TWO PROOFS OF THE RIEMANN HYPOTHESIS

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Abstract. We prove the Riemann hypothesis by a new analytic approach. We first give a new zero-free region of $\zeta(s)$. Using this region, we prove the Riemann hypothesis by two ways. We use a lemma on Dirichlet series and the theorem of de la Vallée Poussin in the first proof. The second one depends on numerical results.

1. Introduction

The Riemann zeta function $\zeta(s)$ is defined on $\sigma > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $s = \sigma + it$. It is analytically continued to a meromorphic function on the whole plane with a pole at $s = 1$. It is well-known that negative even integers are zeros of $\zeta(s)$. Other zeros of $\zeta(s)$ are called complex zeros or nontrivial zeros.

Riemann stated the following statement, the so-called Riemann hypothesis, in [6] in 1859.

The Riemann hypothesis. All nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$.

Hilbert listed it as the eighth problem of his 23 problems in his 1900 address to the Paris International Congress of Mathematicians. This is one of the most important unsolved problems in the twenty-first century.

Nobody has succeeded to prove it up to the present, but many computational results are known. In the early part of the twentieth century, they were obtained by hand computation ([1], [4], [5] and [7]). Numerical computations by computers have permitted us to check the truth of the Riemann hypothesis to extremely large t. We refer to [3] for a history of numerical verifications.

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We will obtain further results with the development of computers and numerical methods. However, we can never reach to the goal in this way. We have to analyze the behaviour of the zeta function or the xi function for larger t than a fixed t_0. For this purpose, we expand the real part and the imaginary part of the integration in a representation of the xi function into sums of powers of $1/t$ in Section 3. The precise representations of these parts are given in Section 4. We give a necessary condition for zeros of the xi function. We obtain a new zero-free region of $\zeta(s)$ using this condition (Theorem 1).

Using Theorem 1, we prove the Riemann hypothesis by two ways. We prepare a lemma on Dirichlet series for the first proof in Section 7. Let B be the supremum of the real parts of nontrivial zeros. Combining our new zero-free region with the theorem of de la Vallée Poussin, we first show $\frac{1}{2} \le B < 1$. We next show that if $\frac{1}{2} < B < 1$, then there is no zero of $\zeta(s)$ on the line $\sigma = B$ by the lemma on Dirichlet series. Then, we conclude $B = \frac{1}{2}$.

The second proof depends on numerical results. We give precise estimates in Section 9. We determine T_0 in Theorem 1 by these estimates. Then, the Riemann hypothesis naturally follows from already known numerical results.

2. Preliminaries

The xi function $\xi(s)$ of Riemann is defined by

$$\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

It is an entire function and satisfies the functional equation $\xi(s) = \xi(1-s)$. It also satisfies $\xi(\overline{s}) = \overline{\xi(s)}$. The set of nontrivial zeros of $\zeta(s)$ coincides with the set of zeros of $\xi(s)$. Therefore, if s is a nontrivial zero of $\zeta(s)$, then $\overline{s}, 1-s$ and $1-\overline{s}$ are also zeros of $\zeta(s)$.

The function $\xi(s)$ has the following integral representation

$$\xi(s) = \frac{1}{2} + \frac{s}{2}(s-1)\int_1^\infty (x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1})\psi(x)dx, \tag{2.1}$$

where $\psi(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$ (for example, see p.22 in [8]).

Let

$$\Phi(s) = \int_1^\infty (x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1})\psi(x)dx.$$

We denote by $R(s) = \mathrm{Re}\Phi(s)$ and $I(s) = \mathrm{Im}\Phi(s)$ the real part and the imaginary part of $\Phi(s)$, respectively. Since

$$\begin{cases} \mathrm{Re}\xi(s) = \dfrac{1}{2} + \dfrac{1}{2}\mathrm{Re}\left(s(s-1)\Phi(s)\right), \\ \mathrm{Im}\xi(s) = \dfrac{1}{2}\mathrm{Im}\left(s(s-1)\Phi(s)\right), \end{cases}$$

we have that $\xi(s) = 0$ if and only if

$$\begin{cases} 1 + (\sigma(\sigma-1) - t^2)R(s) - t(2\sigma-1)I(s) = 0, \\ \quad t(2\sigma-1)R(s) + (\sigma(\sigma-1) - t^2)I(s) = 0. \end{cases} \tag{2.2}$$

3. Functions $R(s)$ and $I(s)$

We have

$$R(s) = \int_1^\infty (x^{\frac{\sigma}{2}-1} + x^{-\frac{\sigma}{2}-\frac{1}{2}}) \cos\left(\frac{t}{2}\log x\right)\psi(x)dx$$

and

$$I(s) = \int_1^\infty (x^{\frac{\sigma}{2}-1} - x^{-\frac{\sigma}{2}-\frac{1}{2}}) \sin\left(\frac{t}{2}\log x\right)\psi(x)dx.$$

First we expand $R(s)$. We set

$$f_0(x) := (x^{\frac{\sigma}{2}} + x^{-\frac{\sigma}{2}+\frac{1}{2}})\psi(x).$$

Using integration by parts and

$$\frac{d}{dx}\left(\frac{2}{t}\sin\left(\frac{t}{2}\log x\right)\right) = x^{-1}\cos\left(\frac{t}{2}\log x\right),$$

we obtain

$$R(s) = -\frac{2}{t}\int_1^\infty \sin\left(\frac{t}{2}\log x\right) f_0'(x)dx.$$

Let $f_1(x) := xf_0'(x)$. Then we have

$$\begin{aligned} \int_1^\infty \sin\left(\frac{t}{2}\log x\right) f_0'(x)dx &= \int_1^\infty x^{-1}\sin\left(\frac{t}{2}\log x\right) f_1(x)dx \\ &= \frac{2}{t}f_1(1) + \frac{2}{t}\int_1^\infty \cos\left(\frac{t}{2}\log x\right) f_1'(x)dx \end{aligned}$$

by

$$\frac{d}{dx}\left(\frac{2}{t}\cos\left(\frac{t}{2}\log x\right)\right) = -x^{-1}\sin\left(\frac{t}{2}\log x\right)$$

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and integration by parts. We define $f_2(x) := xf_1'(x)$. By the same way as above, we obtain

$$R(s) = -\left(\frac{2}{t}\right)^2 f_1(1) + \left(\frac{2}{t}\right)^3 \int_1^\infty \sin\left(\frac{t}{2}\log x\right) f_2'(x)dx. \tag{3.1}$$

We use the same argument for $I(s)$. We define $g_0(x) := (x^{\frac{\sigma}{2}} - x^{-\frac{\sigma}{2}+\frac{1}{2}})\psi(x)$ and $g_k(x) := xg_{k-1}'(x)$ for $k = 1, 2, 3$. Then, we similarly obtain

$$I(s) = -\left(\frac{2}{t}\right)^3 g_2(1) + \left(\frac{2}{t}\right)^4 \int_1^\infty \sin\left(\frac{t}{2}\log x\right) g_3'(x)dx \tag{3.2}$$

by integration by parts.

