FUNCTIONAL COMPOSITION AND GENERALIZATION OF NATURAL NUMBERS BY DECOMPOSITION OF SEMIRING STRUCTURE

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Abstract. In this paper we discover natural numbers in new algebraic object. The entity of the algebraic object which is seen as natural numbers is function, and from the functional composition we can have new generalization and analogs of natural numbers. From observation at the structure of them, we derive a combinatorial theorem which the addition and the multiplication of natural numbers satisfy, and we show that the theorem proves several important formulae in natural numbers.

1. Introduction

Axiomatic composition of natural numbers started from Peano's five axioms and the recursive definition of operation [5, 7]*. The addition is derived from the recursive definition, the multiplication is defined mainly by the recursive definition and the addition, and the laws which the addition and the multiplication should satisfy are proved from the axioms and these definitions. In this paper we do not discuss axiomatic composition of natural numbers or the methods of proving each law, but we assume them and our main theme is that we compose natural numbers with new algebraic object, in other words, we discover natural numbers in new algebraic object.

The algebraic object which is seen as natural numbers is pairs of elements, or we can describe its entity is function; therefore, we call it functional composition of natural numbers. From the functional composition of natural numbers we can have new generalization and analogs of natural numbers. From observation at the structure of them, we derive a combinatorial theorem which the addition and the multiplication of natural numbers satisfy, and we show that the theorem proves several important formulae in natural numbers.

First, from Peano's composition and its modification we can define natural numbers which mean positive integers in this paper as semiring composition as follows, provided that 0 is excluded.

Definition 1.1. *We define addition on a set X. There is an element $e \in X$ which generates X with the addition. For any element $a \in X$, $a + e \neq e$ holds. For any elements $a, b \in X$, $a \neq b \Rightarrow a + e \neq b + e$ holds. Next, we define multiplication on the set X with distributive law, and we assume the existence of an identity element.*

2020 AMS Subject Classification: 11N80, 11R04, 11B75, 11A25.

Key Words and Phrases. Natural numbers, algebraic number theory, combinatorial number theory, divisor summatory function, arithmetic function.

*"Su no Gainen nitsuite" is the Japanese translations of "Formole di logica matematica" and "Sul concetto di numero" by K. Ono and T. Umezawa. They also add good description about the contents in the book.

We should note that the addition and the multiplication satisfy associative law and commutative law as well as Peano's composition, and we can show the element e is the identity element. With this definition, we can describe the multiplication with the addition recursively, because the distributive law works; for any elements $a, b \in X$, $ab = a(e + \cdots + e) = a + \cdots + a$ holds.

The one main purpose of analysis of natural numbers is to analyze the distribution or positioning the power of primes in the multiplicative structure onto the additive structure. However, in Peano's composition the multiplication depends on the addition, and in the semiring composition one element has double properties of both the addition and the multiplication. The addition and the multiplication in the both compositions are not separated; therefore, it is difficult to examine the distributive relation. In spite of these motivations, the simplicity of natural numbers has made their decomposition difficult. It is a typical topic; the answer is simple but the discovery is difficult.

In this paper we improve the semiring composition as separating the additive structure and the multiplicative structure to two sets. We connect the two sets with a distributive function afterward. With this decomposition, we can analyze natural numbers from different aspects and change conditions precisely.

Especially, we will know the multiplication is strongly bound to the sequential structure of the addition by the distributive function. However, the visualization of the distributive and bijective function introduces us to the other conditions of functions, in other words, the other methods of binding. In addition to it, the aspect that a natural number is a pair of elements of the additive set and the multiplicative set lets us think the whole of natural numbers as a more combinatorial object than the two other compositions.

First, we show the functional composition of natural numbers and analyze the structures of the primes. Second, we show a generalized structure of primes, and we confirm we can embed it to the composition. Third, from a variety of analogs we select one generalization of natural numbers which is related to a combinatorial theorem, and we see a specificity of natural numbers with proving the theorem. At the last, we examine that the combinatorial theorem especially proves a unified identity which proves the other fundamental identities and formulae in natural numbers.

2. Functional composition of natural numbers

Definition 2.1. *The symbol X denotes a set, we define multiplication on it, and it has an identity element e.*

We should note that we can derive associative law and commutative law of the multiplication afterwards from the definitions below.

Definition 2.2. *The symbol Y denotes a set, and we define addition on it. There is an element $d_1 \in Y$ which generates Y. For any element $a \in Y$, $a + d_1 \neq d_1$ holds. For any elements $a, b \in Y$, $a \neq b \Rightarrow a + d_1 \neq b + d_1$ holds.*

As well as Peano's composition, the addition in Y satisfies associative law and commutative law. Also, any element a in Y can be described as the sum $a = d_1 + d_1 + \cdots + d_1$ of which the number of terms d_1 is unique for each element a.

