Computing the Sum of Geometric Series based on Algebraic Expression

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Abstract: Nowadays, the growing complexity of mathematical and computational modelling demands the simplicity of mathematical equations to solve today's scientific problems and meet challenges. In this paper, the author introduces the sum of geometric series based on numerical and algebraic expressions. This computational technique can enable the students and scholars to enhance their research knowledge in the domain of computing and mathematical sciences.

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1. Introduction

Geometric series [1-8] played a vital role in differential and integral calculus at the earlier stage of development and still continues as an important part of the study in science, engineering, and management. In this article, the author converts a numerical expression into the sum of geometric series.

2. Numerical Expansions and Geometric Series

Let us find the numerical expansions of 2^n and $\frac{1}{2^n}$ and its geometric series.

$$2^n = 2^n \Rightarrow 2^n = 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 + 1 \Rightarrow \sum_{i=0}^{n-1} 2^i = 2^n - 1. \quad (1)$$

$$\frac{1}{2^n} = \frac{1}{2^n} \Rightarrow \frac{1}{2^n} = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \cdots + \frac{1}{2^{n+k}} + \frac{1}{2^{n+k}} \Rightarrow \sum_{i=1}^{k} \frac{1}{2^{n+i}} = \frac{1}{2^n} - \frac{1}{2^{n+k}}. \quad (2)$$

Also, $2 = 2 \Rightarrow 2 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \frac{1}{2^n} \Rightarrow \sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n} = \frac{2^{n+1} - 1}{2^n}.$

$$1 = 1 \Rightarrow 1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^n} + \frac{1}{2^n} \Rightarrow \sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

$$\frac{1}{2} = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^n} + \frac{1}{2^n} \Rightarrow \sum_{i=2}^{n} \frac{1}{2^i} = \frac{1}{2} - \frac{1}{2^n} = \frac{2^{n-1} - 1}{2^n}.$$

Similarly, we can create the sum of geometric series for other expansion.

$$\frac{1}{2^2} = \frac{1}{2^2} \Rightarrow \sum_{i=3}^{n} \frac{1}{2^i} = \frac{2^{n-2} - 1}{2^n}; \quad \frac{1}{2^3} = \frac{1}{2^3} \Rightarrow \sum_{i=4}^{n} \frac{1}{2^i} = \frac{2^{n-3} - 1}{2^n}; \ \cdots\cdots\cdots;$$

$$\frac{1}{2^{n-1}}=\frac{1}{2^n}+\frac{1}{2^n}\Rightarrow\sum_{i=n}^{n}\frac{1}{2^i}=\frac{1}{2^{n-1}}-\frac{1}{2^n}=\frac{2-1}{2^n};\text{ and }\frac{1}{2^n}=\frac{1}{2^n}\Rightarrow 0=\frac{1-1}{2^n}.$$

Theorem 2.1: $\sum_{i=1}^{n} i\times\frac{1}{2^i}=\frac{1}{2^n}\{2^{n+1}-(n+1)\}$.

Let us prove this theorem as follows:

$$\sum_{i=1}^{n}\frac{1}{2^i}+\sum_{i=2}^{n}\frac{1}{2^i}+\sum_{i=3}^{n}\frac{1}{2^i}+\cdots+\sum_{i=n-1}^{n}\frac{1}{2^i}+\sum_{i=n}^{n}\frac{1}{2^i}=\frac{2^n-1}{2^n}+\frac{2^{n-1}-1}{2^n}+\cdots+\frac{2-1}{2^n}+\frac{1-1}{2^n}.$$

$$\sum_{i=1}^{n} i\times\frac{1}{2^i}=\frac{1}{2^n}\{(1+2+2^2+\cdots+2^{n-1}+2^n)-n\}=\frac{1}{2^n}\left\{\sum_{i=0}^{n}2^i-n\right\}. \quad (3)$$

By substituting the equation (1) in (3), we get

$$\sum_{i=1}^{n} i\times\frac{1}{2^i}=\frac{1}{2^n}(2^{n+1}-1-n)=\frac{1}{2^n}\{2^{n+1}-(n+1)\}.$$

Hence proved.

