

Computation of Novel Binomial Series without Binomial Coefficients

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Abstract: Nowadays, the growing complexity of mathematical and computational modelling demands the simplicity of mathematical equations for solving today's scientific problems and challenges. For this purpose, a new binomial series without binomial coefficients is introduced in the paper for applications of engineering and computing sciences.

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1. Introduction

Geometric series [1,10] played a vital role in differential and integral calculus at the earlier stage of development and still continues as an important part of the study in science, engineering, management and its applications [11]. In this article, a new binomial series without binomial coefficients are computed with binomial theorems.

2. Novel Binomial Series

The author of this article introduces a new binomial series given below.

Theorem 2.1:
$$\sum_{k=0}^n x^k y^{n-k} = \sum_{k=0}^n x^{n-k} y^k = \frac{x^{n+1} - y^{n+1}}{x - y} = \frac{y^{n+1} - x^{n+1}}{y - x}, x \neq y.$$

Proof. Let's prove this theorem using the geometric series with rational numbers.

$$\sum_{k=0}^n \left(\frac{x}{y}\right)^k = \frac{\left(\frac{x}{y}\right)^{n+1} - 1}{\frac{x}{y} - 1} = \left(\frac{x^{n+1} - y^{n+1}}{y^{n+1}}\right) \left(\frac{y}{x - y}\right), x \neq y. \quad (1)$$

By simplifying the equation (1), we get

$$\begin{aligned} y^n \left(1 + \frac{x}{y} + \frac{x^2}{y^2} + \frac{x^3}{y^3} + \dots + \frac{x^{n-1}}{y^{n-1}} + \frac{x^n}{y^n}\right) &= \frac{x^{n+1} - y^{n+1}}{x - y}, x \neq y. \\ y^n + xy^{n-1} + x^2 y^{n-2} + x^3 y^{n-3} + \dots + x^{n-1} y + x^n &= \frac{x^{n+1} - y^{n+1}}{x - y}, x \neq y. \\ \sum_{k=0}^n x^k y^{n-k} &= \frac{x^{n+1} - y^{n+1}}{x - y}, x \neq y. \end{aligned} \quad (2)$$

By rearranging the binomial series (2), we obtain

$$\sum_{k=0}^n x^{n-1} y^k = \frac{y^{n+1} - x^{n+1}}{y - x}, x \neq y. \quad (3)$$

From the binomial series (2) and (3), we conclude that

$$\sum_{k=0}^n x^k y^{n-k} = \sum_{k=0}^n x^{n-k} y^k = \frac{x^{n+1} - y^{n+1}}{x - y} = \frac{y^{n+1} - x^{n+1}}{y - x}, x \neq y. \quad (4)$$

Hence, theorem is proved.

Corollary 2.1: $\sum_{k=0}^n x^k y^{n-k} = \sum_{k=0}^n x^{n-k} y^k = (n+1)x^n, \quad \text{for } x = y.$

Theorem 2.2: $\sum_{i=k}^n x^i y^{n-i} = \sum_{i=k}^n x^{n-i} y^i = \frac{x^{n+1} - x^k y^{n+1-k}}{x - y} = \frac{y^{n+1} - y^k x^{n+1-k}}{y - x}, x \neq y.$

Proof. Let's prove the theorem using the geometric series with rational numbers.

$$\begin{aligned} \sum_{i=k}^n \left(\frac{x}{y}\right)^i &= \frac{\left(\frac{x}{y}\right)^{n+1} - \left(\frac{x}{y}\right)^k}{\frac{x}{y} - 1} = \left(\frac{x^{n+1}}{y^{n+1}} - x^k y^{-k}\right) \left(\frac{y}{x-y}\right) = \frac{1}{y^n} \left(\frac{x^{n+1} - x^k y^{n+1-k}}{x-y}\right). \\ y^n \sum_{i=k}^n \left(\frac{x}{y}\right)^i &= \sum_{i=k}^n x^i y^{n-i} = \frac{x^{n+1} - x^k y^{n+1-k}}{x-y} \Rightarrow \sum_{i=k}^n x^{n-i} y^i = \frac{y^{n+1} - y^k x^{n+1-k}}{y-x}. \\ \therefore \sum_{i=k}^n x^i y^{n-i} &= \sum_{i=k}^n x^{n-i} y^i = \frac{x^{n+1} - x^k y^{n+1-k}}{x-y} = \frac{y^{n+1} - y^k x^{n+1-k}}{y-x}, x \neq y. \end{aligned}$$

Corollary 2.2: $\sum_{i=k}^n x^i y^{n-i} = \sum_{i=k}^n x^{i-k} y^k = (n+1-k)x^n, \quad \text{for } x = y.$

