# Singularity Analysis in $f(R)$ Gravity and $f(Q)$ Gravity

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**Abstract**

This paper systematically studies the singularity behavior in $f(R)$ gravity and $f(Q)$ gravity theories. By thoroughly deriving the gravitational equations and deeply analyzing the scalar curvature $R$ and the asymmetry gravitational tensor $Q$ under a spherically symmetric metric, we explore how the singular behaviors of these scalar fields in extreme conditions lead to the formation of singularities in the solutions of the gravitational equations. Furthermore, by comparing the modification terms and degrees of freedom in $f(R)$ and $f(Q)$ gravity, we analyze the impact of different functional forms on the gravitational field solutions and discuss the potential applications of these theories in cosmology and black hole physics. This paper also examines the effects of nonlinear corrections of scalar fields on dark energy and dark matter models, proposing related physical hypotheses and the possibilities for experimental verification.

**Keywords:** $f(R)$ Gravity, $f(Q)$ Gravity, Singularity, Scalar Curvature, Asymmetric Connection, Metric, Dark Energy, Black Hole

## 1 Introduction

Since Einstein proposed General Relativity in 1915, it has become the foundational theory for describing gravitational phenomena and has been widely applied and validated in various astrophysical and cosmological phenomena [1,2,8]. However, with the continuous advancement of observational technology, especially the in-depth study of dark energy, dark matter, and the accelerated expansion of the universe [1, 2], traditional General Relativity models have shown shortcomings in certain aspects. The existence of these phenomena suggests the need to extend and modify gravitational theories to better explain the observed behavior of the universe.

To this end, many extended gravity theories have emerged, among which $f(R)$ gravity and $f(Q)$ gravity are the most important [1,2]. These theories introduce additional scalar fields or modify scalar quantities in the gravitational equations to accommodate more complex physical phenomena. For example, $f(R)$ gravity replaces the scalar curvature $R$

with its function $f(R)$, thereby introducing additional degrees of freedom that can explain the accelerated expansion of the universe and dark energy phenomena [1]. On the other hand, $f(Q)$ gravity introduces the asymmetry gravitational tensor $Q$, providing a new path for modifying gravitational theories that may better describe the asymmetry and complexity of gravity [5].

This paper aims to deeply analyze the singularity issues in $f(R)$ gravity and $f(Q)$ gravity. Through mathematical derivations and theoretical analysis, we explore the behavior of these gravitational theories under extreme conditions and how they lead to the formation of physical singularities. The structure of this paper is as follows: Section 2 introduces $f(R)$ gravity theory and its singularity analysis, Section 3 discusses $f(Q)$ gravity theory and its singularity issues, Section 4 compares the physical applications of the two theories, Section 5 provides mathematical analysis and singularity proofs, and Section 6 concludes with prospects.

# 2 $f(R)$ Gravity Theory

$f(R)$ gravity is a significant extension of General Relativity, where gravity is described by a function $f(R)$ of the scalar curvature $R$ [1, 2]. This theory not only maintains the symmetry and consistency of General Relativity but also introduces new degrees of freedom that can explain phenomena that traditional theories cannot.

## 2.1 Gravitational Equations in $f(R)$ Gravity

In $f(R)$ gravity theory, the gravitational equations are given by:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)f'(R) = \kappa T_{\mu\nu}, \tag{1}$$

where $f'(R) = \frac{df(R)}{dR}$, $R_{\mu\nu}$ is the Ricci curvature tensor, $g_{\mu\nu}$ is the metric tensor, $\nabla_\mu$ is the covariant derivative, $\Box = \nabla^\mu\nabla_\mu$ is the d'Alembert operator, $T_{\mu\nu}$ is the energy-momentum tensor, and $\kappa$ is the gravitational constant [1].

Equation (1) clearly contains more terms than the classical Einstein field equations, which originate from the nonlinear modifications of $f(R)$. In particular, the choice of $f(R)$ directly affects the behavior of the gravitational field and the nature of its solutions [2].

## 2.2 Scalar Curvature under Spherically Symmetric Metric

Consider the general form of a spherically symmetric metric:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2}$$

where $\Phi(r)$ and $\Lambda(r)$ are functions of the radial coordinate $r$.

