Analytical Solution for the Perimeter of an Ellipse via a New Definite Integral: A Study on Elliptical Cylinder Sections

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Abstract

Introduction: Research on the perimeter of an ellipse has so far only found approximations. This occurs because the integral of the perimeter of an ellipse does not have an antiderivative.

Objective: Therefore, this study aims to find a new definite integral for the perimeter of an ellipse that has a derivative.

Methods: This study observed the relationship between the intersection of an elliptical cylinder, which results in a circle, and the perimeter of the base of the elliptical cylinder.

Results: This study found a new definite integral to obtain the exact formula for the perimeter of an ellipse, which can be solved analytically.

Keywords: Cylinder section, Lateral Surface, Wave Curve, Transformation, Analytical Solution, Definite Integral, Ellipse Perimeter.

1 Introduction

An ellipse is formed when a plane intersects a cone or cylinder at an angle, creating a closed curve. It can also be viewed as a circle that has been stretched vertically or horizontally [1]. This stretching can be explained using coordinate transformations or Affine Transformations [2], [3]. Although this concept of stretching is useful for deriving the area formula of an ellipse, it does not apply to finding its perimeter [4].

For many years, mathematicians have been searching for an exact formula for the perimeter of an ellipse. Calculating the perimeter using integrals is quite challenging. Various methods have been developed to estimate the perimeter [5], [6], [7]. Over time, these approximations have become increasingly accurate, with some even determining the boundaries of the ellipse's perimeter.

One method of finding the perimeter of an ellipse involves transforming the ellipse into a circle while keeping its original perimeter. This can be described as a thread forming an ellipse, which can then be turned into a circle [4]. To achieve this, we need to determine the correct radius for the circle. The radius c of the circle should be such that the perimeter of the circle, $(2\pi c)$, equals the perimeter of the ellipse.

An ellipse can be created by slicing a cylinder with a plane at an angle to its base. When the lateral surface of the cylinder is unwrapped and laid flat, it forms a wave-like curve [8], [9]. This wave has the same wavelength as the perimeter of the cylinder's base. Interestingly, the lengths of the ellipse and the wave curve are identical [10]. However, this study is only for circular cylinders whose sections are ellipses.

The study aims to explore the relationship between the base of an elliptical cylinder and the resulting section, which is a circle. By flattening the lateral surface into a plane, we can observe the resulting section as a wave curve and the perimeter of the base (an ellipse) as a wavelength. The primary focus is to examine the relationship between the area of the wave curve and the wavelength, where the wavelength represents the perimeter of the ellipse. This relationship can yield a new definite integral for the perimeter of an ellipse.

2 Research Methods

This study examines a cylinder with an elliptical base, which is intersected by a plane inclined at an angle α to the base. The base of the cylinder has semi-major and semi-minor axes of lengths b and a, respectively. The intersection of the plane and the cylinder forms circles with radius b. The study describes several steps to derive a definite integral for calculating the perimeter of the ellipse.

a. *Observation on an elliptical cylinder*: the study begins with a cylinder whose base is an ellipse. The semi-major axis is b and the semi-minor axis is a

b. *Cutting the cylinder*: the cylinder is cut by a plane at an angle α to the base. This cut results in sections that are circles with radius b.

c. *Opening the lateral surface*: the lateral surface of the cylinder is cut along its height and then stretched out into a flat plane.

d. *Wave curve formation*: when the lateral surface is flattened, the circular sections form a sine wave curve. The wavelength of this sine wave is equal to the perimeter of the ellipse (the base of the cylinder).

e. *Relationship between area and wavelength*: the study examines the relationship between the area under the sine wave curve and the length of the wave curve. This involves the perimeter of the ellipse.

f. *Definite integral for perimeter*: using the relationship found in the previous step, the study derives a definite integral that represents the exact formula for the perimeter of an ellipse

The GeoGebra application is used to illustrate cylinders and curves during the stages of research.

3 Results

Perimeter of the ellipse:

An ellipse can be transformed into a circle while preserving its perimeter, we obtain the perimeter of the ellipse (P) as

$$P = 2\pi c = 4h\int_0^{\frac{1}{2}\pi} \sin(t)\sqrt{b^2\sin^2(t) + a^2\cos^2(t)}\ dt = \frac{4}{3}\pi\left(\frac{(a^2 + ab + b^2)}{(b + a)}\right)$$

where $h = \sqrt{b^2 - a^2}$ with $b \geq a \geq 0$. Here, c is the radius of the circle, and a and b are the semi-major and semi-minor axes of the ellipse, respectively.

