

Collatz Sequence — 1st Way

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Abstract: A Collatz sequence is a sequence of numbers generated by starting with a positive integer and repeatedly applying two rules: If the number is even, divide it by two, and if the number is odd, multiply it by 3 and add 1

let Collatz Sequence $(n) = S(n)$, Loop of Collatz Sequence $(n) = lS(n)$,

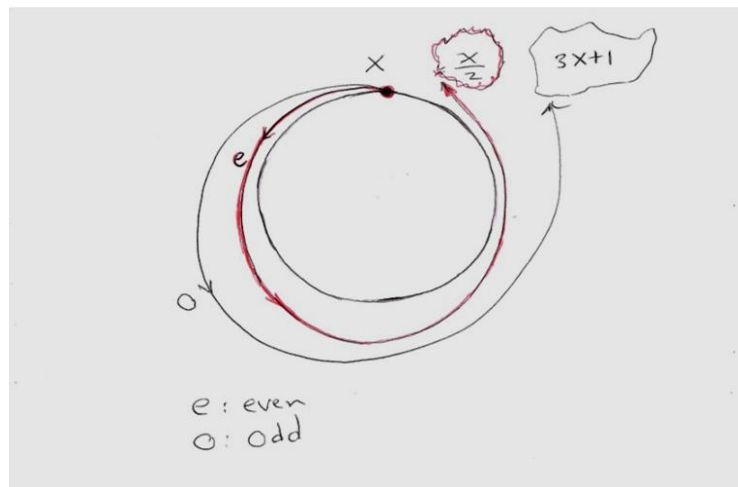
let $r = \text{number of elements of } lS(n)$, & $x, y, z, t, r, k, h, g, m, n \in N_+$

$$[n \text{ even} \rightarrow \frac{n}{2}, \text{ or } n \text{ odd} \rightarrow (3n + 1)] \Leftrightarrow S(n) = \left\{ \frac{n}{2} \text{ or } (3n + 1), \dots, 4, 2, 1 \right\} \Leftrightarrow lS(n) = \{4, 2, 1\}$$

Proof:

1st) Taha's Loop Sketch

A) is $S(n) = \left\{ \frac{n}{2} \text{ or } (3n + 1), \dots, x \right\} \Rightarrow lS(n) = \{x\}, \forall n \in N_+$?

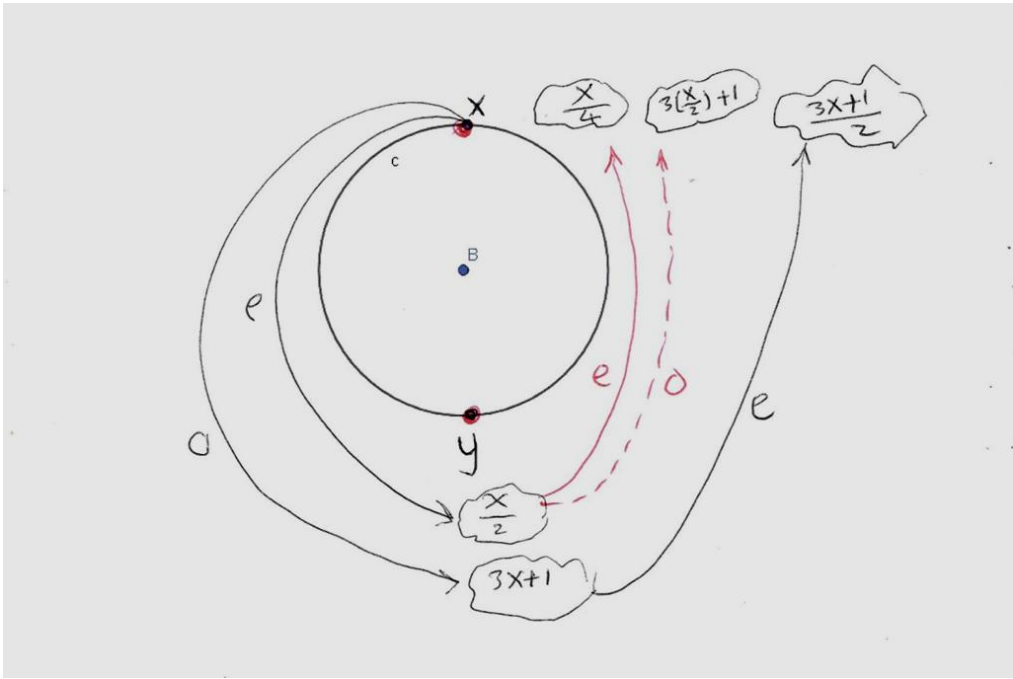


Sketch of $lS(n)$ to find equivalent expressions (Cloud) to x

Cloud	Cloud = x	x
$\frac{x}{2}$	$\frac{x}{2} = x \Rightarrow x = 0 \notin lS(n)$	----
$3x + 1$	$3x + 1 = x \Rightarrow x = -\frac{1}{2} \notin lS(n)$	----

$\therefore lS(n) \neq \{x\} \forall n \in N_+, \text{ when } r = 1$

B) is $S(n) = \left\{\frac{n}{2} \text{ or } (3n + 1), \dots, x, y\right\} \Rightarrow lS(n) = \{x, y\}, \forall n \in N_+$?