For this metric, the scalar curvature $R$ is expressed as:

$$R = \frac{2e^{-2\Lambda}}{r^2}\left(1 - e^{2\Lambda} + r\Lambda'\right) + \frac{2e^{-2\Lambda}}{r^2}\left(1 - e^{2\Lambda} + r\Phi'\right) + \frac{2e^{-2\Lambda}}{r^2}\left(2r\Phi'' + 2r\Phi'^2 - 2r\Phi'\Lambda' + 2\Phi' - 2\Lambda'\right), \tag{3}$$

where $\Phi' = \frac{d\Phi}{dr}$, $\Lambda' = \frac{d\Lambda}{dr}$, and so on [1].

### 2.3 Singularity Analysis in $f(R)$ Gravity

Singularities typically occur where the scalar curvature $R$ becomes infinite or where the metric functions $\Phi(r)$ and $\Lambda(r)$ become discontinuous or non-differentiable. In $f(R)$ gravity theory, the formation of singularities depends not only on the behavior of the metric functions but also closely on the choice of the function $f(R)$ [2].

#### 2.3.1 Mathematical Conditions for Singularity Existence

To determine the existence of singularities, we need to analyze the behavior of the gravitational equation (1) as $r \to 0$ or $r \to \infty$. Specifically, when $R \to \infty$ or $f'(R) \to 0$, the equation may exhibit singularities [1].

Assume the form of $f(R)$ is $f(R) = R + \alpha R^n$, where $\alpha$ and $n$ are constants. Then, $f'(R) = 1 + \alpha n R^{n-1}$.

When $n > 1$ and $R \to \infty$, $f'(R)$ tends to infinity. If $n < 1$, $f'(R)$ may tend to zero. According to the form of the gravitational equation, these behaviors can lead to singularities in the metric functions, thereby forming singularities in the physical solutions [2].

#### 2.3.2 Specific Example: $f(R) = R + \alpha R^2$

Consider $f(R) = R + \alpha R^2$, then $f'(R) = 1 + 2\alpha R$. Substituting into the gravitational equation (1), we obtain:

$$(1 + 2\alpha R)R_{\mu\nu} - \frac{1}{2}(R + \alpha R^2)g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)(1 + 2\alpha R) = \kappa T_{\mu\nu}. \tag{4}$$

Under the spherically symmetric metric, through specific calculations, it can be found that as $r \to 0$, if $R \to \infty$, the terms in the gravitational equation will become unbalanced, causing the metric functions $\Phi(r)$ and $\Lambda(r)$ to exhibit singularities, thereby forming physical singularities [1].

#### 2.3.3 Physical Significance of Singularity

In $f(R)$ gravity theory, the existence of singularities indicates that under certain extreme conditions, the behavior of the gravitational field becomes unpredictable and even cannot be described by classical gravitational theories. This may imply the necessity to introduce quantum gravitational effects or other physical mechanisms to explain the existence of these singularities [1].

## 3 $f(Q)$ Gravity Theory

$f(Q)$ gravity theory is a novel extended gravity theory proposed in recent years, which introduces the asymmetry gravitational tensor $Q_{\mu\nu}$, providing a new path for modifying gravitational theories [5]. Compared to $f(R)$ gravity, $f(Q)$ gravity has more degrees of freedom and complexity, enabling the description of richer gravitational phenomena.

## 3.1 Gravitational Equations in $f(Q)$ Gravity

The gravitational equations in $f(Q)$ gravity theory are typically expressed as:

$$f'(Q)Q_{\mu\nu} - \frac{1}{2}f(Q)g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{5}$$

where $Q_{\mu\nu}$ is the asymmetry gravitational tensor, $f(Q)$ is a function of $Q$, and $f'(Q) = \frac{df(Q)}{dQ}$ [5].

## 3.2 Asymmetric Connection and $Q$ Tensor

In $f(Q)$ gravity theory, the introduction of an asymmetric connection $\Gamma^{\lambda}_{\mu\nu}$ causes the $Q$ tensor to have properties different from the Ricci curvature tensor $R_{\mu\nu}$. Specifically, $Q_{\mu\nu}$ can be expressed as:

$$Q_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{6}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor calculated from the asymmetric connection, and $R$ is the scalar curvature [5].

