$Collatz\ Sequence\ Proof\ (1st\ Way)$

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$Abstract$: A Collatz sequence is a sequence of numbers generated by starting with a positive integer and repeatedly applying two rules: If the number is even, divide it by two, and if the number is odd, multiply it by 3 and add 1

$let\ Collatz\ Sequence\ (n) = S(n), Loop\ of\ Collatz\ Sequence\ (n) = lS(n),$

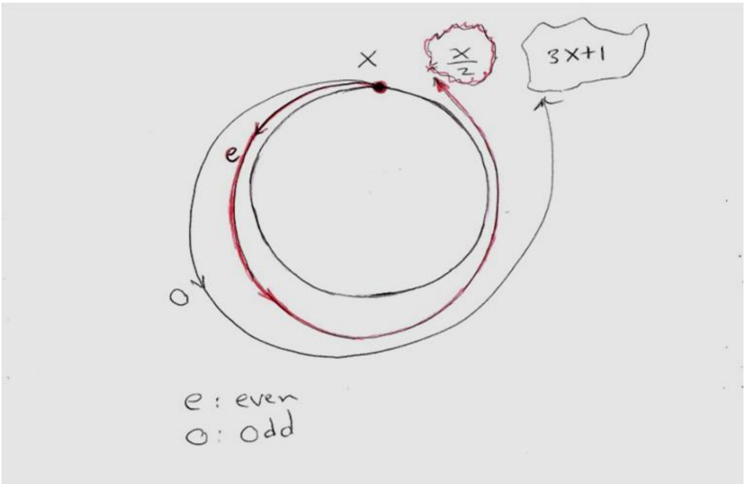
$\boldsymbol{let\ r = number\ of\ elements\ of\ lS(n)}, \&\ x, y, z, t, r, k, h, g, m, n \in N_+$

$$[n\ even \to \frac{n}{2}, or\ n\ odd \to (3n+1)] \Leftrightarrow S(n) = \left\{\frac{n}{2} or\ (3n+1), \ldots, 4,2,1\right\} \Leftrightarrow lS(n) = \{4,2,1\}$$

$Proof$:

$1st)\ Taha's\ Loop\ Sketch$

$A)\ is\ S(n) = \left\{\frac{n}{2} or\ (3n+1), \ldots, x\right\} \Rightarrow lS(n) = \{x\}, \forall n \in N_+?$



$Sketch\ of\ lS(n)\ to\ find\ equivalent\ expressions\ (Cloud)\ to\ x$

$Cloud$	$Cloud = x$	x
$\frac{x}{2}$	$\frac{x}{2} = x \Rightarrow x = 0 \notin lS(\mathrm{n})$	----
$3x+1$	$3x + 1 = x \Rightarrow x = -\frac{1}{2} \notin lS(\mathrm{n})$	----

