

Computation of Novel Binomial Series and Theorems using Multivariable Geometric Series

Chinnaraji Annamalai

School of Management, Indian Institute of Technology, Kharagpur, India

Email: anna@iitkgp.ac.in

<https://orcid.org/0000-0002-0992-2584>

Abstract: Nowadays, the growing complexity of mathematical and computational modelling demands the simplicity of mathematical equations for solving today's scientific problems and challenges. In this paper, the author introduces a novel binomial series and its theorems using the multivariable geometric series, where multivariable geometric series is a geometric series having more than one variable. This computing technique can enable the researchers for further involvement in research and development.

MSC Classification codes: 05A10, 40A05 (65B10)

Keywords: binomial theorem, computation, bivariable

1. Introduction

Geometric series [1,11] played a vital role in differential and integral calculus at the earlier stage of development and still continues as an important part of the study in science, engineering, management and its applications [12]. In this article, a new binomial series without binomial coefficients are computed with binomial theorems.

2. Novel Binomial Series

The author of this article introduces a new binomial series given below.

Theorem 2.1:
$$\sum_{k=0}^n x^k y^{n-k} = \sum_{k=0}^n x^{n-k} y^k = \frac{x^{n+1} - y^{n+1}}{x - y} = \frac{y^{n+1} - x^{n+1}}{y - x}, x \neq y.$$

Proof. Let's prove this theorem using the geometric series as follows.

$$\sum_{k=0}^n \left(\frac{x}{y}\right)^k = \frac{\left(\frac{x}{y}\right)^{n+1} - 1}{\frac{x}{y} - 1} = \left(\frac{x^{n+1} - y^{n+1}}{y^{n+1}}\right) \left(\frac{y}{x - y}\right), x \neq y. \quad (1)$$

By simplifying the equation (1), we get

$$\begin{aligned} y^n \left(1 + \frac{x}{y} + \frac{x^2}{y^2} + \frac{x^3}{y^3} + \cdots + \frac{x^{n-1}}{y^{n-1}} + \frac{x^n}{y^n}\right) &= \frac{x^{n+1} - y^{n+1}}{x - y}, x \neq y. \\ y^n + xy^{n-1} + x^2y^{n-2} + x^3y^{n-3} + \cdots + x^{n-1}y + x^n &= \frac{x^{n+1} - y^{n+1}}{x - y}, x \neq y. \\ \sum_{k=0}^n x^k y^{n-k} &= \frac{x^{n+1} - y^{n+1}}{x - y}, x \neq y. \end{aligned} \quad (2)$$

By rearranging the binomial series (2), we obtain

$$\sum_{k=0}^n x^{n-k} y^k = \frac{y^{n+1} - x^{n+1}}{y - x}, x \neq y. \quad (3)$$

From the binomial series (2) and (3), we conclude that

$$\sum_{k=0}^n x^k y^{n-k} = \sum_{k=0}^n x^{n-k} y^k = \frac{x^{n+1} - y^{n+1}}{x - y} = \frac{y^{n+1} - x^{n+1}}{y - x}, x \neq y. \quad (4)$$

Hence, theorem is proved.

Corollary 2.1: $\sum_{k=0}^n x^k y^{n-k} = \sum_{k=0}^n x^{n-k} y^k = (n+1)x^n, \quad \text{for } x = y.$

Corollary 2.2: $\sum_{k=1}^{n-1} x^k y^{n-k} = \sum_{k=1}^{n-1} x^{n-k} y^k = \frac{x^n y - x y^n}{x - y} = \frac{x y^n - x^n y}{y - x}, x \neq y.$

Let's prove the corollary 2.2 as follows:

$$\begin{aligned} \sum_{k=0}^n x^{n-k} y^k &= x^n + \sum_{k=1}^{n-1} x^{n-k} y^k + y^n = \frac{x^{n+1} - y^{n+1}}{x - y} \\ \sum_{k=1}^{n-1} x^{n-k} y^k &= \frac{x^{n+1} - y^{n+1}}{x - y} - (x^n + y^n) = \frac{x^n y - x y^n}{x - y} \\ \therefore \sum_{k=1}^{n-1} x^k y^{n-k} &= \sum_{k=1}^{n-1} x^{n-k} y^k = \frac{x^n y - x y^n}{x - y} = \frac{x y^n - x^n y}{y - x}, x \neq y. \end{aligned}$$