Definition 2.3. *We define a bijective mapping $f : X \to Y$, and we think the case that for any elements a, b, c in Y, distributive law $f(f^{-1}(a + b)f^{-1}(c)) = f(f^{-1}(a)f^{-1}(c)) + f(f^{-1}(b)f^{-1}(c))$ holds.*

Multiplication is not defined directly on Y, but the relation which can be thought as multiplication is constructed by X and f. We can apply this composition of distributive function to general semiring, ring, and other algebraic objects which have distributive law and analyze them from functional aspects with changing conditions between analogs. The analysis can be more decomposed than normal compositions of distributive law in the one set.

Next, we show the whole object defined by these definitions satisfies the conditions of natural numbers.

Theorem 2.4. *The formula $f(e) = d_1$ holds.*

Proof. We put the index numbers $1, 2, \cdots, n-1, n \in N$ on any element $y \in Y$ along with the number of terms d_1 of the sum $y = d_1 + d_1 + \cdots + d_1$, and $m_y \in N$ denotes the index number of y.

First, if an identity element e in X satisfies $d_1 \neq a = f(e)$, then $f^{-1}(a) = e$ holds. Therefore, $f^{-1}(a)f^{-1}(a) = ee = e$ holds, and we substitute a for $a = d_1+d_1+\cdots+d_1$ in it, provided that $m_a \geq 2$ holds. We apply distributive law repeatedly to the left side of $f(f^{-1}(d_1+d_1+\cdots+d_1)f^{-1}(d_1+d_1+\cdots+d_1)) = f(e)$, and it becomes the m_a^2 times sum of $f(f^{-1}(d_1)f^{-1}(d_1))$. Although the index number of $f(f^{-1}(d_1)f^{-1}(d_1))$ is any number, the index number of the left side of the formula is larger than m_a. Therefore, there exists an element $b \in Y$ such that $m_b > m_a$ and $a \neq b = f(e)$ hold. This contradicts that f is normal mapping and not multivalued mapping. Therefore, an identity element e in X satisfies $d_1 = f(e)$. □

Corollary 2.5. *For any $a, b \in Y$, $f(f^{-1}(a) \cdot f^{-1}(b)) = a + \cdots + a$ holds, provided that the number of the terms a is equal to m_b.*

Proof. The formula $f(f^{-1}(a) \cdot f^{-1}(b))$ can be transformed as follows;

$$\begin{aligned} f(f^{-1}(a) \cdot f^{-1}(b)) &= f(f^{-1}(a) \cdot f^{-1}(d_1 + \cdots + d_1)) \\ &= f(f^{-1}(a) \cdot f^{-1}(d_1)) + \cdots + (f^{-1}(a) \cdot f^{-1}(d_1)) \\ &= a + \cdots + a. \end{aligned}$$

□

With this corollary, we can describe the multiplication by the addition recursively, and the multiplication is determined uniquely. Therefore, these definitions (X, f, Y) are equal to natural numbers.

Next, we confirm the structures of the primes in the above. Let the symbol Z be a subset of the set X such that $Z = f^{-1}(Y_p)$ holds, provided that $Y_p = \{y \in Y \mid m_y \textit{ is prime numbers}\}$. We show Z are irreducible elements and prime elements, or accurately (Z, f) have both of the properties. In the functional composition, these properties are not only composed from the multiplication of Z, but also composed from the function f. The reason is a natural number corresponds to a pair of elements of X and Y. However, Definition 2.3 assumes bijection which excludes being multivalued and not being injective. Therefore, the discussion here becomes simple.

Corollary 2.6. *The pair (Z, f) functions as irreducible elements.*

Proof. For any element $z \in Z$ if z is divisible, there exist $a, b \in Y$ such that $z = f^{-1}(a)f^{-1}(b)$ and $f^{-1}(a), f^{-1}(b) \neq e$ hold. From Theorem 2.4, $a, b \neq d_1$ holds. Also from distributive law, the formula $f(z) = f(f^{-1}(a)f^{-1}(b)) = f(f^{-1}(d_1 + d_1 + \cdots + d_1)f^{-1}(d_1 + d_1 + \cdots + d_1)) = f(f^{-1}(d_1)f^{-1}(d_1)) + f(f^{-1}(d_1)f^{-1}(d_1)) + \cdots + f(f^{-1}(d_1)f^{-1}(d_1)) = d_1 + d_1 + \cdots + d_1$ in which the number of d_1 is $m_a m_b$ holds.