$\therefore lS(\mathrm{n}) \neq \{x\}\ \forall n \in N_+, when\ r = 1$

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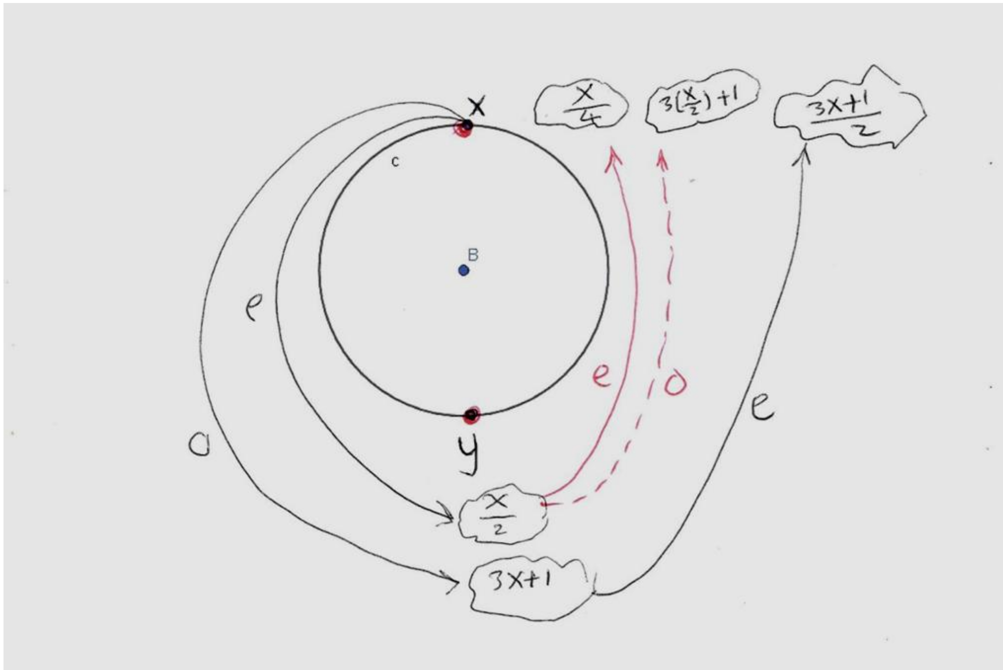
$B)\ is\ S(n) = \left\{\frac{n}{2} or\ (3n+1), \dots, x, y\right\} \Rightarrow lS(n) = \{x, y\}, \forall n \in N_+?$



$Sketch\ of\ lS(n)\ to\ find\ equivalent\ expressions\ (Cloud)\ to\ x$

$Cloud$	$Cloud = x$	x	y
$\frac{x}{4}$	$\frac{x}{4} = x \Rightarrow x = 0 \notin lS(n)$	....	....
$\frac{3x+2}{2}$	$\frac{3x+2}{2} = x \Rightarrow 3x + 2 = 2x \Rightarrow x = -2 \notin lS(n)$	....	....
$\frac{3x+1}{2}$	$\frac{3x+1}{2} = x \Rightarrow x = -1 \notin lS(n)$	....	....

$\therefore lS(n) \neq \{x, y\}\ \forall n \in N_+, when\ r = 2$

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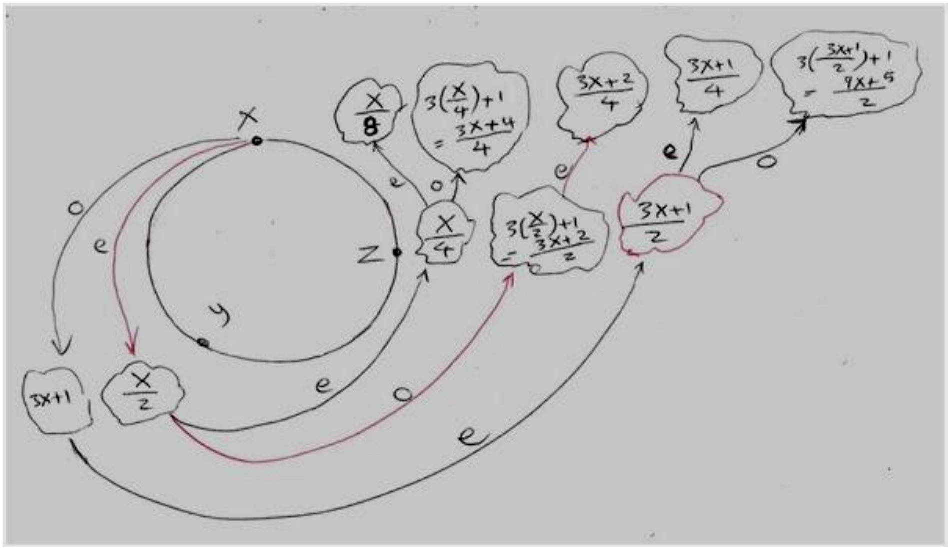
$C)\ is\ S(n) = \left\{\frac{n}{2}\, or\ (3n+1), \ldots, x, y, z\right\} \Rightarrow lS(n) = \{x, y, z\}, \forall n \in N_+?$



