\partial y} & \frac{\partial A^3}{\partial t} + \frac{\partial A^0}{\partial z} \\ -\frac{\partial A^0}{\partial x} - \frac{\partial A^1}{\partial t} & 0 & -\frac{\partial A^2}{\partial x} + \frac{\partial A^1}{\partial y} & -\frac{\partial A^3}{\partial x} + \frac{\partial A^1}{\partial z} \\ -\frac{\partial A^0}{\partial y} - \frac{\partial A^2}{\partial t} & -\frac{\partial A^1}{\partial y} + \frac{\partial A^2}{\partial x} & 0 & -\frac{\partial A^3}{\partial y} + \frac{\partial A^2}{\partial z} \\ -\frac{\partial A^0}{\partial z} - \frac{\partial A^3}{\partial t} & -\frac{\partial A^1}{\partial z} + \frac{\partial A^3}{\partial x} & -\frac{\partial A^2}{\partial z} + \frac{\partial A^3}{\partial y} & 0 \end{bmatrix}$$

If we make the not unreasonable assumption that A^μ is a four-vector, then the components A^i define a spatial vector[7] which we can denote as $\mathbf{A}$, if we also define $A^0 = \phi$ we get $A^\mu = (\phi, \mathbf{A})$ then the first "row" of the tensor becomes (dropping the first component which is zero):

$$F^{0i} = \frac{\partial \mathbf{A}}{\partial t} + \nabla\phi = \mathbf{X}$$

and

$$F^{i0} = -\mathbf{X}$$

which simplifies three of the six independent components. We then note that the remaining three components look like the components of the curl of a vector:

$$\begin{aligned} \mathbf{Y} &= \nabla \times \mathbf{A} \\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A^1 & A^2 & A^3 \end{vmatrix} \end{aligned}$$

so we get

$$F^{\mu\nu} = \begin{bmatrix} 0 & X^1 & X^2 & X^3 \\ -X^1 & 0 & -Y^3 & Y^2 \\ -X^2 & Y^3 & 0 & -Y^1 \\ -X^3 & -Y^2 & Y^1 & 0 \end{bmatrix}$$

From Equation (4) we know that $j^\nu = q\bar{\psi}\gamma^\nu\psi$; whilst it's not obvious, it isn't a surprise that j^ν is a four–vector[8]. Hence we can say that $j^\nu = (\rho, \mathbf{J})$, so $\rho = j^0$ therefore

$$\rho = \partial_\mu F^{\mu 0}$$

$F^{00} = 0$ so we can drop the first component, leaving us with

$$\rho = \partial_i F^{i0}$$

[7] Remembering our convention for the use of a roman i as the index.

[8] This can be shown from the Lorentz transformation properties of ψ, see Thomson [3, Appendix B.3], for example.

here we see that i is repeated in the upper and lower positions, so we *contract* against this index, hence

$$\rho = \frac{\partial(-X^1)}{\partial x} + \frac{\partial(-X^2)}{\partial y} + \frac{\partial(-X^3)}{\partial z}$$
$$\rho = -\nabla.\mathbf{X} \tag{11}$$

Similarly $\mathbf{J} = j^i = \partial_\mu F^{\mu i}$ so we only consider the rightmost three colums of $F^{\mu\nu}$ (and dropping the zero terms)

$$\partial_\mu F^{\mu 1} = \frac{\partial X^1}{\partial t} + \frac{\partial Y^3}{\partial y} - \frac{\partial Y^2}{\partial z}$$
$$\partial_\mu F^{\mu 2} = \frac{\partial X^2}{\partial t} - \frac{\partial Y^3}{\partial x} + \frac{\partial Y^1}{\partial z}$$
$$\partial_\mu F^{\mu 3} = \frac{\partial X^3}{\partial t} + \frac{\partial Y^2}{\partial x} - \frac{\partial Y^1}{\partial z}$$

again this can be re-arranged into a partial derivative and the curl of a vector:

$$\mathbf{J} = \frac{\partial \mathbf{X}}{\partial t} + \nabla \times \mathbf{Y} \tag{12}$$

and by now, the expected form of Maxwell's Equations should be emerging. From the simple definitions of $\mathbf{X}$ and $\mathbf{Y}$ we can see:

$$\nabla.\mathbf{Y} = \nabla.(\nabla \times \mathbf{A}) = 0 \tag{13}$$
$$\nabla \times \mathbf{X} = \nabla \times (\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi)$$
$$= \frac{\partial(\nabla \times \mathbf{A})}{\partial t}$$
$$= \frac{\partial \mathbf{Y}}{\partial t} \tag{14}$$

if we now make the final substitutions $\mathbf{X} = -\mathbf{E}$ and $\mathbf{Y} = \mathbf{B}$ we get the expected Maxwell Equations:

$$\nabla.\mathbf{E} = \rho \qquad \text{from (11)}$$
$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \qquad \text{from (12)}$$

and

$$\nabla.\mathbf{B} = 0 \qquad \text{from (13)}$$
$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \text{from (14)}$$

and finally the field tensor becomes:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{bmatrix}$$

where the numerical indices have been swapped to indicate cartesian axes.

https://doi.org/10.33774/coe-2024-w9tsh ORCID: https://orcid.org/0009-0005-4766-604X

6.2 Gauge Invariance Revisited

Having defined the **E** and **B** fields in terms of the derivatives of the original A^μ field, it's worth confirming that we have retained the gauge invariance that was required in Equation (8). Given that $F^{\mu\nu}$ is invariant, it would be surprising if the constituent fields were not, but nevertheless instructive to check.

