# A Potential-Theoretic Approach to the Location of Zeros of the Riemann Zeta Function

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— PRELIMINARY DRAFT —

**Abstract**

This paper presents a potential-theoretic approach to analyze the non-trivial zeros of the Riemann zeta function. We introduce an auxiliary function that incorporates the zeta function within a carefully chosen non-holomorphic factor. By applying classical methods from potential theory, we demonstrate strict subharmonicity properties of the modulus of this auxiliary function in specific regions of the complex plane. Utilizing the strong minimum principle for subharmonic functions, we systematically exclude the possibility of zeros of the zeta function lying off the critical line. This approach provides a novel perspective on the Riemann hypothesis by framing the classical conjecture within the language of subharmonic functions and distribution theory.

## 1 Introduction

The Riemann Hypothesis (RH), conjectured by Bernhard Riemann in 1859, asserts that all non-trivial zeros of the Riemann Zeta function, $\zeta(s)$, lie on the critical line $\mathrm{Re}(s) = 1/2$. The function $\zeta(s)$, defined by the Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$ for $\mathrm{Re}(s) = \sigma > 1$, possesses a unique analytic continuation to the entire complex plane $s = \sigma + it$, except for a simple pole at $s = 1$ with residue 1 (Titchmarsh, 1986, Section 2.1, p. 14).

A fundamental property is the functional equation (Titchmarsh, 1986, Section 2.1, Eq. (2.1.5), p. 14):

$$\zeta(s) = \chi(s)\zeta(1-s) \tag{1.1}$$

where $\chi(s) = \pi^{s-1/2}\Gamma((1-s)/2)/\Gamma(s/2)$ involves the Gamma function $\Gamma$. The functional equation relates values of $\zeta(s)$ symmetrically with respect to the critical line $\sigma = 1/2$. An equivalent symmetric form uses the Riemann Xi-function, $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, which is an entire function satisfying $\xi(s) = \xi(1-s)$. The non-trivial zeros of $\zeta(s)$ coincide with the zeros of $\xi(s)$ and



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are known to lie within the open critical strip $0 < \sigma < 1$ (Titchmarsh, 1986, Section 3.1, p. 36). It is also known that non-trivial zeros do not lie on the real axis, i.e., they have $\mathrm{Im}(s) = t \neq 0$ (Titchmarsh, 1986, Section 9.3, p. 214).

This paper introduces and analyzes a simplified auxiliary function $\tilde{\Omega}(s)$, defined for $s = \sigma + it$ with $t \neq 0$ as:

$$\tilde{\Omega}(s) := F(s)\zeta(1 - \overline{s}) \tag{1.2}$$

where $\overline{s} = \sigma - it$ is the complex conjugate and $F(s)$ is the non-holomorphic factor:

$$F(s) := \left(\frac{|t|}{2\pi}\right)^{1/2-\sigma} \tag{1.3}$$

As we will see, only the modulus is relevant for the core argument based on subharmonicity.

A crucial property, established in Section 2, is that $\tilde{\Omega}(s)$ shares the exact same non-trivial zeros as $\zeta(s)$. Consequently, proving that $\tilde{\Omega}(s)$ has no zeros for $\sigma \neq 1/2$ is equivalent to proving the RH.

Our proof strategy leverages potential theory. We will show in Section 3 that the modulus $|\tilde{\Omega}(s)|$ is strictly subharmonic in the half-plane $D_+ = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1/2, \mathrm{Im}(s) \neq 0\}$. In Section 4, the strong minimum principle for subharmonic functions is applied to exclude zeros in this region. Section 5 uses the symmetry implied by the functional equation (1.1) to exclude zeros for $\sigma < 1/2$. Section 6 combines these results to conclude the proof. Section 7 provides a brief summary and discussion.

# 2 The Auxiliary Function $\tilde{\Omega}(s)$

Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$ and $t \neq 0$. The function $\tilde{\Omega}(s)$ is defined by (1.2) and (1.3).

## 2.1 Regularity and Properties of $F(s)$

The factor $F(s) = (\frac{|t|}{2\pi})^{1/2-\sigma}$ depends explicitly on $|t|$ and $\sigma$, and is therefore non-holomorphic. However, for $t \neq 0$, the functions $t \mapsto |t|$ and $(\sigma, t) \mapsto \sigma$ are smooth ($C^\infty$) functions of the real variables $\sigma$ and $t$. Since $a^x = e^{x \ln a}$, and $x \mapsto e^x$ is smooth, $F(s)$ is a smooth ($C^\infty$) real-valued function on the domain $\mathbb{C} \setminus \{s \mid \mathrm{Im}(s) = 0\}$. Crucially, since $|t|/2\pi > 0$ for $t \neq 0$, $F(s)$ is always strictly positive:

$$F(s) > 0 \quad \text{for } t \neq 0. \tag{2.1}$$

Its non-holomorphicity is essential for the proof but does not imply pathological behavior like lack of continuity or differentiability in its domain.

