# Computation of Novel Binomial Series and Theorems using Multivariable Geometric Series

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**Abstract:** Nowadays, the growing complexity of mathematical and computational modelling demands the simplicity of mathematical equations for solving today's scientific problems and challenges. In this paper, the author introduces a novel binomial series and its theorems using the multivariable geometric series, where multivariable geometric series is a geometric series having more than one variable. This computing technique can enable the researchers for further involvement in research and development.

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## 1. Introduction

Geometric series [1,11] played a vital role in differential and integral calculus at the earlier stage of development and still continues as an important part of the study in science, engineering, management and its applications [12]. In this article, a new binomial series without binomial coefficients are computed with binomial theorems.

## 2. Novel Binomial Series

The author of this article introduces a new binomial series given below.

***Theorem* 2.1**: $\sum_{k=0}^{n} x^k y^{n-k} = \sum_{k=0}^{n} x^{n-k} y^k = \frac{x^{n+1} - y^{n+1}}{x - y} = \frac{y^{n+1} - x^{n+1}}{y - x}, x \neq y.$

*Proof*. Let's prove this theorem using the geometric series as follows.

$$\sum_{k=0}^{n} \left(\frac{x}{y}\right)^k = \frac{\left(\frac{x}{y}\right)^{n+1} - 1}{\frac{x}{y} - 1} = \left(\frac{x^{n+1} - y^{n+1}}{y^{n+1}}\right)\left(\frac{y}{x - y}\right), x \neq y. \tag{1}$$

By simplifying the equation (1), we get

$$y^n \left(1 + \frac{x}{y} + \frac{x^2}{y^2} + \frac{x^3}{y^3} + \cdots + \frac{x^{n-1}}{y^{n-1}} + \frac{x^n}{y^n}\right) = \frac{x^{n+1} - y^{n+1}}{x - y}, x \neq y.$$

$$y^n + xy^{n-1} + x^2 y^{n-2} + x^3 y^{n-3} + \cdots + x^{n-1} y + x^n = \frac{x^{n+1} - y^{n+1}}{x - y}, x \neq y.$$

$$\sum_{k=0}^{n} x^k y^{n-k} = \frac{x^{n+1} - y^{n+1}}{x - y}, x \neq y. \tag{2}$$

By rearranging the binomial series (2), we obtain



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$$\sum_{k=0}^{n} x^{n-k} y^k = \frac{y^{n+1} - x^{n+1}}{y - x}, x \neq y. \tag{3}$$

From the binomial series (2) and (3), we conclude that

$$\sum_{k=0}^{n} x^k y^{n-k} = \sum_{k=0}^{n} x^{n-k} y^k = \frac{x^{n+1} - y^{n+1}}{x - y} = \frac{y^{n+1} - x^{n+1}}{y - x}, x \neq y. \tag{4}$$

Hence, theorem is proved.

***Corollary* 2.1**: $\sum_{k=0}^{n} x^k y^{n-k} = \sum_{k=0}^{n} x^{n-k} y^k = (n+1)x^n, \quad$ for $x = y$.

***Corollary* 2.2**: $\sum_{k=1}^{n-1} x^k y^{n-k} = \sum_{k=1}^{n-1} x^{n-k} y^k = \frac{x^n y - x y^n}{x - y} = \frac{x y^n - x^n y}{y - x}, x \neq y.$

Let's prove the corollary 2.2 as follows:

$$\sum_{k=0}^{n} x^{n-k} y^k = x^n + \sum_{k=1}^{n-1} x^{n-k} y^k + y^n = \frac{x^{n+1} - y^{n+1}}{x - y}$$

$$\sum_{k=1}^{n-1} x^{n-k} y^k = \frac{x^{n+1} - y^{n+1}}{x - y} - (x^n + y^n) = \frac{x^n y - x y^n}{x - y}$$

$$\therefore \sum_{k=1}^{n-1} x^k y^{n-k} = \sum_{k=1}^{n-1} x^{n-k} y^k = \frac{x^n y - x y^n}{x - y} = \frac{x y^n - x^n y}{y - x}, x \neq y.$$

***Theorem* 2.2**: $\sum_{i=k}^{n} x^i y^{n-i} = \sum_{i=k}^{n} x^{n-i} y^i = \frac{x^{n+1} - x^k y^{n+1-k}}{x - y} = \frac{y^{n+1} - y^k x^{n+1-k}}{y - x}, x \neq y.$

*Proof*. Let's prove the theorem using the geometric series with rational numbers.

