$$Collatz\ Sequence\ Proof\ (1st\ Way)$$

$Abstract$:

A Collatz sequence is a sequence of numbers generated by starting with a positive integer and repeatedly applying two rules:

- If the number is even, divide it by two
- If the number is odd, multiply it by 3 and add 1

$Proof$:

$$Collatz\ SequenceRules: n\ even \rightarrow \frac{n}{2}, or\ n\ odd \rightarrow (3n+1)$$

$$let\ S(n) = Set\ of\ Collatz\ Sequence\ \&\ lS(n) = loop\ of\ S(n)$$

$$S(5) = \{16,8,4,2,1\} \Rightarrow lS(5) = \{4,2,1\}$$

$$S(6) = \{3,10,5,16,8,4,2,1\} \Rightarrow lS(6) = \{4,2,1\}$$

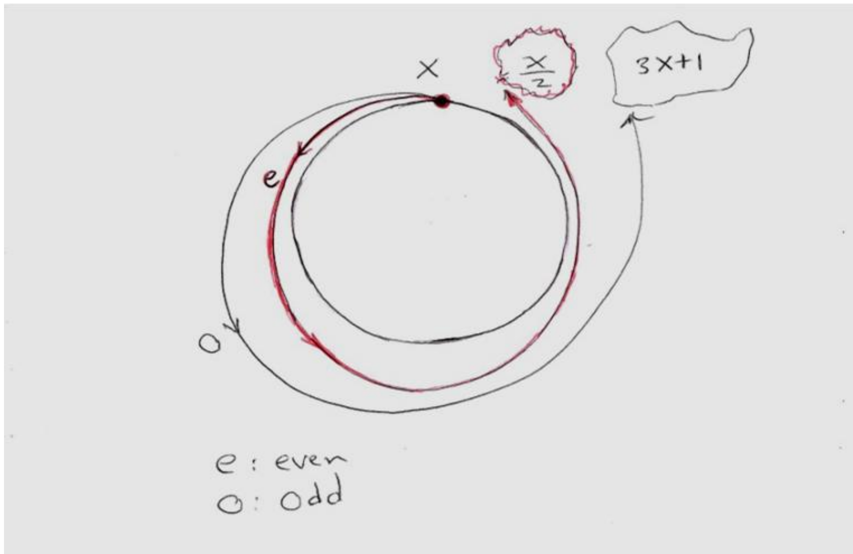
$$let\ \ n, x, y, z, t, u, v, k, r, h \in N_+$$

$$S(n) = \left\{\frac{n}{2} or\ (3n+1), \dots, ?\right\} or\ S(n) = \left\{\frac{n}{2} or\ (3n+1), \dots\right\}$$

$$r = \ number\ of\ elements\ of\ \ lS(n),\ \ x \in S(n), \&\ x \in \ lS(n)$$

https://doi.org/10.33774/coe-2024-c4f7p-v7 **ORCID:** https://orcid.org/0000-0002-8954-9264 Content not peer-reviewed by Cambridge University Press.

$A)$ $let\ lS(n) = \{x\}\ \forall n \in N_+, when\ x \in S(n) = \{\frac{n}{2}\ or\ (3n+1), \dots, x\}$



$lS(n) Collatz\ Taha's\ Sketch\ of\ lS(n)\ to\ find\ equivalent\ expresion\ (Cloud)\ to\ x$