Sketch of $lS(n)$ to find equivalent expressions (Cloud) to x

$Cloud$	$Solve\ equation: Cloud = x$	x	y	z
$\frac{x}{8}$	$\frac{x}{8} = x \Rightarrow x = 0 \notin lS(n)$	---	---	---
$\frac{3x+4}{4}$	$\frac{3x+4}{4} = x \Rightarrow x = 4 \in lS(n)$	4	$\frac{x}{2} = 2$	$\frac{x}{4}$ =1
$\frac{3x+2}{4}$	$\frac{3x+2}{4} = x \Rightarrow x = 2 \in lS(n)$	2	$\frac{x}{2} = 1$	$\frac{3x+2}{2} = 4$
$\frac{3x+1}{4}$	$\frac{3x+1}{4} = x \Rightarrow x = 1 \in lS(n)$	1	$3x+1$ =3(1)+1=4	$\frac{3x+1}{2}$ =2
$\frac{9x+5}{2}$	$\frac{9x+5}{2} = x \Rightarrow x = -\frac{5}{7} \notin lS(n)$	---	---	---

$\therefore lS(n) = \{x, y, z\} = \{4,2,1\}, \forall n \in N_+, when\ r = 3$

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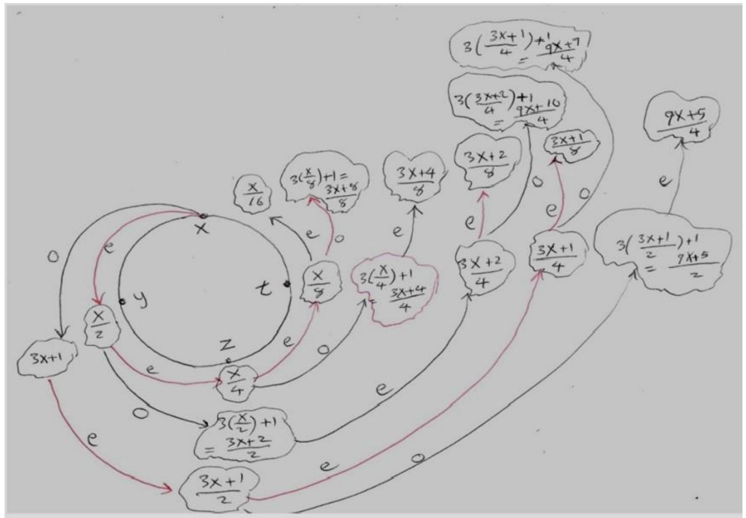
$D)\quad is\ S(n) = \left\{\frac{n}{2}\, or\ (3n+1), \dots, x, y, z, t\right\} \Rightarrow lS(n) = \{x, y, z, t\}, \forall n \in N_+?$



$Cloud$	$Solve\ equation: Cloud = x$	x	y	z	t
$\frac{x}{16}$	$\frac{x}{16} = x \Rightarrow x = 0 \notin lS(n)$	---	---	---	---
$\frac{3x+8}{8}$	$\frac{3x+8}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{3x+4}{8}$	$\frac{3x+4}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{3x+2}{8}$	$\frac{3x+2}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{3x+1}{8}$	$\frac{3x+1}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{9x+10}{4}$	$\frac{9x+10}{4} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{9x+7}{4}$	$\frac{9x+7}{4} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{9x+5}{4}$	$\frac{9x+5}{4} = x \Rightarrow x \notin lS(n)$	---	---	---	---

$\therefore lS(n) \neq \{x, y, z, t\}, \forall n \in N_+, when\ r = 4$

$2nd)\ Taha's\ Loop\ Table$

r *is number of elements of* $lS(n)$, & *let* $x \in lS(n)$

x = e or odd →

$r = 1$	→ $\frac{x}{2}$ = e or odd →	→ $3x + 1$ = e →

$r = 2$	→ $\frac{x}{4}$ =e or odd →	→ $\frac{3x+2}{2}$ =e →	→ $\frac{3x+1}{2}$ =e or odd →

$r = 3$	→ $\frac{x}{8}$ =e or odd →	→ $\frac{3x+4}{4}$ =e →	→ $\frac{3x+2}{4}$ =e or odd →	→ $\frac{3x+1}{4}$ =e or odd →	→ $\frac{9x+5}{2}$ =e →