Theorem 2.2: $\sum_{i=k}^n x^i y^{n-i} = \sum_{i=k}^n x^{n-i} y^i = \frac{x^{n+1} - x^k y^{n+1-k}}{x - y} = \frac{y^{n+1} - y^k x^{n+1-k}}{y - x}, x \neq y.$

Proof. Let's prove the theorem using the geometric series with rational numbers.

$$\begin{aligned} \sum_{i=k}^n \left(\frac{x}{y}\right)^i &= \frac{\left(\frac{x}{y}\right)^{n+1} - \left(\frac{x}{y}\right)^k}{\frac{x}{y} - 1} = \left(\frac{x^{n+1}}{y^{n+1}} - x^k y^{-k}\right) \left(\frac{y}{x - y}\right) = \frac{1}{y^n} \left(\frac{x^{n+1} - x^k y^{n+1-k}}{x - y}\right). \\ y^n \sum_{i=k}^n \left(\frac{x}{y}\right)^i &= \sum_{i=k}^n x^i y^{n-i} = \frac{x^{n+1} - x^k y^{n+1-k}}{x - y} \Rightarrow \sum_{i=k}^n x^{n-i} y^i = \frac{y^{n+1} - y^k x^{n+1-k}}{y - x}. \\ \therefore \sum_{i=k}^n x^i y^{n-i} &= \sum_{i=k}^n x^{n-i} y^i = \frac{x^{n+1} - x^k y^{n+1-k}}{x - y} = \frac{y^{n+1} - y^k x^{n+1-k}}{y - x}, x \neq y. \end{aligned}$$

Corollary 2.3: $\sum_{i=k}^n x^i y^{n-i} = \sum_{i=k}^n x^{n-k} y^k = (n+1-k)x^n, \quad \text{for } x = y.$

Corollary 2.4(i):
$$\sum_{k=0}^n (x+y)^k y^{n-k} = \frac{(x+y)^{n+1} - y^{n+1}}{x}, x \neq 0.$$

Let us prove the corollary 2.4(i) as follows:

Case(i):
$$\sum_{k=0}^n \left(\frac{x+y}{y}\right)^k = \frac{\left(\frac{x+y}{y}\right)^{n+1} - 1}{\frac{x+y}{y} - 1} \Rightarrow \sum_{k=0}^n (x+y)^k y^{n-k} = \frac{(x+y)^{n+1} - y^{n+1}}{x}, x \neq 0.$$

Case(ii):
$$\sum_{k=0}^n \left(\frac{y}{x+y}\right)^k = \frac{\left(\frac{y}{x+y}\right)^{n+1} - 1}{\frac{y}{x+y} - 1} \Rightarrow \sum_{k=0}^n (x+y)^k y^{n-k} = \frac{(x+y)^{n+1} - y^{n+1}}{x}, x \neq 0.$$

Hence proved.

Example: If $x = 1$ and $y = 1$, then
$$\sum_{k=0}^n 3^k 2^{n-k} = 3^{n+1} - 2^{n+1}.$$

Corollary 2.4(ii):
$$\sum_{k=0}^n x^k (x+y)^{n-k} = \frac{(x+y)^{n+1} - x^{n+1}}{y}, y \neq 0$$

Example: If $x = 1$ and $y = 1$, then
$$\sum_{k=0}^n 2^k = 2^{n+1} - 1.$$

Corollary 2.5(i):
$$\sum_{k=0}^n (x-y)^k y^{n-k} = \frac{y^{n+1} - (x+y)^{n+1}}{x}, x \neq 0.$$

Let us prove the corollary 2.5(i) as follows:

$$\sum_{k=0}^n \left(\frac{x-y}{y}\right)^k = \frac{\left(\frac{x-y}{y}\right)^{n+1} - 1}{\frac{x-y}{y} - 1} \Rightarrow \sum_{k=0}^n (x-y)^k y^{n-k} = \frac{y^{n+1} - (x+y)^{n+1}}{x}, x \neq 0.$$

Hence proved.