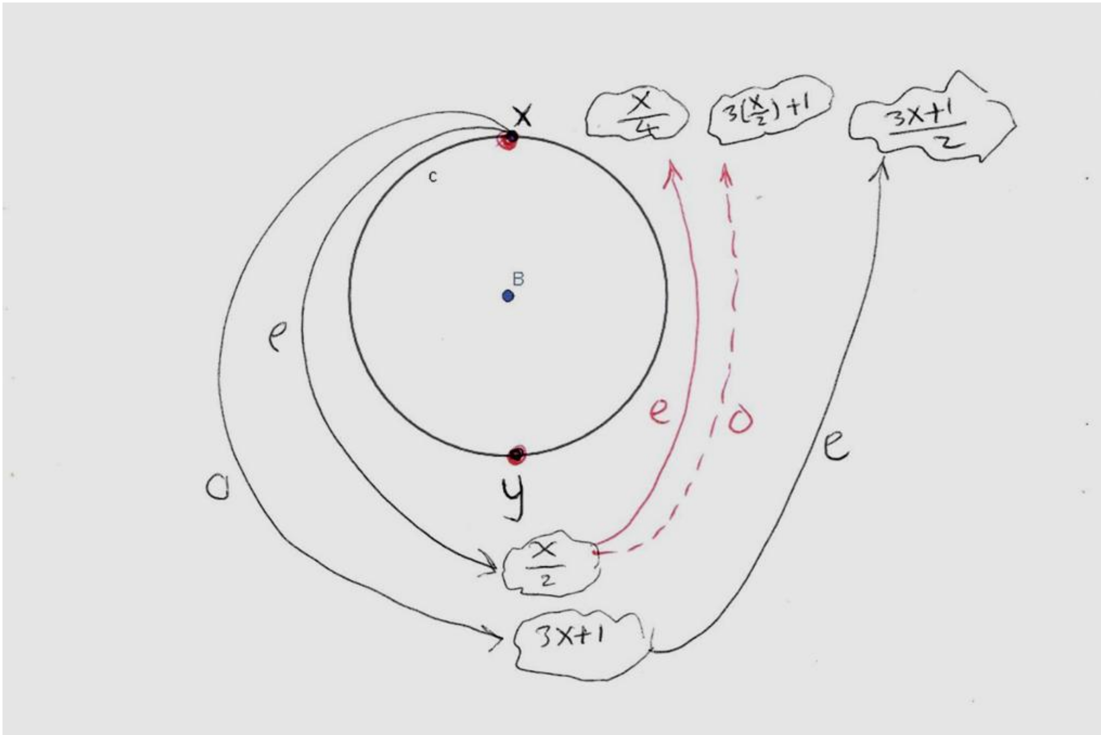
$Cloud$	$Cloud = x$	x
$\frac{x}{2}$	$\frac{x}{2} = x \Rightarrow x = 0 \notin lS(\mathrm{n})$	----
$3x + 1$	$3x + 1 = x \Rightarrow x = -\frac{1}{2} \notin lS(\mathrm{n})$	----

$Taha's\ Chart\ to\ find\ x\ value$

$\therefore lS(\mathrm{n}) \neq \{x\}\ \forall n \in N_+$

https://doi.org/10.33774/coe-2024-c4f7p-v7 **ORCID:** https://orcid.org/0000-0002-8954-9264 Content not peer-reviewed by Cambridge University Press.

$B)\ let\ lS(n) = \{x, y\}\ \forall n \in N_+,\ when\ x, y \in S(n) = \{\frac{n}{2}\ or\ (3n+1), \dots, x, y\}$



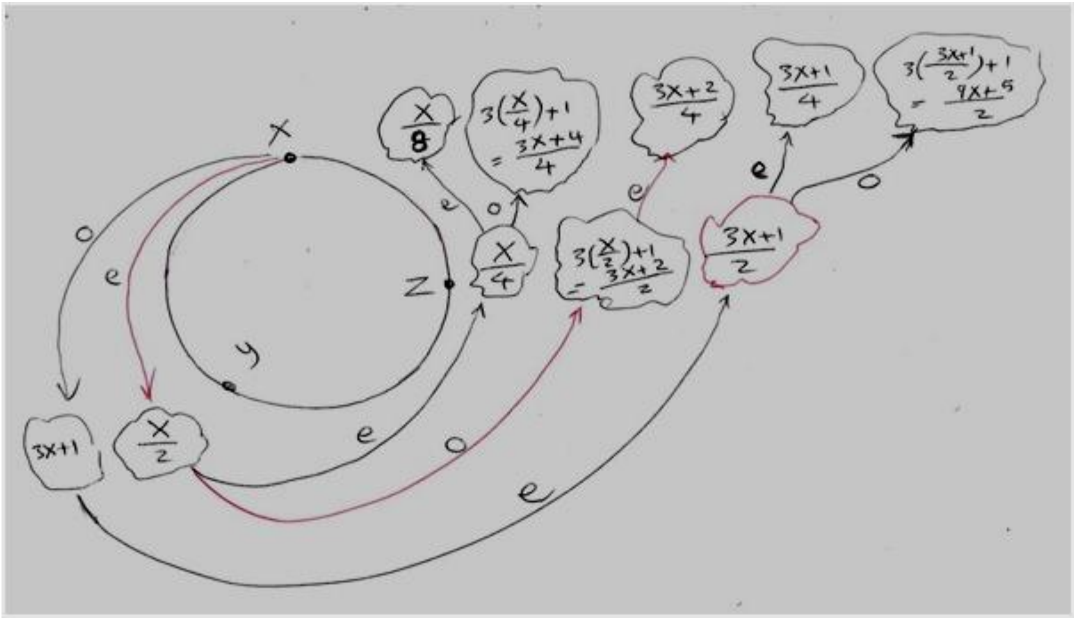
$Taha's\ Sketch\ of\ lS(n)\ to\ find\ equivalent\ expresions\ (Cloud)\ to\ x$