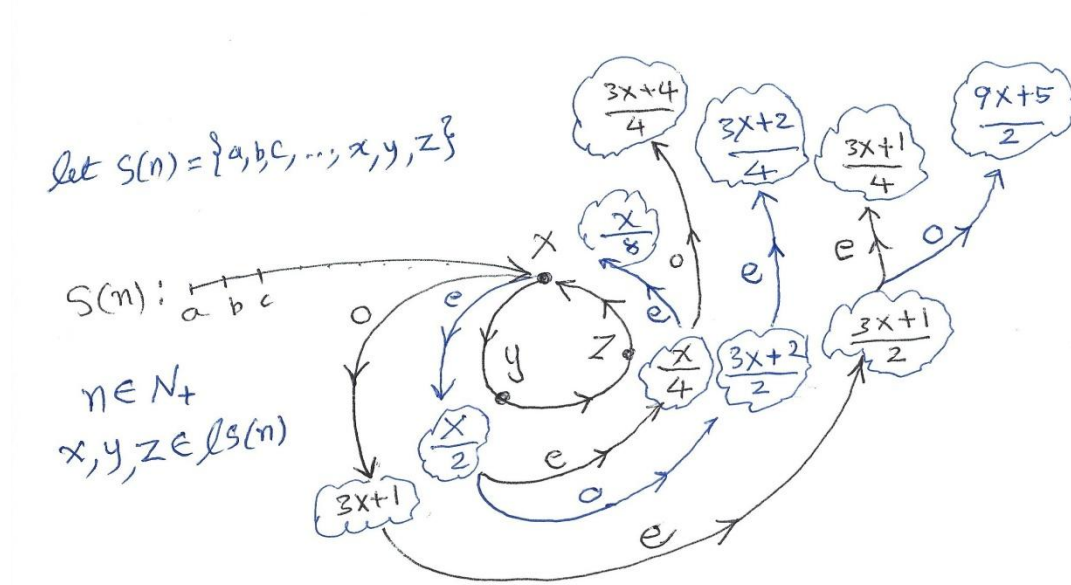


Sketch of $lS(n)$ to find equivalent expressions (Cloud) to x

Cloud	Cloud = x	x	y
$\frac{x}{4}$	$\frac{x}{4} = x \Rightarrow x = 0 \notin lS(n)$		
$\frac{3x + 2}{2}$	$\frac{3x+2}{2} = x \Rightarrow 3x + 2 = 2x \Rightarrow x = -2 \notin lS(n)$		
$\frac{3x + 1}{2}$	$\frac{3x+1}{2} = x \Rightarrow x = -1 \notin lS(n)$		

$\therefore lS(n) \neq \{x, y\} \forall n \in N_+, \text{ when } r = 2$

C) is $S(n) = \left\{ \frac{n}{2} \text{ or } (3n + 1), \dots, x, y, z \right\} \Rightarrow lS(n) = \{x, y, z\}, \forall n \in N_+$?

