Non trivial zeros of the generalized Zeta function

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Abstract

In this paper, the generalized Riemann Zeta function is defined on the right half-plane. In particular, this allows us to prove that for every $\tau \in \mathbb{R}$ there exists at most one point $r \in (0,1)$ such that $|\zeta(r+i\tau)| = 0$.

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1 Main results

Consider the representation of the Riemann Zeta function ζ defined by Abel's summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s\int_1^{+\infty} u^{-1-s}\{u\}du, \quad \Re(s) > 0, \quad \Im(s) \in \mathbb{R}^*,$$

where $\{u\}$ is the fractional part of the real u. Using the integration by parts formula, that can be written as

$$\zeta(s) := -\frac{s}{1-s} - \frac{1}{2} + s(1+s)\int_1^{+\infty} u^{-2-s}\eta_*(u)du, \tag{1}$$

where by Dirichlet's Theorem, the real periodic function $\eta_* : [1,+\infty) \to \mathbb{R}$ is defined as

$$\eta_*(u) := \int_1^u \Big(\frac{1}{2} - \{v\}\Big)dv = \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^2}\Big(1 - \exp(ij2\pi u)\Big), \quad \forall u \geq 1.$$

We denote by $L^\infty := L^\infty([1,+\infty),\mathbb{R})$ the usual set of essentially bounded measurable functions from $[1,+\infty)$ to $\mathbb{R}$. Define the subset $L^\infty_+ \subset L^\infty$, as

$$L^\infty_+ := \Big\{g \in L^\infty : \quad \text{g positive almost everywhere}\Big\}.$$

Define the generalized Zeta function in the following sens

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Definition 1. For every $\eta \in L^\infty_+$, the generalized Zeta function is defined on the right half plane as

$$\zeta[\eta](s) := \int_1^{+\infty} u^{-2-s}\eta(u)du, \quad \Re(s) > 0, \quad \Im(s) \in \mathbb{R}^*.$$

We shall prove the following results:

Theorem 2. *Let be $\eta \in L^\infty_+$ and consider the function $\zeta[\eta]$ given by the Definition (1). For every $(r_*, \tau) \in \mathbb{R}^*_+ \times \mathbb{R}^*$ we have*

$$\Big|\zeta[\eta](r_* + i\tau) - \zeta[\eta](r - i\tau)\Big| > 0, \quad \forall r > r_*.$$

Denote by $B \subset \mathbb{C}$ the critical strip, defined as

$$B := \Big\{ s \in \mathbb{C} : \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^* \Big\}.$$

Notice that for every $s \in B$ we have $f_*(s) = f_*(1 - s)$ where the function $f_* : B \to \mathbb{C}$ is defined as

$$f_*(s) := \frac{1}{1 - s^2} + \frac{1}{2s(1 + s)} = \frac{1}{2}\Big(\frac{1}{1 - s} + \frac{1}{s}\Big), \quad \forall s \in B.$$

Using Equation (1) and the symmetry of the non trivial zeros of the Riemann Zeta function, the following Corollary prove in particular, that for every $\tau \in \mathbb{R}$ there exists at most one point $r \in (0, 1)$ such that $\Big|(r + i\tau)^{-1}(1 + r + i\tau)^{-1}\zeta(r + i\tau)\Big| = 0$.

Corollary 3. *Let be $\eta \in L^\infty_+$ and $f : B \to \mathbb{C}$ be a function that satisfies $f(s) = f(1 - s)$ for every $s \in B$. Suppose that there exists $s_* \in B$ such that $\Re(s_*) \in (0, 2^{-1})$ and such that $\zeta[\eta](s_*) = f(s_*)$. Then $\zeta[\eta](1 - s_*) \neq f(1 - s_*)$.*

2 Proof of the Main results

Proof of the Theorem 2 . Let be $\eta \in L^\infty_+$ and $\tau \in \mathbb{R}^*$ fixed. Let be $r_* > 0$ and $r > r_*$ fixed. In order to simplify the notation, denote

$$\begin{aligned} \Delta_\zeta[\eta](r, r_*, \tau) &:= \zeta[\eta](r_* + i\tau) - \zeta[\eta](r - i\tau), \\ \tilde{\eta}_{r,r_*,\tau}(u) &:= \eta(u)(u^{-r_*} - u^{-(r-2i\tau)}), \quad \forall u \geq 1. \end{aligned}$$

By the definition of the generalized Zeta function given by Definition 1, we have

$$\Delta_\zeta[\eta](r, r_*, \tau) := \int_1^{+\infty} u^{-2-i\tau}\tilde{\eta}_{r,r_*,\tau}(u)du.$$