$Cloud$	$Cloud = x$	x	y
$\frac{x}{4}$	$\frac{x}{4} = x \Rightarrow x = 0 \notin lS(n)$	....	....
$\frac{3x+2}{2}$	$\frac{3x+2}{2} = x \Rightarrow 3x + 2 = 2x \Rightarrow x = -2 \notin lS(n)$	....	....
$\frac{3x+1}{2}$	$\frac{3x+1}{2} = x \Rightarrow x = -1 \notin lS(n)$	....	....

$Taha's\ Chart\ to\ find\ x\ value$

$\therefore lS(n) \neq \{x, y\}\ \forall n \in N_+$

$C)\ let\ lS(n) = \{x, y, z\}\ \forall n \in N_+, \qquad when\ x, y, z \in S(n) = \{\frac{n}{2}\ or\ (3n+1), \dots, x, y, z\}$



$Taha's\ Sketch\ of\ lS(n)\ to\ find\ equivalent\ expresions\ (Cloud) to\ x$

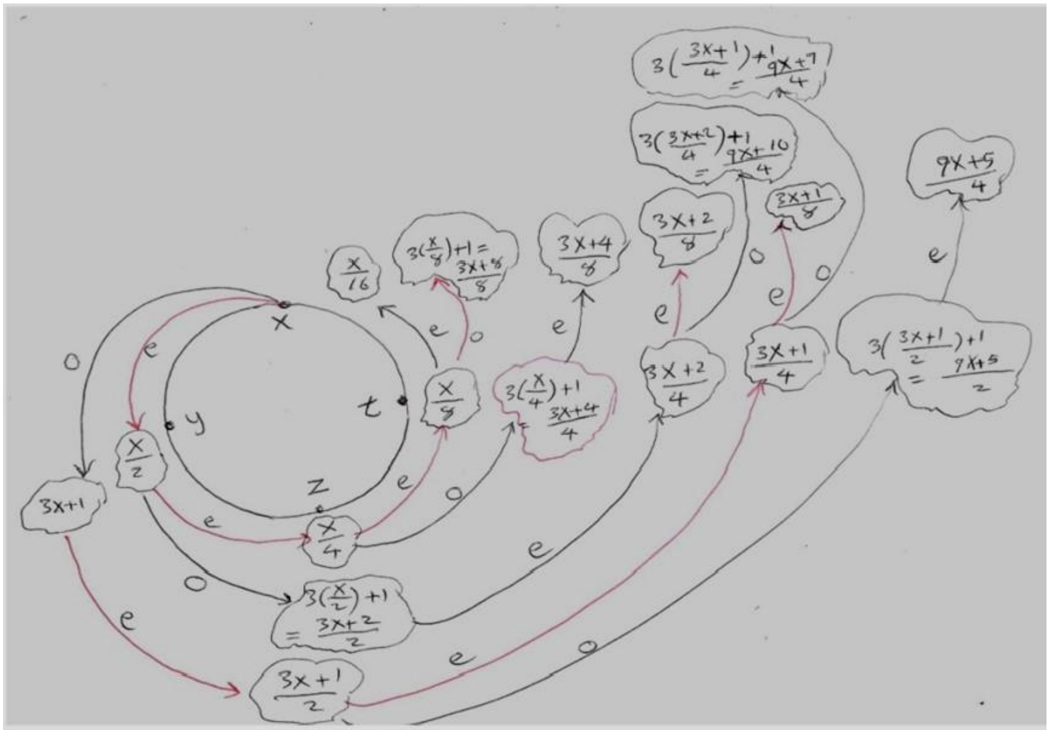
$Cloud$	$Solve\ equation: Cloud = x$	x	y	z
$\frac{x}{8}$	$\frac{x}{8} = x \Rightarrow x = 0 \notin lS(n)$	---	---	---
$\frac{3x+4}{4}$	$\frac{3x+4}{4} = x \Rightarrow x = 4 \in lS(n)$	4	$\frac{x}{2} = 2$	$\frac{x}{4}$ =1
$\frac{3x+2}{4}$	$\frac{3x+2}{4} = x \Rightarrow x = 2 \in lS(n)$	2	$\frac{x}{2} = 1$	$\frac{3x+2}{2} = 4$
$\frac{3x+1}{4}$	$\frac{3x+1}{4} = x \Rightarrow x = 1 \in lS(n)$	1	$3x+1$ =3(1)+1=4	$\frac{3x+1}{2}$ =2
$\frac{9x+5}{2}$	$\frac{9x+5}{2} = x \Rightarrow x = -\frac{5}{7} \notin lS(n)$	---	---	---