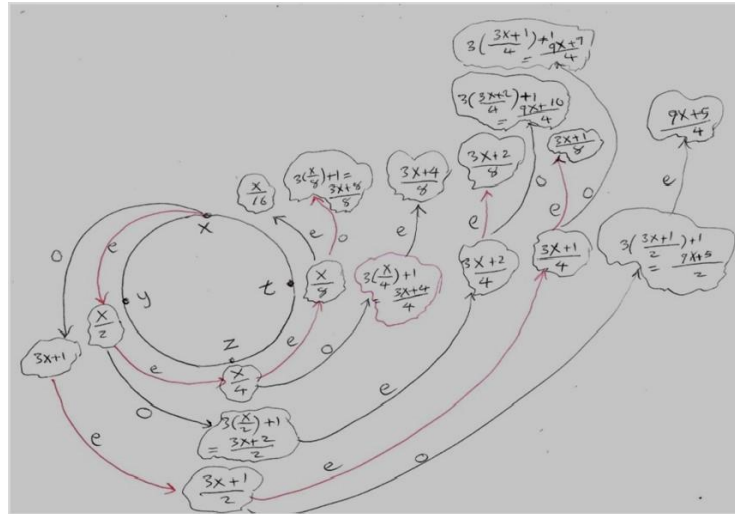


Sketch of $LS(n)$ to find equivalent expressions (Cloud) to x

<i>Cloud</i>	<i>Solve equation: Cloud = x</i>	<i>x</i>	<i>y</i>	<i>z</i>
$\frac{x}{8}$	$\frac{x}{8} = x \Rightarrow x = 0 \notin LS(n)$	---	---	---
$\frac{3x+4}{4}$	$\frac{3x+4}{4} = x \Rightarrow x = 4 \in LS(n)$	4	$\frac{x}{2} = 2$	$\frac{x}{4} = 1$
$\frac{3x+2}{4}$	$\frac{3x+2}{4} = x \Rightarrow x = 2 \in LS(n)$	2	$\frac{x}{2} = 1$	$\frac{3x+2}{2} = 4$
$\frac{3x+1}{4}$	$\frac{3x+1}{4} = x \Rightarrow x = 1 \in LS(n)$	1	$3x + 1$ $= 3(1) + 1 = 4$	$\frac{3x+1}{2} = 2$
$\frac{9x+5}{2}$	$\frac{9x+5}{2} = x \Rightarrow x = -\frac{5}{7} \notin LS(n)$	---	---	---

$$\therefore ls(n) = \{x, y, z\} = \{4, 2, 1\}, \forall n \in N_+, \text{ when } r = 3$$

D) is $S(n) = \left\{ \frac{n}{2} \text{ or } (3n + 1), \dots, x, y, z, t \right\} \Rightarrow lS(n) = \{x, y, z, t\}, \forall n \in N_+$?



Cloud	Solve equation: Cloud = x	x	y	z	t
$\frac{x}{16}$	$\frac{x}{16} = x \Rightarrow x = 0 \notin lS(n)$	---	---	---	---
$\frac{3x+8}{8}$	$\frac{3x+8}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{3x+4}{8}$	$\frac{3x+4}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{3x+2}{8}$	$\frac{3x+2}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{3x+1}{8}$	$\frac{3x+1}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{9x+10}{4}$	$\frac{9x+10}{4} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{9x+7}{4}$	$\frac{9x+7}{4} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{9x+5}{4}$	$\frac{9x+5}{4} = x \Rightarrow x \notin lS(n)$	---	---	---	---

$\therefore lS(n) \neq \{x, y, z, t\}, \forall n \in N_+, \text{ when } r = 4$

Or 2nd) Taha's Loop Table

r is number of elements of $LS(n)$, & let $x \in LS(n)$

$x = e$ or odd 

$r = 1$	 $\frac{x}{2}$ $= e$ or odd 	 $3x + 1$ $= e$ 
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A) $LS(n) \neq \{x\} \forall n \in N_+$, when $r = 1$

$r = 2$	 $\frac{x}{4}$ $= e$ or odd 	 $\frac{3x + 2}{2}$ $= e$ 	 $\frac{3x + 1}{2}$ $= e$ or odd 
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B) $LS(n) \neq \{x, y\} \forall n \in N_+$, when $r = 2$

$r = 3$	 $\frac{x}{8}$ $= e$ or odd 	 $\frac{3x + 4}{4}$ $= e$ 	 $\frac{3x + 2}{4}$ $= e$ or odd 	 $\frac{3x + 1}{4}$ $= e$ or odd 	 $\frac{9x + 5}{2}$ $= e$ 
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C) $LS(n) = \{x, y, z\} = \{1, 2, 4\} \forall n \in N_+$, when $r = 3$

$r = 4$	 $\frac{x}{16}$ $= e$ or odd 	 $\frac{3x + 8}{8}$ $= e$ 	 $\frac{3x + 4}{8}$ $= e$ or odd 	 $\frac{3x + 2}{8}$ $= e$ or odd 	 $\frac{3x + 1}{8}$ $= e$ or odd 	 $\frac{9x + 10}{4}$ $= e$ 
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 $\frac{9x + 7}{4}$ =e 	 $\frac{9x + 5}{4}$ =e or odd 
---	--