We collect here precise forms of $f_k(x), g_k(x)$ and their derivatives for the later use. By a direct calculation, we obtain the following formulas

$$\begin{aligned} f_0'(x) = &\left\{\frac{\sigma}{2}x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)x^{-\frac{\sigma}{2}-\frac{1}{2}}\right\}\psi(x) \\ &+ \left(x^{\frac{\sigma}{2}} + x^{-\frac{\sigma}{2}+\frac{1}{2}}\right)\psi^{(1)}(x), \end{aligned}$$

$$\begin{aligned} f_1(x) = &\left\{\frac{\sigma}{2}x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi(x) \\ &+ \left(x^{\frac{\sigma}{2}+1} + x^{-\frac{\sigma}{2}+\frac{3}{2}}\right)\psi^{(1)}(x), \end{aligned}$$

$$\begin{aligned} f_1'(x) = &\left\{\left(\frac{\sigma}{2}\right)^2 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^2 x^{-\frac{\sigma}{2}-\frac{1}{2}}\right\}\psi(x) \\ &+ \left\{(\sigma+1)x^{\frac{\sigma}{2}} + (-\sigma+2)x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi^{(1)}(x) \\ &+ \left(x^{\frac{\sigma}{2}+1} + x^{-\frac{\sigma}{2}+\frac{3}{2}}\right)\psi^{(2)}(x), \end{aligned}$$

$$\begin{aligned} f_2(x) = &\left\{\left(\frac{\sigma}{2}\right)^2 x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^2 x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi(x) \\ &+ \left\{(\sigma+1)x^{\frac{\sigma}{2}+1} + (-\sigma+2)x^{-\frac{\sigma}{2}+\frac{3}{2}}\right\}\psi^{(1)}(x) \\ &+ \left(x^{\frac{\sigma}{2}+2} + x^{-\frac{\sigma}{2}+\frac{5}{2}}\right)\psi^{(2)}(x), \end{aligned}$$

$$
\begin{aligned}
f_2'(x) = &\left\{ \left(\frac{\sigma}{2}\right)^3 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^3 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{3}{4}\sigma^2 + \frac{3}{2}\sigma + 1\right) x^{\frac{\sigma}{2}} + \left(\frac{3}{4}\sigma^2 - 3\sigma + \frac{13}{4}\right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{3}{2}\sigma + 3\right) x^{\frac{\sigma}{2}+1} + \left(-\frac{3}{2}\sigma + \frac{9}{2}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+2} + x^{-\frac{\sigma}{2}+\frac{5}{2}} \right) \psi^{(3)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_0'(x) = &\left\{ \frac{\sigma}{2} x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right) x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
&+ \left(x^{\frac{\sigma}{2}} - x^{-\frac{\sigma}{2}+\frac{1}{2}} \right) \psi^{(1)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_1(x) = &\left\{ \frac{\sigma}{2} x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\
&+ \left(x^{\frac{\sigma}{2}+1} - x^{-\frac{\sigma}{2}+\frac{3}{2}} \right) \psi^{(1)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_1'(x) = &\left\{ \left(\frac{\sigma}{2}\right)^2 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^2 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ (\sigma+1) x^{\frac{\sigma}{2}} - (-\sigma+2) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+1} - x^{-\frac{\sigma}{2}+\frac{3}{2}} \right) \psi^{(2)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_2(x) = &\left\{ \left(\frac{\sigma}{2}\right)^2 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^2 x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ (\sigma+1) x^{\frac{\sigma}{2}+1} - (-\sigma+2) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(1)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+2} - x^{-\frac{\sigma}{2}+\frac{5}{2}} \right) \psi^{(2)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_2'(x) = &\left\{ \left(\frac{\sigma}{2}\right)^3 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^3 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{3}{4}\sigma^2 + \frac{3}{2}\sigma + 1\right) x^{\frac{\sigma}{2}} - \left(\frac{3}{4}\sigma^2 - 3\sigma + \frac{13}{4}\right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{3}{2}\sigma + 3\right) x^{\frac{\sigma}{2}+1} - \left(-\frac{3}{2}\sigma + \frac{9}{2}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+2} - x^{-\frac{\sigma}{2}+\frac{5}{2}} \right) \psi^{(3)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_3(x) = &\left\{ \left(\frac{\sigma}{2}\right)^3 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^3 x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{3}{4}\sigma^2 + \frac{3}{2}\sigma + 1\right) x^{\frac{\sigma}{2}+1} - \left(\frac{3}{4}\sigma^2 - 3\sigma + \frac{13}{4}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{3}{2}\sigma + 3\right) x^{\frac{\sigma}{2}+2} - \left(-\frac{3}{2}\sigma + \frac{9}{2}\right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(2)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+3} - x^{-\frac{\sigma}{2}+\frac{7}{2}} \right) \psi^{(3)}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
g_3'(x) = &\left\{ \left(\frac{\sigma}{2}\right)^4 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^4 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{1}{2}\sigma^3 + \frac{3}{2}\sigma^2 + 2\sigma + 1\right) x^{\frac{\sigma}{2}} \right. \\
&\quad \left. - \left(-\frac{1}{2}\sigma^3 + 3\sigma^2 - \frac{13}{2}\sigma + 5\right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{3}{2}\sigma^2 + 6\sigma + 7\right) x^{\frac{\sigma}{2}+1} - \left(\frac{3}{2}\sigma^2 - 9\sigma + \frac{29}{2}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
&+ \left\{ (2\sigma+6) x^{\frac{\sigma}{2}+2} - (-2\sigma+8) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+3} - x^{-\frac{\sigma}{2}+\frac{7}{2}} \right) \psi^{(4)}(x).
\end{aligned}
$$

4. Formulas of $\psi^{(k)}(x)$

An important property of the series $\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ is the reciprocal formula

$$
2\psi(x) + 1 = x^{-\frac{1}{2}} \left(2\psi\left(\frac{1}{x}\right) + 1 \right) \tag{4.1}
$$

for $x > 0$ (for example, see (2.6.3) in [8]). The integral representation (2.1) of $\xi(s)$ is derived from the above reciprocal formula.

It is rewritten as

$$\psi(x) = x^{-\frac{1}{2}}\psi\left(\frac{1}{x}\right) + \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}. \tag{4.2}$$

Differentiating (4.2) one after another, we obtain the following equalities

$$\psi^{(1)}(x) = -\frac{1}{2}x^{-\frac{3}{2}}\psi\left(\frac{1}{x}\right) - x^{-\frac{5}{2}}\psi^{(1)}\left(\frac{1}{x}\right) - \frac{1}{4}x^{-\frac{3}{2}}, \tag{4.3}$$

$$\begin{aligned}\psi^{(2)}(x) = \frac{3}{4}x^{-\frac{5}{2}}\psi\left(\frac{1}{x}\right) + 3x^{-\frac{7}{2}}\psi^{(1)}\left(\frac{1}{x}\right)\\ + x^{-\frac{9}{2}}\psi^{(2)}\left(\frac{1}{x}\right) + \frac{3}{8}x^{-\frac{5}{2}}\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\psi^{(3)}(x) = -\frac{15}{8}x^{-\frac{7}{2}}\psi\left(\frac{1}{x}\right) - \frac{45}{4}x^{-\frac{9}{2}}\psi^{(1)}\left(\frac{1}{x}\right)\\ - \frac{15}{2}x^{-\frac{11}{2}}\psi^{(2)}\left(\frac{1}{x}\right) - x^{-\frac{13}{2}}\psi^{(3)}\left(\frac{1}{x}\right)\\ - \frac{15}{16}x^{-\frac{7}{2}}.\end{aligned} \tag{4.5}$$

Next we give formulas of $\psi^{(k)}(1)$. Substituting $x = 1$ in (4.3), we have

$$\frac{1}{4}\psi(1) + \psi^{(1)}(1) = -\frac{1}{8}. \tag{4.6}$$

Similarly, we obtain

$$\frac{15}{4}\psi^{(2)}(1) + \psi^{(3)}(1) = \frac{15}{32}\psi(1) + \frac{15}{64} \tag{4.7}$$

by (4.5) and (4.6).

We can consider $\psi^{(k)}$ for negative integers k. For any $k \in \mathbb{Z}$ we set

$$\psi^{(k)}(x) := (-1)^k\pi^k\sum_{n=1}^{\infty} n^{2k}e^{-\pi n^2 x}.$$

Then we have $\psi(x) = \psi^{(0)}(x)$ and $\dfrac{d}{dx}\psi^{(k)}(x) = \psi^{(k+1)}(x)$. By integration by parts, we obtain

$$\int_1^{\infty}\psi(x)dx = -\psi^{(-1)}(1) \tag{4.8}$$

and

$$\int_1^\infty x^2\psi(x)dx = -\psi^{(-1)}(1) + 2\psi^{(-2)}(1) - 2\psi^{(-3)}(1). \tag{4.9}$$

Furthermore we have

$$\int_1^\infty x^{2k-1}|\psi^{(2k-1)}(x)|dx = \sum_{j=0}^{2k-1}(-1)^j\frac{(2k-1)!}{(2k-1-j)!}\psi^{(2k-2-j)}(1) \tag{4.10}$$

and

$$\int_1^\infty x^{2k}\psi^{(2k)}(x)dx = \sum_{j=0}^{2k}(-1)^{j+1}\frac{(2k)!}{(2k-j)!}\psi^{(2k-1-j)}(1) \tag{4.11}$$

for $k = 1, 2, \ldots$.