$Taha's\ Chart\ to\ find\ x\ value$

$\therefore lS(n) = \{x, y, z\} = \{4,2,1\}, \forall n \in N_+$

https://doi.org/10.33774/coe-2024-c4f7p-v7 **ORCID:** https://orcid.org/0000-0002-8954-9264

$D)\ let\ lS(n) = \{x, y, z, t\}\ \forall n \in N_+, when\ x, y, z, t \in S(n) = \{\frac{n}{2}\ or\ (3n+1), \ldots, x, y, z, t\}$



$Taha's\ Sketch\ of\ lS(n)\ to\ find\ equivalent\ expresions\ (Cloud)\ to\ x$

https://doi.org/10.33774/coe-2024-c4f7p-v7 **ORCID:** https://orcid.org/0000-0002-8954-9264

$Cloud$	$Solve\ equation: Cloud = x$	x	y	z	t
$\frac{x}{16}$	$\frac{x}{16} = x \Rightarrow x = 0 \notin lS(n)$	---	---	---	---
$\frac{3x+8}{8}$	$\frac{3x+8}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{3x+4}{8}$	$\frac{3x+4}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{3x+2}{8}$	$\frac{3x+2}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{3x+1}{8}$	$\frac{3x+1}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{9x+10}{4}$	$\frac{9x+10}{4} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{9x+7}{4}$	$\frac{9x+7}{4} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{9x+5}{4}$	$\frac{9x+5}{4} = x \Rightarrow x \notin lS(n)$	---	---	---	---

$Taha's\ Chart\ to\ find\ x\ value$

$\therefore lS(n) \neq \{x, y, z, t\}\ \forall n \in N_+$

https://doi.org/10.33774/coe-2024-c4f7p-v7 **ORCID:** https://orcid.org/0000-0002-8954-9264 Content not peer-reviewed by Cambridge University Press. **License:** All Rights Reserved

Taha's Chart to find Clouds

$r \text{ is number of elements of } lS(n), x \in S(n), \& \ x \in lS(n)$

x = e or odd →

$r = 1$	→ $\frac{x}{2}$ = e or odd →	→ $3x + 1$ = e →

$r = 2$	→ $\frac{x}{4}$ =e or odd →	→ $\frac{3x+2}{2}$ =e →	→ $\frac{3x+1}{2}$ =e or odd →

$r = 3$	→ $\frac{x}{8}$ =e or odd →	→ $\frac{3x+4}{4}$ =e →	→ $\frac{3x+2}{4}$ =e or odd →	→ $\frac{3x+1}{4}$ =e or odd →	→ $\frac{9x+5}{2}$ =e →

$r = 4$	→ $\frac{x}{16}$ =e or odd →	→ $\frac{3x+8}{8}$ =e →	→ $\frac{3x+4}{8}$ =e or odd →	→ $\frac{3x+2}{8}$ =e or odd →	→ $\frac{3x+1}{8}$ =e or odd →	→ $\frac{9x+10}{4}$ =e →

https://doi.org/10.33774/coe-2024-c4f7p-v7 **ORCID:** https://orcid.org/0000-0002-8954-9264 Content not peer-reviewed by Cambridge University Press.