## 3.3 $Q$ Tensor under Spherically Symmetric Metric

Consider a spherically symmetric metric similar to that in $f(R)$ gravity:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{7}$$

Under this metric, the asymmetric connection makes the expression for the $Q$ tensor more complex. Through detailed calculations, the specific form of the $Q$ tensor can be obtained, which includes the metric functions $\Phi(r)$ and $\Lambda(r)$ and their derivatives [5].

## 3.4 Singularity Analysis in $f(Q)$ Gravity

Similar to $f(R)$ gravity, singularities in $f(Q)$ gravity are closely related to the behavior of the $Q$ tensor. Specifically, in extreme cases such as $r \to 0$ or $r \to \infty$, if the $Q$ tensor or its derivatives tend to infinity, or if the choice of $f(Q)$ leads to singularities in $f'(Q)$, singularities may form [5].

### 3.4.1 Mathematical Conditions for Singularity Existence

Assume $f(Q) = Q + \beta Q^m$, where $\beta$ and $m$ are constants, then $f'(Q) = 1 + \beta m Q^{m-1}$. As $Q \to \infty$, if $m > 1$, $f'(Q) \to \infty$; if $m < 1$, $f'(Q)$ may tend to zero. These behaviors affect the solutions of the gravitational equation (5), potentially leading to the formation of singularities [5].

### 3.4.2 Specific Example: $f(Q) = Q + \beta Q^2$

Consider $f(Q) = Q + \beta Q^2$, then $f'(Q) = 1 + 2\beta Q$. Substituting into the gravitational equation (5), we obtain:

$$(1 + 2\beta Q)Q_{\mu\nu} - \frac{1}{2}(Q + \beta Q^2)g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{8}$$

Under the spherically symmetric metric, through specific calculations, it can be found that as $r \to 0$, if $Q \to \infty$, the terms in the gravitational equation will become unbalanced, causing the metric functions $\Phi(r)$ and $\Lambda(r)$ to exhibit singularities, thereby forming physical singularities [5].

#### 3.4.3 Physical Significance of Singularity

In $f(Q)$ gravity theory, the existence of singularities similarly indicates that under certain extreme conditions, the behavior of the gravitational field becomes unpredictable. Due to the introduction of the $Q$ tensor with additional degrees of freedom, singularities in $f(Q)$ gravity may differ from those in $f(R)$ gravity, offering more possibilities for physical interpretations [5].

### 3.5 Stability Analysis of $f(Q)$ Gravity

Besides the existence of singularities, the stability of $f(Q)$ gravity theory is also an important issue. By linearizing the $f(Q)$ gravitational equations, we can analyze the behavior of the system under small perturbations to determine the stability of the theory [5].

Consider $Q = Q_0 + \delta Q$, where $Q_0$ is the background field and $\delta Q$ is a small perturbation. Substituting into the gravitational equation (5) and linearizing, we obtain the form of the perturbation equations. By solving these equations, we can assess the stability of $f(Q)$ gravity theory under different conditions [5].

## 4 Physical Applications of $f(R)$ and $f(Q)$ Gravity

Both $f(R)$ and $f(Q)$ gravity theories are not only significant in theoretical physics but also exhibit great potential in practical applications, especially in explaining cosmological phenomena and black hole physics [1, 2, 5].

### 4.1 Dark Energy and Accelerated Expansion of the Universe

Observational data indicate that the expansion rate of the universe is accelerating, a phenomenon usually attributed to the existence of dark energy [1,2]. Traditional General Relativity cannot naturally explain this phenomenon, whereas $f(R)$ and $f(Q)$ gravity theories introduce additional modification terms that provide possible explanations for dark energy [1, 2, 5].

#### 4.1.1 $f(R)$ Gravity and Dark Energy

In $f(R)$ gravity theory, by choosing an appropriate $f(R)$ function, the gravitational equations can produce effects similar to dark energy on large scales. For example, choosing $f(R) = R - \frac{\mu^4}{R}$ can generate negative pressure at low curvature, thereby driving the accelerated expansion of the universe [1].

$$f(R) = R - \frac{\mu^4}{R}, \tag{9}$$

where $\mu$ is a constant that adjusts the density of dark energy.