We give representations of $R(s)$ and $I(s)$ more precisely, determining $f_1(1)$ and $g_2(1)$ in (3.1) and (3.2) respectively. We obtain

$$\begin{aligned} f_1(1) &= \left\{\frac{\sigma}{2} + \left(-\frac{\sigma}{2} + \frac{1}{2}\right)\right\}\psi(1) + 2\psi^{(1)}(1) \\ &= 2\left(\frac{1}{4}\psi(1) + \psi^{(1)}(1)\right) \\ &= -\frac{1}{4} \end{aligned}$$

by the formula of $f_1(x)$ in Section 3 and (4.6). Then we have

$$-\left(\frac{2}{t}\right)^2 f_1(1) = \frac{1}{t^2}. \tag{4.12}$$

Therefore, it follows from (3.1) and (4.12) that

$$R(s) = \frac{1}{t^2} + \alpha(s)\frac{1}{t^3}, \tag{4.13}$$

where

$$\alpha(s) = 2^3\int_1^\infty \sin\left(\frac{t}{2}\log x\right) f_2'(x)dx. \tag{4.14}$$

Using the same argument as above for $I(s)$, we obtain

$$I(s) = (2\sigma - 1)\frac{1}{t^3} + \beta(s)\frac{1}{t^4}, \tag{4.15}$$

where

$$\beta(s) = 2^4\int_1^\infty \sin\left(\frac{t}{2}\log x\right) g_3'(x)dx. \tag{4.16}$$

Lemma 1. *There exists a positive constant M such that*

$$|\alpha(s)| < M$$

for $\frac{1}{2} \le \sigma \le 1$ *and* $t \in \mathbb{R}$.

Proof. By (4.14), we have

$$|\alpha(s)| \le 2^3 \int_1^\infty |f_2'(x)|\, dx.$$

From the formula of $f_2'(x)$ in Section 3, it follows that

$$|f_2'(x)| < c_0\psi(x) + c_1 x|\psi^{(1)}(x)| + c_2 x^2 \psi^{(2)}(x) + 2x^3|\psi^{(3)}(x)|,$$

where c_0, c_1 and c_2 are positive constants. Integrating the above inequality, we obtain the desired inequality by (4.8), (4.10) and (4.11). □

We also have a similar result for $\beta(s)$ in (4.16).

Lemma 2. *There exists a positive constant L such that*

$$\frac{|\beta(s)|}{2\sigma - 1} < L$$

for $\frac{1}{2} < \sigma \le 1$ *and* $t \in \mathbb{R}$.

Proof. We rewrite $g_3'(x)$ in Section 3 as

$$\begin{aligned} g_3'(x) = &\sum_{j=0}^{3} \left(A_j(\sigma) x^{\frac{\sigma}{2}-1+j} - B_j(\sigma) x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \right) \psi^{(j)}(x) \\ &+ \left(x^{\frac{\sigma}{2}+3} - x^{-\frac{\sigma}{2}+\frac{7}{2}} \right) \psi^{(4)}(x). \end{aligned} \tag{4.17}$$

Now we have

$$\begin{aligned} &A_j(\sigma) x^{\frac{\sigma}{2}-1+j} - B_j(\sigma) x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \\ &\quad = \left(A_j(\sigma) - B_j(\sigma)\right) x^{\frac{\sigma}{2}-1+j} + B_j(\sigma) \left(x^{\frac{\sigma}{2}-1+j} - x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \right). \end{aligned} \tag{4.18}$$

It holds that

$$\begin{aligned} &x^{\frac{\sigma}{2}-1+j} - x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \\ &\quad = \sum_{n=1}^{\infty} \frac{1}{n!} \left\{ \left(\frac{\sigma}{2} - 1 + j \right)^n - \left(-\frac{\sigma}{2} - \frac{1}{2} + j \right)^n \right\} (\log x)^n. \end{aligned}$$

We have

$$\left(\frac{\sigma}{2}-1+j\right)^n-\left(-\frac{\sigma}{2}-\frac{1}{2}+j\right)^n = \left(\sigma-\frac{1}{2}\right)\sum_{k=0}^{n-1}\left(\frac{\sigma}{2}-1+j\right)^{n-1-k}\left(-\frac{\sigma}{2}-\frac{1}{2}+j\right)^k. \tag{4.19}$$

Since

$$\left|\frac{\sigma}{2}-1\right|<\frac{3}{4}<1 \quad \text{and} \quad \left|-\frac{\sigma}{2}-\frac{1}{2}\right|\le 1$$

for σ with $\frac{1}{2}<\sigma\le 1$, we obtain

$$\left|\left(\frac{\sigma}{2}-1\right)^n-\left(-\frac{\sigma}{2}-\frac{1}{2}\right)^n\right|\le\left(\sigma-\frac{1}{2}\right)n$$

by (4.19). Therefore, we have

$$\begin{aligned}\left|x^{\frac{\sigma}{2}-1}-x^{-\frac{\sigma}{2}-\frac{1}{2}}\right| &\le\left(\sigma-\frac{1}{2}\right)(\log x)\sum_{n=1}^{\infty}\frac{1}{(n-1)!}(\log x)^{n-1}\\ &=\left(\sigma-\frac{1}{2}\right)(\log x)x.\end{aligned}$$

Noting

$$\left|\frac{\sigma}{2}\right|\le\frac{1}{2} \quad \text{and} \quad \left|-\frac{\sigma}{2}-\frac{1}{2}+1\right|<\frac{1}{4}<\frac{1}{2},$$

we similarly obtain

$$\begin{aligned}\left|x^{\frac{\sigma}{2}}-x^{-\frac{\sigma}{2}-\frac{1}{2}+1}\right| &\le\left(\sigma-\frac{1}{2}\right)(\log x)\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(\frac{\log x}{2}\right)^{n-1}\\ &=\left(\sigma-\frac{1}{2}\right)(\log x)x^{\frac{1}{2}}.\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}\left|x^{\frac{\sigma}{2}-1+j}-x^{-\frac{\sigma}{2}-\frac{1}{2}+j}\right| &= x^{j-1}\left|x^{\frac{\sigma}{2}}-x^{-\frac{\sigma}{2}-\frac{1}{2}+1}\right|\\ &\le\left(\sigma-\frac{1}{2}\right)(\log x)x^{-\frac{1}{2}+j}\end{aligned}$$

for $j=2,3,4$.

Moreover, we have the following estimates for any σ with $\frac{1}{2} < \sigma \leq 1$

$$|A_0(\sigma) - B_0(\sigma)| = \frac{1}{2^4}(2\sigma - 1)\left|2\sigma^2 - 2\sigma + 1\right| \\ \leq \frac{1}{2^4}(2\sigma - 1),$$

$$|B_0(\sigma)| < B_0\left(\frac{1}{2}\right) = \frac{1}{2^8},$$

$$|A_1(\sigma) - B_1(\sigma)| = \frac{1}{2}(2\sigma - 1)\left|\sigma^2 - \sigma + 8\right| \\ \leq 2^2(2\sigma - 1),$$

$$|B_1(\sigma)| < B_1\left(\frac{1}{2}\right) = \frac{39}{2^4},$$

$$|A_2(\sigma) - B_2(\sigma)| = \frac{15}{2}(2\sigma - 1),$$

$$|B_2(\sigma)| < B_2\left(\frac{1}{2}\right) = \frac{89}{2^3},$$

$$|A_3(\sigma) - B_3(\sigma)| = 2(2\sigma - 1) \quad \text{and} \quad |B_3(\sigma)| < B_3\left(\frac{1}{2}\right) = 7.$$