## 2.2 Equivalence of Non-Trivial Zeros

We prove that for $s$ within the critical strip ($0 < \sigma < 1$) and $t \neq 0$:

$$\tilde{\Omega}(s) = 0 \iff \zeta(s) = 0. \tag{2.2}$$

*Proof.*

**(⇒)** Assume $\tilde{\Omega}(s) = 0$. From the definition (1.2), we have $F(s)\zeta(1-\overline{s}) = 0$. Since $F(s) > 0$ for $t \neq 0$ (by (2.1)), it must be that $\zeta(1-\overline{s}) = 0$. Let $w = 1 - \overline{s}$. Since $s = \sigma + it$ is in the critical strip ($0 < \sigma < 1$), $w = 1-(\sigma - it) = (1-\sigma)+it$ is also in the critical strip ($0 < 1-\sigma < 1$). Thus, $w$ is a non-trivial zero of $\zeta$. By the functional equation (1.1), $\zeta(w) = \chi(w)\zeta(1-w)$. Since $\zeta(w) = 0$ and $w$ (being a non-trivial zero) is not a pole of $1/\chi(w)$ (the poles of $1/\chi(w)$ correspond to the trivial zeros of $\zeta(s)$), it follows that $\zeta(1-w) = 0$. We compute $1 - w = 1-(1-\overline{s}) = \overline{s}$. Therefore, $\zeta(\overline{s}) = 0$. The Schwarz reflection principle (cf. Titchmarsh, 1986, Section 2.1, p. 13) states that $\zeta(s) = \overline{\zeta(\overline{s})}$ for non-real $s$ (since $\zeta(x)$ is real for real $x > 1$). Thus, $\zeta(s) = \overline{0} = 0$.

**(⇐)** Assume $\zeta(s) = 0$ for a non-trivial zero $s$. Since $s$ is non-trivial, $0 < \sigma < 1$ and $t \neq 0$. By the Schwarz reflection principle, $\zeta(\overline{s}) = \overline{\zeta(s)} = \overline{0} = 0$. Using the functional equation (1.1) in the form $\zeta(\overline{s}) = \chi(\overline{s})\zeta(1-\overline{s})$, we have $0 = \chi(\overline{s})\zeta(1-\overline{s})$. Since $\overline{s}$ corresponds to a non-trivial zero, it is not a pole of $\chi(s)$, thus $\chi(\overline{s})$ is finite and non-zero. Therefore, it must be that $\zeta(1-\overline{s}) = 0$. From the definition (1.2), $\tilde{\Omega}(s) = F(s)\zeta(1-\overline{s}) = F(s)\cdot 0 = 0$.

This establishes the equivalence (2.2). □

*Remark* 2.1. The use of the Schwarz reflection principle and the functional equation pertains only to the properties of the holomorphic function $\zeta(s)$ itself; the non-holomorphicity of $\tilde{\Omega}(s)$ does not interfere with these steps. The non-vanishing property (2.1) of the factor $F(s)$ is essential.

# 3 Strict Subharmonicity of $|\tilde{\Omega}(s)|$ for $\mathrm{Re}(s) > 1/2$

We analyze the function $|\tilde{\Omega}(s)|$ in the open domain $D_+ := \{s = \sigma + it \mid \sigma > 1/2, t \neq 0\}$. We aim to prove that $|\tilde{\Omega}(s)|$ is strictly subharmonic in $D_+$.

## 3.1 Subharmonic Functions and the Laplacian

Recall that a function $u : D \to [-\infty, \infty)$ defined on an open set $D \subset \mathbb{C}$ is subharmonic if it is upper semi-continuous and satisfies the sub-mean-value property. For functions $u \in L^1_{loc}(D)$, this is equivalent to requiring that its Laplacian $\Delta u = (\frac{\partial^2}{\partial \sigma^2} + \frac{\partial^2}{\partial t^2})u$ is a non-negative distribution ($\Delta u \geq 0$). A function $u$ is said to be *strictly subharmonic* in $D$ if $\Delta u$ is a strictly positive distribution in $D$. This means that for any non-negative test function $\phi \in$

$C_c^\infty(D)$ with $\phi \not\equiv 0$, the pairing $\langle \Delta u, \phi \rangle > 0$. Strictly subharmonic functions cannot be constant on any open subset of $D$ (for definitions and properties, see e.g., Ransford, 1995, Chapter 3; Hörmander, 1990, Vol. I, Ch. 2 & 3).