$\rightarrow$ $\frac{9x+7}{4}$ =e $\rightarrow$	$\rightarrow$ $\frac{9x+5}{4}$ =e or odd $\rightarrow$

$r=5$	$\rightarrow$ $\frac{x}{32}$ =e or odd *	$\rightarrow$ $\frac{3x+16}{16}$ =e or odd $	$\rightarrow$ $\frac{3x+8}{16}$ =e or odd !	$\rightarrow$ $\frac{3x+4}{16}$ =e or odd @	$\rightarrow$ $\frac{3x+2}{16}$ =e or odd #	$\rightarrow$ $\frac{3x+1}{16}$ =e or odd %
$\rightarrow$ $\frac{9x+20}{8}$ =e @@	$\rightarrow$ $\frac{9x+14}{8}$ =e &	$\rightarrow$ $\frac{9x+11}{8}$ =e ((	$\rightarrow$ $\frac{9x+10}{8}$ =e or odd {{	$\rightarrow$ $\frac{9x+7}{8}$ =e or odd [[	$\rightarrow$ $\frac{9x+5}{8}$ =e or odd **	$\rightarrow$ $\frac{27x+19}{4}$ =e ***

$r=6$	* $\frac{x}{64}$	* $\frac{3x+32}{32}$	$ $\frac{3x+16}{32}$	! $\frac{3x+8}{32}$	@ $\frac{3x+4}{32}$	# $\frac{3x+2}{32}$	% $\frac{9x+19}{16}$
% $\frac{3x+1}{32}$	$ $\frac{9x+64}{16}$	{{ $\frac{27x+38}{8}$	@ $\frac{9x+28}{16}$	# $\frac{9x+22}{16}$	% $\frac{3x+1}{32}$	$ $\frac{9x+64}{16}$	! $\frac{9x+40}{16}$
@@ $\frac{9x+20}{16}$		& $\frac{9x+14}{16}$	(($\frac{9x+11}{16}$	{{ $\frac{9x+10}{16}$	[[$\frac{9x+7}{16}$	[[$\frac{27x+29}{16}$	** $\frac{9x+5}{16}$
** $\frac{27x+23}{8}$		*** $\frac{27x+19}{8}$					

Taha's Table to find lS(n) Pattern

$r = 1 = 3(0) + 1$ $k = 0$	$x = \frac{3^0 x}{2^{1-0}} \Rightarrow x = 0 \notin N_+, 0 \notin lS(n) \Rightarrow lS(n) \neq \{x\},\ \forall n \in N_+$
$r = 2 = 3(0) + 2$ $k = 0$	$x = \frac{3^0 x}{2^{2-0}} \Rightarrow x = \frac{1}{3} \notin N_+ \Rightarrow x, y \notin lS(n) \Rightarrow lS(n) \neq \{\ x, y\},\ \forall n \in N_+$
$r = 3 = 3(1)$ $k = 1$	$x = \frac{3^1 x + 3^0 2^0}{2^{3-1}} \Rightarrow x = 1, y = 4, z = 2 \Rightarrow lS(n) = \{4,2,1\},\ \forall n \in N_+$
$r = 4 = 3(1) + 1$ $k = 1$	$x = \frac{3^1 x + 3^0 2^0}{2^{4-1}} \Rightarrow\ x \notin N_+, \forall n \in N_+$
$r = 5 = 3(1) + 2$ $k = 1$	$x = \frac{3^1 x + 3^0 2^0}{2^{5-1}} \Rightarrow\ x \notin N_+, \forall n \in N_+$
$r = 6 = 3(2)$ $k = 2$	$x = \frac{3^2 x + 3(2^0) + 3^0 2^2}{2^{6-2}} \Rightarrow x = 1, y = 4, z = 2 \Rightarrow\ lS(n) = \{x, y, z\},$ $\forall n \in N_+$
$r = 9 = 3(3)$ $k = 3$	$x = \frac{3^3 x + 3^2(2^0) + 3^1(2^2) + 3^0 2^4}{2^{9-3}} \Rightarrow x = 1, y = 4, z = 2 \Rightarrow lS(n) = \{1,4,2\},$ $\forall n \in N_+$
$r = 12 = 3(4)$ $k = 4$	$x = \frac{3^4 x + 3^3(2^0) + 3^2(2^2) + 3(2^4) + 3^0 2^6}{2^{12-4}} \Rightarrow x = 1, y = 4, z = 2,$ $\forall n \in N_+$
$r = 15 = 3(5)$ $k = 5$	$x = \frac{3^5 x + 3^4(2^0) + 3^3(2^2) + 3^2(2^4) + 3(2^6) + 3^0 2^8}{2^{15-5}} \Rightarrow x = 1, y = 4,$ Z=2 $\forall n \in N_+$

$$Taha's\ Pattern\ to\ find\ lS(n)$$

$i)\ if\ r = 3(k)$ $\qquad\qquad$ $Part\ A:$

$$let\ x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + (3^0)2^{2(k-1)}}{2^{r-k}} = 1 \Rightarrow$$

$$x(2^{r-k} - 3^k) = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + (3^0)2^{2(k-1)} = g$$

