Disproving the Euler Product equal to the Riemann Zeta Function

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Abstract: The author of this article disproves the Euler product equal to the Riemann zeta function that the earlier mathematicians proved, and also provides the main reason why the Euler product is not equal to the Riemann zeta function.

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1. Introduction

The Riemann zeta function $\zeta(s)$ for all complex numbers such as $s = a + ib$ is defined by

$$\zeta(s) = \sum n^{-s} = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + \cdots, \qquad \forall\, \mathbb{R}e(s) > 1.$$

Where $\mathbb{R}e(s)$ is the real part of all complex numbers.

The Euler product representation for the Riemann zeta function is given below:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{i=1}^{\infty} \frac{1}{1 - p_i^{-s}}, \qquad \forall\, \mathbb{R}e(s) > 1.$$

2. Proof of the Euler Product for the Riemann Zeta Function

The earlier mathematicians proved the Euler product equal to the Riemann zeta function [1-5] as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{i=1}^{\infty} \frac{1}{1 - \frac{1}{p_i^s}}, \qquad (1)$$

where $\prod_{i=1}^{\infty} \frac{1}{1 - p_i^{-s}}$ is the Euler product and p_i is the i^{th} prime.

The proof of Equation (1) is shown below:

The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} \cdots$

$$\frac{1}{2^s}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{12^s} + \frac{1}{14^s} + \frac{1}{16^s} + \cdots$$

$$\zeta(s) - \frac{1}{2^s}\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{15^s} + \frac{1}{17^s} + \cdots$$

$$i.e., \left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + + \frac{1}{15^s} + \cdots$$

$$\frac{1}{3^s}\left(1-\frac{1}{2^s}\right)\zeta(s)=\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\frac{1}{27^s}+\frac{1}{33^s}+\frac{1}{39^s}+\frac{1}{45^s}+\cdots$$
$$\left(1-\frac{1}{2^s}\right)\zeta(s)-\frac{1}{3^s}\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\cdots$$
$$i.e.,\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\frac{1}{19^s}+\cdots$$

If the above processes are continued, the multiples of prime factors such as 2, 3, 4, 5, 7, ⋯, are removed.

$$\cdots\cdots\cdots\left(1-\frac{1}{13^s}\right)\left(1-\frac{1}{11^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1. \qquad (2)$$

$$i.e.,\prod_{k=1}^{\infty}\left(1-\frac{1}{p_k^s}\right)\zeta(s)=1\Rightarrow\zeta(s)=\prod_{k=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_k^s}\right)}.$$

Hence, proved.

3. Disproving the Euler Product for the Riemann Zeta Function

The author of this article disproves the Euler product equal to the Riemann zeta function $\zeta(s)$ that the earlier mathematicians proved.

If $s=1,$ then $\zeta(1)$ denotes the harmonic series.

$$\zeta(1)=\sum_{n=1}^{\infty}\frac{1}{n^s}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\cdots \qquad (3)$$

Also, by substituting $s=1$ in Equation (2), we obtain

$$\cdots\times\frac{10}{11}\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times\frac{1}{2}\zeta(1)=1\ \Rightarrow\zeta(1)=\frac{2}{1}\times\frac{3}{2}\times\frac{5}{4}\times\frac{7}{6}\times\frac{11}{10}\times\cdots=\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i}\right)}. \ (4)$$

From the equations (3)and (4), $\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i}\right)}$ is greater than $\zeta(1)$ obviously.

Similarly, we compute the Euler product for the Riemann zeta function for $s=2$ as follows. The Basel problem [8] indicates that $\zeta(2)\cong 1.644934$.

$$\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i^2}\right)}=\frac{4}{3}\times\frac{9}{8}\times\frac{25}{24}\times\frac{49}{48}\times\frac{121}{120}\times\cdots>2 \text{ and } \prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i^2}\right)}>\zeta(2).$$

If we continue the above process up to $s=n,$ then $\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i^n}\right)}>\zeta(n).$

$$\therefore\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{1}{p_i^s}\right)}>\zeta(s).$$

From the above result, we conclude that the Euler product is not equal to the Riemann zeta function.

4. The Main Reason why the Euler Product is not equal to the Riemann Zeta Function

In two infinite series, we must apply arithmetic operations such as addition and subtraction, only on the corresponding numbers to get the resultant series correctly. Otherwise, the resultant series might be obtained wrongly.

For instance, let us consider the following infinite series (harmonic series):

$$\boldsymbol{S} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots \quad (5)$$

$$\frac{1}{2}\boldsymbol{S} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{16} + \frac{1}{18} + \cdots \quad (6)$$

Let's apply the arithmetic operation 'subtraction' with the corresponding numbers of Series (5) and Series (6) as follows:

$$\boldsymbol{S} - \frac{1}{2}\boldsymbol{S} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10}\right) + \left(\frac{1}{6} - \frac{1}{12}\right) + \cdots \quad (7)$$

The resultant series: $\frac{1}{2}\boldsymbol{S} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \cdots$ (8)

The arithmetic operation on Series (7) is right. So, the resultant series (8) is true.

Let's apply the arithmetic operation 'subtraction' without following the corresponding numbers as follows:

$$\boldsymbol{S} - \frac{1}{2}\boldsymbol{S} = 1 + \left(\frac{1}{2} - \frac{1}{2}\right) + \frac{1}{3} + \left(\frac{1}{4} - \frac{1}{4}\right) + \frac{1}{5} + \left(\frac{1}{6} - \frac{1}{6}\right) + \frac{1}{7} + \cdots \quad (9)$$

The resultant series: $\frac{1}{2}\boldsymbol{S} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$ (10)

The arithmetic operation on Series (9) is wrong. So, the resultant series (10) is false.

Let us compare the corresponding numbers between Series (8) and Series (10) as follows:

$$\frac{1}{2} < 1, \frac{1}{4} < \frac{1}{3}, \frac{1}{6} < \frac{1}{5}, \frac{1}{8} < \frac{1}{7}, \frac{1}{10} < \frac{1}{9}, \frac{1}{12} < \frac{1}{11}, \cdots \Rightarrow \sum_{i=1}^{\infty} \frac{1}{2i} < \sum_{i=1}^{\infty} \frac{1}{2i - 1}. \quad (11)$$

By comparison of corresponding numbers between the two series (8) and (10), we conclude that the sum of Series (8) is greater than the sum of Series (10).

Thus, it is concluded the value of Riemann zeta function is always less than the value of Euler product for all complex numbers as follows:

$$\zeta(1) < \prod_{i=1}^{\infty} \frac{1}{\left(1 - \frac{1}{p_i}\right)}, \zeta(2) < \prod_{i=1}^{\infty} \frac{1}{\left(1 - \frac{1}{p_i^2}\right)}, \zeta(3) < \prod_{i=1}^{\infty} \frac{1}{\left(1 - \frac{1}{p_i^3}\right)}, \zeta(4) < \prod_{i=1}^{\infty} \frac{1}{\left(1 - \frac{1}{p_i^4}\right)}, \cdots$$

$$\text{and } \zeta(s) < \prod_{i=1}^{\infty} \frac{1}{\left(1 - \frac{1}{p_i^s}\right)}, \qquad \forall\ \mathbb{R}e(s). \tag{12}$$

Therefore, the proof of the Euler product equal to the Riemann zeta function is false because the earlier mathematicians had followed the arithmetic operation like Series (9).

5. Conclusion

In the article, the author disproved the Euler product representation for the Riemann zeta function that the earlier mathematicians proved.

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