Then, it follows from (4.17), (4.18) and the above inequalities that

$$\begin{aligned} \frac{|g_3'(x)|}{2\sigma - 1} < &\left(\frac{1}{2^4}x^{\frac{\sigma}{2}-1} + \frac{1}{2^9}(\log x)x\right)\psi(x) \\ &+ \left(2^2x^{\frac{\sigma}{2}} + \frac{39}{2^5}(\log x)x^{\frac{1}{2}}\right)\left|\psi^{(1)}(x)\right| \\ &+ \left(\frac{15}{2}x^{\frac{\sigma}{2}+1} + \frac{89}{2^4}(\log x)x^{\frac{3}{2}}\right)\psi^{(2)}(x) \\ &+ \left(2x^{\frac{\sigma}{2}+2} + \frac{7}{2}(\log x)x^{\frac{5}{2}}\right)\left|\psi^{(3)}(x)\right| \\ &+ \frac{1}{2}(\log x)x^{\frac{7}{2}}\psi^{(4)}(x). \end{aligned}$$

Since $x^{\frac{\sigma}{2}-1} \le 1$ and $\log x < 2x^{\frac{1}{2}}$ for $x \ge 1$, we have

$$\begin{aligned}\frac{|g_3'(x)|}{2\sigma-1} < {} & \frac{1}{2^4}\psi(x) + \frac{1}{2^8}x^2\psi(x)\\ & + \left(2^2 + \frac{39}{2^4}\right) x \left|\psi^{(1)}(x)\right|\\ & + \left(\frac{15}{2} + \frac{89}{2^3}\right) x^2\psi^{(2)}(x)\\ & + (2+7)x^3 \left|\psi^{(3)}(x)\right|\\ & + x^4\psi^{(4)}(x).\end{aligned}$$

Integrating it, we obtain the conclusion. □

5. Necessary condition

Take any a with $\frac{1}{2} < a < 1$ and fix it. We give a necessary condition for zeros of $\zeta(s)$ when $a < \sigma \le 1$.

Any $s = \sigma + it$ with $a < \sigma$ has the following representation

$$s = a + r\cos\theta + ir\sin\theta, \tag{5.1}$$

where $r = \sqrt{(\sigma-a)^2 + t^2}$ and $\theta = \arctan\left(\frac{t}{\sigma - a}\right)$. By (4.13), (4.15) and Lemmas 1 and 2, there exists $T_0 > 0$ such that $R(s) > 0$ and $|I(s)| > 0$ for any $s = \sigma + it$ with $\frac{1}{2} < \sigma \le 1$ and $|t| > T_0$. We assume $|t| > T_0$. By (5.1), we obtain

$$t(2\sigma - 1) = r^2\sin 2\theta + r(2a-1)\sin\theta$$

and

$$\sigma(\sigma-1) - t^2 = r^2\cos 2\theta + r(2a-1)\cos\theta + a(a-1).$$

Then, we have

$$\begin{aligned}1 &+ \left(\sigma(\sigma-1) - t^2\right) R(s) - t(2\sigma-1)I(s)\\ &= 1 + r^2R(s)\cos 2\theta + r(2a-1)R(s)\cos\theta + a(a-1)R(s)\\ &\quad - r^2I(s)\sin 2\theta - r(2a-1)I(s)\sin\theta\\ &= 1 + r^2\sqrt{R(s)^2 + I(s)^2}\cos\left(2\theta + \arctan\frac{I(s)}{R(s)}\right)\\ &\quad + r(2a-1)R(s)\cos\theta - r(2a-1)I(s)\sin\theta + a(a-1)R(s)\\ &= 1 + r^2\sqrt{R(s)^2 + I(s)^2}\cos\left(2\theta + \arctan\frac{I(s)}{R(s)}\right)\\ &\quad + r(2a-1)\sqrt{R(s)^2 + I(s)^2}\cos\left(\theta + \arctan\frac{I(s)}{R(S)}\right) + a(a-1)R(s)\end{aligned}$$

and

$$\begin{aligned}t(2\sigma-1)R(s)+\left(\sigma(\sigma-1)-t^2\right)I(s)\\&=r^2\sqrt{R(s)^2+I(s)^2}\sin\left(2\theta+\arctan\frac{I(s)}{R(s)}\right)\\&\quad+r(2a-1)\sqrt{R(s)^2+I(s)^2}\sin\left(\theta+\arctan\frac{I(s)}{R(s)}\right)+a(a-1)I(s).\end{aligned}$$

Therefore, the equations (2.2) is equivalent to

$$\begin{cases}r^2|\Phi(s)|\cos\left(2\theta+\arctan\frac{I(s)}{R(s)}\right)\\\qquad=-r(2a-1)|\Phi(s)|\cos\left(\theta+\arctan\frac{I(s)}{R(s)}\right)\\\qquad\quad-a(a-1)R(s)-1,\\r^2|\Phi(s)|\sin\left(2\theta+\arctan\frac{I(s)}{R(s)}\right)\\\qquad=-r(2a-1)|\Phi(s)|\sin\left(\theta+\arctan\frac{I(s)}{R(s)}\right)\\\qquad\quad-a(a-1)I(s),\end{cases}\tag{5.2}$$

where $|\Phi(s)|=\sqrt{R(s)^2+I(s)^2}$. Squaring each equation in (5.2), we obtain

$$\begin{aligned}&r^4|\Phi(s)|^2\cos^2\left(2\theta+\arctan\frac{I(s)}{R(s)}\right)\\&\quad=r^2(2a-1)^2|\Phi(s)|^2\cos^2\left(\theta+\arctan\frac{I(s)}{R(s)}\right)\\&\qquad+2r(2a-1)a(a-1)|\Phi(s)|R(s)\cos\left(\theta+\arctan\frac{I(s)}{R(s)}\right)\\&\qquad+2r(2a-1)|\Phi(s)|\cos\left(\theta+\arctan\frac{I(s)}{R(s)}\right)\\&\qquad+a^2(a-1)^2R(s)^2+2a(a-1)R(s)+1\end{aligned}\tag{5.3}$$

and

$$\begin{aligned} & r^4|\Phi(s)|^2 \sin^2\left(2\theta + \arctan\frac{I(s)}{R(s)}\right) \\ & \quad = r^2(2a-1)^2|\Phi(s)|^2\sin^2\left(\theta+\arctan\frac{I(s)}{R(s)}\right) \\ & \qquad + 2r(2a-1)a(a-1)|\Phi(s)|I(s)\sin\left(\theta+\arctan\frac{I(s)}{R(s)}\right) \\ & \qquad + a^2(a-1)^2 I(s)^2. \end{aligned} \tag{5.4}$$

We add (5.3) to (5.4). Then, we have

$$\begin{aligned} & r^4|\Phi(s)|^2 \\ & \quad = r^2(2a-1)^2|\Phi(s)|^2 \\ & \qquad + 2r(2a-1)a(a-1)|\Phi(s)|^2\sin\left(\theta+\arctan\frac{I(s)}{R(s)}+\arctan\frac{R(s)}{I(s)}\right) \\ & \qquad + 2r(2a-1)|\Phi(s)|\cos\left(\theta+\arctan\frac{I(s)}{R(s)}\right) \\ & \qquad + a^2(a-1)^2|\Phi(s)|^2 + 2a(a-1)R(s) + 1. \end{aligned} \tag{5.5}$$

Hence, we obtain the following proposition.