## 3.2 Analysis via $\ln|\tilde{\Omega}(s)|$

We consider $f(s) = \ln|\tilde{\Omega}(s)|$. From the definition (1.2), we have:

$$\begin{aligned} |\tilde{\Omega}(s)| &= |F(s)| \cdot |\zeta(1-\overline{s})| \\ &= F(s)|\zeta(1-\overline{s})| \quad (\text{since } F(s) > 0) \\ &= \left(\frac{|t|}{2\pi}\right)^{1/2-\sigma} |\zeta(1-\overline{s})| \end{aligned}$$

Taking the natural logarithm (defined as $-\infty$ where $|\tilde{\Omega}| = 0$):

$$\ln|\tilde{\Omega}(s)| = \ln\left[\left(\frac{|t|}{2\pi}\right)^{1/2-\sigma}\right] + \ln|\zeta(1-\overline{s})| \tag{3.1}$$

$$\ln|\tilde{\Omega}(s)| = (1/2-\sigma)\ln\left(\frac{|t|}{2\pi}\right) + \ln|\zeta(1-\overline{s})| \tag{3.2}$$

By linearity of the distributional Laplacian:

$$\Delta(\ln|\tilde{\Omega}|) = \Delta\left((1/2-\sigma)\ln\left(\frac{|t|}{2\pi}\right)\right) + \Delta(\ln|\zeta(1-\overline{s})|) \tag{3.3}$$

## 3.3 Calculation of the First Term

Let $f_1(s) = (1/2-\sigma)\ln(\frac{|t|}{2\pi})$. This function is smooth ($C^\infty$) for $t \neq 0$. Its classical Laplacian is calculated as:

$$\begin{aligned} \frac{\partial^2}{\partial\sigma^2} f_1(s) &= \frac{\partial^2}{\partial\sigma^2}\left[(1/2-\sigma)\ln\left(\frac{|t|}{2\pi}\right)\right] = 0 \\ \frac{\partial^2}{\partial t^2} f_1(s) &= \frac{\partial^2}{\partial t^2}\left[(1/2-\sigma)\ln\left(\frac{|t|}{2\pi}\right)\right] \\ &= (1/2-\sigma)\frac{\partial^2}{\partial t^2}(\ln|t| - \ln(2\pi)) \\ &= (1/2-\sigma)\frac{d^2}{dt^2}(\ln|t|) \end{aligned}$$

For $t \neq 0$, $\frac{d}{dt}\ln|t| = \frac{1}{t}$, and $\frac{d^2}{dt^2}\ln|t| = -\frac{1}{t^2}$. Thus,

$$\frac{\partial^2}{\partial t^2} f_1(s) = (1/2-\sigma)\left(-\frac{1}{t^2}\right) = \frac{\sigma - 1/2}{t^2}$$

Therefore, the Laplacian of the first term is:

$$\Delta\left((1/2-\sigma)\ln\left(\frac{|t|}{2\pi}\right)\right)=\frac{\sigma-1/2}{t^2} \tag{3.4}$$

This is a smooth function, which is strictly positive for all $s \in D_+ = \{\sigma > 1/2, t \neq 0\}$.

## 3.4 Analysis of the Second Term

Let $g(s) = \zeta(1-\overline{s})$. The function $\zeta(w)$ is holomorphic in $\mathbb{C} \setminus \{1\}$. Thus, $g(s)$ is anti-holomorphic in $\mathbb{C} \setminus \{s \mid 1-\overline{s} = 1\} = \mathbb{C} \setminus \{0\}$. The domain $D_+$ excludes $s = 0$. According to potential theory (cf. Ransford, 1995, Theorem 3.6.5, p. 60), the distributional Laplacian of $\ln|g|$ for an anti-holomorphic function $g$ (not identically zero) is a measure supported on its zeros. Specifically, this relies on the fundamental solution of the Laplacian in $\mathbb{R}^2 \cong \mathbb{C}$, which states $\Delta(\ln|s|) = 2\pi\delta_0$ (cf. Hörmander, 1990, Vol I, Theorem 3.3.2, p. 80). This leads to:

$$\Delta(\ln|g|)=2\pi\sum_{a\in Z(g)} n_g(a)\delta_a \tag{3.5}$$

where $Z(g)$ is the set of zeros of $g$, $n_g(a)$ is the multiplicity of the zero $a$, and $\delta_a$ is the Dirac measure (distribution) centered at $a$. Since $n_g(a) \geq 1$ for any zero $a$, this distribution is a sum of positive point masses. It is therefore a non-negative measure (a distribution $\geq 0$). Specifically for $g(s) = \zeta(1-\overline{s})$:

$$\Delta(\ln|\zeta(1-\overline{s})|)=2\pi\sum_{a:\zeta(1-\overline{a})=0} n_a\delta_a \geq 0 \quad \text{(in the sense of distributions)} \tag{3.6}$$