$r = 4$	→ $\frac{x}{16}$ =e or odd →	→ $\frac{3x+8}{8}$ =e →	→ $\frac{3x+4}{8}$ =e or odd →	→ $\frac{3x+2}{8}$ =e or odd →	→ $\frac{3x+1}{8}$ =e or odd →	→ $\frac{9x+10}{4}$ =e →

→ $\frac{9x+7}{4}$ =e →	→ $\frac{9x+5}{4}$ =e or odd →

$E)\ is\ S(n) = \left\{\frac{n}{2} or\ (3n+1), \dots, x, y, z, t, u\right\} \Rightarrow lS(n) = \{x, y, z, t, u\}?\ \forall n \in N_{+},$

$r = 5$	→ $\frac{x}{32}$ =e or odd *	→ $\frac{3x+16}{16}$ =e or odd $	→ $\frac{3x+8}{16}$ =e or odd !	→ $\frac{3x+4}{16}$ =e or odd @	→ $\frac{3x+2}{16}$ =e or odd #	→ $\frac{3x+1}{16}$ =e or odd %
→ $\frac{9x+20}{8}$ =e @@	→ $\frac{9x+14}{8}$ =e &	→ $\frac{9x+11}{8}$ =e ((	→ $\frac{9x+10}{8}$ =e or odd {{	→ $\frac{9x+7}{8}$ =e or odd [[	→ $\frac{9x+5}{8}$ =e or odd **	→ $\frac{27x+19}{4}$ =e ***

$\therefore lS(n) \neq \{x, y, z, t, u\}\ \forall n \in N_{+}, when\ r = 5$

$F)\ is\ S(n) = \left\{\frac{n}{2} or\ (3n+1), \dots, x, y, z, t, u, v\right\} \Rightarrow lS(n) = \{x, y, z, t, u, v\}?\ \forall n \in N_{+},$

$r = 6$	* $\frac{x}{64}$	* $\frac{3x+32}{32}$	$ $\frac{3x+16}{32}$	! $\frac{3x+8}{32}$	@ $\frac{3x+4}{32}$	# $\frac{3x+2}{32}$	% $\frac{9x+19}{16}$
% $\frac{3x+1}{32}$	$ $\frac{9x+64}{16}$	{{ $\frac{27x+38}{8}$	@ $\frac{9x+28}{16}$	# $\frac{9x+22}{16}$	% $\frac{3x+1}{32}$	$ $\frac{9x+64}{16}$	! $\frac{9x+40}{16}$
@@ $\frac{9x+20}{16}$		& $\frac{9x+14}{16}$	(($\frac{9x+11}{16}$	{{ $\frac{9x+10}{16}$	[[$\frac{9x+7}{16}$	[[$\frac{27x+29}{16}$	** $\frac{9x+5}{16}$
** $\frac{27x+23}{8}$		*** $\frac{27x+19}{8}$					

$\therefore lS(n) = \{x, y, z, t, u, v\} = \{4,2,1,4,2,1\} = \{4,2,1\}\ \forall n \in N_{+}, when\ r = 6$