$\in N_+ \Rightarrow\ x(2^{r-k} - 3^k) = g, \&\ x = 1 \Rightarrow\ (2^{r-k} - 3^k) = g$

$ii)\ if\ r = 3(k+1) = 3k + 3,$

$$is\ x = \frac{3^{k+1}x + 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \cdots + 3^{k-k+1}(2^{2k-2}) + (3^0)2^{2k}}{2^{r-(k+1)}} = 1\ ?$$

$Proof:$

$$x = \frac{3^{k+1}x + 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \cdots + 3^{k-k+1}(2^{2k-2}) + (3^0)2^{2k}}{2^{r-(k+1)}} \Rightarrow$$

$$\left(2^{r-(k+1)} - 3^{k+1}\right)x = 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \cdots + 3^{k-k+1}(2^{2k-2}) + (3^0)2^{2k} \Rightarrow$$

$$(2^{3k+3-k-1} - 3^{k+1})x = 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \cdots + 3^{k-k+1}(2^{2k-2}) + (3^0)2^{2k} \Rightarrow$$

$$(2^{2k+2} - 3^{k+1})x = 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \cdots + 3^{k-k+1}(2^{2k-2}) + (3^0)2^{2k} \dots eq1$$

$let\ k + 1 = m \Rightarrow 2k + 2 = 2m, 2k = 2m - 2 = 2(m-1), \&\ k = m - 1 \Rightarrow$

$$(2^{2m} - 3^m)x = 3^{m-1}(2^0) + 3^{m-2}(2^2) + 3^{m-3}(2^4) + \cdots + 3^1\left(2^{2(m-1)-2}\right) + (3^0)2^{2(m-1)} \dots eq1$$

$this\ pattern\ is\ same\ as\ above\ (i) \Rightarrow gx = g \Rightarrow x = 1 \Rightarrow lS(n) = \{1,4,2\}$

$or\ we\ can\ use\ LHS\ \&\ RHS\ below:$

$is\ (2^{2k+2} - 3^{k+1}) = 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \cdots + 3^{k-k+1}(2^{2k-2}) + 2^{2k}?$

$let\ k = 4\ \Rightarrow$

LHS = $(2^{10} - 3^5) = 781,\ \&$

$RHS =\ 3^4(2^0) + 3^3(2^2) + 3^2(2^4) + 3(2^6) + 2^8 = 781$

$\therefore eq1: (781)x = 781 \Rightarrow\ x = 1 \Rightarrow\ y = 4, z = 2 \Rightarrow$

LHS = $(2^{10} - 3^5) = 781,\ \&$

$RHS =\ 3^4(2^0) + 3^3(2^2) + 3^2(2^4) + 3(2^6) + 2^8 = 781$

$\therefore eq1: (781)x = 781 \Rightarrow\ x = 1 \Rightarrow\ y = 4, z = 2 \Rightarrow$

$lS(n) = \{4,2,1\}, \forall n\ \in N_+\ when\ r \in \{3,6,9,12, \dots, 3(k), 3(k+1), \dots\}$

https://doi.org/10.33774/coe-2024-c4f7p-v7 **ORCID:** https://orcid.org/0000-0002-8954-9264

$Part\ B$:

$let\ \ r = 3(k) + h, h < 3, k, h \in N_+$

$i) if\ \ r = 3(k) + 1$, *when h=1*

$$\therefore\ x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)}}{2^{r-k}} \Rightarrow$$

$$(2^{r-k} - 3^k)x = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)} = g$$

$$\because (2^{3k+1-k} - 3^k) = (2^{2k+1} - 3^k) > 2^{2k} - 3^k = g \Rightarrow$$

$$(2^{2k+1} - 3^k)x = g \Rightarrow x = \frac{g}{2^{2k+1} - 3^k} \notin N_+ \Rightarrow\ lS(n) = \emptyset, \forall n\ \in N_+\ when$$

$$r \in \{1,2,3+1, 6+1, \ldots, [3(k)+1]\}$$

$ii)\ if\ \ r = 3(k) + 2$, *when h=2*

$$\therefore\ x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)}}{2^{r-k}} \Rightarrow$$

$$(2^{r-k} - 3^k)x = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)} = g$$

$$\because (2^{3k+2-k} - 3^k) = (2^{2k+2} - 3^k) > 2^{2k} - 3^k = g \Rightarrow$$