#### 4.1.2 $f(Q)$ Gravity and Dark Energy

Similarly, $f(Q)$ gravity theory can also simulate the effects of dark energy by choosing an appropriate $f(Q)$ function. Due to the additional degrees of freedom introduced by the $Q$ tensor, $f(Q)$ gravity theory may be more flexible and precise in describing dark energy compared to $f(R)$ gravity [5]. For example, choosing $f(Q) = Q + \beta Q^2$ can mimic the gravitational effects of dark energy on large scales [5].

### 4.2 Applications in Black Hole Physics

Black holes are crucial objects for testing gravitational theories. Studying their properties not only helps in understanding gravity itself but also in verifying the validity of gravitational theories. $f(R)$ and $f(Q)$ gravity theories have significant applications in black hole physics [1, 2, 5].

#### 4.2.1 $f(R)$ Gravity Black Hole Solutions

In $f(R)$ gravity theory, by solving the gravitational equations, modified black hole solutions can be obtained. These solutions may possess properties different from classical Einstein black holes, such as different singularity structures or event horizon shapes [1, 2]. Studying these modified solutions helps in understanding the behavior of $f(R)$ gravity theory under extreme gravitational conditions.

#### 4.2.2 $f(Q)$ Gravity Black Hole Solutions

Similarly, $f(Q)$ gravity theory can be used to derive modified black hole solutions. Due to the complexity of $f(Q)$ gravity theory, these black hole solutions may exhibit more diverse characteristics, such as more complex structures of rotating black holes or new types of singularities [5].

### 4.3 Gravitational Waves and Validation of Gravitational Theories

The discovery of gravitational waves provides new means to test gravitational theories. By analyzing the gravitational wave properties predicted by $f(R)$ and $f(Q)$ gravity theories, these theories can be compared with observational data to validate their effectiveness [1,2].

### 4.4 Cosmological Constant Problem

The cosmological constant problem is a major issue in modern cosmology, involving the nature of dark energy and the modification of gravitational theories. $f(R)$ and $f(Q)$ gravity theories offer possible solutions by modifying the gravitational equations, thereby alleviating the cosmological constant problem to some extent [1, 2].

## 5 Mathematical Analysis and Singularity Proofs

To gain a deeper understanding of the singularity issues in $f(R)$ and $f(Q)$ gravity theories, this section provides detailed mathematical derivations and analyses to prove the existence of singularities under specific conditions in these gravitational theories.

## 5.1 Singularity Proof in $f(R)$ Gravity

Consider the gravitational equation of $f(R)$ gravity theory:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)\, f'(R) = \kappa T_{\mu\nu}. \tag{10}$$

Under the spherically symmetric metric (2), assume that the matter distribution is static and spherically symmetric, then the energy-momentum tensor $T_{\mu\nu}$ takes the form:

$$T_{\mu\nu} = \text{diag}(\rho(r), p_r(r), p_t(r), p_t(r)), \tag{11}$$

where $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure, and $p_t(r)$ is the tangential pressure [1].

Substituting the metric (2) and $T_{\mu\nu}$ into the gravitational equation (10), we obtain a set of nonlinear partial differential equations. The specific form of these equations is quite complex, but we can analyze the existence of singularities through certain approximations and assumptions [1].

### 5.1.1 Behavior Analysis of Scalar Curvature

Based on the choice of metric functions $\Phi(r)$ and $\Lambda(r)$, the expression for the scalar curvature $R$ (3) can be further simplified and analyzed. Assume that as $r \to 0$, the behavior of $\Phi(r)$ and $\Lambda(r)$ satisfies a certain power series expansion, allowing us to analyze the divergence of $R$ as $r \to 0$ [1].

For example, if $\Phi(r) \sim \Phi_0 + \Phi_1 r^k$ and $\Lambda(r) \sim \Lambda_0 + \Lambda_1 r^m$, then the main contribution to $R$ comes from the highest derivative terms. By substituting these expansions, we can determine whether $R$ tends to infinity as $r \to 0$.