Since $a, b \neq d_1$, $m_a, m_b \neq 1$ holds. This contradicts $m_{f(z)}$ is a prime number from the definition of the set Z. □

Corollary 2.7. *The pair (Z, f) functions as prime elements and unique factorization.*

Proof. We think the case that for any element $z \in Z$, there exist $a, b \in Y$ which satisfy $z \mid f^{-1}(a)f^{-1}(b)$. From $z \mid f^{-1}(a)f^{-1}(b)$, there exists $c \in Y$ which satisfies $f^{-1}(f(z))f^{-1}(c) = f^{-1}(a)f^{-1}(b)$. As same as the proof of Corollary 2.6, $d_1 + d_1 + \cdots + d_1 = f(f^{-1}(f(z))f^{-1}(c)) = f(f^{-1}(a)f^{-1}(b)) = d_1 + d_1 + \cdots + d_1$ holds, and the left side of the number of terms d_1 is $m_{f(z)}m_c$ and the right side of the number of terms d_1 is $m_a m_b$. Therefore, $m_{f(z)}m_c = m_a m_b$ holds. Since $m_{f(z)}$ is a prime number from the definition of the set Z, the divisor formula $m_{f(z)}|m_a$ or $m_{f(z)}|m_b$ holds.

When $m_{f(z)}|m_a$ holds, there exists $d \in Y$ such that $m_{f(z)}m_d = m_a$ and $f^{-1}(f(z))f^{-1}(d) = f^{-1}(a)f^{-1}(d_1)$ hold from the reverse consideration above. Therefore, $z|f^{-1}(a)$ holds. As same as this, when $m_{f(z)}|m_b$ holds, $z|f^{-1}(b)$ holds. □

From the confirmation above, Definition 2.2, 2.3 are the strong conditions to construct the primes. Therefore, if we would like to analyze the behavior or the distribution of a generalized structure of primes, we must ease the conditions.

Next, in this section we show conditions of one generalized structure of primes, but we do not ease Definition 2.2, 2.3 here. Instead, we just confirm we can embed it into the functional composition and it becomes prime numbers. We see another generalized structure of primes and one generalization of natural numbers of which the conditions of Definition 2.2, 2.3 are eased in the next section. They fit the purpose of consideration in the third section more than Definition 2.8, 2.9 below.

Definition 2.8. *The symbol Z denotes a set, and we define multiplication on it. We think the case that the product is not included in Z.*

When Galois defined permutation group, he mainly took the condition "closed" for it [6, pp.22-23]. Similarly, we often think a set closed with any operations, but prime numbers have the feature of "not closed entirely". In addition to the condition, if the set Z is a generator set of a set X, Z becomes all the irreducible elements of the set X (Lemma 2.10). The condition "not closed entirely" is also a necessary condition of a set of irreducible elements. Although this generalization does not directly affect the following discussion in this paper, we show this definition because it is one of the essential features of prime numbers.

Definition 2.9. *The symbol X denotes a set which Z generates, and we add an identity element to X.*

Lemma 2.10. *The elements of Z are irreducible elements.*

Proof. We assume $z \in Z$ is divisible as $z = ab$, provided $a, b \in X$ and $a, b \neq e$. Since Z generates X, the elements a and b are the products of elements of Z. Therefore, z is also the product of elements of Z, but this contradicts the product of Z is not included in Z. □

In addition to these two definitions, we add Definition 2.2, 2.3. We should note again that these definitions consist natural numbers, and we just confirm the one generalized structure of primes Definition 2.8, 2.9 can be embedded into the functional composition of natural numbers. The irreducible elements are enough to show the correspondences. Therefore, we do not need to assume the property of prime elements or unique factorization on Definition 2.9 above.

Theorem 2.11. *The elements of Z correspond to the elements in Y of which the index numbers are prime numbers with one-to-one correspondence by f.*

Proof. First, we assume f distributes an element in Z to an element in Y of which the index number is a composite number. In other words, there is $z \in Z$ such that $a = f(z)$ holds and m_a is a composite number. From a composite number, there exist $b, c \in Y$ which are not d_1 and satisfy $m_a = m_b m_c$. We think about $f(f^{-1}(b)f^{-1}(c))$, and substitute $b = d_1 + d_1 + \cdots + d_1$ and $c = d_1 + d_1 + \cdots + d_1$ into it. By repeatedly applying distributive law, $f(f^{-1}(b)f^{-1}(c)) = d_1 + d_1 + \cdots + d_1$ of which the number of terms d_1 is $m_b m_c$ holds. From $m_a = m_b m_c$, the right side is a. Therefore, $f^{-1}(b)f^{-1}(c) = f^{-1}(a) = z$ holds. This contradicts that $z \in Z$ is indivisible from Lemma 2.10.