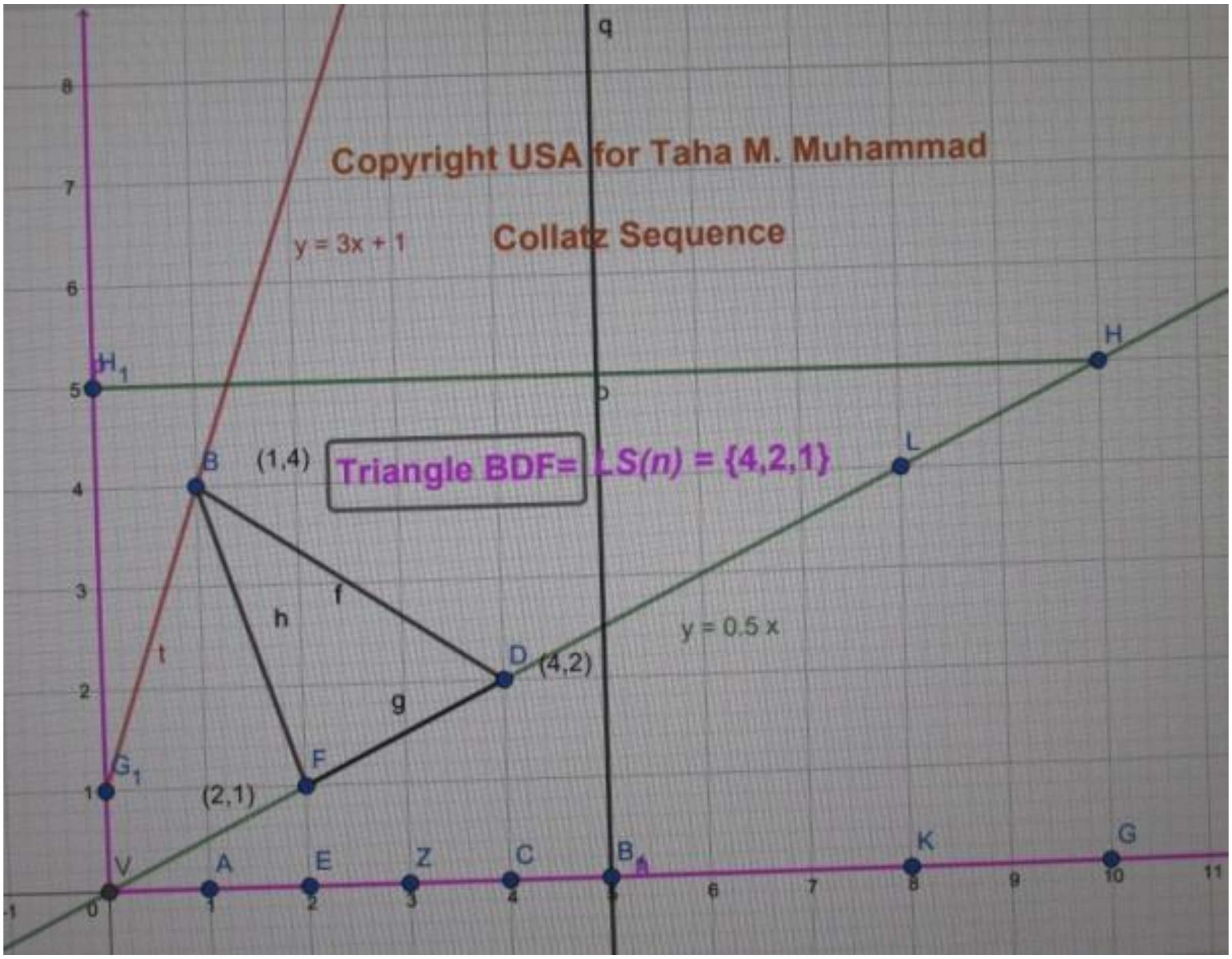
$$(2^{2k+2} - 3^k)x = g \Rightarrow x = \frac{g}{2^{2k+2} - 3^k} \notin N_+ \Rightarrow\ lS(n) = \emptyset, \forall n\ \in N_+\ when$$

$$r \in \{1,2,3+2, 6+2, \ldots, [3(k)+2]\}$$

$\therefore by Part\ B)\ i\ \&\ ii \Rightarrow x \notin N_+ \Rightarrow\ lS(n) = \emptyset, \forall n\ \in N_+\ when\ r = 3k + h, h = 1, or\ h = 2.$