$Pattren\ To\ find\ Value\ of\ x \in lS(n)$ for $r \in N_+$

$r = 1 = 3(0) + 1$ $k = 0$	$x = \frac{3^0 x}{2^{1-0}} \Rightarrow x = 0 \notin N_+, 0 \notin lS(n) \Rightarrow lS(n) \neq \{x\},\ \forall n \in N_+$
$r = 2 = 3(0) + 2$ $k = 0$	$x = \frac{3^0 x}{2^{2-0}} \Rightarrow x = \frac{1}{3} \notin N_+ \Rightarrow x, y \notin lS(n) \Rightarrow lS(n) \neq \{\ x, y\},\ \forall n \in N_+$
$r = 3 = 3(1)$ $k = 1$	$x = \frac{3^1 x + 3^0 2^0}{2^{3-1}} \Rightarrow x = 1, y = 4, z = 2 \Rightarrow lS(n) = \{4,2,1\}, \forall\ n \in N_+$
$r = 4 = 3(1) + 1$ $k = 1$	$x = \frac{3^1 x + 3^0 2^0}{2^{4-1}} \Rightarrow\ x \notin N_+ \Rightarrow\ lS(n) \neq \{\ x, y, z, t\}, \forall n \in N_+$
$r = 5 = 3(1) + 2$ $k = 1$	$x = \frac{3^1 x + 3^0 2^0}{2^{5-1}} \Rightarrow\ x \notin N_+ \Rightarrow\ lS(n) \neq \{\ x, y, z, t, u\}, \forall n \in N_+$
$r = 6 = 3(2)$ $k = 2$	$x = \frac{3^2 x + 3(2^0) + 3^0 2^2}{2^{6-2}} \Rightarrow x = 1, y = 4, z = 2 \Rightarrow\ lS(n) = \{1,4,2\},\ \forall n \in N_+$
$r = 7 = 3(2) + 1$ $k = 2$	$x = \frac{3^2 x + 3(2^0) + 3^0 2^2}{2^{7-2}} \Rightarrow x = \frac{7}{23} \notin N_+ \Rightarrow$ $lS(n) \neq \{\ x, y, z, t, u, v\}, \forall n \in N_+$
$r = 9 = 3(3)$ $k = 3$	$x = \frac{3^3 x + 3^2(2^0) + 3^1(2^2) + 3^0 2^4}{2^{9-3}} \Rightarrow x = 1, y = 4, z = 2 \Rightarrow lS(n) = \{1,4,2\},$ $\forall n \in N_+$
$r = 12 = 3(4)$ $k = 4$	$x = \frac{3^4 x + 3^3(2^0) + 3^2(2^2) + 3(2^4) + 3^0 2^6}{2^{12-4}} \Rightarrow x = 1 \Rightarrow$ $lS(n) = \{1,4,2\}, \forall n \in N_+$

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$Equation\ to\ find\ x\ \in lS(n)$

$Part\ A$:

$i)\ if\ r = 3(k),$

$$is\ x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + (3^0)2^{2(k-1)}}{2^{r-k}} = 1?$$

$Proof:$

$$\therefore x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + (3^0)2^{2(k-1)}}{2^{3k-k}} \Rightarrow$$

$(2^{2k} - 3^k)x = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + (3^0)2^{2(k-1)} \ldots eq1.$

$let\ k = 4$

$\therefore (2^{2k} - 3^k) = (2^8 - 3^4) = 175 \ldots (\ Coefficient\ of\ x\ in\ LHS\ of\ eq1)$

$3^3(2^0) + 3^2(2^2) + 3^1(2^4) + 3^0(2^6) = 175 \ldots (RHS\ of\ eq1)$

$\therefore 175x = 175 \Rightarrow x = 1 \Rightarrow\ lS(n) = \{4,2,1\}, \forall n\ \in N_+\ when\ r = 3(k)$

$ii)\ if\ r = 3(k+1),$

$$is\ x = \frac{3^{k+1}x + 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \cdots + 3^{k-k+1}(2^{2k-2}) + (3^0)2^{2k}}{2^{r-(k+1)}} = 1\ ?$$

$Proof$:

$$x = \frac{3^{k+1}x + 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \cdots + 3^{k-k+1}(2^{2k-2}) + (3^0)2^{2k}}{2^{3(k+1)-(k+1)}} \Rightarrow$$

$(2^{2k+2} - 3^{k+1})x = 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \cdots + 3^{k-k+1}(2^{2k-2}) + (3^0)2^{2k} \ldots eq2.$

$let\ k = 4$

$\therefore (2^{2k+2} - 3^{k+1}) = (2^{10} - 3^5) = 781 \ldots (\ Coefficient\ of\ x\ in\ LHS\ of\ eq2)$

$3^4(2^0) + 3^3(2^2) + 3^2(2^4) + 3^1(2^6) + 3^0(2^8) = 781 \ldots (RHS\ of\ eq2)$

$\therefore 781x = 781 \Rightarrow x = 1 \Rightarrow\ lS(n) = \{4,2,1\}, \forall n\ \in N_+\ when\ r = 3(k+1)$

$by\ i\ \&\ ii \Rightarrow\ lS(n) = \{4,2,1\}, \forall n\ \in N_+\ when\ r\ is\ divisible\ by\ 3.$

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$Part\ B:$

$i)\ if\ \ r = 3(k) + 1, eq1\ below:$

$$is\ x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)}}{2^{r-k}} \notin N_+,$$

$\forall n \in N_+$?