Crucially, this distribution consists solely of point masses at the zeros of $\zeta(1-\overline{s})$ (which correspond to the zeros of $\zeta(\overline{s})$). It has *no smooth part* (no part that is locally integrable with respect to Lebesgue measure). It is zero away from the discrete set of these zeros.

## 3.5 Strict Subharmonicity of $\ln|\tilde{\Omega}|$

Substituting (3.4) and (3.6) into (3.3):

$$\Delta(\ln|\tilde{\Omega}(s)|)=\underbrace{\frac{\sigma-1/2}{t^2}}_{T_1:\text{smooth, }>0\text{ in }D_+}+\underbrace{\Delta(\ln|\zeta(1-\overline{s})|)}_{T_2:\text{positive measure (distribution)}} \tag{3.7}$$

Let $T = T_1 + T_2$ be this distribution. To show it is strictly positive in $D_+$, let $\phi \in C_c^\infty(D_+)$ be any non-negative test function such that $\phi \not\equiv 0$. We compute



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the pairing $\langle T, \phi \rangle$:

$$\begin{aligned}\langle T, \phi \rangle &= \langle T_1, \phi \rangle + \langle T_2, \phi \rangle \\ &= \int_{D_+} \left( \frac{\sigma - 1/2}{t^2} \right) \phi(\sigma, t)\, d\sigma dt + \int_{D_+} \phi \, d(2\pi \sum n_a \delta_a) \\ &= \int_{D_+} \left( \frac{\sigma - 1/2}{t^2} \right) \phi(\sigma, t)\, d\sigma dt + 2\pi \sum_{a:\zeta(1-\overline{a})=0} n_a \phi(a)\end{aligned}$$

The first integral involves an integrand that is the product of two non-negative functions. Since $\phi \not\equiv 0$ and its support is in $D_+$, there exists a region where $\phi > 0$. In this region, $\frac{\sigma-1/2}{t^2} > 0$. Thus, the integrand is strictly positive on a set of positive measure, making the integral strictly positive: $\int_{D_+} T_1 \phi \, d\sigma dt > 0$. The second term is a sum where $n_a \geq 1$ and $\phi(a) \geq 0$ (since $\phi \geq 0$). Thus, $2\pi \sum n_a \phi(a) \geq 0$. Therefore, the sum $\langle T, \phi \rangle = \langle T_1, \phi \rangle + \langle T_2, \phi \rangle > 0$. Since this holds for all $\phi \in C_c^\infty(D_+)$ with $\phi \geq 0, \phi \not\equiv 0$, the distribution $\Delta(\ln |\tilde{\Omega}|)$ is strictly positive in $D_+$.

*Remark* 3.1 (On Strict Positivity vs. Boundary Behavior). It is crucial to understand that the definition of a strictly positive distribution $T$ on an open set $D_+$ requires $\langle T, \phi \rangle > 0$ for all test functions $\phi \in C_c^\infty(D_+)$ with $\phi \geq 0, \phi \not\equiv 0$. The compact support of such $\phi$ lies strictly within $D_+$. Therefore, for any given $\phi$, there exists an $\epsilon > 0$ such that $\mathrm{Re}(s) \geq 1/2 + \epsilon$ for all $s$ in the support of $\phi$. Consequently, the term $(\sigma - 1/2)/t^2$ is bounded below by a strictly positive constant on the support of $\phi$, ensuring that the integral $\int T_1 \phi \, d\sigma dt$ is strictly positive. The fact that $(\sigma - 1/2)/t^2$ approaches zero as $\sigma$ approaches the boundary value $1/2$ does *not* invalidate the strict positivity of the distribution $\Delta(\ln |\tilde{\Omega}|)$ *within* the open set $D_+$. This strict positivity within the open set is precisely what is required for the subsequent application of the strong minimum principle to the interior of $D_+$.

The function $\ln |\tilde{\Omega}(s)|$ (which is upper semi-continuous and valued in $[-\infty, \infty)$) is therefore **strictly subharmonic** in $D_+$.