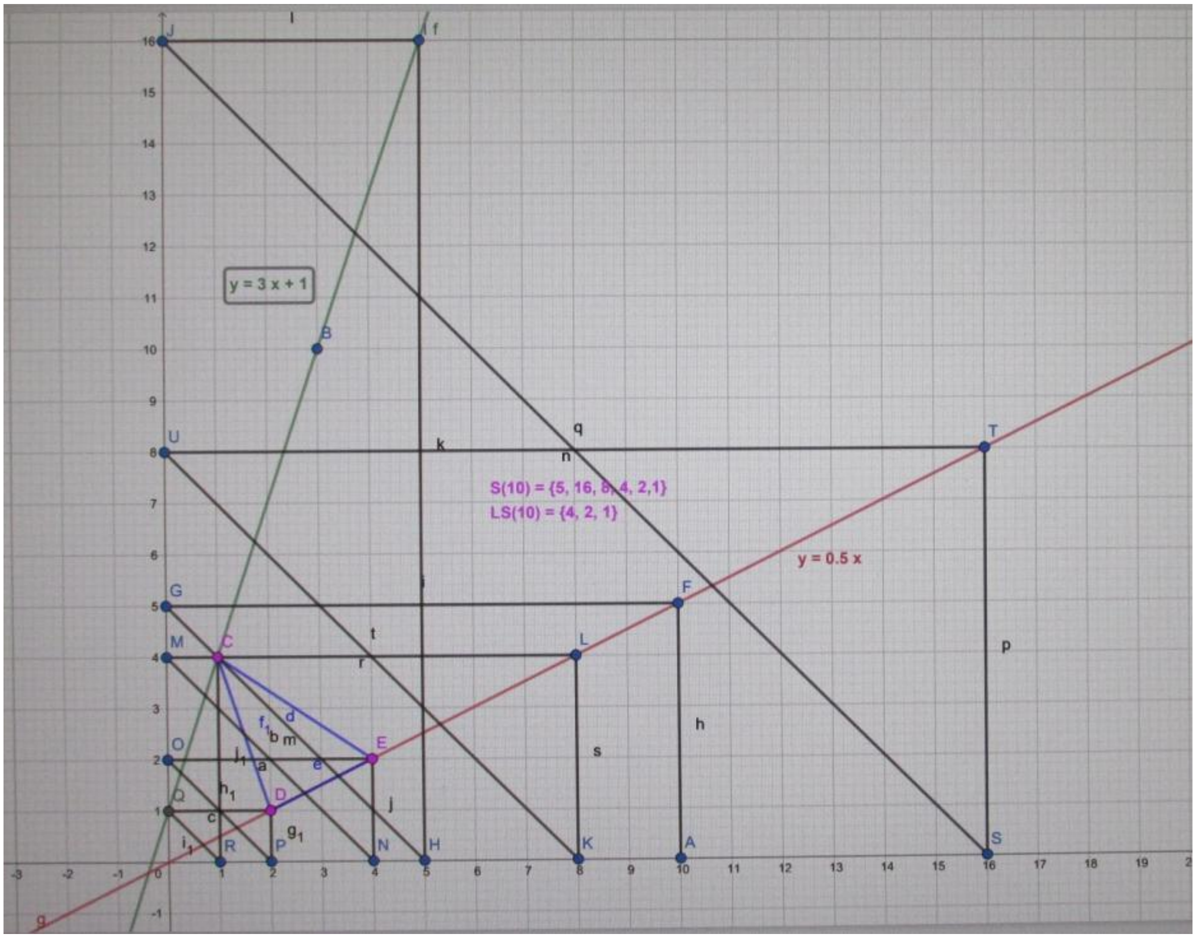
$\therefore by\ part\ A\ \&\ part\ B \Rightarrow lS(n) = \{1,2,4\} \cup \emptyset = \{1,2,4\}, \forall n \in N_+ \Rightarrow$

$the\ loop\ Collatz\ Sequence\ of\ each\ natural\ number\ is = \{1,2,4\}$



$Taha's\ Graph\ to\ find\ S(n) and\ lS(n)$

https://doi.org/10.33774/coe-2024-c4f7p-v7 **ORCID:** https://orcid.org/0000-0002-8954-9264 Content not peer-reviewed by Cambridge University Press. **License:** All Rights Reserved



$Taha's\ Graph\ to\ find\ S(10) and\ lS(10)$