$Proof:$

$$x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)}}{2^{(3k+1)-k}}$$

$$(2^{2k+1} - 3^k)x = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)} \ldots eq1$$

$let\ k = 4$

$\therefore (2^{2k+1} - 3^k) = (2^9 - 3^4) = 431 \ldots (\ Coefficient\ of\ x\ in\ \ LHS\ of\ eq1)$

$RHS\ of\ eq1: 2^3(2^0) + 3^2(2^2) + 3^1(2^4) + 3^0(2^6) = 156$

$\therefore 431x = 156 \Rightarrow x \notin N_+ \ \Rightarrow\ \nexists\ lS(n), \forall n\ \in N_+\ when\ r = 3(k) + 1$

$ii)\ if\ \ r = 3(k) + 2, is\ eq2\ below:$

$$x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)}}{2^{r-k}} \notin N_+?$$

$Proof:$

$$x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)}}{2^{(3k+2)-k}}$$

$$(2^{2k+2} - 3^k)x = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)} \ldots eq2$$

$let\ k = 4$

$\therefore (2^{2k+2} - 3^k) = (2^{10} - 3^4) = 943 \ldots (Coefficient\ of\ x\ in\ LHS\ of\ eq2)$

$RHS\ of\ eq2:\ 2^3(2^0) + 3^2(2^2) + 3^1(2^4) + 3^0(2^6)$

$\therefore 943x = 156 \Rightarrow x \notin N_+ \ \Rightarrow\ \nexists\ lS(n), \forall n\ \in N_+\ when\ r = 3(k) + 2$

$by\ i\ \&ii: \nexists\ lS(n), \forall n\ \in N_+\ when\ r = 3(k) + h, h \in \{1,2\}$

$\therefore by\ parts\ A\ \&\ \ B \Rightarrow lS(n) = \{1,2,4\}, \forall n \in N_+$

$Final\ Conclusion:$

$1 -\ \ any\ loop\ number\ (r/3) \in N_+ \ \Rightarrow\ \ lS(n) = \{1,2,4\}, \forall n \in N_+$

$2 -\ \ any\ loop\ number\ (r/3) \notin N_+ \ \ \Rightarrow lS(n)\ does\ not\ exist.$

$\therefore lS(n) = \{1,2,4\}, \forall n \in N_+$

$$The\ Equation\ in\ Brief$$

$$\because x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + (3^0)2^{2(k-1)}}{2^{r-k}}$$

$$\therefore x = \frac{3^k x + [3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + (3^0)2^{2(k-1)}]}{2^{r-k}}$$

$$\therefore x = \frac{3^k x + [\sum_{i=0}^{k-1} 3^{k-1-i}\, 2^{2i}]}{2^{r-k}}$$

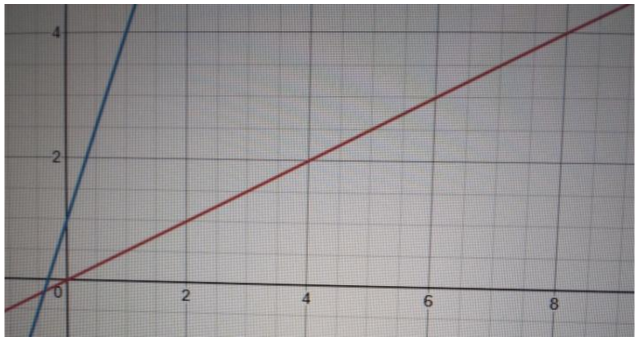
$$let\ S = \sum_{i=0}^{k-1} 3^{k-1-i}\, 2^{2i}$$