On the other hand, if f distributes an element in $\overline{Z} \subset X$ to an element in Y of which the index number is a prime number, then there exit $a, b, c \in Y$ such that $f(f^{-1}(b)f^{-1}(c)) = a$ and $b, c \neq d_1$ hold, and m_a is a prime number. With thinking as the above, $m_b m_c = m_a$ holds, and this contradicts that m_a is a prime number. At the last, the function f is bijection; therefore, the proposition holds.

□

Distribution of f from each element in Z to each element in Y of which the index number is a prime number is arbitrary. However, if we fix it, fixing the limited distribution of f only from Z to Y, f distributes all of the elements X to Y uniquely because of the restriction from Definition 2.2, 2.3, in other words, uniqueness of prime factorization.

The merit of this functional composition is that we are able to analyze natural numbers with separating the additive structure and the multiplicative structure. Especially, we can think analogs and compare them with natural numbers to analyze their properties.

In the next section, from a variety of analogs we show one example which has mainly functional and multiplicative structure but does not have bijection and distributive law on a function and does not have sequential structure on an additive set. Its purpose is to see a specificity of natural numbers which is related to a combinatorial theorem holds on them.

3. One generalization of natural numbers

From the conditions above we can generalize natural numbers. We have shown one of the essential features of prime numbers "not closed entirely" in Definition 2.8. The other essential features of prime numbers are the structure of direct product and the structure of power set.

Concretely, when we think the set of the powers of a prime number $X_p = \{p^n \mid n > 0 \wedge p \textit{ is a prime number.}\}$, X_p corresponds to the power set

$$\{\{\emptyset\}, \{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \cdots\}$$

which is the subset of the power set of $\{x_1, x_2, x_3, \cdots\}$ with *gcd* corresponding to the intersection and *lcm* corresponding to the union. Therefore, when the symbol X denotes the direct product of X_p: $X := X_p \times X_p \times X_p \times \cdots$, the direct product X has the structure of power set in each component. The direct product X corresponds to the structure of the multiplication set composed by prime numbers. The structure of power sets in X becomes a basis of divisor, multiple, *gcd*, and *lcm*. With the pair of the distributive function and the sequential addition set (f, Y), the structure of power sets in X also becomes a basis of remainder.

Moreover, when we exclude 1 from X_p and the symbol X_p' denotes $X_p' = \{p^n \mid n > 1 \wedge p \textit{ is a prime number.}\}$, the set of the powers of a prime number X_p' is equal to the sequential addition set Y defined by Definition 2.2. Therefore, the direct product X has multidimensional linearity. From these considerations, we abstract the property of direct product and the property of power set to the next definition.

Definition 3.1. *We define the set X as follows:*

$$X := \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \times \cdots ;$$

the symbol $\mathbb{X}_i$ denotes any subset of the power set of any set Z_i. The power set $\mathbb{X}_i$ is a unital magma which means $\mathbb{X}_i$ is closed by an operation and has an identity element $\{\emptyset\} \in \mathbb{X}_i$.

The set Z_i corresponds to a prime number, and the power set $\mathbb{X}_i$ corresponds to the power of a prime number. Operations on each component $\mathbb{X}_i$ compose an operation on X naturally. We can also define the other operation *gcd* on X composed from the intersections of each component $\mathbb{X}_i$.

Definition 3.2. *The set Y is a magma which means Y is closed by an operation and does not have an identity element.*

This operation corresponds to the addition in natural numbers.

Definition 3.3. *The symbol f denotes a mapping from X to Y:*

$$f : X \to Y.$$

They are not necessarily infinite. The entire of these three definitions are the generalization of natural numbers. It means we can make natural numbers by adding appropriate conditions to the above; for example, if we add conditions on Definition 3.1, we can change the direct product set X to the multiplication set of natural numbers.

First, the conditions are setting Z_i and $\mathbb{X}_i$ as follows:

$$\begin{aligned} Z_i &= \{x_1, x_2, x_3, \cdots\}, \\ \mathbb{X}_i &= \{\{\emptyset\} = \boldsymbol{e}, \{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \cdots\}; \end{aligned}$$

and setting an operation corresponding $\mathbb{X}_i$ to the power set of a prime number $\{1, p, p^2, p^3, \cdots\}$. As mentioned above, we can also introduce the operation by the correspondence with Definition 2.2 provided that we should add 0 to the definition.

