A REMARK ON THE RIEMANN ZETA FUNCTION

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Abstract. We prove that if the number of nontrivial zeros of the Riemann zeta function which are not on the critical line is finite, then every nontrivial zero is on the critical line.

1. Introduction

The Riemann zeta function $\zeta(s)$ is defined on $\sigma > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $s = \sigma + it$. It is analytically continued to a meromorphic function on the whole plane with a pole at $s = 1$. It is well-known that negative even integers are zeros of $\zeta(s)$. Other zeros of $\zeta(s)$ are called complex zeros or nontrivial zeros.

Riemann stated the following statement, the so-called Riemann hypothesis, in [6] in 1859.

The Riemann hypothesis. All nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$.

Hilbert listed it as the eighth problem of his 23 problems in his 1900 address to the Paris International Congress of Mathematicians. This is one of the most important unsolved problems in the twenty-first century.

Nobody has succeeded to prove it up to the present, but many computational results are known. In the early part of the twentieth century, they were obtained by hand computation ([1], [4], [5] and [7]). Numerical computations by computers have permitted us to check the truth of the Riemann hypothesis to extremely large t. We refer to [3] for a history of numerical verifications.

In this paper, we give a remark that if the number of nontrivial zeros of the Riemann zeta function which are not on the critical line is finite,

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then every nontrivial zero is on the critical line. First we prepare a lemma on Dirichlet series. Combining this lemma and the theorem of de la Vallée Poussin, we obtain our result.

2. Lemma on Dirichlet series

This section is devoted to the following lemma which is a variant of Lemma 3.12 in [8].

Lemma 1. *We assume that a Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is absolutely convergent for $\sigma > 1$. Take $\sigma_0 < 1$ and $c > 0$ with $\sigma_0 + c > 1$. Let $g(s)$ be an entire function with finite number of zeros $Z_g = \{\alpha_1, \ldots, \alpha_N\}$ such that $\sigma_0 < \mathrm{Re}(\alpha_j) < \sigma_0 + c$ $(j = 1, \ldots, N)$ and $|g(s)| \to \infty$ as $s \to \infty$. Then, there exist meromorphic functions $h_n(s)$ $(1 \leq n < x)$ on $\mathbb{C}$ whose poles are at most $\alpha_1, \ldots, \alpha_N$ and all simple such that

$$\sum_{n<x} \frac{a_n}{n^s} \left(\frac{1}{g(s)} + h_n(s) \right) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{f(s+w)}{g(s+w)} \frac{x^w}{w} dw + \sum_{n<x} \frac{a_n}{n^s} R_n(c,x,T)(s) + \sum_{x<n} \frac{a_n}{n^s} Q_n(c,x,T)(s) \tag{2.1}$$

for any $s \in D(\sigma_0, T) \setminus Z_g$, where a positive number x is not an integer, T is a positive number with $|\mathrm{Im}(\alpha_j)| < \frac{T}{4}$ $(j = 1, \ldots, N)$, $D(\sigma_0, T) = \{\sigma + it; \sigma_0 \leq \sigma, |t| < \frac{T}{4}\}$, and $R_n(c,x,T)(s)$ and $Q_n(c,x,T)(s)$ are holomorphic functions depending only on $g(s), n, c, x, T$ and satisfying

$$|R_n(c,x,T)(s)| < \frac{M}{A\pi T} \left(\frac{x}{n} \right)^c \quad \text{and} \quad |Q_n(c,x,T)(s)| < \frac{M}{A\pi T} \left(\frac{x}{n} \right)^c \tag{2.2}$$

on $D(\sigma_0, T)$ for some constants A and M. Therefore, the right side of (2.1) converges absolutely and uniformly on $D(\sigma_0, T)$.

Proof. Take any $s \in D(\sigma_0, T)$. A function $\frac{1}{g(s+w)} \left(\frac{x}{n} \right)^w \frac{1}{w}$ of w has poles at $w = 0, \alpha_1 - s, \ldots, \alpha_N - s$. It has the residues $\frac{1}{g(s)}$ and $a_j \left(\frac{x}{n} \right)^{\alpha_j - s} \frac{1}{\alpha_j - s}$ at $w = 0$ and $\alpha_j - s$ respectively, where a_j is the residue of $\frac{1}{g(s)}$ at α_j. We define a meromorphic function

$$h_n(s) := \sum_{j=1}^{N} a_j \left(\frac{x}{n} \right)^{\alpha_j - s} \frac{1}{\alpha_j - s}.$$