D) $lS(n) \neq \{x, y, z, t\} \forall n \in N_+, \text{ when } r = 4$

$r = 5$	 $\frac{x}{32}$ =e or odd *	 $\frac{3x + 16}{16}$ =e or odd \$	 $\frac{3x + 8}{16}$ =e or odd !	 $\frac{3x + 4}{16}$ =e or odd @	 $\frac{3x + 2}{16}$ =e or odd #	 $\frac{3x + 1}{16}$ =e or odd %
 $\frac{9x + 20}{8}$ =e @@	 $\frac{9x + 14}{8}$ =e &	 $\frac{9x + 11}{8}$ =e ((	 $\frac{9x + 10}{8}$ =e or odd {{	 $\frac{9x + 7}{8}$ =e or odd [[	 $\frac{9x + 5}{8}$ =e or odd **	 $\frac{27x + 19}{4}$ =e ****

E) $lS(n) \neq \{x, y, z, t, u\} \forall n \in N_+, \text{ when } r = 5$

$r = 6$	*	*	\$	!	@	#	%
	$\frac{x}{64}$	$\frac{3x + 32}{32}$	$\frac{3x + 16}{32}$	$\frac{3x + 8}{32}$	$\frac{3x + 4}{32}$	$\frac{3x + 2}{32}$	$\frac{9x + 19}{16}$
%	\$	{{	@	#	%	\$	!
$\frac{3x + 1}{32}$	$\frac{9x + 64}{16}$	$\frac{27x + 38}{8}$	$\frac{9x + 28}{16}$	$\frac{9x + 22}{16}$	$\frac{3x + 1}{32}$	$\frac{9x + 64}{16}$	$\frac{9x + 40}{16}$
@@	&	((	{{	[[	[[	**	
$\frac{9x + 20}{16}$	$\frac{9x + 14}{16}$	$\frac{9x + 11}{16}$	$\frac{9x + 10}{16}$	$\frac{9x + 7}{16}$	$\frac{27x + 29}{16}$	$\frac{9x + 5}{16}$	
**	***						
$\frac{27x + 23}{8}$	$\frac{27x + 19}{8}$						

F) $lS(n) = \{x, y, z, t, u, v\} = \{4, 2, 1, 4, 2, 1\} = \{4, 2, 1\} \forall n \in N_+, \text{ when } r = 6$

Pattren To find Value of $x \in lS(n)$ for $r \in N_+$

$r = 1 = 3(0) + 1$ $k = 0$	$x = \frac{3^0 x}{2^{1-0}} \Rightarrow x = 0 \notin N_+, 0 \notin lS(n) \Rightarrow lS(n) \neq \{x\}, \forall n \in N_+$
$r = 2 = 3(0) + 2$ $k = 0$	$x = \frac{3^0 x}{2^{2-0}} \Rightarrow x = \frac{1}{3} \notin N_+ \Rightarrow x, y \notin lS(n) \Rightarrow lS(n) \neq \{x, y\}, \forall n \in N_+$
$r = 3 = 3(1)$ $k = 1$	$x = \frac{3^1 x + 3^0 2^0}{2^{3-1}} \Rightarrow x = 1, y = 4, z = 2 \Rightarrow lS(n) = \{4, 2, 1\}, \forall n \in N_+$
$r = 4 = 3(1) + 1$ $k = 1$	$x = \frac{3^1 x + 3^0 2^0}{2^{4-1}} \Rightarrow x \notin N_+ \Rightarrow lS(n) \neq \{x, y, z, t\}, \forall n \in N_+$
$r = 5 = 3(1) + 2$ $k = 1$	$x = \frac{3^1 x + 3^0 2^0}{2^{5-1}} \Rightarrow x \notin N_+ \Rightarrow lS(n) \neq \{x, y, z, t, u\}, \forall n \in N_+$
$r = 6 = 3(2)$ $k = 2$	$x = \frac{3^2 x + 3(2^0) + 3^0 2^2}{2^{6-2}} \Rightarrow x = 1, y = 4, z = 2 \Rightarrow lS(n) = \{1, 4, 2\}, \forall n \in N_+$
$r = 7 = 3(2) + 1$ $k = 2$	$x = \frac{3^2 x + 3(2^0) + 3^0 2^2}{2^{7-2}} \Rightarrow x = \frac{7}{23} \notin N_+ \Rightarrow$ $lS(n) \neq \{x, y, z, t, u, v\}, \forall n \in N_+$
$r = 9 = 3(3)$ $k = 3$	$x = \frac{3^3 x + 3^2(2^0) + 3^1(2^2) + 3^0 2^4}{2^{9-3}} \Rightarrow x = 1, y = 4, z = 2 \Rightarrow lS(n) = \{1, 4, 2\},$ $\forall n \in N_+$
$r = 12 = 3(4)$ $k = 4$	$x = \frac{3^4 x + 3^3(2^0) + 3^2(2^2) + 3(2^4) + 3^0 2^6}{2^{12-4}} \Rightarrow x = 1 \Rightarrow$ $lS(n) = \{1, 4, 2\}, \forall n \in N_+$

