Combinatorial System: Coefficients, Identities, and Generating Functions

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Abstract: This paper formalizes the unique counting structures introduced by Chinnaraji Annamalai, specifically focusing on what he terms the Annamalai Binomial Coefficient and the resulting power series known as the Combinatorial Geometric Series (CGS). The coefficient is demonstrated to be mathematically equivalent to a widely recognized form of the standard binomial coefficient, often used in problems involving choices with repetition. The study highlights how the CGS, which is constructed through the process of multiple, iterative summations of the basic geometric series, acts as a powerful generating function for this sequence of coefficients. For an infinite series, the resulting closed-form expression is a remarkably simple reciprocal power of the factor (one minus the variable). Furthermore, the framework provides clear formulas for the product of multiple finite geometric series, detailing how a key part of the numerator acts to effectively truncate the infinite series, thereby ensuring the result accurately reflects the finite limits of the original problem. By emphasizing these explicit recursive counting relationships, Annamalai's combinatorial system provides a valuable computational tool for established counting results and offers practical applications in modern technological domains like cybersecurity and machine learning.

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1. Introduction

Combinatorics, the study of discrete structures, relies heavily on fundamental tools such as binomial coefficients [1-5] and generating functions. The standard binomial coefficient ($\binom{n}{k}$ is central to counting subsets and coefficients in the binomial expansion [6-9]. Similarly, the geometric series $\sum x^n = (1-x)^{-1}$ is the generating function for the sequence of all ones. This paper examines the binomial structures proposed by Annamalai, which defines a novel binomial coefficient, V_n^r, and derives a related power series, the Combinatorial Geometric Series [16-19]. The significance of this framework lies in emphasizing the recursive and product relationships of these coefficients, providing an alternative but equivalent pathway to known results in combinatorial enumeration [20, 21].

2. Binomial Coefficient

Annamalai defines a binomial coefficient V_n^r for non-negative integers n and r. The definition for V_n^r is given by the product form:

$$V_n^r = \frac{(n+1)(n+2)(n+3)\cdots(n+r)}{r!} = \prod_{i=1}^{r} \frac{n+i}{i}$$

The identity between Annamalai's coefficient [10-13] and the standard binomial coefficient is established as follows:

$$V_n^r = \frac{(n+r)!}{n!\,r!} = \binom{n+r}{r}$$

Initial conditions are defined as $V_0^r = V_n^0 = V_0^0 = 1$.

A key property of this coefficient is symmetry:

$$V_n^r = V_r^n \Rightarrow \binom{n+r}{r} = \binom{n+r}{n}$$

3. Combinatorial Geometric Series and Identities

The combinatorial geometric series (CGS) is a power series that arises from the multiple summations of the extended geometric series. The r^{th} order CGS is defined by

$$\sum_{i=0}^{n} V_i^r x^i = \sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} \sum_{i_3=i_2}^{n} \cdots\cdots\cdots \sum_{i_r=i_{r-1}}^{n} x^{i_r}$$

Annamalai's binomial theorem states that multiple summations of extended geometric series with binomial coefficients form a binomial series:

$$\sum_{i=0}^{n} V_i^{r+1} x^i = \sum_{i=0}^{n} V_i^r x^i + \sum_{i=1}^{n} V_{i-1}^r x^i + \sum_{i=2}^{n} V_{i-2}^r x^i + \cdots\cdots\cdots + \sum_{i=n-1}^{n} V_{i-(n-1)}^r x^i + \sum_{i=n}^{n} V_{i-n}^r x^i$$

By grouping terms based on the coefficient V_k^r and re-expressing the sums as geometric series, we get:

$$\sum_{i=0}^{n} V_i^{r+1} x^i = V_0^r \sum_{i=0}^{n} x^i + V_1^r \sum_{i=1}^{n} x^i + V_2^r \sum_{i=2}^{n} x^i + V_3^r \sum_{i=3}^{n} x^i + \cdots\cdots\cdots + V_{n-1}^r \sum_{i=n-1}^{n} x^i + V_n^r \sum_{i=n}^{n} x^i$$

Substituting the standard formula for the geometric summation,$\sum_{i=k}^{n-1} x^i = \frac{x^n - x^k}{x-1}$, yields the final series form:

$$\sum_{i=0}^{n-1} V_i^{r+1} x^i = \frac{1}{x-1} \sum_{i=0}^{n-1} V_i^r x^i \left(x^{n-i} - 1\right),\ \forall\, x \neq 1$$

A key recursive relationship is that the sum of successive coefficients of order r is equal to the next higher-order coefficient, V_n^{r+1}.

$$\sum_{i=0}^{n} V_i^r = V_0^r + V_1^r + V_2^r + V_3^r + \cdots + V_{n-1}^r + V_n^r = V_n^{r+1}.$$

4. Product of Multiple Geometric Series

Annamalai's work [14-18] also addresses the product of r identical finite geometric series.

$$\underbrace{\left(\sum_{i=0}^{n-1} x^i\right)\left(\sum_{i=0}^{n-1} x^i\right)\left(\sum_{i=0}^{n-1} x^i\right)\cdots\cdots\left(\sum_{i=0}^{n-1} x^i\right)}_{r\ times} = \left(\sum_{i=0}^{n-1} x^i\right)^r = \left(\frac{1-x^n}{1-x}\right)^r = \frac{(1-x^n)^r}{(1-x)^r}$$

The binomial expansion of $(1-x^n)^r$ is presented as:

$$(1-x^n)^r = \sum_{k=0}^{r}\binom{r}{k}(-x^n)^k = \sum_{k=0}^{r}\binom{r}{k}\left((-1)x^n\right)^k = \sum_{k=0}^{r}\binom{r}{k}(-1)^k x^{kn}$$

A generalized form of the infinite geometric series is also noted:

$$\sum_{i=0}^{\infty} V_i^{r-1} x^i = \frac{1}{(1-x)^r}.$$

5. Generating Function

The generating function for the closed-form expression is given by:

$$N(x) = \sum_{k=0}^{r}\binom{r}{k}(-1)^k x^{kn} = (1-x^n)^r$$

The generating function [20, 21] for the infinite sum of the coefficients $\sum_{i=0}^{\infty} V_i^r\, x^i$ is mentioned below:

$$G_r(x) = \sum_{i=0}^{\infty} V_i^{r-1} x^i = \sum_{i=0}^{\infty}\binom{i+r}{i} x^i$$

This is the generating function for the sequence of coefficients $\binom{r}{r}$, $\binom{r+1}{r}$, $\binom{r+2}{r}$, $\cdots\cdots$

It is a well-known result that the generating function for these coefficients is denoted by:

$$\sum_{i=0}^{\infty} V_i^r x^i = \frac{1}{(1-x)^{r+1}}$$

This identity holds as a convergent power series if and only if $|x| < 1$.

6. Conclusion

Annamalai's work provides a significant extension to classical combinatorial and geometric series theory. By introducing the extended geometric series and the optimized binomial coefficient V_n^r, the framework establishes new identities, theorems, and series representations. The derived results, including the Annamalai's Binomial Theorem and the Annamalai Series, along with the understanding of the underlying generative function, are foundational tools that will be useful for future research and development in computational mathematics.

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