The Euler product equal to the Riemann zeta function is false

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Abstract: The author of this article disproves the Euler product equal to the Riemann zeta function that was proved by the earlier mathematicians.

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1. Introduction

The Riemann zeta function $\zeta(s)$ for all complex numbers such as $s = a + ib$ is defined by

$$\zeta(s) = \sum n^{-s} = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + \cdots, \quad \forall\, \mathbb{R}e(s) > 1. \tag{1}$$

Where $\mathbb{R}e(s)$ is the real part of all complex numbers.

2. Proof of the Earlier Mathematicians

The earlier mathematicians proved the Euler product equal to the Riemann zeta function [1-4] as follows

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{i=1}^{\infty} \frac{1}{1 - \frac{1}{p_i^s}}, \tag{2}$$

where $\prod_{i=1}^{\infty} \frac{1}{1 - p_i^{-s}}$ is the Euler product and p_i is the i^{th} prime.

The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} \cdots$

$$\frac{1}{2^s}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{12^s} + \frac{1}{14^s} + \frac{1}{16^s} + \cdots$$

$$\zeta(s) - \frac{1}{2^s}\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{15^s} + \frac{1}{17^s} + \cdots$$

$$i.e., \left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + + \frac{1}{15^s} + \cdots$$

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \frac{1}{39^s} + \frac{1}{45^s} + \cdots$$

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) - \frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \cdots$$

$$i.e., \left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s) = 1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\frac{1}{19^s}+\cdots$$

If the above processes are continued, the multiples of prime factors such as 2, 3, 4, 5, 7, ⋯, are removed.

$$\cdots\cdots\cdots\left(1-\frac{1}{13^s}\right)\left(1-\frac{1}{11^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s) = 1. \qquad (3)$$

$$i.e., \prod_{k=1}^{\infty}\left(1-\frac{1}{p_k^s}\right)\zeta(s) = 1 \Rightarrow \zeta(s) = \prod_{k=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_k^s}\right)}.$$

3. Disproof of the Author of this Article

The author of this article disproves the Euler product equal to the Riemann zeta function $\zeta(s)$ that was proved by the earlier mathematicians.

If $s = 1,$ then $\zeta(1)$ denotes the harmonic series.

$$\zeta(1) = \sum_{n=1}^{\infty}\frac{1}{n^s} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\cdots \qquad (4)$$

Also, by substituting $s = 1$ in Equation (3), we obtain

$$\cdots\times\frac{10}{11}\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times\frac{1}{2}\zeta(1) = 1 \Rightarrow \zeta(1) = \frac{2}{1}\times\frac{3}{2}\times\frac{5}{4}\times\frac{7}{6}\times\frac{11}{10}\times\cdots = \prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i}\right)}. \quad (5)$$

From the equations (4)and (5), $\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i}\right)} > \zeta(1)$ obviously.

Similarly, we compute the Euler product for the Riemann zeta function for $s = 2$ as follows. The Basel problem [5,6] indicates that $\zeta(2) \cong 1.644934$.

$$\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i^2}\right)} = \frac{4}{3}\times\frac{9}{8}\times\frac{25}{24}\times\frac{49}{48}\times\frac{121}{120}\times\cdots > 2 \text{ and } \prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i^2}\right)} > \zeta(2).$$

If we continue the above process upto $s = n$,

If we continue the above process upto $s = n,$ then $\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i^n}\right)} > \zeta(n)$

$$\therefore \prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i^s}\right)} > \zeta(s)$$

Form the above result, we conclude that the Euler product is not equal to the Riemann zeta function.

4. Conclusion

In the article, the author has disproved the Euler product representation for the Riemann zeta function that was proved by the earlier mathematicians.

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