Proof:

The following steps will guide us in deriving a definite integral for the perimeter of an ellipse. In the final part, we will determine the exact formula for calculating the perimeter of the ellipse.

3.1 Sine Waves on Lateral Surface of Elliptical Cylinder

If an elliptical cylinder has a base in the shape of an ellipse with semi-major axis (a) and semi-minor axis (b), it can be cut by a plane to produce a circular section with radius (b). The plane intersects the cylinder at an angle such that the slope (m) is given by $m = \tan\alpha = GB/O'G = b/a$. This intersection results in a circle with radius (b).

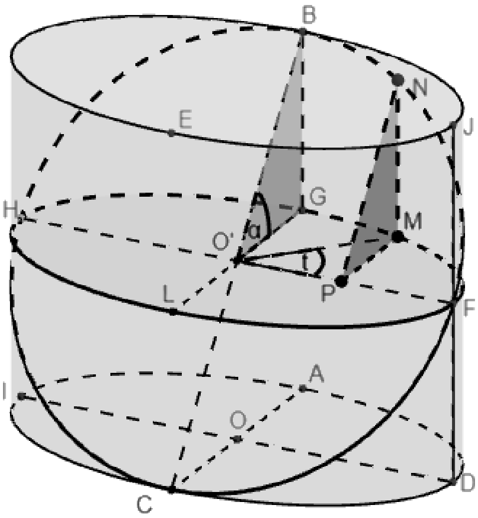

Figure 1. Section of an elliptical cylinder with the resulting section being a circle

Look at the Figure 1. There is an ellipse $E = \widehat{DAIC}$, an ellipse $E_1 = \widehat{FGHL}$, and circle $L = \widehat{FBHC}$. We also get that

$$OD = O'F = b \tag{1}$$
$$OA = O'G = a \tag{2}$$
$$m\angle PO'M = t \tag{3}$$
$$O'B = b \tag{4}$$
$$h = BG = \sqrt{O'B^2 - O'G^2} = \sqrt{b^2 - a^2} \tag{5}$$
$$PM = b\sin(t) \tag{6}$$
$$OP = b\cos(t) \tag{7}$$

Note that $\Delta O'GB \sim \Delta PM$ (the proof of the angle-angle-angle theorem can be seen in (Jupri 2021)), then

$$\frac{O'G}{PM} = \frac{GB}{MN} \tag{8}$$

Substitute equations (2), (5), and (6) into equation (8) then

$$\frac{a}{a\sin(t)} = \frac{h}{MN} \tag{9}$$

We get

$$MN = h\frac{a\sin(t)}{a} \tag{10}$$

Simplify

$$MN = h\sin(t) \tag{11}$$

If the lateral surface is cut along $\overline{DJ}$ then stretched horizontally as shown in the image below.

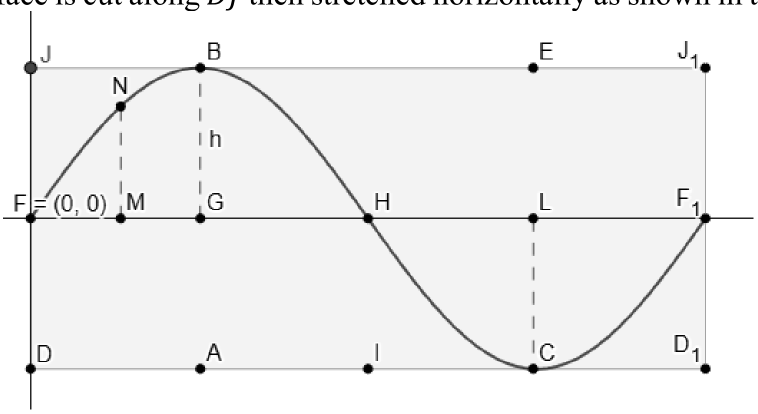

Figure 2. The lateral surface is stretched into a plane

Look at figure 2, the wavelength is the same as the perimeter of the base of the elliptical cylinder is the same (the perimeter of the ellipse), namely

$$P = FF_1 \tag{12}$$

The result of the cut along $\overline{DJ}$ yields $\overleftrightarrow{DJ}$ and $\overleftrightarrow{D_1J_1}$. Point N can move along the sine curve with a distance from the line FF_1, namely

$$MN = h \sin(t)$$

If we determine $x = \overleftrightarrow{FF_1}$, $y = \overleftrightarrow{DJ}$, and $F(0,0)$ then we get

$$y = h \sin(t) \tag{13}$$

and

$$x = P(t) \tag{14}$$

with $P(t)$ is the perimeter of the base of the elliptical cylinder (perimeter of the ellipse) for $0 \leq t \leq 2\pi$.