Second, for the set $\{x_1, x_2, x_3, \cdots, x_n\}$ in $\mathbb{X}_i$, if we permit the index $n = +\infty$, the natural numbers also become to include $+\infty$. To avoid this occasion, we can simply add the condition that any n is selectable, but only a finite n is selectable.

At the last, if we permit the maximum index i of $\mathbb{X}_i$ to be $+\infty$ and we permit to freely select any component other than $\boldsymbol{e}$ in the elements of X, X becomes to an uncountable set. To avoid this case, we can add the condition that the number of the components which do not take $\boldsymbol{e}$ is only finite. In other words, we can make a new set $X^{'}$ from X by taking the elements of which the number of the components is only finite.

This discussion is the same as follows. We think the set of real numbers S such that the integer part of each number is 0. Next, we take out the numbers from S such that the number of digits of the decimal part of each number is finite, and the symbol T denotes the set. $T \subset \mathbb{Q}$ satisfies; therefore, T is countable. In short, permitting to take any finite number of components from the infinite number of components and permitting the infinite number of components are different.

In addition to the restricted Definition 3.1 above, if we restrict Definition 3.2, 3.3 to Definition 2.2, 2.3, these definitions compose natural numbers.

Next, from this generalization we can think the relation of the following two sets A and B. Let a, b, and c denote elements of X; and d and e denote elements of Y. When c moves in the subset C of X, the set A and the set B are determined by the triple of d, e, and b which satisfy the following formulae.

Definition 3.4. *The set A is defined as follows:*

$$A = \{\ (d, e, b) \mid \exists c \in C,\ ab = c \wedge f(a) = d + e \wedge gcd(f^{-1}(d), f^{-1}(e)) = \boldsymbol{e}\ \}.$$

Definition 3.5. *The set B is defined as follows:*

$$B = \{\ (f(\tfrac{f^{-1}(d)}{b}), f(\tfrac{f^{-1}(e)}{b}), b) \mid \exists c \in C,\ f(c) = d + e \wedge gcd(f^{-1}(d), f^{-1}(e)) = b\ \}.$$

In general $A = B$ does not hold. One simple counterexample is here; we define

$$\begin{aligned}
&X = \mathbb{X}_1 = \{\boldsymbol{e}, \{x_1\}, \{x_1, x_2\}\} = \{\boldsymbol{e}, \boldsymbol{x_1}, \boldsymbol{x_2}\},\\
&\boldsymbol{x_i x_i} = \boldsymbol{x_j},\ \boldsymbol{x_i x_j} = \boldsymbol{x_i},\\
&Y = \{y_1, y_2, y_3\},\\
&y_i + y_i = y_{i+1},\ y_i + y_j = y_i,\ \textit{provided } y_4 = y_1,\\
&\textit{and } f(\boldsymbol{x_i}) = y_{i+1},\ \textit{provided } \boldsymbol{e} = \boldsymbol{x_0}.
\end{aligned}$$

Now, we think $C = \{\boldsymbol{e}\}$, and then

$$\begin{aligned}
A &= \{(y_1, y_2, \boldsymbol{e}),\ (y_1, y_3, \boldsymbol{e})\},\\
B &= \{(y_1, y_2, \boldsymbol{e}),\ (y_1, y_3, \boldsymbol{e}),\ (y_1, y_1, \boldsymbol{x_2})\}
\end{aligned}$$

holds; therefore, $A \neq B$ holds.

When the set C, accurately the set $\{(c, f(c)) \mid c \in C\}$, is equal to the set $\{1, 2, 3, \cdots, n\}$ on natural numbers, $A = B$ holds as a combinatorial theorem which is in the beginning of the next section. The set A corresponds to the first set of the theorem, and the set B corresponds to the third set of the theorem.

4. A combinatorial theorem and related identities and formulae

Theorem 4.1. *The next combinatorial formula holds:*

$$\begin{aligned}
&\{(a, b, c) \mid 1 \leq k \leq n\ \wedge\ (a + b)c = k\ \wedge\ gcd(a, b) = 1\}\\
=\ &\{(a, b, c) \mid 1 \leq k \leq n\ \wedge\ a + b = k\ \wedge\ gcd(a, b) = 1\ \wedge\ 1 \leq c \leq [\frac{n}{k}]\}\\
=\ &\{(\frac{a}{c}, \frac{b}{c}, c) \mid 1 \leq k \leq n\ \wedge\ a + b = k\ \wedge\ gcd(a, b) = c\},
\end{aligned}$$

provided that we think the pairs $(a, b) = (k, 0)$ and define $gcd(k, 0) := k$.