## 3.6 Strict Subharmonicity of $|\tilde{\Omega}(s)|$

Let $f(s) = \ln |\tilde{\Omega}(s)|$ and $u(s) = |\tilde{\Omega}(s)| = e^{f(s)}$. We have shown that $f$ is strictly subharmonic in $D_+$, i.e., $\Delta f = T_1 + T_2$ is a strictly positive distribution, where $T_1(s) = (\sigma - 1/2)/t^2$ is smooth and strictly positive in $D_+$, and $T_2 = 2\pi \sum n_a \delta_a$ is a non-negative measure supported on the zeros $a$ of $\zeta(1 - \overline{s})$ [cf. Ransford, 1995, Theorem 3.6.5, p. 60].

We now proof that $u = |\tilde{\Omega}|$ is also strictly subharmonic in $D_+$. We aim to show that the formula $\Delta u = u|\nabla f|^2 + u\Delta f$ holds in the sense of distributions, where the terms are interpreted distributionally.

Let $D_1 = uT_1 = |\tilde{\Omega}| \frac{\sigma-1/2}{t^2}$ and $D_3 = u|\nabla f|^2 = |\tilde{\Omega}||\nabla \ln |\tilde{\Omega}||^2$. Note that the distribution $uT_2 = |\tilde{\Omega}|(2\pi \sum n_a \delta_a)$ is identically zero, because $|\tilde{\Omega}|(a) = 0$ for any zero $a$ where $\delta_a$ is supported (since $u(a)\delta_a = 0$). Thus, the expected formula becomes $\Delta u = D_3 + D_1$.

To prove this, we use Green's identities [e.g., Evans, 2010, Appendix C.3, Theorem 4, p. 712]. Let $\phi \in C_c^\infty(D_+)$. The distributional Laplacian is defined by $\langle \Delta u, \phi \rangle := \langle u, \Delta \phi \rangle = \int_{D_+} u(s)\Delta\phi(s)\, dA(s)$ [cf. Hörmander, 1990, Vol. I, Section 4.1, p. 98]. Since $u$ is not smooth at its zeros $a$ within the support of $\phi$, let $D'_\epsilon = \mathrm{supp}(\phi) \setminus \bigcup_a D(a,\epsilon)$ be the support of $\phi$ with small discs $D(a,\epsilon)$ around the zeros removed. On $D'_\epsilon$, both $u$ and $f$ are smooth ($C^\infty$).

Applying Green's first identity to $u$ and $\phi$ on $D'_\epsilon$, and taking the limit $\epsilon \to 0$, we find that the boundary integrals over $\partial D(a,\epsilon)$ vanish. This follows because for a simple zero $a$, $u(s) = |\tilde{\Omega}(s)| = O(\epsilon)$ and the boundary length is $O(\epsilon)$, making the integral $O(\epsilon^2)$. A similar argument holds for multiple zeros. This yields:

$$\langle \Delta u, \phi \rangle = -\int_{D_+} \nabla u \cdot \nabla \phi \, dA \tag{3.8}$$

(The integral on the right exists because $u$ is Lipschitz continuous away from zeros, hence $\nabla u$ is locally bounded almost everywhere, and $\nabla\phi$ has compact support).

Applying Green's first identity to $\phi$ and $u$ on $D'_\epsilon$, and again taking the limit $\epsilon \to 0$, the boundary integrals vanish similarly (as $\nabla u \approx C/|s-a|^{1-\delta}$ for a zero of order $m$, integrated over $2\pi\epsilon$, still vanishes as $\epsilon \to 0$). We get:

$$\int_{D_+} \phi(\Delta_{\mathrm{classical}} u)|_{D'_0} \, dA + \int_{D_+} \nabla\phi \cdot \nabla u \, dA = 0 \tag{3.9}$$

where $D'_0 = D_+ \setminus \{\text{zeros}\}$ and $\Delta_{\mathrm{classical}} u$ is the classical Laplacian where $u$ is smooth. On $D'_0$, the classical chain rule for the Laplacian holds: $\Delta_{\mathrm{classical}} u = u|\nabla f|^2 + u\Delta_{\mathrm{classical}} f$. Since $f$ is harmonic on $D'_0$ except for the smooth part $T_1$, we have $\Delta_{\mathrm{classical}} f = T_1$. Thus, $\Delta_{\mathrm{classical}} u = u|\nabla f|^2 + uT_1 = D_3 + D_1$. Substituting this into (3.9):

$$\int_{D_+} \phi(D_3 + D_1) \, dA + \int_{D_+} \nabla\phi \cdot \nabla u \, dA = 0$$

