Non trivial zeros of the Riemann zeta function

W. Oukil
Faculty of Mathematics.
University of Science and Technology Houari Boumediene.
BP 32 EL ALIA 16111 Bab Ezzouar, Algiers, Algeria.

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Abstract

In this paper, we consider the representation of the Riemann zeta function ζ defined by Abel's summation formula. We show that: if $|\zeta(s)| = 0$ then $|\zeta(1-s)| \neq 0$ for any point s in the critical strip except the critical line.

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1 Main results

Consider the representation of the Riemann zeta function ζ defined by Abel's summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s\int_1^{+\infty} u^{-1-s}\{u\}du, \quad s \neq 1, \quad \Re(s) > 0, \quad \Im(s) \in \mathbb{R}, \tag{1}$$

where $\{u\}$ is the fractional part of the real u. Denote by $B \subset \mathbb{C}$ the critical strip except the critical line, defined as

$$B := \Big\{ s \in \mathbb{C} : \quad \Re(s) \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \quad \Im(s) \in \mathbb{R} \Big\}.$$

We shall prove the following result:

Theorem 1. *Consider the Riemann zeta function ζ given by Equation* (1)*. We have*

$$\Big| \frac{\zeta(s)}{s(1+s)} - \frac{\zeta(1-s)}{(1-s)(2-s)} \Big| > 0, \quad \forall s \in B.$$

Proof. Consider the Equation (1), using the integration by parts formula, that gives

$$\frac{\zeta(z)}{z(1+z)} = -\frac{1}{2}\Big(\frac{1}{1-z} + \frac{1}{z}\Big) + \int_1^{+\infty} u^{-2-z}\eta(u)du,, \quad \forall z \in B. \tag{2}$$

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where by Dirichlet's Theorem, the real 1-periodic function $\eta : [1, +\infty) \to \mathbb{R}$ is defined as

$$\eta(u) := \int_1^u \Big(\frac{1}{2} - \{v\}\Big) dv = \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^2}\Big(1 - \exp(ij2\pi u)\Big), \quad \forall u \geq 1. \tag{3}$$

Let be $s \in B$. From Equation (2) we have

$$\frac{\zeta(s)}{s(1+s)} - \frac{\zeta(1-s)}{(1-s)(2-s)} = \int_1^{+\infty} \Big(u^{-2-s} - u^{-3+s}\Big)\eta(u)du. \tag{4}$$

For every $n > 2$ define

$$\omega(s,n) := 2\frac{\int_1^n \Big(u^{-2-s} - u^{-3+s}\Big)\eta(u)du}{\int_1^n \Big(u^{-2-s} - u^{-3+s}\Big)du}. \tag{5}$$

Equation (4) can be written as

$$\Big(\int_1^n \Big(u^{-2-s} - u^{-3+s}\Big)du\Big)^{-1}\left(\frac{\zeta(s)}{s(1+s)} - \frac{\zeta(1-s)}{(1-s)(2-s)}\right) = \omega(s,n),$$

Then

$$n^{2+s}\omega(s,n) = \Big(\int_1^n \Big(u^{-2-s} - u^{-3+s}\Big)du\Big)^{-1}\left(\frac{\zeta(s)}{s(1+s)} - \frac{\zeta(1-s)}{(1-s)(2-s)}\right),$$

Prove the present theorem by contradiction. Suppose that

$$\Big|\frac{\zeta(s)}{s(1+s)} - \frac{\zeta(1-s)}{(1-s)(2-s)}\Big| = 0. \tag{6}$$

Then

$$|\omega(s,n)| = 0, \quad \forall n > 2. \tag{7}$$

By definition of $\omega(s,n)$ in Equation (5) we have

$$x_s(n) = 2\int_1^n \Big(u^{-2-s} - u^{-3+s}\Big)\eta(u)du, \quad n \geq 2,$$

where in order to simplify the notation we denoted the sequence $\{x_s(n)\}_n$ as

$$x_s(n) := \omega(s,n)\int_1^n \Big(u^{-2-s} - u^{-3+s}\Big)du, \quad n \geq 2. \tag{8}$$

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We can write

$$x_s(n) = 2\int_1^{+\infty} \Big(u^{-2-s} - u^{-3+s}\Big)\eta(u)du - 2\int_n^{+\infty} \Big(u^{-2-s} - u^{-3+s}\Big)\eta(u)du, \quad n \geq 2. \tag{9}$$

By the hypothesis (6) and Equation (4), we have

$$\Big|\int_1^{+\infty} \Big(u^{-2-s} - u^{-3+s}\Big)\eta(u)du\Big| = 0,$$

it follows,

$$x_s(n) = -2\int_n^{+\infty} \Big(u^{-2-s} - u^{-3+s}\Big)\eta(u)du, \quad n \geq 2. \tag{10}$$

From Equation (3) we have

$$\int_1^u \Big(\eta(v) - \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^2}\Big)dv = -i\sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^3}\exp(ij2\pi u), \quad \forall u \geq 1.$$

By consequence,

$$\int_k^{k+1} \Big(\eta(v) - \frac{1}{12}\Big)dv = 0, \quad \forall k \in \mathbb{N}, \quad \text{and} \quad \Big|\int_1^u \Big(\eta(v) - \frac{1}{12}\Big)dv\Big| \leq \frac{1}{2^2\pi^3}\zeta(3), \quad \forall u \geq 1.$$

Using the integration by parts formula, we have

$$\begin{aligned}\int_n^{+\infty} \eta(u)\Big(u^{-2-s} - u^{-3+s}\Big)du = &\frac{1}{12}\Big(\frac{n^{-1-s}}{1+s} - \frac{n^{-2+s}}{2-s}\Big)\\ &- \int_n^{+\infty}\int_k^u \Big(\eta(v) - \frac{1}{12}\Big)dv\Big((2+s)u^{-3-s} - (3-s)u^{-4+s}\Big)du.\end{aligned}$$

Since $s \in B$, then

$$\lim_{n\to+\infty}\left|\Big(\frac{n^{-1-s}}{1+s} - \frac{n^{-2+s}}{2-s}\Big)^{-1}\int_n^{+\infty} \eta(u)\Big(u^{-2-s} - u^{-3+s}\Big)du\right| = \frac{1}{12}.$$

Thanks to Equations (10) we obtain

$$\lim_{n\to+\infty}\Big|\Big(\frac{n^{-1-s}}{1+s} - \frac{n^{-2+s}}{2-s}\Big)^{-1}x_s(n)\Big| = \frac{1}{12}.$$

By definition of $x_s(n_m)$ in Equation (8), we get

$$\Big|\int_1^{+\infty} \Big(u^{-2-s} - u^{-3+s}\Big)du\Big|\lim_{n\to+\infty}\Big|\Big(\frac{n^{-1-s}}{1+s} - \frac{n^{-2+s}}{2-s}\Big)^{-1}\omega(s,n)\Big| = \frac{1}{12},$$

We obtained a contradiction with Equation (7). □

References

[1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).