If $n < x$, then we obtain

$$\frac{1}{2\pi i}\left(\int_{-\infty-iT}^{c-iT}+\int_{c-iT}^{c+iT}+\int_{c+iT}^{-\infty+iT}\right)\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w} = \frac{1}{g(s)}+h_n(s)$$

by the residue theorem. Let

$$M := \sup\left\{\left|\frac{1}{g(s)}\right| ; s=u+it, -\infty<u<\infty, \frac{3}{4}T\le|t|\le\frac{5}{4}T\right\}.$$

Then we have $0 < M < \infty$ by the assumption of $g(s)$. Therefore, the following estimation holds

$$\left|\frac{1}{2\pi i}\int_{-\infty-iT}^{c-iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}\right| < \frac{M}{2\pi T}\frac{(x/n)^c}{\log(x/n)}.$$

Similarly we have

$$\left|\frac{1}{2\pi i}\int_{c+iT}^{-\infty+iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}\right| < \frac{M}{2\pi T}\frac{(x/n)^c}{\log(x/n)}.$$

We define

$$R_n(c,x,T)(s) := \frac{1}{2\pi i}\left(\int_{-\infty-iT}^{c-iT}+\int_{c+iT}^{-\infty+iT}\right)\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}.$$

Then we obtain

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w} = \frac{1}{g(s)}+h_n(s)-R_n(c,x,T)(s) \tag{2.3}$$

and

$$|R_n(c,x,T)(s)| < \frac{M}{\pi T}\frac{(x/n)^c}{\log(x/n)}$$

for $s \in D(\sigma_0, T)$.

For $x < n$, we similarly obtain

$$\frac{1}{2\pi i}\left(\int_{\infty+iT}^{c+iT}+\int_{c+iT}^{c-iT}+\int_{c-iT}^{\infty-iT}\right)\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}=0,$$

because there is no residue term. If we set

$$Q_n(c,x,T)(s) := \frac{1}{2\pi i}\left(\int_{c+iT}^{\infty+iT}+\int_{\infty-iT}^{c-iT}\right)\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w},$$

then we have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w} = -Q_n(c,x,T)(s). \tag{2.4}$$

We also obtain

$$|Q_n(c,x,T)(s)| < \frac{M}{\pi T}\frac{(x/n)^c}{|\log(x/n)|}$$

for $s \in D(\sigma_0, T)$ by the same way as above.

From (2.3) and (2.4), it follows that

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{f(s+w)}{g(s+w)}\frac{x^w}{w}dw &= \sum_{n<x}\frac{a_n}{n^s}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}\\ &\quad+\sum_{x<n}\frac{a_n}{n^s}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}\\ &= \sum_{n<x}\frac{a_n}{n^s}\left(\frac{1}{g(s)}+h_n(s)\right)\\ &\quad-\sum_{n<x}\frac{a_n}{n^s}R_n(c,x,T)(s)\\ &\quad-\sum_{x<n}\frac{a_n}{n^s}Q_n(c,x,T)(s).\end{aligned}$$

Then we obtain (2.1). Furthemore, we can take $A > 0$ such that $|\log(x/n)| > A$ for any $n \in \mathbb{N}$, by the assumption of x. Hence, we also obtain (2.2). □

3. Result

Assume that the number of nontrivial zeros of $\zeta(s)$ which are not on the critical line is finite. Then, the following Proposition is immediate.

Proposition 1. *Under the above assumption, there exists $T_0 > 0$ such that any $s = \sigma + it$ with $\frac{1}{2} < \sigma < 1$ and $t > T_0$ is not a zero of $\zeta(s)$.*

Let

$$B := \sup\{\beta; \zeta(\beta + i\gamma) = 0, \gamma \neq 0\}.$$

The Riemann hypothesis states $B = \frac{1}{2}$. We recall the theorem of de la Vallée Poussin ([2]) which says that there is a constant $A > 0$ such that $\zeta(s)$ is not zero for

$$\sigma \geq 1 - \frac{A}{\log t} \quad (t > t_0),$$

where t_0 is some positive constant. We may restate it as follows: if $s = \sigma + it$ $(t > t_0)$ satisfies

$$t \leq \exp\left(\frac{A}{1-\sigma}\right), \tag{3.1}$$

then $\zeta(s) \neq 0$.

Proposition 2. *It holds that* $\frac{1}{2} \leq B < 1$.