3.2 Area Under the Sinus Curve

The area under the sine curve we calculate as

$$A = 4\int_0^{\frac{1}{2}\pi} y \; dx = 4\int_0^{\frac{1}{2}\pi} h \sin(t) \; dx \tag{15}$$

for $0 \leq t \leq 2\pi$.

Because $x = P(t)$, then $dx = P'(t)\,dt$. Equation (15) becomes

$$A = 4\int_0^{\frac{1}{2}\pi} h \sin(t)\, P'(t)\, dt \tag{16}$$

Because the perimeter of E_1 is

$$P = 4\int_0^{\frac{1}{2}\pi} P'(t) \; dt = 4\int_0^{\frac{1}{2}\pi} \sqrt{b^2 \sin^2(t) + a^2 \cos^2(t)} \; dt \tag{17}$$

From equation (17), we get

$$P'(t)\, dt = \sqrt{b^2 \sin^2(t) + a^2 \cos^2(t)} \; dt \tag{18}$$

We substitute equation (18) into equation (15) obtained

$$A = 4\int_0^{\frac{1}{2}\pi} h \sin(t)\, P'(t)\, dt = 4\int_0^{\frac{1}{2}\pi} h \sin(t) \sqrt{b^2 \sin^2(t) + a^2 \cos^2(t)} \; dt \tag{19}$$

3.3 Alternative Methods to Formulate the Area Under the Sine Curve

Look at Figure 2, the lateral surface of the elliptical cylinder is rolled back by attaching line DJ to D_1J_1, points D and D_1 meet, as do points J and J_1. This reformation results in a circular cylinder. The newly formed cylinder has a circular base with radius c. The perimeter of the base of this circular cylinder is the same as the perimeter of the base of the original ellipse. The formula for the base of this circular cylinder of a circle is given by:

$$x = P(t) = ct \tag{20}$$

for $0 \leq t \leq 2\pi$. Here, c is the radius of the circle.

We can search for A in another way i.e.

$$A = 4\int_0^{\frac{1}{2}\pi} h \sin(t)\, dx \tag{21}$$

because $dx = K'(t) = c\;dt$ then

$$A = 4\int_0^{\frac{1}{2}\pi} h \sin(t)\, K'(t)\, dt = 4\int_0^{\frac{1}{2}\pi} h \sin(t)\, c\; dt \tag{22}$$

The result is

$$A = 4ch \int_0^{\frac{1}{2}\pi} \sin(t)\, dt = 4ch \tag{23}$$

3.4 Integral for Radius c

An ellipse can be transformed into a circle with radius c. To find the value of c, substitute equation (23) into (19).

$$4ch = 4 \int_0^{\frac{1}{2}\pi} h \sin(t) \sqrt{b^2 \sin^2(t) + a^2 \cos^2(t)}\ dt \tag{24}$$

Simplify

$$c = \int_0^{\frac{1}{2}\pi} \sin(t) \sqrt{b^2 \sin^2(t) + a^2 \cos^2(t)}\ dt \tag{25}$$

3.5 Finding Integral Result for *c*

Because $\sin^2 t = 1 - \cos^2 t$, equation (21) becomes

$$c = \int_0^{\frac{1}{2}\pi} \sin(t)\ \sqrt{a^2 \cos^2(t) + b^2(1 - \cos^2(t))}\, dt \tag{26}$$

Simplify

$$c = \int_0^{\frac{1}{2}\pi} \sin(t)\ \sqrt{(a^2 - b^2)\cos^2(t) + b^2}\, dt \tag{27}$$

$$c = \int_0^{\frac{1}{2}\pi} b \sin(t)\ \sqrt{\frac{a^2 - b^2}{b^2}\cos^2(t) + 1\, dt} \tag{28}$$

Solution for equation (28)

$$c = \int b \sin(t)\ \sqrt{\frac{a^2 - b^2}{b^2}\cos^2(t) + 1\, dt} \tag{29}$$

Substitute

$$u = \frac{(a^2 - b^2)\cos(t)}{b^2} + 1 \rightarrow du = -\frac{(a^2 - b^2)\sin(t)}{b^2}\ dt \tag{30}$$

We get

$$c = \frac{b^3}{b^2 - a^2} \int \sqrt{u}\ du \tag{31}$$

Now solving

$$\int \sqrt{u}\ du \tag{12}$$

Apply power rule

$$\int \sqrt{u}\ du = \frac{u^{n+1}}{n+1} \text{ with } n = \frac{1}{2} \tag{33}$$