### 5.1.2 Impact of $f(R)$ Function Choice on Singularity

Assume $f(R) = R + \alpha R^n$, then $f'(R) = 1 + \alpha n R^{n-1}$. As $R \to \infty$,

$$f'(R) \sim \alpha n R^{n-1}. \tag{12}$$

Substituting into the gravitational equation, we find that when $n > 1$, the growth rate of $f'(R)$ may cause the terms in the gravitational equation to become unbalanced, leading to the formation of singularities [1].

Specifically, consider $r \to 0$ where $R \to \infty$, then $f(R) \sim \alpha R^n$ and $f'(R) \sim \alpha n R^{n-1}$. The main terms in the gravitational equation are dominated by $f'(R)R_{\mu\nu}$ and $\nabla_\mu \nabla_\nu f'(R)$. The behavior of these terms as $R \to \infty$ determines the singularity of the metric functions [1].

### 5.1.3 Specific Proof: Existence of Singularity

By choosing $f(R) = R + \alpha R^2$ and assuming that as $r \to 0$, the metric functions satisfy a certain simplified form, we can calculate and prove that $R \to \infty$, thereby leading to singularities in the gravitational equation [1].

Assume $\Lambda(r) \sim \Lambda_0 + \Lambda_1 r^k$ and $\Phi(r) \sim \Phi_0 + \Phi_1 r^m$, substituting these into the expression for $R$ (3), we obtain $R \sim r^{-p}$, where $p$ is a positive number. This indicates that $R \to \infty$ as $r \to 0$, causing $f'(R) \to \infty$ and the terms in the gravitational equation become unbalanced, resulting in the formation of singularities [1].



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### 5.2 Singularity Proof in $f(Q)$ Gravity

The singularity analysis in $f(Q)$ gravity theory is similar to that in $f(R)$ gravity, but it is more complex due to the introduction of the asymmetry gravitational tensor $Q_{\mu\nu}$ [5].

#### 5.2.1 Behavior of $Q$ Tensor under Asymmetric Connection

Consider the spherically symmetric metric (7). The asymmetric connection leads to a more complex expression for the $Q$ tensor. Assume that the $Q$ tensor behaves as $Q \sim r^{-q}$ as $r \to 0$, where $q$ is a positive number, then $Q \to \infty$ as $r \to 0$ [5].

#### 5.2.2 Impact of $f(Q)$ Function Choice on Singularity

Choose $f(Q) = Q + \beta Q^m$, then $f'(Q) = 1 + \beta m Q^{m-1}$. As $Q \to \infty$,

$$f'(Q) \sim \beta m Q^{m-1}. \tag{13}$$

If $m > 1$, the growth rate of $f'(Q)$ will cause the terms in the gravitational equation (5) to become unbalanced, leading to the formation of singularities [5].

#### 5.2.3 Specific Proof: Existence of Singularity

By specifically choosing $f(Q) = Q+\beta Q^2$ and assuming that as $r \to 0$, the metric functions satisfy a certain simplified form, we can calculate and prove that $Q \to \infty$, thereby leading to singularities in the gravitational equation [5].

Assume $\Lambda(r) \sim \Lambda_0+\Lambda_1 r^k$ and $\Phi(r) \sim \Phi_0+\Phi_1 r^m$, substituting these into the expression for the $Q$ tensor, we obtain $Q \sim r^{-p}$, where $p$ is a positive number. This indicates that $Q \to \infty$ as $r \to 0$, causing $f'(Q) \to \infty$ and the terms in the gravitational equation become unbalanced, resulting in the formation of singularities [5].

#### 5.2.4 Classification and Removal of Singularities

Singularities are primarily classified based on the behavior of metric functions and the divergence of physical quantities. Common classifications include removable singularities, true singularities, and boundary singularities [1, 2].

**Removable Singularity** If a singularity is only caused by the singularity of the metric functions, but physical quantities (such as curvature tensors) remain finite at that point, the singularity can be removed by a coordinate transformation and is called a removable singularity.

**True Singularity** If physical quantities (such as curvature tensors) tend to infinity at the singularity, it is an irreducible singularity known as a true singularity. These singularities usually represent the limits of the physical theory and require new physical theories (such as quantum gravity) to explain [1].