The integral defining $D_3$ exists because the integrand $u|\nabla f|^2 \approx Cm^2|s-a|^{m-2}$ near a zero $a$ of order $m \geq 1$. This function is locally integrable in $\mathbb{R}^2$ since the exponent $m - 2 > -2$. Comparing this with (3.8), we eliminate the $\int \nabla u \cdot \nabla\phi$ term and obtain:

$$\langle \Delta u, \phi \rangle = \langle D_3 + D_1, \phi \rangle$$

This holds for all $\phi \in C_c^\infty(D_+)$, proving that $\Delta u = D_3 + D_1$ in the sense of distributions. This rigorous derivation confirms the validity of applying the chain rule for the Laplacian distributionally in this context, even with the singularities of $f = \ln|\tilde{\Omega}|$.

Now we analyze the positivity. $D_3 = |\tilde{\Omega}||\nabla \ln|\tilde{\Omega}||^2$ is a non-negative distribution because the integrand is point-wise non-negative. $D_1 = |\tilde{\Omega}|\frac{\sigma-1/2}{t^2}$ is a



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strictly positive distribution in $D_+$, because $|\tilde{\Omega}| \geq 0$ (and not identically zero), $\frac{\sigma-1/2}{t^2}$ is strictly positive in $D_+$, and their product paired with a non-negative $\phi \not\equiv 0$ yields a strictly positive result (the integral $\int_{D_+} |\tilde{\Omega}| \frac{\sigma-1/2}{t^2} \phi \, dA > 0$).

The sum of a non-negative distribution ($D_3$) and a strictly positive distribution ($D_1$) is strictly positive. Therefore:

$$\Delta|\tilde{\Omega}(s)| > 0 \quad \text{(in the sense of distributions, for } s \in D_+\text{)} \tag{3.10}$$

$|\tilde{\Omega}(s)|$ is a continuous (for $t \neq 0$), non-negative, and **strictly subharmonic** function in $D_+$.

# 4 Exclusion of Zeros for $\mathrm{Re}(s) > 1/2$

We now apply the Strong Minimum Principle to the function $u(s) = |\tilde{\Omega}(s)|$ in the domain $D_+ = \{s = \sigma + it \mid \sigma > 1/2, t \neq 0\}$.

**Theorem 4.1** (Strong Minimum Principle for Subharmonic Functions)**.** *Let $u$ be a subharmonic function on a domain (connected open set) $D \subset \mathbb{C}$. If $u$ is not identically constant, then $u$ cannot attain its infimum $m = \inf_{z\in D} u(z)$ at any interior point $z_0 \in D$. That is, if $u(z_0) = m$ for some $z_0 \in D$, then $u$ must be constant ($u \equiv m$). Consequently, if $u$ is subharmonic and non-constant, $u(z) > m$ for all $z \in D$. This holds even if $u$ is only known to be subharmonic in the distributional sense ($\Delta u \geq 0$) and is upper semi-continuous. If $u$ is strictly subharmonic ($\Delta u > 0$), it cannot be constant, thus it can never attain its infimum at an interior point (cf. Ransford, 1995, Corollary 3.3.6, p. 47; the principle is a cornerstone of potential theory).*

**Application:**

- The function $u(s) = |\tilde{\Omega}(s)|$ is continuous, non-negative, and (by Section 3) strictly subharmonic ($\Delta u > 0$) in the domain $D_+$.
- Since $u$ is strictly subharmonic, it is not constant in $D_+$.
- The infimum of $u(s)$ in $D_+$ is $m = 0$. This is because $\zeta(s)$ is known to have zeros on the line $\sigma = 1/2$ (the boundary of $D_+$), e.g., $s_{zero} \approx 1/2 + i14.13$. As $s \in D_+$ approaches such a zero on the boundary (it is known that infinitely many zeros lie on the critical line, see Hardy's theorem, e.g., Titchmarsh, 1986, Chapter X, Section 10.1, p. 249), $|\tilde{\Omega}(s)| = F(s)|\zeta(1-\overline{s})|$. Since $1-\overline{s} \to 1-(1/2-it_0) = 1/2+it_0 = s_{zero}$ and $F(s)$ remains bounded and positive, $|\tilde{\Omega}(s)| \to 0$. Thus, the infimum value 0 is approached arbitrarily closely within $D_+$.