Proof. Since

$$\exp\left(\frac{A}{1-\sigma}\right) \longrightarrow \infty \quad \text{as} \quad \sigma \longrightarrow 1-0,$$

there exists $\sigma_0 < 1$ such that $T_0 < \exp\left(\frac{A}{1-\sigma}\right)$ for $\sigma_0 < \sigma < 1$, where T_0 is the constant in Proposition 1. Then, there is no zero of $\zeta(s)$ in a region $\sigma_0 < \sigma$ by Proposition 1 and the theorem of de la Vallée Poussin. □

Proposition 3. *If* $\frac{1}{2} < B < 1$, *then there is no zero of* $\zeta(s)$ *on the line* $\sigma = B$.

Proof. If the function $\zeta(s)$ has a zero on $\sigma = B$, then the number of zeros of $\zeta(s)$ on $\sigma = B$ is finite by Proposition 1. Let $Z_B = \{\rho_1, \overline{\rho_1}, \ldots, \rho_N, \overline{\rho_N}\}$ be the set of zeros of $\zeta(s)$ on the line $\sigma = B$. By Proposition 1, there exists $\delta_0 > 0$ such that there are no zeros of $\zeta(s)$ except Z_B in the set $\{\sigma + it; B - \delta_0 \leq \sigma, -\infty < t < \infty\}$.

It is well-known that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for $\sigma > 1$. We define an entire function $g(s)$ by

$$g(s) := \prod_{j=1}^{N} (s - \rho_j)(s - \overline{\rho_j}).$$

Then, its zeros are Z_B, and $|g(s)| \longrightarrow \infty$ as $s \to \infty$. We take $T > 0$ such that $|\mathrm{Im}(\rho_j)| < \frac{T}{4}$ for $j = 1, \ldots, N$. Putting $\sigma_0 = B - \delta_0$ and $c = 2$, we apply Lemma 1. We take $x > 1$ which is not an integer, and fix it. Then we have

$$\begin{aligned} \sum_{n<x} \frac{\mu(n)}{n^s} \left(\frac{1}{g(s)} + h_n(s)\right) &= \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w) g(s+w)} \frac{x^w}{w} dw \\ &\quad + \sum_{n<x} \frac{\mu(n)}{n^s} R_n(2, x, T)(s) \\ &\quad + \sum_{x<n} \frac{\mu(n)}{n^s} Q_n(2, x, T)(s) \end{aligned} \tag{3.2}$$

for $s \in D(\sigma_0, T) \setminus Z_B$, where $h_n(s)$ is a meromorphic function on $\mathbb{C}$ whose poles are at most Z_B and all simple.

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We consider the integral of $\frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}$ along $C = C_0 + C_1 + C_2 + C_3$, where C_0, C_1, C_2 and C_3 are segments from $2 - iT$ to $2 + iT$, from $2 + iT$ to $-\frac{\delta_0}{2} + iT$, from $-\frac{\delta_0}{2} + iT$ to $-\frac{\delta_0}{2} - iT$ and from $-\frac{\delta_0}{2} - iT$ to $2 - iT$, respectively. We set

$$D_0 := \left\{\sigma + it; B - \frac{1}{4}\delta_0 < \sigma < B + \frac{1}{4}\delta_0, |t| < \frac{T}{4}\right\}.$$

Then $D_0 \subset D(\sigma_0, T)$ and $Z_B \subset D_0$. For any $s \in D_0 \setminus Z_B$, the poles of $\frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}$ in a domain surrounded by C are $w = 0, \rho_1 - s, \overline{\rho_1} - s, \dots, \rho_N - s$ and $\overline{\rho_N} - s$. The residue of $\frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}$ at $w = 0$ is $\frac{1}{\zeta(s)g(s)}$. Let a_j and b_j be the residues of $\frac{1}{\zeta(s)g(s)}$ at ρ_j and $\overline{\rho_j}$ respectively. Then, the residues of $\frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}$ at $\rho_j - s$ and $\overline{\rho_j} - s$ are $a_j\frac{x^{\rho_j - s}}{\rho_j - s}$ and $b_j\frac{x^{\overline{\rho_j} - s}}{\overline{\rho_j} - s}$ respectively. We define

$$Q(s) := \sum_{j=1}^{N}\left(a_j\frac{x^{\rho_j - s}}{\rho_j - s} + b_j\frac{x^{\overline{\rho_j} - s}}{\overline{\rho_j} - s}\right).$$