Proof. Every positive integer $1 \leq l \leq N$ can be formed $l = (m + n)d$, provided $gcd(m, n) = 1$. The symbol T_N denotes the set of these triples (m, n, d). In the case, we think $m + n$ and d are divisors of l, the first formula is derived. In the case, we think $(m + n)d$ is multiples of $m + n$, the middle formula is derived. In the case, we think $md + nd$ is all the pair of sums less than or equal to N, the last formula is derived. Therefore, the three sets are equal to the same triples of the set T_N. □

We show another proof as follows.

Proof. We assign the symbol J to the first set, the symbol K to the second set, and the symbol L to the third set. First, we show a proof of $J \subset K$. For any element $(a,b,c) \in J$, if we set $a+b=k'$, then $1 \leq k' \leq n$ holds. Since $k'c = k \leq n$ and the max integer of x which the inequality $k'x \leq n$ holds at is $[\frac{n}{k'}]$, $c \leq [\frac{n}{k'}]$ holds. Second, we show a proof of $K \subset J$. For any element $(a,b,c) \in K$, $(a+b)c = kc \leq k[\frac{n}{k}] \leq k\frac{n}{k} = n$ holds. Therefore, the element (a,b,c) satisfies the conditions of the set J. Third, we show a proof of $J \subset L$. For any element $(a,b,c) \in J$, $(a+b)c = ac+bc$ holds. If we set $ac = a'$ and $bc = b'$, then $1 \leq a'+b' = k \leq n$ and $gcd(a',b') = c$ hold. Therefore, $(a,b,c) = (\frac{a'}{c}, \frac{b'}{c}, c) \in L$ holds. At the last, we show a proof of $L \subset J$. For any element $(\frac{a}{c}, \frac{b}{c}, c) \in L$, if we set $a = a'c$ and $b = b'c$, then $gcd(a',b') = 1$ holds. Since $(a'+b')c = a+b = k \leq n$ holds, $(\frac{a}{c}, \frac{b}{c}, c) = (a', b', c) \in J$ holds. □

The sets J, K, L are determined by a variable $n \in N$. Now J_n denotes the determined J by $n \in N$, and $\mathbb{J}$ denotes a family of the sets J_n: $\mathbb{J} = \{J_n \mid n \in N\}$. The family of sets $\mathbb{K}$ and $\mathbb{L}$ are also defined as same as $\mathbb{J}$. In general, we can think a variety of mappings F from $\mathbb{J}, \mathbb{K}, \mathbb{L}$ with saving the equality $J_n = K_n = L_n$ as $F(J_n) = F(K_n) = F(L_n)$.

Our next interest is whether we can find a mapping of having good properties or not. Especially, the next summatory identity has good properties. Theorem 4.1, a combinatorial theorem in natural numbers, seems as trivial as just seeing one object from different aspects. Therefore, in the first look the theorem does not yield any important results, but it works like one of the principles of natural numbers.

Corollary 4.2. *The next summatory identity holds:*

$$\begin{aligned} & \sum_{k=1}^{n} \sum_{\substack{(a+b)c=k \\ gcd(a,b)=1}} f(a,b) \cdot g(c) \\ = & \sum_{k=1}^{n} \sum_{\substack{a+b=k \\ gcd(a,b)=1}} f(a,b) \sum_{1 \leq i \leq [\frac{n}{k}]} g(i) \\ = & \sum_{a+b \leq n} f(\frac{a}{gcd(a,b)}, \frac{b}{gcd(a,b)}) \cdot g(gcd(a,b)). \end{aligned}$$

Proof. We put the triples of J_n, K_n, and L_n into $f(x,y) \cdot g(z)$, and we take each sum of them. Since the equality $J_n = K_n = L_n$ holds, when the symbol F denotes the mapping of this operation, the equality $F(J_n) = F(K_n) = F(L_n)$ also holds. □

This is expansion of the next identity in arithmetic summatory function [1, p.65]:

$$\sum_{k=1}^{n} \sum_{d|k} f(d) g(\frac{k}{d}) = \sum_{k=1}^{n} f(k) \sum_{1 \leq i \leq [\frac{n}{k}]} g(i).$$

Corollary 4.3. *The next formulae hold:*

$$\begin{aligned}\sum_{k=1}^{n}\sigma_x(k) &= \sum_{k=1}^{n} k^x\left[\frac{n}{k}\right] = \sum_{a+b\leq n}\frac{\left(\frac{a+b}{gcd(a,b)}\right)^x}{\varphi\left(\frac{a+b}{gcd(a,b)}\right)};\\ &\textit{especially},\\ \sum_{k=1}^{n}\sigma(k) &= \sum_{k=1}^{n} k\left[\frac{n}{k}\right] = \sum_{a+b\leq n}\frac{\frac{a+b}{gcd(a,b)}}{\varphi\left(\frac{a+b}{gcd(a,b)}\right)},\\ &\textit{and}\\ \sum_{k=1}^{n}\tau(k) &= \sum_{k=1}^{n}\left[\frac{n}{k}\right] = \sum_{a+b\leq n}\frac{1}{\varphi\left(\frac{a+b}{gcd(a,b)}\right)}.\end{aligned}$$