Proposition 1. *Let $\frac{1}{2} < a < 1$. If $\xi(s) = 0$ for some $s = \sigma + it$ with $a < \sigma \le 1$ and $|t| > T_0$, then the equation (5.5) holds at s.*

6. New zero-free region

Theorem 1. *Any $s = \sigma + it$ with $\frac{1}{2} < \sigma \le 1$ and $|t| > T_0$ is not a zero of $\zeta(s)$.*

Proof. Suppose that $s = \sigma + it$ with $\frac{1}{2} < \sigma \le 1$ and $t > T_0$ is a zero of $\xi(s)$. We take a such that $\frac{1}{2} < a < \sigma$. Then, the equation (5.5) holds at s by Proposition 1. Since $\overline{s} = \sigma - it$ is also a zero of $\xi(s)$, it holds at $\overline{s}$. If $s = \sigma + it = a + r\cos\theta + ir\sin\theta$, then we have

$\overline{s} = a + r\cos(-\theta) + ir\sin(-\theta)$ and $I(\overline{s}) = -I(s)$. Therefore, we obtain

(6.1)

$$\begin{aligned} r^4|\Phi(s)|^2 \\ &= r^2(2a-1)^2|\Phi(s)|^2 \\ &\quad - 2r(2a-1)a(a-1)|\Phi(s)|^2 \sin\left(\theta + \arctan\frac{I(s)}{R(s)} + \arctan\frac{R(s)}{I(s)}\right) \\ &\quad + 2r(2a-1)|\Phi(s)|\cos\left(\theta + \arctan\frac{I(s)}{R(s)}\right) \\ &\quad + a^2(a-1)^2|\Phi(s)|^2 + 2a(a-1)R(s) + 1 \end{aligned}$$

by (5.5) for $\overline{s}$. From (5.5) and (6.1), it follows that

(6.2)

$$4r(2a-1)a(a-1)|\Phi(s)|^2\sin\left(\theta + \arctan\frac{I(s)}{R(s)} + \arctan\frac{R(s)}{I(s)}\right) = 0.$$

We have

$$\arctan\frac{I(s)}{R(s)} + \arctan\frac{R(s)}{I(s)} = \frac{\pi}{2},$$

because $I(s)/R(s) > 0$. Since $0 < \theta < \frac{\pi}{2}$, we obtain $\sin\left(\theta + \frac{\pi}{2}\right) \neq 0$. Then, the equation (6.2) does not hold. This completes the proof. □

7. Lemma on Dirichlet series

This section is devoted to the following lemma which is a variant of Lemma 3.12 in [8].

Lemma 3. *We assume that a Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is absolutely convergent for $\sigma > 1$. Take $\sigma_0 < 1$ and $c > 0$ with $\sigma_0 + c > 1$. Let $g(s)$ be an entire function with finite number of zeros $Z_g = \{\alpha_1, \ldots, \alpha_N\}$ such that $\sigma_0 < \mathrm{Re}(\alpha_j) < \sigma_0 + c$ $(j = 1, \ldots, N)$ and $|g(s)| \to \infty$ as $s \to \infty$. Let x be a non-integral number with $x > 1$. Then, there exist meromorphic functions $h_n(s)$ $(1 \leq n < x)$ on $\mathbb{C}$ whose poles are at most $\alpha_1, \ldots, \alpha_N$ and all simple such that

$$(7.1)\quad \sum_{n<x} \frac{a_n}{n^s}\left(\frac{1}{g(s)} + h_n(s)\right) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{f(s+w)}{g(s+w)}\frac{x^w}{w}dw + \sum_{n<x}\frac{a_n}{n^s}R_n(c,x,T)(s) + \sum_{x<n}\frac{a_n}{n^s}Q_n(c,x,T)(s)$$

for any $s \in D(\sigma_0, T) \setminus Z_g$, where T is a positive number with $|\mathrm{Im}(\alpha_j)| < \frac{T}{4}$ $(j = 1, \dots, N)$, $D(\sigma_0, T) = \{\sigma + it; \sigma_0 \le \sigma, |t| < \frac{T}{4}\}$, and $R_n(c, x, T)(s)$ and $Q_n(c, x, T)(s)$ are holomorphic functions depending only on $g(s), n, c, x, T$ and satisfying
(7.2)
$$|R_n(c, x, T)(s)| < \frac{M}{A\pi T}\left(\frac{x}{n}\right)^c \quad \text{and} \quad |Q_n(c, x, T)(s)| < \frac{M}{A\pi T}\left(\frac{x}{n}\right)^c$$
on $D(\sigma_0, T)$ for some constants A and M. Therefore, the right side of (7.1) converges absolutely and uniformly on $D(\sigma_0, T)$.

Proof. Take any $s \in D(\sigma_0, T)$. A function $\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w \frac{1}{w}$ of w has poles at $w = 0, \alpha_1 - s, \dots, \alpha_N - s$. It has the residues $\frac{1}{g(s)}$ and $a_j\left(\frac{x}{n}\right)^{\alpha_j - s}\frac{1}{\alpha_j - s}$ at $w = 0$ and $\alpha_j - s$ respectively, where a_j is the residue of $\frac{1}{g(s)}$ at α_j. We define a meromorphic function
$$h_n(s) := \sum_{j=1}^{N} a_j\left(\frac{x}{n}\right)^{\alpha_j - s}\frac{1}{\alpha_j - s}.$$

If $n < x$, then we obtain
$$\frac{1}{2\pi i}\left(\int_{-\infty - iT}^{c - iT} + \int_{c - iT}^{c + iT} + \int_{c + iT}^{-\infty + iT}\right)\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w \frac{dw}{w} = \frac{1}{g(s)} + h_n(s)$$
by the residue theorem. Let
$$M := \sup\left\{\left|\frac{1}{g(s)}\right|; s = u + it, -\infty < u < \infty, \frac{3}{4}T \le |t| \le \frac{5}{4}T\right\}.$$
Then we have $0 < M < \infty$ by the assumption of $g(s)$. Therefore, the following estimation holds
$$\left|\frac{1}{2\pi i}\int_{-\infty - iT}^{c - iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w \frac{dw}{w}\right| < \frac{M}{2\pi T}\frac{(x/n)^c}{\log(x/n)}.$$
Similarly we have
$$\left|\frac{1}{2\pi i}\int_{c + iT}^{-\infty + iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w \frac{dw}{w}\right| < \frac{M}{2\pi T}\frac{(x/n)^c}{\log(x/n)}.$$
We define
$$R_n(c, x, T)(s) := \frac{1}{2\pi i}\left(\int_{-\infty - iT}^{c - iT} + \int_{c + iT}^{-\infty + iT}\right)\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w \frac{dw}{w}.$$

Then we obtain

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}=\frac{1}{g(s)}+h_n(s)-R_n(c,x,T)(s) \tag{7.3}$$

and

$$|R_n(c,x,T)(s)|<\frac{M}{\pi T}\frac{(x/n)^c}{\log(x/n)}$$

for $s\in D(\sigma_0,T)$.

For $x<n$, we similarly obtain

$$\frac{1}{2\pi i}\left(\int_{\infty+iT}^{c+iT}+\int_{c+iT}^{c-iT}+\int_{c-iT}^{\infty-iT}\right)\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}=0,$$

because there is no residue term. If we set

$$Q_n(c,x,T)(s):=\frac{1}{2\pi i}\left(\int_{c+iT}^{\infty+iT}+\int_{\infty-iT}^{c-iT}\right)\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w},$$

then we have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}=-Q_n(c,x,T)(s). \tag{7.4}$$

We also obtain

$$|Q_n(c,x,T)(s)|<\frac{M}{\pi T}\frac{(x/n)^c}{|\log(x/n)|}$$

for $s\in D(\sigma_0,T)$ by the same way as above.

From (7.3) and (7.4), it follows that

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{f(s+w)}{g(s+w)}\frac{x^w}{w}dw &= \sum_{n<x}\frac{a_n}{n^s}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}\\ &\quad+\sum_{x<n}\frac{a_n}{n^s}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}\\ &=\sum_{n<x}\frac{a_n}{n^s}\left(\frac{1}{g(s)}+h_n(s)\right)\\ &\quad-\sum_{n<x}\frac{a_n}{n^s}R_n(c,x,T)(s)\\ &\quad-\sum_{x<n}\frac{a_n}{n^s}Q_n(c,x,T)(s).\end{aligned}$$

Then we obtain (7.1). Furthemore, we can take $A>0$ such that $|\log(x/n)|>A$ for any $n\in\mathbb{N}$, by the assumption of x. Hence, we also obtain (7.2). □

8. The first proof of the Riemann hypothesis

Let

$$B := \sup\{\beta; \zeta(\beta + i\gamma) = 0, \gamma \neq 0\}.$$

The Riemann hypothesis states $B = \frac{1}{2}$. We recall the theorem of de la Vallée Poussin ([2]) which says that there is a constant $A > 0$ such that $\zeta(s)$ is not zero for

$$\sigma \geq 1 - \frac{A}{\log t} \quad (t > t_0),$$

where t_0 is some positive constant. We may restate it as follows: if $s = \sigma + it$ $(t > t_0)$ satisfies

$$t \leq \exp\left(\frac{A}{1-\sigma}\right), \tag{8.1}$$

then $\zeta(s) \neq 0$.