Corollary 2.5(ii):
$$\sum_{k=0}^n (y-x)^k y^{n-k} = \frac{y^{n+1} - (y-x)^{n+1}}{x}, x \neq 0.$$

Let us prove the corollary 2.5(ii) as follows:

$$\sum_{k=0}^n \left(\frac{x-y}{-y}\right)^k = \sum_{k=0}^n \left(\frac{y-x}{y}\right)^k = \frac{\left(\frac{y-x}{y}\right)^{n+1} - 1}{\frac{y-x}{y} - 1} \Rightarrow \sum_{k=0}^n (x-y)^k y^{n-k} = \frac{y^{n+1} - (y-x)^{n+1}}{x}.$$

Hence proved.

Theorem 2.3: $\prod_{k=0}^n (x^{2^k} + y^{2^k}) = \frac{x^{2^{n+1}} - y^{2^{n+1}}}{x - y}, x \neq y.$

Proof. Let's prove this theorem as follows.

$$\begin{aligned} \frac{x^2 - y^2}{x - y} &= \frac{(x + y)(x - y)}{x - y} = x + y. \\ \frac{x^{2^2} - y^{2^2}}{x - y} &= \frac{x^4 - y^4}{x - y} = \frac{(x^2 + y^2)(x^2 - y^2)}{x - y} = (x^2 + y^2)(x + y). \\ \frac{x^{2^3} - y^{2^3}}{x - y} &= \frac{x^8 - y^8}{x - y} = (x^4 + y^4)(x^2 + y^2)(x + y) = (x^{2^2} + y^{2^2})(x^{2^1} + y^{2^1})(x^{2^0} + y^{2^0}). \end{aligned}$$

We can continue the same process up to 2^{n+1} .

$$\begin{aligned} \frac{x^{2^3} - y^{2^3}}{x - y} &= \frac{x^8 - y^8}{x - y} = (x^{2^n} + y^{2^n})(x^{2^{n-1}} + y^{2^{n-1}}) \cdots (x^{2^1} + y^{2^1})(x^{2^0} + y^{2^0}). \\ \therefore \prod_{k=0}^n (x^{2^k} + y^{2^k}) &= \frac{x^{2^{n+1}} - y^{2^{n+1}}}{x - y}, x \neq y. \end{aligned}$$

Hence proved.

Corollary 2.6: $\prod_{k=0}^{n-1} x^{2^k} = x^{2^n-1}.$

Case i: Let's prove the corollary 2.6 using the theorem 2.3.

$$\prod_{k=0}^{n-1} (x^{2^k} + 0^{2^k}) = \frac{x^{2^n} - 0}{x - 0} \Rightarrow \prod_{k=0}^{n-1} x^{2^k} = \frac{x^{2^n}}{x} = x^{2^n-1}.$$

Case II: Let's prove the corollary 2.6 using the geometric series.

$$\prod_{k=0}^{n-1} x^{2^k} = x^1 \times x^2 \times x^{2^2} \times x^{2^3} \times \cdots \times x^{2^{n-1}} = x^{1+2+2^2+2^3+\cdots+2^{n-1}} = x^{\frac{2^n-1}{2-1}} = x^{2^n-1}.$$

Hence proved.

Corollary 2.7: If $x = y$, then $\prod_{k=0}^n (x^{2^k} + y^{2^k}) = 2 \prod_{k=0}^n x^{2^k} = 2x^{2^{n+1}-1}.$

Corollary 2.8: $\sum_{i=0}^{2^{n+1}-1} x^i y^{n-i} = \prod_{k=0}^n (x^{2^k} + y^{2^k}) = \frac{x^{2^{n+1}} - y^{2^{n+1}}}{x - y}, x \neq y.$

Examples for corollary 2.8:

(i). Let $n = 0$.