Proof. We put $f(a,b)=\frac{(a+b)^x}{\varphi(a+b)}$ and $g(c)=1$ on Corollary 4.2. □

The formula $\sum_{k=1}^{n}\tau(k)=\sum_{k=1}^{n}\left[\frac{n}{k}\right]$ is well-known in a basis of divisor summatory function [2]. The corollary above shows rich relationships between divisor function, floor function, Euler's totient function. Especially, Euler's totient function appears in the reciprocals.

Corollary 4.4. *The next formulae hold:*

$$\begin{aligned}\sigma_x(n) &= \sum_{a+b=n}\frac{\left(\frac{a+b}{gcd(a,b)}\right)^x}{\varphi\left(\frac{a+b}{gcd(a,b)}\right)};\\ &\textit{especially},\\ \sigma(n) &= \sum_{a+b=n}\frac{\frac{a+b}{gcd(a,b)}}{\varphi\left(\frac{a+b}{gcd(a,b)}\right)},\\ &\textit{and}\\ \tau(n) &= \sum_{a+b=n}\frac{1}{\varphi\left(\frac{a+b}{gcd(a,b)}\right)}.\end{aligned}$$

Proof. On Corollary 4.3, the left side and the right side of the formulae can be separated to the variable k of $1\leq k\leq n$ with not depending on n. Therefore, we can take their differences. □

We have known the next identity in gcd-sum function [4, 3, 8]:

$$\sum_{k=1}^{n} f(gcd(k,n))=\sum_{d|n} f(d)\varphi\left(\frac{n}{d}\right).$$

When we define $\varphi_x(n):=\frac{\varphi(n)}{n^x}$ and set

$$f(i)=\frac{1}{\varphi_x\left(\frac{n}{i}\right)},$$

then we can also have Corollary 4.4. As the background of it, we can derive the identity from Corollary 4.2 as follows.

Corollary 4.5. *The next formulae hold:*

$$\begin{aligned}\sum_{d|n} f(d)\cdot g\left(\frac{n}{d}\right) &= \sum_{k=1}^{n} f(gcd(k,n))\cdot\frac{g\left(\frac{n}{gcd(k,n)}\right)}{\varphi\left(\frac{n}{gcd(k,n)}\right)} = \sum_{k=1}^{n} f\left(\frac{n}{gcd(k,n)}\right)\cdot\frac{g(gcd(k,n))}{\varphi\left(\frac{n}{gcd(k,n)}\right)},\\ &\textit{especially},\\ \sum_{d|n} f(d)\varphi\left(\frac{n}{d}\right) &= \sum_{k=1}^{n} f(gcd(k,n)) = \sum_{k=1}^{n} f\left(\frac{n}{gcd(k,n)}\right)\cdot\frac{\varphi(gcd(k,n))}{\varphi\left(\frac{n}{gcd(k,n)}\right)}.\end{aligned}$$

https://doi.org/10.33774/coe-2024-7z62l-v4 **ORCID:** https://orcid.org/0000-0003-2389-3588

Proof. From Corollary 4.2,

$$\sum_{\substack{(a+b)c=n\\ gcd(a,b)=1}} f(a,b)\cdot g(c) = \sum_{a+b=n} f(\frac{a}{gcd(a,b)},\frac{b}{gcd(a,b)})\cdot g(gcd(a,b))$$

holds. When we set $f(a,b)=\frac{h(a+b)}{\varphi(a+b)}$,

$$\sum_{\substack{(a+b)c=n\\ gcd(a,b)=1}} \frac{h(a+b)}{\varphi(a+b)}\cdot g(c) = \sum_{a+b=n} \frac{h(\frac{a+b}{gcd(a,b)})}{\varphi(\frac{a+b}{gcd(a,b)})}\cdot g(gcd(a,b))$$

holds. Next, we think $a+b=d$ in the left formula and $a=k$ in the right formula, and we can replace h to f again:

$$\sum_{d|n} f(d)\cdot g(\frac{n}{d}) = \sum_{k=1}^{n} \frac{f(\frac{n}{gcd(k,n)})}{\varphi(\frac{n}{gcd(k,n)})}\cdot g(gcd(k,n)).$$

In the left formula, we can exchange f and g. Therefore,

$$\sum_{d|n} f(d)\cdot g(\frac{n}{d}) = \sum_{k=1}^{n} \frac{g(\frac{n}{gcd(k,n)})}{\varphi(\frac{n}{gcd(k,n)})}\cdot f(gcd(k,n))$$

holds. Now, we think $g(x)=\varphi(x)$, and the identities are proved. □

The left formula of Corollary 4.5 is Dirichlet convolution, and the identity is also expansion of gcd-sum function.