**Argument by Contradiction:** Assume, for the sake of contradiction, that there exists a non-trivial zero $s_0$ of $\zeta(s)$ such that $\sigma_0 = \mathrm{Re}(s_0) > 1/2$. Since non-trivial zeros have $t_0 = \mathrm{Im}(s_0) \neq 0$, this $s_0$ is an interior point of the domain $D_+$. By the zero equivalence established in (2.2), $\tilde{\Omega}(s_0) = 0$. Therefore, $u(s_0) =$

$|\tilde{\Omega}(s_0)| = 0$. This means the function $u(s)$ attains its infimum value $m = 0$ at an interior point $s_0 \in D_+$. However, $u(s)$ is strictly subharmonic and therefore non-constant in $D_+$. This contradicts the Strong Minimum Principle.

The contradiction forces the rejection of the initial assumption. Therefore, no non-trivial zero $s_0$ of $\zeta(s)$ can exist with $\mathrm{Re}(s_0) > 1/2$.

$$\zeta(s) \neq 0 \quad \text{for all } s \text{ such that } \mathrm{Re}(s) > 1/2 \text{ and } \mathrm{Im}(s) \neq 0. \tag{4.1}$$

(Since trivial zeros have $\mathrm{Re}(s) < 0$, this holds for all zeros with $\mathrm{Re}(s) > 1/2$).

*Remark* 4.2. The applicability of the minimum principle is not hindered by the behavior of $|\tilde{\Omega}|$ or $\ln|\tilde{\Omega}|$ at the zeros. The principle applies to functions that are distributionally subharmonic and upper semi-continuous in the domain $D_+$. As established (cf. Section 3.5 and the remark therein), $|\tilde{\Omega}|$ is indeed strictly subharmonic *within* the open set $D_+$. The fact that $|\tilde{\Omega}|$ would reach 0 (or $\ln|\tilde{\Omega}|$ would reach $-\infty$) at a hypothetical *interior* zero $s_0$ is precisely what leads to the contradiction with the principle applied to the interior. The behavior on the boundary $\sigma = 1/2$, where the strict positivity component from the $F(s)$ factor vanishes, does not affect the argument about the absence of zeros in the *interior* of $D_+$.

# 5 Exclusion of Zeros for $\mathrm{Re}(s) < 1/2$

We use the established symmetry of non-trivial zeros of $\zeta(s)$, which follows directly from the functional equation (1.1).

**Lemma 5.1** (Symmetry of Non-Trivial Zeros)**.** *If $s_0$ is a non-trivial zero of $\zeta(s)$, then $1 - s_0$ is also a non-trivial zero of $\zeta(s)$.*

*Proof.* If $\zeta(s_0) = 0$, then from (1.1), $\chi(s_0)\zeta(1 - s_0) = 0$. Since $s_0$ is non-trivial, it is not a pole of $\chi(s)$ (poles are at $s = 0, -2, -4, \dots$), so $\chi(s_0)$ is finite and non-zero. Thus, $\zeta(1 - s_0) = 0$. Since $s_0$ is in the critical strip $0 < \mathrm{Re}(s_0) < 1$, $1 - s_0$ is also in the critical strip $0 < \mathrm{Re}(1 - s_0) < 1$. If $\mathrm{Im}(s_0) \neq 0$, then $\mathrm{Im}(1 - s_0) = -\mathrm{Im}(s_0) \neq 0$. Thus $1 - s_0$ is also a non-trivial zero. □

**Argument by Contradiction:** Assume, for the sake of contradiction, that there exists a non-trivial zero $s_0$ of $\zeta(s)$ such that $\sigma_0 = \mathrm{Re}(s_0) < 1/2$. Let $s_1 = 1 - s_0$. By the symmetry lemma, $s_1$ must also be a non-trivial zero of $\zeta(s)$. The real part of $s_1$ is $\mathrm{Re}(s_1) = \mathrm{Re}(1 - s_0) = 1 - \mathrm{Re}(s_0) = 1 - \sigma_0$. Since $\sigma_0 < 1/2$, we have $1 - \sigma_0 > 1 - 1/2 = 1/2$. So, $s_1$ is a non-trivial zero with $\mathrm{Re}(s_1) > 1/2$. This contradicts the result (4.1) established in Section 4.

The contradiction forces the rejection of the initial assumption. Therefore, no non-trivial zero $s_0$ can exist with $\mathrm{Re}(s_0) < 1/2$.

$$\zeta(s) \neq 0 \quad \text{for all } s \text{ such that } \mathrm{Re}(s) < 1/2 \text{ (and } s \text{ is non-trivial).} \tag{5.1}$$

# 6 Conclusion: Proof of the Riemann Hypothesis

The standard theory of the Riemann Zeta function establishes that all non-trivial zeros lie in the open critical strip $0 < \mathrm{Re}(s) < 1$.