Theorem 2.2: $\sum_{i=0}^{n-1}(i+1)\times 2^i=(n-1)2^n+1$.

Let us prove this theorem as follows:

$$2^n=2^n\Rightarrow 2^n=2^{n-1}+2^{n-2}+2^{n-3}+\cdots+2^k+2^k\Rightarrow\sum_{i=k}^{n-1}2^i=2^n-2^k. \quad (4)$$

By substituting $k=0,1,2,3,\cdots,n-1$ in the equation (4), we obtain

$$\sum_{i=0}^{n-1}2^i=2^n-1;\ \sum_{i=1}^{n-1}2^i=2^n-2;\sum_{i=2}^{n-1}2^i=2^n-2^2,\cdots,\sum_{i=n-1}^{n-1}2^i=2^n-2^{n-1}. \quad (5)$$

Adding all the equations in (5), we get

$$\sum_{i=0}^{n-1}2^i+\sum_{i=1}^{n-1}2^i+\sum_{i=2}^{n-1}2^i+\cdots+\sum_{i=n-1}^{n-1}2^i=2^n-1+2^n-1+2^n-2^{n-1}+\cdots+2^n-2^{n-1}.$$

$$\sum_{i=0}^{n-1}(i+1)2^i=n2^n-(1+2+2^2+2^3+\cdots+2^{n-1})=n2^n-\sum_{i=0}^{n-1}2^i=n2^n-(2^n-1). \quad (6)$$

From the equation (6), we conclude that

$$\sum_{i=0}^{n-1}(i+1)2^i=(n-1)2^n+1.$$

Hence proved.

Let x^n be a positive integer. The algebraic expression for the equation $x^n=x^n$ is shown below:

$$x^n=\overbrace{x^{n-1}+x^{n-1}+x^{n-1}+\cdots+x^{n-1}}^{x\text{ positive integers}}=x^{n-1}(x)=x^{n-1}(x-1)+x^{n-1}$$
$$=x^{n-1}(x-2)+x^{n-1}+x^{n-1}=x^{n-1}(x-3)+x^{n-1}+x^{n-1}+x^{n-1}=\cdots$$

If x^n be a positive real number, then we can simply write as $x^n = (x-1)x^{n-1} + x^{n-1}$.

Also, $x^n = (x-1)x^{n-1} + (x-1)x^{n-2} + (x-1)x^{n-3} + \cdots + (x-1)x^{k+1} + (x-1)x^k + x^k$.

$$\therefore \sum_{i=k}^{n-1} x^i = \frac{x^n - x^k}{x-1}; \sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x-1}; \sum_{i=1}^{n-1} x^i = \frac{x^n - x}{x-1}; \sum_{i=2}^{n-1} x^i = \frac{x^n - x^2}{x-1}, \cdots$$

Where $x \neq 1$.

$\boldsymbol{Theorem\ 2.3}$: $\displaystyle\sum_{i=0}^{n-1}(i+1)x^i = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}, x \neq 1.$

Let prove this theorem as follows:

$$\sum_{i=0}^{n-1} x^i + \sum_{i=1}^{n-1} x^i + \sum_{i=2}^{n-1} x^i + \cdots + \sum_{i=n-1}^{n-1} x^i = \frac{x^n - 1}{x-1} + \frac{x^n - x}{x-1} + \frac{x^n - x^2}{x-1} + \cdots + \frac{x^n - x^{n-1}}{x-1}.$$

$$\sum_{i=0}^{n-1}(i+1)x^i = \frac{1}{x-1}\{nx^n - (1 + x + x^2 + x^3 + \cdots + x^{n-1})\} = \frac{1}{x-1}\left\{nx^n - \frac{x^n - 1}{x-1}\right\}.$$

By simplifying the above expression, we get

$$\sum_{i=0}^{n-1}(i+1)x^i = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}, x \neq 1.$$

Hence proved.

3. Conclusion

In this article, the author has introduced the sums of geometric series using numerical expressions. This idea can enable the researchers to be involved in the scientific research on computing and mathematical sciences.

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