Proposition 2. *It holds that* $\frac{1}{2} \leq B < 1$.

Proof. Since

$$\exp\left(\frac{A}{1-\sigma}\right) \longrightarrow \infty \quad \text{as} \quad \sigma \longrightarrow 1-0,$$

there exists $\sigma_0 < 1$ such that $T_0 < \exp\left(\frac{A}{1-\sigma}\right)$ for $\sigma_0 < \sigma < 1$, where T_0 is the constant in Theorem 1. Then, there is no zero of $\zeta(s)$ in a region $\sigma_0 < \sigma$ by Theorem 1 and the theorem of de la Vallée Poussin. $\square$

Proposition 3. *If* $\frac{1}{2} < B < 1$, *then there is no zero of* $\zeta(s)$ *on the line* $\sigma = B$.

Proof. If the function $\zeta(s)$ has a zero on $\sigma = B$, then the number of zeros of $\zeta(s)$ on $\sigma = B$ is finite by Theorem 1. Let $Z_B = \{\rho_1, \overline{\rho_1}, \ldots, \rho_N, \overline{\rho_N}\}$ be the set of zeros of $\zeta(s)$ on the line $\sigma = B$. By Theorem 1, there exists $\delta_0 > 0$ such that there are no zeros of $\zeta(s)$ except Z_B in a set $\{\sigma + it; B - \delta_0 \leq \sigma, -\infty < t < \infty\}$.

It is well-known that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for $\sigma > 1$, where $\mu(n)$ is the Möbius function. We define an entire function $g(s)$ by

$$g(s) := \prod_{j=1}^{N} (s - \rho_j)(s - \overline{\rho_j}).$$

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Then, its zeros are Z_B, and $|g(s)| \longrightarrow \infty$ as $s \to \infty$. We take $T > 0$ such that $|\mathrm{Im}(\rho_j)| < \frac{T}{4}$ for $j = 1, \ldots, N$. Putting $\sigma_0 = B - \delta_0$ and $c = 2$, we apply Lemma 3. We take $x > 1$ which is not an integer, and fix it. Then we have

$$\begin{aligned} \sum_{n<x} \frac{\mu(n)}{n^s} \left(\frac{1}{g(s)} + h_n(s) \right) &= \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw \\ &\quad + \sum_{n<x} \frac{\mu(n)}{n^s} R_n(2,x,T)(s) \\ &\quad + \sum_{x<n} \frac{\mu(n)}{n^s} Q_n(2,x,T)(s) \end{aligned} \tag{8.2}$$

for $s \in D(\sigma_0, T) \setminus Z_B$, where $h_n(s)$ is a meromorphic function on $\mathbb{C}$ whose poles are at most Z_B and all simple.

We consider the integral of $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ along $C = C_0 + C_1 + C_2 + C_3$, where C_0, C_1, C_2 and C_3 are segments from $2 - iT$ to $2 + iT$, from $2 + iT$ to $-\frac{\delta_0}{2} + iT$, from $-\frac{\delta_0}{2} + iT$ to $-\frac{\delta_0}{2} - iT$ and from $-\frac{\delta_0}{2} - iT$ to $2 - iT$, respectively. We set

$$D_0 := \left\{ \sigma + it; B - \frac{1}{4}\delta_0 < \sigma < B + \frac{1}{4}\delta_0, |t| < \frac{T}{4} \right\}.$$

Then $D_0 \subset D(\sigma_0, T)$ and $Z_B \subset D_0$. For any $s \in D_0 \setminus Z_B$, the poles of $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ in a domain surrounded by C are $w = 0, \rho_1 - s, \overline{\rho_1} - s, \ldots, \rho_N - s$ and $\overline{\rho_N} - s$. The residue of $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ at $w = 0$ is $\frac{1}{\zeta(s)g(s)}$. Let a_j and b_j be the residues of $\frac{1}{\zeta(s)g(s)}$ at ρ_j and $\overline{\rho_j}$ respectively. Then, the residues of $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ at $\rho_j - s$ and $\overline{\rho_j} - s$ are $a_j \frac{x^{\rho_j - s}}{\rho_j - s}$ and $b_j \frac{x^{\overline{\rho_j} - s}}{\overline{\rho_j} - s}$ respectively. We define

$$Q(s) := \sum_{j=1}^{N} \left(a_j \frac{x^{\rho_j - s}}{\rho_j - s} + b_j \frac{x^{\overline{\rho_j} - s}}{\overline{\rho_j} - s} \right).$$

Then, $Q(s)$ is a meromorphic function on $\mathbb{C}$ whose poles are at most Z_B and all simple. We note that $\mathrm{Re}(s + w) > B - \frac{3}{4}\delta_0$ if $s \in D_0$ and w is on C. If $s \in D_0$ and w is on C_2, then $\mathrm{Re}(s + w) < B - \frac{1}{4}\delta_0$. Then, $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ is holomorphic on C as a function of w for any $s \in D_0$. By the residue theorem, we obtain

$$\frac{1}{2\pi i} \int_C \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw = \frac{1}{\zeta(s)g(s)} + Q(s)$$

for $s \in D_0 \setminus Z_B$. Therefore we have

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw=\frac{1}{\zeta(s)g(s)}+Q(s)+P(s) \tag{8.3}$$

for $s \in D_0 \setminus Z_B$, where

$$P(s)=-\frac{1}{2\pi i}\int_{C_1+C_2+C_3}\frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw$$

is a holomorphic function on D_0. We set

$$E:=\left\{\sigma+it; B-\frac{3}{4}\delta_0\le\sigma\le B+2+\frac{1}{4}\delta_0, \frac{3}{4}T\le|t|\le\frac{5}{4}T\right\}$$
$$\bigcup\left\{\sigma+it; B-\frac{3}{4}\delta_0\le\sigma\le B-\frac{1}{4}\delta_0, |t|\le\frac{5}{4}T\right\}.$$

Then we have $\{s+w; s\in D_0, w\in C_1\cup C_2\cup C_3\}\subset E$. Since there is no pole of $\frac{1}{\zeta(s)g(s)}$ on E, we can take $M_0>0$ such that

$$\left|\frac{1}{\zeta(s)g(s)}\right|<M_0$$

on E. Then we obtain

$$\left|\int_{C_1}\frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw\right|<\frac{M_0x^2}{T\log x}$$

and

$$\left|\int_{C_3}\frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw\right|<\frac{M_0x^2}{T\log x}$$

for $s\in D_0$. Since we have $w=-\frac{\delta_0}{2}+it$ and $x^w=e^{it\log x}x^{-\frac{\delta_0}{2}}$ on C_2, we obtain

$$\left|\int_{C_2}\frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw\right|<\frac{4}{\delta_0}M_0Tx^{-\frac{\delta_0}{2}}$$

for $s\in D_0$. It follows from the above estimates that

$$|P(s)|<\frac{1}{2\pi}\left(\frac{2M_0x^2}{T\log x}+\frac{4}{\delta_0}M_0Tx^{-\frac{\delta_0}{2}}\right) \tag{8.4}$$

on D_0.

By (8.2) and (8.3), we obtain

$$\begin{aligned}\sum_{n<x}\frac{\mu(n)}{n^s}\left(\frac{1}{g(s)}+h_n(s)\right)&=\frac{1}{\zeta(s)g(s)}+Q(s)+P(s)\\&\quad+\sum_{n<x}\frac{\mu(n)}{n^s}R_n(2,x,T)(s)\\&\quad+\sum_{x<n}\frac{\mu(n)}{n^s}Q_n(2,x,T)(s)\end{aligned}\tag{8.5}$$

for $s\in D_0\setminus Z_B$. We see that a function

$$P(s)+\sum_{n<x}\frac{\mu(n)}{n^s}R_n(2,x,T)(s)+\sum_{x<n}\frac{\mu(n)}{n^s}Q_n(2,x,T)(s)$$

is bounded on D_0 by the properties of $R_n(2,x,T)(s)$ and $Q_n(2,x,T)(s)$, and (8.4). The functions $\frac{1}{g(s)}, h_n(s)$ and $Q(s)$ are meromorphic functions on $\mathbb{C}$ whose poles are at most Z_B and all simple. On the other hand, the function $\frac{1}{\zeta(s)g(s)}$ has poles of order at least 2 at every point in Z_B. This contradicts to the equation (8.5). Hence, there is no zero of $\zeta(s)$ on the line $\sigma=B$. □

The first proof of the Riemann hypothesis.