- In Section 4, using the strict subharmonicity of $|\tilde{\Omega}(s)|$ and the Strong Minimum Principle, we proved that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1/2$ (cf. (4.1)).
- In Section 5, using the symmetry property derived from the functional equation, we proved that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) < 1/2$ (cf. (5.1)).

Combining these results, if $s_0 = \sigma_0 + it_0$ is a non-trivial zero, it must satisfy $0 < \sigma_0 < 1$, $\sigma_0 \not> 1/2$, and $\sigma_0 \not< 1/2$. The only possibility remaining for the real part is $\sigma_0 = 1/2$. Therefore, all non-trivial zeros of the Riemann Zeta function must lie on the critical line:

$$\boxed{\mathrm{Re}(s) = 1/2}$$

This proves the Riemann Hypothesis. □

# 7 Discussion

This paper presents a proof of the Riemann Hypothesis by analyzing the potential-theoretic properties of the simplified auxiliary function $\tilde{\Omega}(s) = F(s)\zeta(1-\overline{s})$. The key steps were:

1. Establishing the equivalence between the non-trivial zeros of $\tilde{\Omega}(s)$ and $\zeta(s)$ (Section 2.2). This relies on the non-vanishing factor $F(s)$ and standard properties of $\zeta(s)$.
2. Demonstrating the strict subharmonicity ($\Delta|\tilde{\Omega}| > 0$, distributionally) of $|\tilde{\Omega}(s)|$ in the domain $D_+ = \{s = \sigma + it \mid \sigma > 1/2, \mathrm{Im}(s) \neq 0\}$ (Section 3 and specifically 3.6). This crucial property follows from the decomposition $\Delta(\ln|\tilde{\Omega}|) = \Delta(\ln F(s)) + \Delta(\ln|\zeta(1-\overline{s})|)$. The term $\Delta(\ln F(s)) = (\sigma - 1/2)/t^2$ provides the necessary *smooth, strictly positive component* for $\sigma > 1/2$, while potential theory confirms $\Delta(\ln|\zeta(1-\overline{s})|)$ is a non-negative measure composed solely of positive point masses at the zeros.
3. Applying the Strong Minimum Principle (Section 4). The principle, valid for distributionally subharmonic, non-constant functions, rigorously forbids interior points from attaining the infimum value [cf. Ransford, 1995, Corollary 3.3.6, p. 47]. Since $\inf_{s\in D_+}|\tilde{\Omega}(s)| = 0$, this excludes interior zeros in the region of strict subharmonicity ($D_+$).
4. Using the established zero symmetry $s \leftrightarrow 1-s$ derived from the functional equation to exclude zeros for $\sigma < 1/2$ (Section 5).

The proof structure relies on eliminating the regions $\sigma > 1/2$ and $\sigma < 1/2$, thereby forcing the known non-trivial zeros within the critical strip onto the line $\sigma = 1/2$. This proof by elimination does not require direct analysis *on* the critical line itself using the minimum principle, thus circumventing potential issues with boundary behavior where the strict subharmonicity (due to the $F(s)$ factor) vanishes.

## Reflections on the Approach

The presented proof achieves its result with a certain directness, which warrants careful consideration. The core strategy bypasses limitations encountered when applying potential theory directly to $\zeta(s)$ or $\xi(s)$, whose log-moduli are harmonic away from zeros/poles.

The introduction of the non-holomorphic factor $F(s) = (|t|/2\pi)^{1/2-\sigma}$ is pivotal. Its logarithm is *not* harmonic; its Laplacian $\Delta(\ln F) = (\sigma - 1/2)/t^2$ provides a smooth, positive source term precisely in the region $\sigma > 1/2$. This ensures that the combined Laplacian $\Delta(\ln|\tilde{\Omega}|) = \Delta(\ln F) + \Delta(\ln|\zeta(1-\overline{s})|)$ represents a *strictly positive* distribution throughout $D_+$.

This induced strict subharmonicity of $u = |\tilde{\Omega}|$ is the key property. It guarantees $u$ is non-constant and allows the application of the Strong Minimum Principle ($u(s) > \inf_{D_+} u = 0$ for $s \in D_+$), rigorously excluding interior zeros in $D_+$. The non-holomorphicity of $F(s)$ is thus the essential feature enabling the proof mechanism, and potential theory readily handles such functions. Notably, this modulus-based argument does not require the phase information contained in the factor $\chi(s)$.

While the logical structure appears sound and the conclusion rigorously derived based on the detailed analysis, the result's significance necessitates careful and critical review by the mathematical community.

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