We get

$$\int \sqrt{u}\ du = \frac{2u^{\frac{3}{2}}}{3} \tag{34}$$

Plug in solve integrals

$$c = \frac{b^3}{b^2 - a^2} \int \sqrt{u}\ du = \frac{2b^3 u^{\frac{3}{2}}}{3(b^2 - a^2)} \tag{35}$$

Undo Substitution $u = [(a^2 - b^2)\cos(t)/b^2] + 1$

$$c = \frac{2b^3\left(\frac{(a^2-b^2)\cos(t)}{b^2}+1\right)^{\frac{3}{2}}}{3(b^2-a^2)} \tag{36}$$

This problem is solved:

$$c = \int b\sin(t)\sqrt{\frac{a^2-b^2}{b^2}\cos^2(t)+1}\,dt = \frac{2b^3\left(\frac{(a^2-b^2)\cos(t)}{b^2}+1\right)^{\frac{3}{2}}}{3(b^2-a^2)} + C \tag{37}$$

We get

$$c = \int_0^{\frac{1}{2}\pi} \sin(t)\sqrt{(a^2-b^2)\cos^2(t)+b^2}\,dt = \frac{2(a^2+ab+b^2)}{3(b+a)} \tag{38}$$

The steps of the integral in equation (38) can be seen in the appendix.

Perimeter of The Ellipse

Since an ellipse can be transformed into a circle, we obtain the circumference of the ellipse as

$$K = 2\pi c = \frac{4}{3}\pi\left(\frac{(a^2+ab+b^2)}{(b+a)}\right) \tag{39}$$

with c being the radius of the circle, and a and b being the semi-major and semi-minor axes of the ellipse, respectively.

4 Conclusions

The definite integral that yields the exact formula for the perimeter of the ellipse in this article still needs to be verified for its accuracy. This review could include whether this formula falls within the previously established upper and lower bounds for the perimeter of the ellipse. Additionally, it could be examined whether this formula can calculate with the same level of accuracy as the approximations previously found.

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APPENDIX

$$c = \int_0^{\frac{1}{2}\pi} \sin(t)\,\sqrt{(a^2 - b^2)\cos^2(t) + b^2}\,dt = \frac{2(a^2 + ab + b^2)}{3(b + a)}$$

Proof:

It has been proven that

$$c = \int b\sin(t)\,\sqrt{\frac{a^2 - b^2}{b^2}\cos^2(t) + 1}\,dt = \frac{2b^3\left(\frac{(a^2 - b^2)\cos(t)}{b^2} + 1\right)^{\frac{3}{2}}}{3(b^2 - a^2)} + C$$

The integral for $0 \le t \le \pi/2$ is as follows

When $t = \pi/2$, we get

$$\frac{2b^3\left(\frac{(a^2 - b^2)\cos(\pi/2)}{b^2} + 1\right)^{\frac{3}{2}}}{3(b^2 - a^2)} = \frac{2b^3}{3(b^2 - a^2)}$$

When $t = \pi/2$, we get

$$\frac{2b^3\left(\frac{(a^2 - b^2)\cos(0)}{b^2} + 1\right)^{\frac{3}{2}}}{3(b^2 - a^2)} = \frac{2b^3\left(\frac{(a^2 - b^2)}{b^2} + 1\right)^{\frac{3}{2}}}{3(b^2 - a^2)}$$

$$= \frac{2b^3\left(\frac{(a^2 - b^2) + b^2}{b^2}\right)^{\frac{3}{2}}}{3(b^2 - a^2)}$$

$$= \frac{2b^3\left(\frac{a^2}{b^2}\right)^{\frac{3}{2}}}{3(b^2 - a^2)}$$

$$= \frac{2a^3}{3(b^2 - a^2)}$$

So, the integral for $0 \le t \le \pi/2$ is

$$c = \frac{2b^3}{3(b^2 - a^2)} - \frac{2a^3}{3(b^2 - a^2)}$$

Simplify

$$c = \frac{2b^3 - 2a^3}{3(b^2 - a^2)}$$

$$= \frac{2(a^2 + ab + b^2)(a - b)}{3(b + a)(b - a)}$$

$$= \frac{2(a^2 + ab + b^2)}{3(b + a)}$$

or

$$= \frac{2[(a + b)^2 - ab]}{3(b + a)} = \frac{2}{3}\left((a + b) - \frac{ab}{(a + b)}\right)$$