Corollary 4.6. *The next formula is equivalent to Corollary 4.5:*

$$\sum_{d|n} f(d) = \sum_{k=1}^{n} \frac{f(gcd(k,n))}{\varphi(\frac{n}{gcd(k,n)})} = \sum_{k=1}^{n} \frac{f(\frac{n}{gcd(k,n)})}{\varphi(\frac{n}{gcd(k,n)})}.$$

Proof. We set $g(x)=1$ on Corollary 4.5. On the other hand, we set $f(x)=h(x)\cdot g(\frac{n}{x})$ on this corollary and replace h to f again. □

Corollary 4.7. *The next formula is equivalent to Corollary 4.5 and Corollary 4.6:*

$$\sum_{d|n} f(d)\cdot g(d) \;=\; \sum_{k=1}^{n} \frac{f(gcd(k,n))\cdot g(gcd(k,n))}{\varphi(\frac{n}{gcd(k,n)})} = \sum_{k=1}^{n} \frac{f(\frac{n}{gcd(k,n)})\cdot g(\frac{n}{gcd(k,n)}))}{\varphi(\frac{n}{gcd(k,n)})}.$$

Proof. We set $f(x)=h(x)\cdot g(x)$ on Corollary 4.6 and replace h to f again. On the other hand, we set $g(x)=1$ on this corollary. □

Corollary 4.8. *The next formula holds:*

$$\sigma(n) \;=\; \sum_{k=1}^{n} \frac{gcd(k,n)}{\varphi(\frac{n}{gcd(k,n)})} = \sum_{k=1}^{n} \frac{\frac{n}{gcd(k,n)}}{\varphi(\frac{n}{gcd(k,n)})}.$$

Proof. We set $f(x)=x$ on Corollary 4.6. □

We should compare Corollary 4.8 with Corollary 4.4, and we should note that $\sum_{k=1}^{n}\frac{n}{gcd(k,n)}$ does not necessarily equal $\sum_{k=1}^{n} gcd(k,n)$. The next corollary is also derived from Corollary 4.5.

Corollary 4.9. *The next formulae hold:*

$$\begin{aligned}\sum_{d|n} f(d)\cdot g(\frac{n}{d})\cdot\varphi(d) &= \sum_{k=1}^{n} f(gcd(k,n))\cdot g(\frac{n}{gcd(k,n)})\cdot\frac{\varphi(gcd(k,n))}{\varphi(\frac{n}{gcd(k,n)})}\\ &= \sum_{k=1}^{n} f(\frac{n}{gcd(k,n)})\cdot g(gcd(k,n)),\end{aligned}$$

especially,

$$\sum_{d|n} f(d)\cdot\varphi(d) = \sum_{k=1}^{n} f(gcd(k,n))\cdot\frac{\varphi(gcd(k,n))}{\varphi(\frac{n}{gcd(k,n)})} = \sum_{k=1}^{n} f(\frac{n}{gcd(k,n)}).$$

Proof. We think $f(x)=h(x)\cdot\varphi(x)$ and insert it to Corollary 4.5; and we replace h to f again. Next, we think $g(x)=1$. □

In the last, we can have the next elementary formula:

$$\begin{aligned}\frac{n(n+1)}{2} &= \sum_{k=1}^{n} k = \sum_{k=1}^{n}\sum_{d|k}\varphi(d)\\ &= \sum_{k=1}^{n}\sum_{1\le i\le[\frac{n}{k}]}\varphi(i) = \sum_{k=1}^{n}\varphi(k)[\frac{n}{k}]\\ &= \sum_{k=1}^{n}\sum_{a+b=k}\frac{\varphi(gcd(a,b))}{\varphi(\frac{a+b}{gcd(a,b)})} = \sum_{k=1}^{n}\sum_{a+b=k} 1;\end{aligned}$$

therefore, the formula

$$n=\sum_{d|n}\varphi(d)=\sum_{k=1}^{n}\frac{\varphi(gcd(k,n))}{\varphi(\frac{n}{gcd(k,n)})}$$

is given.

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