Then, $Q(s)$ is a meromorphic function on $\mathbb{C}$ whose poles are at most Z_B and all simple. We note that $\mathrm{Re}(s + w) > B - \frac{3}{4}\delta_0$ if $s \in D_0$ and w is on C. If $s \in D_0$ and w is on C_2, then $\mathrm{Re}(s + w) < B - \frac{1}{4}\delta_0$. Then, $\frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}$ is holomorphic on C as a function of w for any $s \in D_0$. By the residue theorem, we obtain

$$\frac{1}{2\pi i}\int_C \frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw = \frac{1}{\zeta(s)g(s)} + Q(s)$$

for $s \in D_0 \setminus Z_B$. Therefore we have

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw = \frac{1}{\zeta(s)g(s)} + Q(s) + P(s) \tag{3.3}$$

for $s \in D_0 \setminus Z_B$, where

$$P(s) = -\frac{1}{2\pi i}\int_{C_1+C_2+C_3} \frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw$$

is a holomorphic function on D_0. We set

$$E := \left\{\sigma + it; B - \frac{3}{4}\delta_0 \le \sigma \le B + 2 + \frac{1}{4}\delta_0, \frac{3}{4}T \le |t| \le \frac{5}{4}T\right\}$$
$$\bigcup \left\{\sigma + it; B - \frac{3}{4}\delta_0 \le \sigma \le B - \frac{1}{4}\delta_0, |t| \le \frac{5}{4}T\right\}.$$

Then we have $\{s+w; s \in D_0, w \in C_1 \cup C_2 \cup C_3\} \subset E$. Since there is no pole of $\frac{1}{\zeta(s)g(s)}$ on E, we can take $M_0 > 0$ such that

$$\left|\frac{1}{\zeta(s)g(s)}\right| < M_0$$

on E. Then we obtain

$$\left|\int_{C_1} \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw\right| < \frac{M_0 x^2}{T \log x}$$

and

$$\left|\int_{C_3} \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw\right| < \frac{M_0 x^2}{T \log x}$$

for $s \in D_0$. Since we have $w = -\frac{\delta_0}{2} + it$ and $x^w = e^{it \log x} x^{-\frac{\delta_0}{2}}$ on C_2, we obtain

$$\left|\int_{C_2} \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw\right| < \frac{4}{\delta_0} M_0 T x^{-\frac{\delta_0}{2}}$$

for $s \in D_0$. It follows from the above estimates that

$$|P(s)| < \frac{1}{2\pi}\left(\frac{2M_0 x^2}{T \log x} + \frac{4}{\delta_0} M_0 T x^{-\frac{\delta_0}{2}}\right) \tag{3.4}$$

on D_0.

By (3.2) and (3.3), we obtain

$$\begin{aligned}\sum_{n<x} \frac{\mu(n)}{n^s}\left(\frac{1}{g(s)} + h_n(s)\right) = {} & \frac{1}{\zeta(s)g(s)} + Q(s) + P(s) \\ & + \sum_{n<x} \frac{\mu(n)}{n^s} R_n(2,x,T)(s) \\ & + \sum_{x<n} \frac{\mu(n)}{n^s} Q_n(2,x,T)(s)\end{aligned} \tag{3.5}$$

for $s \in D_0 \setminus Z_B$. We see that a function

$$P(s) + \sum_{n<x} \frac{\mu(n)}{n^s} R_n(2,x,T)(s) + \sum_{x<n} \frac{\mu(n)}{n^s} Q_n(2,x,T)(s)$$

is bounded on D_0 by the properties of $R_n(2,x,T)(s)$ and $Q_n(2,x,T)(s)$, and (3.4). The functions $\frac{1}{g(s)}, h_n(s)$ and $Q(s)$ are meromorphic functions on $\mathbb{C}$ whose poles are at most Z_B and all simple. On the other hand, the function $\frac{1}{\zeta(s)g(s)}$ has poles of order at least 2 at every point in Z_B. This contradicts to the equation (3.5). Hence, there is no zero of $\zeta(s)$ on the line $\sigma = B$. □

Theorem 1. *Assume that the number of nontrivial zeros of $\zeta(s)$ which are not on the critical line is finite. Then, the Riemann hypothesis is true.*

Proof. We may assume $\frac{1}{2} \leq B < 1$ by Proposition 2. Suppose that $\frac{1}{2} < B < 1$. Then there is no zero of $\zeta(s)$ on the line $\sigma = B$ by Proposition 3. Hence, we can take B' with $\frac{1}{2} < B' < B$ such that $\zeta(s) \neq 0$ for $B' < \sigma$ by Proposition 1. This contradicts to the definition of B. Thus we conclude $B = \frac{1}{2}$. $\square$

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