$$\therefore x = \frac{3^k x + S}{2^{r-k} - 3^k}$$

$$i)\ if\ x = \frac{3^k x + S}{2^{r-k} - 3^k} \Rightarrow x \in N_+ \Rightarrow x\ is\ an\ element\ of\ lS(n), \forall n \in N_+$$

$$ii)\ if\ x = \frac{3^k x + S}{2^{r-k} - 3^k} \Rightarrow x \notin N_+ \Rightarrow lS(n)\ does\ not\ exist, \forall n \in N_+$$

$$Graph\ of\ Collatz\ Sequence\ to\ find\ S(n)\ and\ lS(n),\},\forall n\in N_+$$



$$\boldsymbol{y_1} = \boldsymbol{n/2}, \boldsymbol{n} \in N_{\boldsymbol{even}}, \& \boldsymbol{n} \in x - axis$$

$$\boldsymbol{y_2} = \boldsymbol{3n+1}, \boldsymbol{n} \in N_{\boldsymbol{odd}}, n \in x - axis$$

$$\mathbf{4}\ \boldsymbol{by}\ \boldsymbol{y_1} \to \mathbf{2}\ \mathbf{by}\ \boldsymbol{y_1} \to \mathbf{1}\ \boldsymbol{by}\ \boldsymbol{y_2} \to \mathbf{4}$$

$$\therefore \boldsymbol{lS}(\boldsymbol{n}) = \{\mathbf{4},\mathbf{2},\mathbf{1}\}, \forall n \in N_+$$

Collatz Solution 1st Way

Engineering for proving Collatz Sequence using loops

$$r = number\ of\ elements\ in\ lS(n), n, r \in N_+$$

$$S(n) = \left\{\frac{n}{2} or\ (3n+1), \ldots, x\right\} \Rightarrow lS(n) \neq \{x\}, \forall n \in N_+, r = 1$$

$$S(n) = \left\{\frac{n}{2} or\ (3n+1), \ldots, x, y\right\} \Rightarrow lS(n) \neq \{x,y\}, \forall n \in N_+, r = 2$$

$$S(n) = \left\{\frac{n}{2} or\ (3n+1), \ldots, x, y, z\right\} \Rightarrow lS(n) = \{x,y,z\} = \{4,2,1\}\ \forall n \in N_+, r = 3$$

$$S(n) = \left\{\frac{n}{2} or\ (3n+1), \ldots, x, y, z, t\right\} \Rightarrow lS(n) \neq \{x,y,z,t\}, \forall n \in N_+, r = 4$$

$$S(n) = \left\{\frac{n}{2} or\ (3n+1), \ldots, x, y, z, t, u\right\} \Rightarrow lS(n) \neq \{x,y,z,t,u\}, \forall n \in N_+, r = 5$$

$$S(n) = \left\{\frac{n}{2} or\ (3n+1), \ldots, x, y, z, t, u, v\right\} \Rightarrow lS(n) = \{4,2,1,4,2,1\} = \{4,2,1\}, \forall n \in N_+, r = 6$$

$$S(n) = \left\{\frac{n}{2} or\ (3n+1), \ldots, x, y, z, t, u, v, g\right\} \Rightarrow lS(n) \neq \{x,y,z,t,u,v,g\}, \forall n \in N_+, r = 7$$

$$\left(\frac{r}{3}\right) \in N_+ \Rightarrow (r = 3k) \Rightarrow lS(n) = \{4,2,1\}, \forall n \in N_+, \& r \in N_{3k}, k \in N_+$$

$$\left(\frac{r}{3}\right) \notin N_+ \Rightarrow (r = 3k + h, \& h \in \{1,2\}) \Rightarrow lS(n)\ does\ not\ exist, \forall n \in N_+, \& r \in N_{3k+h}$$

$$\because N_+ = N_{r=3k} \cup N_{r=3k+h}$$

$$\therefore S(n) = \left\{\frac{n}{2} or\ (3n+1), \ldots, 4,2,1\right\} \Rightarrow lS(n) = \{4,2,1\}\ the\ only\ loop\ \forall n \in N_+$$

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