Application of Geometric Series and Maclaurin Series Relating to Taylor Series

Chinnaraji Annamalai
School of Management, Indian Institute of Technology, Kharagpur, India
Email: anna@iitkgp.ac.in
https://orcid.org/0000-0002-0992-2584

Abstract: This paper presents a novel idea to compute the Maclaurin series and Taylor series and also provides application of the geometric series and the Maclaurin series.

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1. Introduction

In the earlier days, geometric series [1-20] with positive exponents served as a vital role in the development of differential and integral calculus [38] and as an introduction to Maclaurin series, Taylor series and Fourier series [44]. Geometric series have significant applications in physics, engineering, biology, economics, finance, management, queueing theory, computer science and medicine [6]. Also, the product of geometric series [21-37] with prime numbers plays a vital in the sum of natural numbers and harmonic series [39-44].

2. Maclaurin Series

Taylor series is stated as follows:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $a = 0$ in the Taylor series, the Taylor series becomes the Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Let us prove these series using the following power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

If $x = 0$ in the above power series, $f(0) = a_0$.

Now, let us differentiate the power series as follows:

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$. If $x = 0$ in $f'(x)$, $f'(0) = a_1$.

By differentiating $f'(x)$, we get

$f''(x) = 2a_2 + 3 \times 2a_3 x + 4 \times 3a_4 x^2 + 5 \times 4a_5 x^3 + \cdots$

If we do the same process again and again, we conclude that

$$a_0 = f(0);\ a_1 = f'(0);\ a_2 = \frac{f''(0)}{2!};\ a_3 = \frac{f''(0)}{3!};\ \cdots;\ a_n = \frac{f^{(n)}(0)}{n!}.$$

Now, we obtain the Maclaurin series by substituting the values of $a_0,\ a_1,\ a_2,\ a_3,$ etc.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n.$$

By making the power series into general form, we get

$$f(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 + \cdots$$

By substituting $b_n = \dfrac{f^{(n)}(a)}{n!}$ for $n = 0, 1, 2, 3, \cdots$, in the general power series, we get

Taylor series: $f(x) = f(a) + \dfrac{f'(a)}{1!}(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \dfrac{f'''(a)}{3!}(x-a)^3 + \cdots$

3. Application of Geometric Series and Maclaurin Series

Theorem 3.1: The sum of alternative harmonic series is equal to $ln\,2$.

$$i.e. \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = ln\,2.$$

Case i: This theorem is proved using the geometric series.

Let $S = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots$

$$\Rightarrow xS = x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \cdots$$

Then, $(S - xS) = 1 \Rightarrow (1-x)S = 1 \Rightarrow \dfrac{1}{1-x} = S.$

$$i.e. \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots$$

Integrating on both sides with respect to x:

$$\int \frac{1}{1-x}dx = \int (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots)dx.$$

First, let us find the solution of $\displaystyle\int \frac{1}{1-x}dx.$

Let $u = 1 - x;\ du = -dx;\ dx = -du.$

Then, $\displaystyle\int \frac{1}{1-x}dx = -\int \frac{1}{u}du = -\,ln\,u = -\,ln(1-x).$

Now, $\displaystyle\int (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots)dx = -\,ln(1-x).$

$$\Rightarrow x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \cdots = -\,ln(1-x).$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{x^n}{n}\cdots = -\,ln(1-x) \Rightarrow -\sum_{n=1}^{\infty}\frac{x^n}{n} = ln(1-x).$$

Substituting $x = -1$ on both sides:

$$-\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = ln(1-(-1)) \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = ln\,2 \textbf{ OR } \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = ln\,2.$$

Case ii: Again, this theorem is proved using the Maclaurin series

Let us consider the function: $f(x) = \ln(1+x)$.

If $x = 0$ in $(x) = \ln(1+x)$, then $f(0) = ln(1+0) = ln\,1 = 0$.

Now, by differentiating the function $f(x) = \ln(1+x)$ and using $x = 0$, we obtain

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$$f'(x) = \frac{1}{1+x} = (1+x)^{-1} \text{ and } f'(0) = \frac{1}{1+0} = (1+0)^{-1} = 1.$$
$$f''(x) = -(1+x)^{-2} \text{ and } f''(0) = -(1+0)^{-1} = -1 = -(1!).$$
$$f'''(x) = -2(1+x)^{-3} \text{ and } f'''(0) = -2(1+0)^{-3} = -2 = -(2!).$$
$$f'^{v}(x) = (-2) \times (-3)(1+x)^{-4} \text{ and } f'^{v}(0) = 3!\,(1+0)^{-3} = 3!.$$
$$f^{v}(x) = (-2) \times (-3) \times (-4)(1+x)^{-5} \text{ and } f^{v}(0) = -(4!).$$

Similarly, we can continue this process infinitely.

Maclaurin series: $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

Then, $ln(1+x) = 0 + \frac{1}{1!}x - \frac{1!}{2!}x^2 + \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4 + \frac{4!}{5!}x^5 - \frac{5!}{6!}x^6 + \cdots$

$$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}x^3 - \frac{x^4}{4}x^4 + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

By substituting $x = 1$ in the series $ln(1+x)$, we conclude that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = ln\,2.$$

4. Conclusion

In the article, the author expressed application of the geometric series and Maclaurin series. This idea can enable the scientific researchers for further involvement in research and development.

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