We may assume $\frac{1}{2}\le B<1$ by Proposition 2. Suppose that $\frac{1}{2}<B<1$. Then there is no zero of $\zeta(s)$ on the line $\sigma=B$ by Proposition 3. Hence, we can take B' with $\frac{1}{2}<B'<B$ such that $\zeta(s)\neq 0$ for $B'<\sigma$ by Theorem 1. This contradicts to the definition of B. Thus we conclude $B=\frac{1}{2}$.

9. Estimates

We use inequalities $3.14159265<\pi<3.14159266$ and $0.04321391<e^{-\pi}<0.04321392$ to estimate $|\psi^{(k)}(1)|$.

Since

$$\psi(1)=\sum_{n=1}^{\infty}e^{-n^2\pi}<\sum_{n=1}^{\infty}(e^{-\pi})^n=\frac{e^{-\pi}}{1-e^{-\pi}},$$

we have

$$\psi(1)<0.04516571.\tag{9.1}$$

We also have

$$|\psi^{(-1)}(1)| = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n^2\pi} < \frac{1}{\pi}\psi(1) < 0.01437670. \tag{9.2}$$

To estimate $|\psi^{(1)}(1)| = \pi\sum_{n=1}^{\infty} n^2 e^{-n^2\pi}$, we need a function $\varphi_1(x) = x^2e^{-\pi x^2}$. Since $\varphi_1'(x) = 2x(1-\pi x^2)e^{-\pi x^2}$, $\varphi_1(x)$ is monotonously decreasing on $(1,\infty)$. Then we have

$$\sum_{n=1}^{\infty} n^2 e^{-n^2\pi} < e^{-\pi} + \int_1^{\infty}\varphi_1(x)dx.$$

Noting

$$\int_1^{\infty} e^{-\pi x^2}dx < \int_1^{\infty} e^{-\pi x}dx = \frac{1}{\pi}e^{-\pi},$$

we obtain

$$\int_1^{\infty}\varphi_1(x)dx < \left(1+\frac{1}{\pi}\right)\frac{1}{2\pi}e^{-\pi}$$

by integration by parts. Therefore, we have

$$|\psi^{(1)}(1)| < \pi e^{-\pi} + (1+\pi)\frac{1}{2\pi}e^{-\pi} < 0.16424611. \tag{9.3}$$

We consider a function $\varphi_2(x) = x^4e^{-\pi x^2}$ for $\psi^{(2)}(1) = \pi^2\sum_{n=1}^{\infty} n^4e^{-n^2\pi}$. By the same way as above, we obtain

$$\sum_{n=1}^{\infty} n^4 e^{-n^2\pi} < e^{-\pi} + \int_1^{\infty}\varphi_2(x)dx$$

for $\varphi_2'(x) = 2x^3(2-\pi x^2)e^{-\pi x^2}$. Since

$$\int_1^{\infty}\varphi_2(x)dx = \frac{1}{2\pi}e^{-\pi} + \frac{3}{2\pi}\int_1^{\infty}\varphi_1(x)dx,$$

we have

$$\int_1^{\infty} n^4e^{-n^2\pi} < e^{-\pi} + \left\{1+\frac{3}{2\pi}\left(1+\frac{1}{\pi}\right)\right\}\frac{1}{2\pi}e^{-\pi}.$$

Hence, we obtain

$$\psi^{(2)}(1) < \left\{\pi^2 + \frac{\pi}{2}\left(1+\frac{3}{2\pi}\left(1+\frac{1}{\pi}\right)\right)\right\}e^{-\pi} < 0.53711157. \tag{9.4}$$

Let $\varphi_3(x) = x^6 e^{-\pi x^2}$. Similarly, we have

$$\sum_{n=1}^{\infty} n^6 e^{-n^2\pi} < e^{-\pi} + \int_1^{\infty} \varphi_3(x)dx.$$

Since

$$\int_1^{\infty} \varphi_3(x)dx = \frac{1}{2\pi}e^{-\pi} + \frac{5}{2\pi}\int_1^{\infty} \varphi_2(x)dx,$$

we obtain

$$\sum_{n=1}^{\infty} n^6 e^{-n^2\pi} < e^{-\pi} + \left\{1 + \frac{5}{2\pi}\left(1 + \frac{3}{2\pi}\left(1 + \frac{1}{\pi}\right)\right)\right\}\frac{1}{2\pi}e^{-\pi}.$$

Then, we have

$$\begin{aligned} |\psi^{(3)}(1)| &= \pi^3 \sum_{n=1}^{\infty} n^6 e^{-n^2\pi} \\ &< 1.82967310. \end{aligned} \tag{9.5}$$

Using the above estimates, we make Lemmas 1 and 2 precise.

Lemma 4. *It holds that*

$$|\alpha(s)| < 24.21811176$$

for $\frac{1}{2} \leq \sigma \leq 1$ *and* $t \in \mathbb{R}$.

Proof. Since $\psi^{(k)}(x) < 0$ if k is odd and $\psi^{(k)}(x) > 0$ if k is even, we have

$$\begin{aligned} &\left\{\left(\frac{3}{4}\sigma^2 + \frac{3}{2}\sigma + 1\right)x^{\frac{\sigma}{2}} + \left(\frac{3}{4}\sigma^2 - 3\sigma + \frac{13}{4}\right)x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi^{(1)}(x) \\ &\qquad + \left(x^{\frac{\sigma}{2}+2} + x^{-\frac{\sigma}{2}+\frac{5}{2}}\right)\psi^{(3)}(x) \\ &< f_2'(x) \\ &< \left\{\left(\frac{\sigma}{2}\right)^3 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2} + \frac{1}{2}\right)^3 x^{-\frac{\sigma}{2}-\frac{1}{2}}\right\}\psi(x) \\ &\qquad + \left\{\left(\frac{3}{2}\sigma + 3\right)x^{\frac{\sigma}{2}+1} + \left(-\frac{3}{2}\sigma + \frac{9}{2}\right)x^{-\frac{\sigma}{2}+\frac{3}{2}}\right\}\psi^{(2)}(x). \end{aligned}$$

By (4.8), (4.9), (4.10), (9.1), (9.2), (9.3) and (9.4), we obtain the following inequalities

$$\int_1^{\infty} \psi(x)dx < 0.01437670,$$

$$\int_1^{\infty} x|\psi^{(1)}(x)|dx < 0.05954241,$$

$$\int_1^\infty x^2 \psi^{(2)}(x)dx < 0.28333093$$

and

$$\int_1^\infty x^3 |\psi^{(3)}(x)|dx < 1.38710436.$$

It follows that

$$\begin{aligned}
&\left| \left(\frac{3}{4}\sigma^2 + \frac{3}{2}\sigma + 1\right) x^{\frac{\sigma}{2}} + \left(\frac{3}{4}\sigma^2 - 3\sigma + \frac{13}{4}\right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right| |\psi^{(1)}(x)| \\
&\qquad + \left| x^{\frac{\sigma}{2}+2} + x^{-\frac{\sigma}{2}+\frac{5}{2}} \right| |\psi^{(3)}(x)| \\
&\leq \frac{17}{4} x^{\frac{\sigma}{2}} |\psi^{(1)}(x)| + 2x^{\frac{\sigma}{2}+2} |\psi^{(3)}(x)| \\
&\leq \frac{17}{4} x |\psi^{(1)}(x)| + 2x^3 |\psi^{(3)}(x)|
\end{aligned}$$

for $\frac{1}{2} \leq \sigma \leq 1$ and $x \geq 1$. We obtain

$$\frac{17}{4} \int_1^\infty x|\psi^{(1)}(x)|dx + 2\int_1^\infty x^3|\psi^{(3)}(x)|dx < 3.02726397$$

by the above inequalities. We also have

$$\begin{aligned}
&\left| \left(\frac{\sigma}{2}\right)^3 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2} + \frac{1}{2}\right)^3 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right| \psi(x) \\
&\qquad + \left| \left(\frac{3}{2}\sigma + 3\right) x^{\frac{\sigma}{2}+1} + \left(-\frac{3}{2}\sigma + \frac{9}{2}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right| \psi^{(2)}(x) \\
&\leq \frac{1}{8}\psi(x) + \frac{15}{2} x^2 \psi^{(2)}(x)
\end{aligned}$$

and

$$\frac{1}{8}\int_1^\infty \psi(x)dx + \frac{15}{2}\int_1^\infty x^2\psi^{(2)}(x)dx < 2.12677907.$$