$$\sum_{i=k}^{n} \left(\frac{x}{y}\right)^i = \frac{\left(\frac{x}{y}\right)^{n+1} - \left(\frac{x}{y}\right)^k}{\frac{x}{y} - 1} = \left(\frac{x^{n+1}}{y^{n+1}} - x^k y^{-k}\right)\left(\frac{y}{x - y}\right) = \frac{1}{y^n}\left(\frac{x^{n+1} - x^k y^{n+1-k}}{x - y}\right).$$

$$y^n \sum_{i=k}^{n} \left(\frac{x}{y}\right)^i = \sum_{i=k}^{n} x^i y^{n-i} = \frac{x^{n+1} - x^k y^{n+1-k}}{x - y} \Rightarrow \sum_{i=k}^{n} x^{n-i} y^i = \frac{y^{n+1} - y^k x^{n+1-k}}{y - x}.$$

$$\therefore \sum_{i=k}^{n} x^i y^{n-i} = \sum_{i=k}^{n} x^{n-i} y^i = \frac{x^{n+1} - x^k y^{n+1-k}}{x - y} = \frac{y^{n+1} - y^k x^{n+1-k}}{y - x}, x \neq y.$$

***Corollary* 2.3**: $\sum_{i=k}^{n} x^i y^{n-i} = \sum_{i=k}^{n} x^{n-k} y^k = (n+1-k)x^n, \quad$ for $x = y$.



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$$\boldsymbol{Corollary\ 2.4(i)}: \sum_{k=0}^{n}(x+y)^k y^{n-k} = \sum_{k=0}^{n} y^k\,(x+y)^{n-k} = \frac{(x+y)^{n+1}-y^{n+1}}{x}, x \neq 0.$$

Let us prove the corollary 2.4($i$) as follows:

$$\boldsymbol{Case(i)}: \sum_{k=0}^{n}\left(\frac{x+y}{y}\right)^k = \frac{\left(\frac{x+y}{y}\right)^{n+1}-1}{\frac{x+y}{y}-1} \Rightarrow \sum_{k=0}^{n}(x+y)^k y^{n-k} = \frac{(x+y)^{n+1}-y^{n+1}}{x}, x \neq 0.$$

$$\boldsymbol{Case(ii)}: \sum_{k=0}^{n}\left(\frac{y}{x+y}\right)^k = \frac{\left(\frac{y}{x+y}\right)^{n+1}-1}{\frac{y}{x+y}-1} \Rightarrow \sum_{k=0}^{n}(x+y)^k y^{n-k} = \frac{(x+y)^{n+1}-y^{n+1}}{x}, x \neq 0.$$

Hence proved.

Example: If $x = 1$ and $y = 1$, then $\sum_{k=0}^{n} 3^k 2^{n-k} = 3^{n+1} - 2^{n+1}$.

$$\boldsymbol{Corollary\ 2.4(ii)}: \sum_{k=0}^{n} x^k (x+y)^{n-k} = \sum_{k=0}^{n}(x+y)^k x^{n-k} = \frac{(x+y)^{n+1}-x^{n+1}}{y}, y \neq 0$$

Example: If $x = 1$ and $y = 1$, then $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$.

$$\boldsymbol{Corollary\ 2.5(i)}: \sum_{k=0}^{n}(x-y)^k x^{n-k} = \sum_{k=0}^{n} x^k (x-y)^{n-k} = \frac{x^{n+1}-(x-y)^{n+1}}{y}, x \neq 0.$$

Let us prove the corollary 2.5($i$) as follows:

$$\sum_{k=0}^{n}\left(\frac{x-y}{x}\right)^k = \frac{\left(\frac{x-y}{x}\right)^{n+1}-1}{\frac{x-y}{x}-1} \Rightarrow \sum_{k=0}^{n}(x-y)^k x^{n-k} = \frac{x^{n+1}-(x-y)^{n+1}}{y}, x \neq 0.$$

Hence proved.

$$\boldsymbol{Corollary\ 2.5(ii)}: \sum_{k=0}^{n}(y-x)^k y^{n-k} = \sum_{k=0}^{n} y^k (y-x)^{n-k} = \frac{y^{n+1}-(y-x)^{n+1}}{x}, x \neq 0.$$

Let us prove the corollary 2.5($ii$) as follows:

$$\sum_{k=0}^{n}\left(\frac{x-y}{-y}\right)^k = \sum_{k=0}^{n}\left(\frac{y-x}{y}\right)^k = \frac{\left(\frac{y-x}{y}\right)^{n+1}-1}{\frac{y-x}{y}-1} \Rightarrow \sum_{k=0}^{n}(x-y)^k x^{n-k} = \frac{y^{n+1}-(y-x)^{n+1}}{x}.$$

Hence proved.



