The main reason why the Euler product is not equal to the Riemann Zeta function

Chinnaraji Annamalai
School of Management, Indian Institute of Technology, Kharagpur, India
Email: anna@iitkgp.ac.in
https://orcid.org/0000-0002-0992-2584

Abstract: The author of this article has already disproved the Euler product equal to the Riemann zeta function that the earlier mathematicians proved. This article presents the main reason why the Euler product is not equal to the Riemann zeta function.

MSC Classification codes: 11R59, 65D20

Keywords: computation, infinite product, zeta function

1. Introduction

The Riemann zeta function $\zeta(s)$ for all complex numbers such as $s = a + ib$ is defined by

$$\zeta(s) = \sum n^{-s} = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + \cdots, \quad \forall\, \mathbb{R}e(s) > 1. \tag{1}$$

Where $\mathbb{R}e(s)$ is the real part of all complex numbers.

2. Proof of the Earlier Mathematicians

The earlier mathematicians proved the Euler product equal to the Riemann zeta function [1-5] as follows

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{i=1}^{\infty} \frac{1}{1 - \frac{1}{p_i^s}}, \tag{2}$$

where $\prod_{i=1}^{\infty} \frac{1}{1 - p_i^{-s}}$ is the Euler product and p_i is the i^{th} prime.

The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} \cdots$

$$\frac{1}{2^s}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{12^s} + \frac{1}{14^s} + \frac{1}{16^s} + \cdots$$

$$\zeta(s) - \frac{1}{2^s}\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{15^s} + \frac{1}{17^s} + \cdots$$

$$i.e., \left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + + \frac{1}{15^s} + \cdots$$

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \frac{1}{39^s} + \frac{1}{45^s} + \cdots$$

$$\left(1-\frac{1}{2^s}\right)\zeta(s)-\frac{1}{3^s}\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\cdots$$
$$i.e.,\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\frac{1}{19^s}+\cdots$$

If the above processes are continued, the multiples of prime factors such as 2, 3, 4, 5, 7, ⋯, are removed.

$$\cdots\cdots\cdots\left(1-\frac{1}{13^s}\right)\left(1-\frac{1}{11^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1. \qquad (3)$$

$$i.e.,\prod_{k=1}^{\infty}\left(1-\frac{1}{p_k^s}\right)\zeta(s)=1\Rightarrow\zeta(s)=\prod_{k=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_k^s}\right)}.$$

3. Disproof by the Author of this Article

The author of this article disproves the Euler product equal to the Riemann zeta function $\zeta(s)$ that the earlier mathematicians proved.

If $s=1,$ then $\zeta(1)$ denotes the harmonic series.

$$\zeta(1)=\sum_{n=1}^{\infty}\frac{1}{n^s}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\cdots \qquad (4)$$

Also, by substituting $s=1$ in Equation (3), we obtain

$$\cdots\times\frac{10}{11}\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times\frac{1}{2}\zeta(1)=1\ \Rightarrow\zeta(1)=\frac{2}{1}\times\frac{3}{2}\times\frac{5}{4}\times\frac{7}{6}\times\ \frac{11}{10}\times\cdots=\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i}\right)}. \quad (5)$$

From the equations (4)and (5), $\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i}\right)}\ >\ \zeta(1)$ obviously.

Similarly, we compute the Euler product for the Riemann zeta function for $s=2$ as follows. The Basel problem [8] indicates that $\zeta(2)\cong 1.644934$.

$$\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i^2}\right)}=\frac{4}{3}\times\frac{9}{8}\times\frac{25}{24}\times\frac{49}{48}\times\frac{121}{120}\times\cdots>2 \text{ and } \prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i^2}\right)}>\zeta(2).$$

If we continue the above process up to $s=n,$

If we continue the above process up to $s=n,$ then $\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i^n}\right)}>\zeta(n).$

$$\therefore\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i^s}\right)}>\zeta(s).$$

From the above result, we conclude that the Euler product is not equal to the Riemann zeta function.

4. The Main Reason for Wrong Proof by the Earlier Mathematicians

In two infinite series, we must apply arithmetic operations such as addition and subtraction, only on the corresponding numbers to get the resultant series correctly. Otherwise, the resultant series might be obtained wrongly.

For instance, let us consider the following infinite series (harmonic series):

$$\boldsymbol{S} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots \tag{6}$$

$$\frac{1}{2}\boldsymbol{S} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{16} + \frac{1}{18} + \cdots \tag{7}$$

In Series (6) and Series (7), apply the arithmetic operation 'subtraction' on the corresponding numbers as follows:

$$\boldsymbol{S} - \frac{1}{2}\boldsymbol{S} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10}\right) + \left(\frac{1}{6} - \frac{1}{12}\right) + \cdots \tag{8}$$

The resultant series: $\frac{1}{2}\boldsymbol{S} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \cdots$ (9)

The arithmetic operation on Series (8) is the right process. So, the resultant series (9) is true.

In Series (6) and Series (7), apply the arithmetic operation 'subtraction' without the corresponding numbers as follows:

$$\boldsymbol{S} - \frac{1}{2}\boldsymbol{S} = 1 + \left(\frac{1}{2} - \frac{1}{2}\right) + \frac{1}{3} + \left(\frac{1}{4} - \frac{1}{4}\right) + \frac{1}{5} + \left(\frac{1}{6} - \frac{1}{6}\right) + \frac{1}{7} + \cdots \tag{10}$$

The resultant series: $\frac{1}{2}\boldsymbol{S} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$ (11)

The arithmetic operation on Series (10) is the wrong process. So, the resultant series (11) is false.

Note that the sum of Series (11) is greater than the sum of Series (9).

Therefore, the proof for the Euler product equal to the Riemann zeta function is false because the earlier mathematicians had followed the arithmetic operation on Series (10).

This is the main reason why the Euler product is not equal to the Riemann zeta function [6, 7].

5. Conclusion

In the article, the author disproved the Euler product representation for the Riemann zeta function that the earlier mathematicians proved.

References

[1] Annamalai, C. (2024) Computation of the Euler Product Representation for the Riemann Zeta Function. *SSRN Electronic Journal*. https://doi.org/10.2139/ssrn.4876146.

[2] Annamalai, C. (2024) A Simple Proof of the Euler Product for the Riemann Zeta Function. *TechRxiv*. https://doi.org/10.36227/techrxiv.171995399.91029590/v1.

[3] Annamalai, C. (2024) Computation of the Riemann Zeta Function for deriving the Euler Product. *TechRxiv*. https://doi.org/10.36227/techrxiv.172055418.84189178/v1.

[4] Annamalai, C. (2024) Representation of the Euler Product for the Riemann Zeta Function. *Cambridge Open Engage*. https://doi.org/10.33774/coe-2024-p15hn-v2.

[5] Annamalai, C. (2024) The Euler Product is not equal to the Riemann Zeta Function. *TechRxiv*. https://doi.org/10.36227/techrxiv.172226916.62616068/v1.

[6] Annamalai, C. (2024) The main reason for why the Euler product not equal to the Zeta function. *Zenodo*. https://doi.org/10.5281/zenodo.13150587.

[7] Annamalai, C. (2024) What is wrong with the sum of infinite series? *ResearchGate*. https://doi.org/10.13140/RG.2.2.22290.88005.

[8] Wikipedia: https://en.wikipedia.org/wiki/Basel_problem.