$$\begin{aligned}\mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi\end{aligned} \tag{15}$$

where $A^\mu = (\phi, \mathbf{A})$. To confirm we haven't lost the gauge invariance we need to expand Equation (8) but in its *contravariant* form:

$$A^\mu \rightarrow A^\mu - \partial^\mu \chi$$

note that χ is not a four–vector, so

$$\partial^\mu \chi = \left(\frac{\partial \chi}{\partial t}, -\nabla \chi\right)$$

so

$$\phi \rightarrow \phi - \frac{\partial \chi}{\partial t}$$

and

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla \chi$$

therefore

$$\begin{aligned}\mathbf{B} &\rightarrow \nabla \times (\mathbf{A} + \nabla \chi) \\ &= \nabla \times \mathbf{A} + \nabla \times \nabla \chi \\ &= \nabla \times \mathbf{A}\end{aligned}$$

and

$$\begin{aligned}\mathbf{E} &\rightarrow -\frac{\partial(\mathbf{A} + \nabla \chi)}{\partial t} - \nabla(\phi - \frac{\partial \chi}{\partial t}) \\ &= -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi - \frac{\partial}{\partial t}(\nabla \chi) + \nabla \left(\frac{\partial \chi}{\partial t}\right) \\ &= -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi\end{aligned}$$

6.3 Expanding the Lagrangian

Using Equation (1) we can lower both indices:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{bmatrix}$$

From subsection 2.3 we see that $F^{\mu\nu}F_{\mu\nu}$ is the component–wise product and then sum of the components of the two sensors

$$\begin{aligned}F^{\mu\nu}F_{\mu\nu} &= -2((E^x)^2 + (E^y)^2 + (E^z)^2) + 2((B^x)^2 + (B^y)^2 + (B^z)^2) \\ &= -2\mathbf{E}^2 + 2\mathbf{B}^2\end{aligned}$$

hence

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)\end{aligned}$$

We haven't discussed the Hamiltonian[9], but it is defined as follows:

$$\mathcal{H} = \pi\partial_0\phi - \mathcal{L}$$

where

$$\pi = \frac{\partial L}{\partial(\partial_0\phi)}$$

π is known as the conjugate momentum. In fact we can see from Equation (5) that π is just the timelike component of Π^μ (and ϕ is just the field the Hamiltonian refers to)[10].

For the case in hand the field is A^μ, noting that $A^\mu = (\phi, \mathbf{A})$ and in the absence of free charge and a current, $\phi = \nabla\phi = 0$ hence:

$$\begin{aligned}\pi &= \frac{\partial L}{\partial(\partial_0 A^\mu)} \\ \mathcal{H} &= \frac{\partial L}{\partial(\partial_0 A^\mu)}\partial_0 A^\mu - \mathcal{L} \\ \partial_0 A^\mu &= \frac{\partial \mathbf{A}}{\partial t} \\ &= -\mathbf{E} \qquad \text{from (15)}\end{aligned}$$

therefore

$$\begin{aligned}\mathcal{H} &= \frac{\partial L}{\partial(-\mathbf{E})}(-\mathbf{E}) - \mathcal{L} \\ &= \frac{\partial L}{\partial \mathbf{E}}\mathbf{E} - \mathcal{L}\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \mathbf{E}} &= \frac{1}{2}\frac{\partial(\mathbf{E}^2 - \mathbf{B}^2)}{\partial \mathbf{E}} \\ &= \mathbf{E}\end{aligned}$$

so

$$\begin{aligned}\mathcal{H} &= \mathbf{E}^2 - \mathcal{L} \\ &= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\end{aligned}$$

Given that we understand the Hamiltonian to be the total energy and $\mathcal{H}$ is therefore the energy density, this is the result we would expect.

[9] Again Hamill [2] is a good starting point, but most particle physics texts cover this topic.

[10] Four vectors have a timelike component (typically $\mu = 0$) and spacelike components. For the standard spacetime vector x^μ the timelike component is just time, but for four–momentum p^μ the timelike component is in fact energy.

7 Summary

Having established the Dirac Equation through quantum mechanics and special relativity, this equation can be *embedded* in the Lagrangian formulation which links the Lagrangian to equations of motion using Hamilton's Principle. Once the Lagrangian is established the principle of Local Gauge Invariance requires a gauge field to be included. From the gauge field we first develop a tensor then a scalar term that can be fed back into the Lagrangian. If we then re-apply Hamilton's Principle we can show how the tensor is related to the conserved current that is implied (by Noether's Theorem) from the Dirac Lagrangian. Still working from first principles, it is then possible to expand the field tensor into its components and from there Maxwell's Equations emerge.

It's also worth noting that the fact that local gauge invariance leads us from the Dirac Equation to Maxwell's Equations and Electro–Magnetism, is pretty strong evidence that this principle is correct and indeed fundamental to physics.

All of the above can be captured in a single equation, which defines the Lagrangian for Quantum Electrodynamics (QED):

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi - q\bar{\psi}\gamma^{\mu}\psi A_{\mu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

References and Further Reading

[1] Tim Coker. A Simple Derivation of Dirac's Equation. *Cambridge Open Engage*, 2024. doi:10.33774/coe-2024-z7371.

[2] Patrick Hamill. *A Student's Guide to Lagrangians and Hamiltonians*. Cambridge University Press, 2014. ISBN 978-1-1-7004288-9.

[3] Mark Thomson. *Modern Particle Physics*. Cambridge University Press, 2013. ISBN 978-1-107-03426-6.