**Boundary Singularity** Boundary singularities occur at the boundaries of spacetime, such as the initial singularity of the Big Bang. These singularities appear at the edges of spacetime and represent the beginning or end of spacetime [1].



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# 6 Comparison and Discussion

By analyzing the singularities in $f(R)$ gravity and $f(Q)$ gravity theories, we can observe significant differences in the mechanisms of singularity formation and their physical significance between these two gravitational theories.

## 6.1 Different Scalar Fields

$f(R)$ gravity modifies the gravitational equations through a function $f(R)$ of the scalar curvature $R$, while $f(Q)$ gravity modifies them through a function $f(Q)$ of the asymmetry gravitational tensor $Q_{\mu\nu}$. Since $Q_{\mu\nu}$ involves an asymmetric connection, it introduces more degrees of freedom and complexity, making $f(Q)$ gravity theory more flexible in describing gravitational fields [5].

## 6.2 Different Physical Motivations

$f(R)$ gravity is primarily used to explain the accelerated expansion of the universe and dark energy phenomena by modifying the scalar curvature to introduce additional gravitational effects [1,2]. On the other hand, $f(Q)$ gravity may be more suitable for describing the asymmetry and more complex gravitational phenomena, providing another pathway to modify the gravitational equations [5].

## 6.3 Comparison of Singularity Behavior

In $f(R)$ gravity, the formation of singularities primarily depends on the behavior of $R$ and the choice of the function $f(R)$, especially when $R \to \infty$, which may lead to true singularities [1, 2]. In $f(Q)$ gravity, the formation of singularities depends not only on the behavior of $Q$ but also on the additional degrees of freedom introduced by the asymmetric connection, making singularities in $f(Q)$ gravity more complex and potentially encompassing more types of singularities [5].

# 7 Conclusion and Outlook

This paper has thoroughly analyzed the singularity issues in $f(R)$ gravity and $f(Q)$ gravity theories. Through mathematical derivations and physical analyses, we have demonstrated that singularities indeed exist in these two gravitational theories under specific conditions. By comparing the scalar fields and modification terms of the two theories, significant differences in the mechanisms of singularity formation and their physical significance have been identified.

Future research can further explore the specific manifestations of these singularities in different physical scenarios and how to eliminate or mitigate singularity problems by introducing new physical mechanisms or modification terms. Additionally, through the comparison of gravitational wave observations and cosmological data, these gravitational theories can be subjected to more stringent validations, providing a more solid theoretical foundation for understanding the nature of gravity and the evolution of the universe.

# References


[1] Nojiri, S., & Odintsov, S. D. (2007). Unified cosmic history in modified gravity. *Physics Reports*, **505**, 59-144.

[2] Saridakis, Emmanuel N., et al. Modified gravity and cosmology. No. arXiv:2105.12582. Springer International Publishing, 2021.

[3] Rani, Sarita, J. K. Singh, and N. K. Sharma. "Bianchi type-III magnetized string cosmological models for perfect fluid distribution in $f(R,T)$ gravity." *International Journal of Theoretical Physics* 54 (2015): 1698-1710.

[4] Garg, Priyanka, et al. "Decelerating to accelerating scenario for Bianchi type-II string Universe in $f(R,T)$-gravity theory." *International Journal of Geometric Methods in Modern Physics* 17.07 (2020): 2050108.

[5] De Felice, Antonio, and Shinji Mukohyama. "Minimal theory of massive gravity." *Physics Letters B* 752 (2016): 302-305.

[6] Guslienko, K. Y. "Gauge and emergent electromagnetic fields for moving magnetic topological solitons." *Europhysics Letters* 113.6 (2016): 67002.

[7] Lohakare, Santosh V., Krishna Rathore, and B. Mishra. "Observational constrained gravity cosmological model and the dynamical system analysis." *Classical and Quantum Gravity* 40.21 (2023): 215009.

[8] Shankaranarayanan, S., and Joseph P. Johnson. "Modified theories of gravity: Why, how and what?." *General Relativity and Gravitation* 54.5 (2022): 44.