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For every $y>0$ and $x>0$, by using the change of variable $u \to t^{y^{-1}}$, we have

$$\begin{aligned}
\Delta_{\zeta}[\eta](r,r_*,\tau) &= y^{-1}\int_1^{+\infty} t^{-1-y^{-1}(1+i\tau)}\tilde{\eta}_{r,r_*,\tau}(t^{y^{-1}})dt \\
&= y^{-1}\int_1^{+\infty} t^{-2-y^{-1}-x}\tilde{\eta}_{r,r_*,\tau}(t^{y^{-1}})dt \\
&+ y^{-1}\int_1^{+\infty} t^{-2-y^{-1}-x}\tilde{\eta}_{r,r_*,\tau}(t^{y^{-1}})\Big((t^{-1})^{-1-x+y^{-1}i\tau}-1\Big)dt \\
&= y^{-1}\int_1^{+\infty} t^{-2-y^{-1}-x}\tilde{\eta}_{r,r_*,\tau}(t^{y^{-1}})dt \\
&+ y^{-1}\int_1^{+\infty} t^{-2-y^{-1}-x}\tilde{\eta}_{r,r_*,\tau}(t^{y^{-1}})\sum_{n=1}^{+\infty}\frac{1}{n!}\Big(\prod_{k=1}^{n}\Big(k+x-y^{-1}i\tau\Big)\Big)(1-t^{-1})^n dt.
\end{aligned}$$

Implies

$$\Delta_{\zeta}[\eta](r,r_*,\tau) = y^{-1}t_{r,r_*,\tau}(y,x,0) + y^{-1}\sum_{n=1}^{+\infty}\frac{t_{r,r_*,\tau}(y,x,n)}{n!}\prod_{k=1}^{n}\Big(k+x-y^{-1}i\tau\Big),$$

where in order to simplify the notation, we denoted

$$t_{r,r_*,\tau}(y,x,n) := \int_1^{+\infty} t^{-2-y^{-1}-x}\eta(t^{y^{-1}})\Big(t^{-y^{-1}r_*}-t^{-y^{-1}(r-2i\tau)}\Big)(1-t^{-1})^n dt, \tag{2}$$

That gives

$$\begin{aligned}
\Gamma(x-iy^{-1}\tau)\Delta_{\zeta}[\eta](r,r_*,\tau) &= y^{-1}t_{r,r_*,\tau}(y,x,0)\Gamma(x-iy^{-1}\tau) \\
&+ y^{-1}\sum_{n=1}^{+\infty}\frac{t_{r,r_*,\tau}(y,x,n)}{n!}\Gamma(n+x-iy^{-1}\tau),
\end{aligned} \tag{3}$$

where Γ is the Gamma function. Since

$$\begin{aligned}
\Gamma(x-iy^{-1}\tau) &= \int_1^{+\infty}\exp(-u)u^{-1+x-iy^{-1}\tau}du + \int_0^1 \exp(-u)u^{-1+x-iy^{-1}\tau}du \\
&= \int_1^{+\infty}\exp(-u)u^{-1+x-iy^{-1}\tau}du + \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\frac{1}{n+x-iy^{-1}\tau},
\end{aligned}$$

Then for every $y>0$ we have

$$\begin{aligned}
&\lim_{x\to 0}\Big|\Gamma(x-iy^{-1}\tau)\Big| = \varphi(y), \\
&\qquad \varphi(y) := \Big|\int_1^{+\infty}\exp(-u)u^{-1-iy^{-1}\tau}du + \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\frac{1}{n-iy^{-1}\tau}\Big|.
\end{aligned} \tag{4}$$

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Equation (3) Implies

$$\begin{aligned}\varphi(y)\Big|\Delta_\zeta[\eta](r,r_*,\tau)\Big| &\geq y^{-1}\varphi(y)|t_{r,r_*,\tau}(y,0,0))| - y^{-1}\sum_{n=1}^{+\infty}|t_{r,r_*,\tau}(y,0,n)|\frac{1}{n!}|\Gamma(n-iy^{-1}\tau)| \\ &\geq y^{-1}\varphi(y)|t_{r,r_*,\tau}(y,0,0)| - y^{-1}\sum_{n=1}^{+\infty}|t_{r,r_*,\tau}(y,0,n)|\frac{\Gamma(n)}{n!} \\ &= y^{-1}\varphi(y)|t_{r,r_*,\tau}(y,0,0)| - y^{-1}\sum_{n=1}^{+\infty}n^{-1}|t_{r,r_*,\tau}(y,0,n)|\end{aligned}$$