$$\sum_{i=0}^{2^{0+1}-1} x^i y^{n-i} = \sum_{i=0}^{2-1} x^i y^{n-i} = \sum_{i=0}^1 x^i y^{n-i} = x + y = \frac{x^2 - y^2}{x - y}.$$

$$\prod_{k=0}^0 (x^{2^k} + y^{2^k}) = x + y = \frac{x^2 - y^2}{x - y}. \quad \therefore \sum_{i=0}^{2^{0+1}-1} x^i y^{n-i} = \prod_{k=0}^0 (x^{2^k} + y^{2^k}) = \frac{x^2 - y^2}{x - y}.$$

(ii). Let $n = 1$.

$$\sum_{i=0}^{2^{1+1}-1} x^i y^{n-i} = \sum_{i=0}^3 x^i y^{n-i} = x^3 + x^2 y + x y^2 + y^3 = \frac{x^{2^2} - y^{2^2}}{x - y} = \frac{x^4 - y^4}{x - y}.$$

$$\prod_{k=0}^1 (x^{2^k} + y^{2^k}) = (x + y)(x^2 + y^2) = x^3 + x^2 y + x y^2 + y^3 = \frac{x^4 - y^4}{x - y}.$$

$$\therefore \sum_{i=0}^{2^{1+1}-1} x^i y^{n-i} = \prod_{k=0}^1 (x^{2^k} + y^{2^k}) = \frac{x^{2^2} - y^{2^2}}{x - y} = \frac{x^4 - y^4}{x - y}.$$

3. Conclusion

In this article, a novel binomial series and theorems have been introduced for mathematical and computational application. This idea can enable the researchers for further involvement in the scientific research.

References

- [1] Annamalai, C. (2023) Binomial Geometric Series. *Cambridge Open Engage, Cambridge University Press*. <https://doi.org/10.33774/coe-2023-072fg>.
- [2] Annamalai, C. (2023) Geometric Progression-Based Binomial Series for Computing Application. *SSRN Electronic Journal*. <http://dx.doi.org/10.2139/ssrn.4591123>.
- [3] Annamalai, C. (2023) Computational Technique for Geometric Series with Radicals. *TechRxiv, IEEE*. <https://doi.org/10.36227/techrxiv.24311773.v1>.
- [4] Annamalai, C. (2023) Computation of Algebraic Expressions and Geometric Series with Radicals. *TechRxiv, IEEE*. <https://doi.org/10.36227/techrxiv.24311773.v2>.
- [5] Annamalai, C. (2023) Binomial Geometric Series for Computational Application. *Cambridge Open Engage, Cambridge University Press*. <https://doi.org/10.33774/coe-2023-kmgvt>.
- [6] Annamalai, C. (2024) Computation of Novel Binomial Series using Geometric Series. *Cambridge Open Engage, Cambridge University Press*. <https://doi.org/10.33774/coe-2023-3ch85-v5>.
- [7] Annamalai, C. (2018) Annamalai's Computing Model for Algorithmic Geometric Series and Its Mathematical Structures. *Journal of Mathematics and Computer Science*, 3(1),1-6 <https://doi.org/10.11648/j.mcs.20180301.11>.

- [8] Annamalai, C. (2017) Analysis and Modelling of Annamalai Computing Geometric Series and Summability. *Mathematical Journal of Interdisciplinary Sciences*, 6(1), 11-15. <https://doi.org/10.15415/mjis.2017.61002>.
- [9] Annamalai, C. (2018) Novel Computation of Algorithmic Geometric Series and Summability. *Journal of Algorithms and Computation*, 50(1), 151-153. <https://www.doi.org/10.22059/JAC.2018.68866>.
- [10] Annamalai, C. (2024) Computation of Novel Binomial Series using Bivariable Geometric Series. *Cambridge Open Engage, Cambridge University Press*. <https://doi.org/10.33774/coe-2024-vsv7f>.
- [11] Annamalai, C. (2024) Computation of Geometric Series on Numerical Expansions. *Cambridge Open Engage, Cambridge University Press*. <https://doi.org/10.33774/coe-2024-sb8lc>.
- [12] Annamalai C (2010) “Application of Exponential Decay and Geometric Series in Effective Medicine”, *Advances in Bioscience and Biotechnology*, Vol. 1(1), pp 51-54. <https://doi.org/10.4236/abb.2010.11008>.