Theorem 2.3: $\prod_{k=0}^n (x^{2^k} + y^{2^k}) = \frac{x^{2^{n+1}} - y^{2^{n+1}}}{x - y}, x \neq y.$

Proof. Let's prove this theorem as follows.

$$\begin{aligned} \frac{x^2 - y^2}{x - y} &= \frac{(x+y)(x-y)}{x-y} = x + y. \\ \frac{x^{2^2} - y^{2^2}}{x - y} &= \frac{x^4 - y^4}{x - y} = \frac{(x^2 + y^2)(x^2 - y^2)}{x - y} = (x^2 + y^2)(x + y). \\ \frac{x^{2^3} - y^{2^3}}{x - y} &= \frac{x^8 - y^8}{x - y} = (x^4 + y^4)(x^2 + y^2)(x + y) = (x^{2^2} + y^{2^2})(x^{2^1} + y^{2^1})(x^{2^0} + y^{2^0}). \end{aligned}$$

We can continue the same process up to 2^{n+1} .

$$\begin{aligned} \frac{x^{2^3} - y^{2^3}}{x - y} &= \frac{x^8 - y^8}{x - y} = (x^{2^n} + y^{2^n})(x^{2^{n-1}} + y^{2^{n-1}}) \cdots (x^{2^1} + y^{2^1})(x^{2^0} + y^{2^0}). \\ \therefore \prod_{k=0}^n (x^{2^k} + y^{2^k}) &= \frac{x^{2^{n+1}} - y^{2^{n+1}}}{x - y}, x \neq y. \end{aligned}$$

Corollary 2.3: $\prod_{k=0}^{n-1} x^{2^k} = x^{2^n-1}.$

Let's prove the corollary 2.3 using the theorem 2.3.

$$\prod_{k=0}^{n-1} (x^{2^k} + 0^{2^k}) = \frac{x^{2^n} - 0}{x - 0} \Rightarrow \prod_{k=0}^{n-1} x^{2^k} = \frac{x^{2^n}}{x} = x^{2^n-1}.$$

3. Conclusion

In this article, a novel binomial series and theorems have been introduced for mathematical and computational application. Also, this idea can enable the researchers for further involvement in the scientific research.

References

- [1] Annamalai, C. (2023) Binomial Geometric Series. *Cambridge Open Engage, Cambridge University Press*. <https://doi.org/10.33774/coe-2023-072fg>.
- [2] Annamalai, C. (2023) Geometric Progression-Based Binomial Series for Computing Application. *SSRN Electronic Journal*. <http://dx.doi.org/10.2139/ssrn.4591123>.
- [3] Annamalai, C. (2023) Computational Technique for Geometric Series with Radicals. *TechRxiv, IEEE*. <https://doi.org/10.36227/techrxiv.24311773.v1>.
- [4] Annamalai, C. (2023) Computation of Algebraic Expressions and Geometric Series with Radicals. *TechRxiv, IEEE*. <https://doi.org/10.36227/techrxiv.24311773.v2>.
- [5] Annamalai, C. (2023) Binomial Geometric Series for Computational Application. *OSF Preprints*. <https://doi.org/10.31219/osf.io/aubje>.
- [6] Annamalai, C. (2023) Novel Binomial Series without Binomial Coefficients. *OSF Preprints*. <https://doi.org/10.31219/osf.io/yd7p9>.
- [7] Annamalai, C. (2018) Annamalai's Computing Model for Algorithmic Geometric Series and Its Mathematical Structures. *Journal of Mathematics and Computer Science*, 3(1),1-6 <https://doi.org/10.11648/j.mcs.20180301.11>.
- [8] Annamalai, C. (2017) Analysis and Modelling of Annamalai Computing Geometric Series and Summability. *Mathematical Journal of Interdisciplinary Sciences*, 6(1), 11-15. <https://doi.org/10.15415/mjis.2017.61002>.
- [9] Annamalai, C. (2018) Novel Computation of Algorithmic Geometric Series and Summability. *Journal of Algorithms and Computation*, 50(1), 151-153. <https://www.doi.org/10.22059/JAC.2018.68866>.
- [10] Annamalai, C. (2023) Binomial Series without Binomial Coefficients. *OSF Preprints*. <https://doi.org/10.31219/osf.io/yd7p9>.
- [11] Annamalai C (2010) "Application of Exponential Decay and Geometric Series in Effective Medicine", *Advances in Bioscience and Biotechnology*, Vol. 1(1), pp 51-54. <https://doi.org/10.4236/abb.2010.11008>.