From the definition of the sequence $(t_{r,r_*,\tau}(y,x,n))_n$ in Equation (2), we obtained

$$\begin{aligned}\varphi(y)\Big|\Delta_\zeta[\eta](r,r_*,\tau)\Big| &\geq y^{-1}\varphi(y)\int_1^{+\infty}t^{-2-y^{-1}}\eta(t^{y^{-1}})\Big(t^{-y^{-1}r_*}-t^{-y^{-1}r}\Big)dt \\ &\quad - y^{-1}\int_1^{+\infty}t^{-2-y^{-1}}\eta(t^{y^{-1}})\Big|t^{-y^{-1}r_*}-t^{-y^{-1}(r-2i\tau)}\Big|\sum_{n=1}^{+\infty}n^{-1}(1-t^{-1})^n dt\end{aligned}$$

Use the fact for $|z|<1$ we have

$$\sum_{n=1}^{+\infty}n^{-1}z^n = -\ln(1-z),$$

by consequence,

$$\begin{aligned}\varphi(y)\Big|\Delta_\zeta[\eta](r,r_*,\tau)\Big| &\geq y^{-1}\varphi(y)\int_1^{+\infty}t^{-2-y^{-1}}\eta(t^{y^{-1}})\Big(t^{-y^{-1}r_*}-t^{-y^{-1}r}\Big)dt \\ &\quad - y^{-1}\int_1^{+\infty}t^{-2-y^{-1}}\eta(t^{y^{-1}})\ln(t)\Big|t^{-y^{-1}r_*}-t^{-y^{-1}(r-2i\tau)}\Big|dt.\end{aligned}$$

Use the change of variable $t^{y^{-1}}\to u$, we get

$$\begin{aligned}\varphi(y)\Big|\Delta_\zeta[\eta](r,r_*,\tau)\Big| &\geq \varphi(y)\int_1^{+\infty}u^{-2-y}\eta(u)\Big(u^{-r_*}-u^{-r}\Big)dt \\ &\quad - y\int_1^{+\infty}u^{-2-y}\eta(u)\ln(u)\Big|u^{-r_*}-u^{-(r-2i\tau)}\Big|dt.\end{aligned}$$

In other words

$$\Big|\Delta_\zeta[\eta](r,r_*,\tau)\Big| \;\geq \int_1^{+\infty}u^{-2-y}\eta(u)\Big(u^{-r_*}-u^{-r}-\frac{y}{\varphi(y)}\ln(u)\Big|u^{-r_*}-u^{-(r-2i\tau)}\Big|\Big)dt. \tag{5}$$

By definition of φ in Equation (4), we have

$$\begin{aligned}\lim_{y\to+\infty}y^{-1}\varphi(y) &= \lim_{y\to+\infty}\Big|y^{-1}\int_1^{+\infty}\exp(-u)u^{-1-iy^{-1}\tau}du+\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\frac{y^{-1}}{n-iy^{-1}\tau}\Big| \\ &= |\tau|^{-1},\end{aligned}$$

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in other words,

$$\lim_{y\to+\infty} \frac{y}{\varphi(y)} = |\tau|.$$

Given that there exists $u_{r,r_*,\tau} > 1$ such that

$$u^{-r_*} - u^{-r} - |\tau| \ln(u) \left| u^{-r_*} - u^{-(r-2i\tau)} \right| > 0, \quad \forall u \in (1, u_{r,r_*,\tau}),$$

Equation (5) implies

$$\begin{aligned} \lim_{y\to+\infty} (u_{r,r_*,\tau})^y \left| \Delta_\zeta[\eta](r, r_*, \tau) \right| \\ \geq \int_1^{u_{r,r_*,\tau}} u^{-2}\eta(u) \left(u^{-r_*} - u^{-r} - |\tau| \ln(u) \left| u^{-r_*} - u^{-(r-2i\tau)} \right| \right) dt > 0. \end{aligned}$$

□

Proof of the Corollary 3 . Let be $\eta \in L_+^\infty$ fixed. Let $f : B \to \mathbb{C}$ be a function that satisfies $f(s) = f(1-s)$ for every $s \in B$. Suppose that there exists $s_* \in B$ such that $\Re(s_*) \in (0, 2^{-1})$ and such that $\zeta[\eta](s_*) = f(s_*)$. By the Theorem 2,

$$\begin{aligned} |\zeta[\eta](1-s_*) - f(1-s_*)| &= |\zeta[\eta](1-s_*) - f(1-s_*) - \zeta[\eta](s_*) + f(s_*)| \\ &= |\zeta[\eta](1-s_*) - \zeta[\eta](s_*)| \neq 0. \end{aligned}$$

□

References

[1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).