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$\boldsymbol{Theorem\ 2.3}$: $\prod_{k=0}^{n}\left(x^{2^k}+y^{2^k}\right)=\frac{x^{2^{n+1}}-y^{2^{n+1}}}{x-y}, x\neq y.$

Proof. Let's prove this theorem as follows.

$$\frac{x^2-y^2}{x-y}=\frac{(x+y)(x-y)}{x-y}=x+y.$$

$$\frac{x^{2^2}-y^{2^2}}{x-y}=\frac{x^4-y^4}{x-y}=\frac{(x^2+y^2)(x^2-y^2)}{x-y}=(x^2+y^2)(x+y).$$

$$\frac{x^{2^3}-y^{2^3}}{x-y}=\frac{x^8-y^8}{x-y}=(x^4+y^4)(x^2+y^2)(x+y)=\left(x^{2^2}+y^{2^2}\right)\left(x^{2^1}+y^{2^1}\right)\left(x^{2^0}+y^{2^0}\right).$$

We can continue the same process up to $2^{n+1}$.

$$\frac{x^{2^3}-y^{2^3}}{x-y}=\frac{x^8-y^8}{x-y}=\left(x^{2^n}+y^{2^n}\right)\left(x^{2^{n-1}}+y^{2^{n-1}}\right)\cdots\left(x^{2^1}+y^{2^1}\right)\left(x^{2^0}+y^{2^0}\right).$$

$$\therefore\prod_{k=0}^{n}\left(x^{2^k}+y^{2^k}\right)=\frac{x^{2^{n+1}}-y^{2^{n+1}}}{x-y}, x\neq y.$$

Hence proved.

$\boldsymbol{Corollary\ 2.6}$: $\prod_{k=0}^{n-1}x^{2^k}=x^{2^n-1}.$

***Case i***: Let's prove the corollary 2.6 using the theorem 2.3.

$$\prod_{k=0}^{n-1}\left(x^{2^k}+0^{2^k}\right)=\frac{x^{2^n}-0}{x-0}\Rightarrow\prod_{k=0}^{n-1}x^{2^k}=\frac{x^{2^n}}{x}=x^{2^n-1}.$$

***Case II:*** Let's prove the corollary 2.6 using the geometric series.

$$\prod_{k=0}^{n-1}x^{2^k}=x^1\times x^2\times x^{2^2}\times x^{2^3}\times\cdots\times x^{2^{n-1}}=x^{1+2+2^2+2^3+\cdots+2^{n-1}}=x^{\frac{2^n-1}{2-1}}=x^{2^n-1}.$$

Hence proved.

$\boldsymbol{Corollary\ 2.7}$: If $x=y$, then $\prod_{k=0}^{n}\left(x^{2^k}+y^{2^k}\right)=2\prod_{k=0}^{n}x^{2^k}=2x^{2^n-1}.$

$\boldsymbol{Corollary\ 2.8}$: $\sum_{i=0}^{2^{n+1}-1}x^iy^{n-i}=\prod_{k=0}^{n}\left(x^{2^k}+y^{2^k}\right)=\frac{x^{2^{n+1}}-y^{2^{n+1}}}{x-y}, x\neq y.$

Examples for corollary 2.8:

$(i)$. Let $n=0$.

$$\sum_{i=0}^{2^{0+1}-1}x^iy^{n-i}=\sum_{i=0}^{2-1}x^iy^{n-i}=\sum_{i=0}^{1}x^iy^{n-i}=x+y=\frac{x^2-y^2}{x-y}.$$

$$\prod_{k=0}^{0}\left(x^{2^k}+y^{2^k}\right)=x+y=\frac{x^2-y^2}{x-y}. \quad \therefore \sum_{i=0}^{2^{0+1}-1} x^i y^{n-i}=\prod_{k=0}^{0}\left(x^{2^k}+y^{2^k}\right)=\frac{x^2-y^2}{x-y}.$$

$(ii)$. Let $n=1$.

$$\sum_{i=0}^{2^{1+1}-1} x^i y^{n-i}=\sum_{i=0}^{3} x^i y^{n-i}=x^3+x^2y+xy^2+y^3=\frac{x^{2^2}-y^{2^2}}{x-y}==\frac{x^4-y^4}{x-y}.$$

$$\prod_{k=0}^{1}\left(x^{2^k}+y^{2^k}\right)=(x+y)(x^2+y^2)=x^3+x^2y+xy^2+y^3=\frac{x^4-y^4}{x-y}.$$

$$\therefore \sum_{i=0}^{2^{1+1}-1} x^i y^{n-i}=\prod_{k=0}^{1}\left(x^{2^k}+y^{2^k}\right)=\frac{x^{2^2}-y^{2^2}}{x-y}=\frac{x^4-y^4}{x-y}.$$

## 3. Conclusion

In this article, a novel binomial series and theorems have been introduced for mathematical and computational application. This idea can enable the researchers for further involvement in the scientific research.

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