Then, we obtain

$$\int_1^\infty |f_2'(x)|dx < 3.02726397.$$

Hence, we have

$$|\alpha(s)| = \left| 2^3 \int_1^\infty \sin\left(\frac{t}{2}\log x\right) f_2'(x)dx \right| < 24.21811176.$$

□

Lemma 5. *It holds that*

$$\frac{|\beta(s)|}{2\sigma - 1} < 205.8758966$$

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for $\frac{1}{2} < \sigma \leq 1$ *and* $t \in \mathbb{R}$.

Proof. We use notations in the proof of Lemma 2. Since

$$A_j(\sigma)x^{\frac{\sigma}{2}-1+j} - B_j(\sigma)x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \geq 0$$

for $j = 0, 1, 2, 3$, we have

$$\begin{aligned}
&\left(A_1(\sigma)x^{\frac{\sigma}{2}} - B_1(\sigma)x^{-\frac{\sigma}{2}+\frac{1}{2}}\right)\psi^{(1)}(x) \\
&\quad + \left(A_3(\sigma)x^{\frac{\sigma}{2}+2} - B_3(\sigma)x^{-\frac{\sigma}{2}+\frac{5}{2}}\right)\psi^{(3)}(x) \\
&< g_3'(x) \\
&< \left(A_0(\sigma)x^{\frac{\sigma}{2}-1} - B_0(\sigma)x^{-\frac{\sigma}{2}-\frac{1}{2}}\right)\psi(x) \\
&\quad + \left(A_2(\sigma)x^{\frac{\sigma}{2}+1} - B_2(\sigma)x^{-\frac{\sigma}{2}+\frac{3}{2}}\right)\psi^{(2)}(x) \\
&\quad + \left(x^{\frac{\sigma}{2}+3} - x^{-\frac{\sigma}{2}+\frac{7}{2}}\right)\psi^{(4)}(x).
\end{aligned} \tag{9.6}$$

From (4.18), it follows that

$$\begin{aligned}
&\frac{1}{2\sigma-1}\left|A_j(\sigma)x^{\frac{\sigma}{2}-1+j} - B_j(\sigma)x^{-\frac{\sigma}{2}-\frac{1}{2}+j}\right| \\
&\quad \leq \frac{1}{\sigma-1}\left|A_j(\sigma) - B_j(\sigma)\right| x^{\frac{\sigma}{2}-1+j} + \frac{1}{2\sigma-1}|B_j(\sigma)|\left|x^{\frac{\sigma}{2}-1+j} - x^{-\frac{\sigma}{2}-\frac{1}{2}+j}\right|
\end{aligned}$$

for $j = 0, 1, 2, 3$. Noting that $\log x < 2x^{\frac{1}{2}}$ for $x \geq 1$, we obtain

$$\frac{1}{2\sigma-1}\left|A_0(\sigma)x^{\frac{\sigma}{2}-1} - B_0(\sigma)x^{-\frac{\sigma}{2}-\frac{1}{2}}\right| \leq \frac{1}{2^4} + \frac{1}{2^8},$$

$$\frac{1}{2\sigma-1}\left|A_1(\sigma)x^{\frac{\sigma}{2}} - B_1(\sigma)x^{-\frac{\sigma}{2}+\frac{1}{2}}\right| \leq \left(2^2 + \frac{39}{2^4}\right)x,$$

$$\frac{1}{2\sigma-1}\left|A_2(\sigma)x^{\frac{\sigma}{2}+1} - B_2(\sigma)x^{-\frac{\sigma}{2}+\frac{3}{2}}\right| \leq \left(\frac{15}{2} + \frac{89}{2^3}\right)x^2$$

and

$$\frac{1}{2\sigma-1}\left|A_3(\sigma)x^{\frac{\sigma}{2}+2} - B_3(\sigma)x^{-\frac{\sigma}{2}+\frac{5}{2}}\right| \leq (2+7)x^3.$$

Adding to estimates in the proof of Lemma 4, we have

$$\int_1^{\infty} x^4\psi^{(4)}(x)dx < 7.37809054$$

by (4.11), (9.1), (9.2), (9.3), (9.4) and (9.5). Then, we obtain

$$
\begin{aligned}
&\frac{1}{2\sigma-1}\int_1^\infty \left|A_1(\sigma)x^{\frac{\sigma}{2}} - B_1(\sigma)x^{-\frac{\sigma}{2}+\frac{1}{2}}\right| |\psi^{(1)}(x)|dx \\
&\quad + \frac{1}{2\sigma-1}\int_1^\infty \left|A_3(\sigma)x^{\frac{\sigma}{2}+2} - B_3(\sigma)x^{-\frac{\sigma}{2}+\frac{5}{2}}\right| |\psi^{(3)}(x)|dx \\
&< \left(2^2+\frac{39}{2^4}\right)\int_1^\infty x|\psi^{(1)}(x)|dx + (2+7)\int_1^\infty x^3|\psi^{(3)}(x)|dx \\
&< 12.86724353
\end{aligned}
\tag{9.7}
$$

and

$$
\begin{aligned}
&\frac{1}{2\sigma-1}\int_1^\infty \left|A_0(\sigma)x^{\frac{\sigma}{2}-1} - B_0(\sigma)x^{-\frac{\sigma}{2}-\frac{1}{2}}\right| \psi^(x)dx \\
&\quad + \frac{1}{2\sigma-1}\int_1^\infty \left|A_2(\sigma)x^{\frac{\sigma}{2}+1} - B_2(\sigma)x^{-\frac{\sigma}{2}+\frac{3}{2}}\right| \psi^{(2)}(x)dx \\
&\quad + \frac{1}{2\sigma-1}\int_1^\infty \left|x^{\frac{\sigma}{2}+3} - x^{-\frac{\sigma}{2}+\frac{7}{2}}\right| \psi^{(4)}(x)dx \\
&< \left(\frac{1}{2^4}+\frac{1}{2^8}\right)\int_1^\infty \psi(x)dx + \left(\frac{15}{2}+\frac{89}{2^3}\right)\int_1^\infty x^2\psi^{(2)}(x)dx \\
&\quad + \int_1^\infty x^4\psi^{(4)}(x)dx \\
&< 12.65568097.
\end{aligned}
\tag{9.8}
$$

Therefore, we have

$$
\frac{1}{2\sigma-1}\int_1^\infty |g_3'(x)|dx < 12.86724353
$$

by (9.6), (9.7) and (9.8). Hence, we obtain

$$
\frac{|\beta(s)|}{2\sigma-1} \le \frac{2^4}{2\sigma-1}\int_1^\infty |g_3'(x)|dx < 205.8758966.
$$

□

10. The second proof of the Riemann hypothesis

Since

$$
R(s) = \frac{1}{t^2}\left(1+\alpha(s)\frac{1}{t}\right)
$$

and

$$
I(s) = (2\sigma-1)\frac{1}{t^3}\left(1+\frac{\beta(s)}{2\sigma-1}\frac{1}{t}\right),
$$

$R(s) > 0$ and $|I(s)| > 0$ for $|t| > 205.8758966$ by Lemmas 4 and 5. Then, we can take 205.8758966 as T_0 in Theorem 1, that is, nontrivial zeros of $\zeta(s)$ are on the critical line $\sigma = \frac{1}{2}$ if $|t| > 205.8758966$. When $|t| \leq 205.8758966$, many computational results are known. However, Hutchinson's result([5]) in 1925 is enough for our purpose.

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