Equation to find $x \in lS(n)$

Part A:

i) if $r = 3(k)$,

$$is\ x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + (3^0)2^{2(k-1)}}{2^{r-k}} = 1?$$

Proof:

$$\therefore x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + (3^0)2^{2(k-1)}}{2^{3k-k}} \Rightarrow$$

$$(2^{2k} - 3^k)x = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + (3^0)2^{2(k-1)} \dots eq1.$$

let $k = 4$, & substitute it in two expressions

(Coefficient of x in LHS of eq1) & (RHS of eq1)

$$\therefore (2^{2k} - 3^k) = (2^8 - 3^4) = 175 \dots (\text{Coefficient of } x \text{ in LHS of eq1})$$

$$\& 3^3(2^0) + 3^2(2^2) + 3^1(2^4) + 3^0(2^6) = 175 \dots (\text{RHS of eq1})$$

$$\therefore 175x = 175 \dots eq1 \Rightarrow x = 1 \Rightarrow lS(n) = \{4, 2, 1\}, \forall n \in N_+ \text{ when } r = 3(k)$$

ii) if $r = 3(k + 1)$,

$$is\ x = \frac{3^{k+1}x + 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + (3^0)2^{2k}}{2^{r-(k+1)}} = 1?$$

Proof:

$$x = \frac{3^{k+1}x + 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + (3^0)2^{2k}}{2^{3(k+1)-(k+1)}} \Rightarrow$$

$$(2^{2k+2} - 3^{k+1})x = 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + (3^0)2^{2k} \dots eq2.$$

let $k = 4$ & substitute it in two expressions

(Coefficient of x in LHS of eq2) & (RHS of eq2)

$$\therefore (2^{2k+2} - 3^{k+1}) = (2^{10} - 3^5) = 781 \dots (\text{Coefficient of } x \text{ in LHS of eq2})$$

$$\& 3^4(2^0) + 3^3(2^2) + 3^2(2^4) + 3^1(2^6) + 3^0(2^8) = 781 \dots (\text{RHS of eq2})$$

$$\therefore 781x = 781 \dots eq2 \Rightarrow x = 1 \Rightarrow lS(n) = \{4, 2, 1\}, \forall n \in N_+ \text{ when } r = 3(k + 1)$$

by i & ii $\Rightarrow lS(n) = \{4, 2, 1\}, \forall n \in N_+ \text{ when } r \text{ is divisible by } 3.$

Part B:

i) if $r = 3(k) + 1$, eq1 below:

$$\text{is } x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + 3^0 2^{2(k-1)}}{2^{r-k}} \notin N_+,$$

$\forall n \in N_+$?

Proof:

$$x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + 3^0 2^{2(k-1)}}{2^{(3k+1)-k}}$$

$$(2^{2k+1} - 3^k)x = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + 3^0 2^{2(k-1)} \dots \text{eq1}$$

let $k = 4$ & substitute it:

$$\therefore (2^{2k+1} - 3^k) = (2^9 - 3^4) = 431 \dots (\text{Coefficient of } x \text{ in LHS of eq1})$$

$$\text{RHS of eq1: } 3^3(2^0) + 3^2(2^2) + 3^1(2^4) + 3^0(2^6) = 175$$

$$\therefore 431x = 175 \Rightarrow x \notin N_+ \Rightarrow \nexists lS(n), \forall n \in N_+ \text{ when } r = 3(k) + 1$$

ii) if $r = 3(k) + 2$, is eq2 below:

$$x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + 3^0 2^{2(k-1)}}{2^{r-k}} \notin N_+?$$

Proof:

$$x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + 3^0 2^{2(k-1)}}{2^{(3k+2)-k}}$$

$$(2^{2k+2} - 3^k)x = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + 3^0 2^{2(k-1)} \dots \text{eq2}$$

let $k = 4$ & substitute it:

$$\therefore (2^{2k+2} - 3^k) = (2^{10} - 3^4) = 943 \dots (\text{Coefficient of } x \text{ in LHS of eq2})$$

$$\text{RHS of eq2: } 2^3(2^0) + 3^2(2^2) + 3^1(2^4) + 3^0(2^6)$$

$$\therefore 943x = 156 \dots \text{eq2} \Rightarrow x \notin N_+ \Rightarrow \nexists lS(n), \forall n \in N_+ \text{ when } r = 3(k) + 2$$

by i & ii: $\nexists lS(n), \forall n \in N_+ \text{ when } r = 3(k) + h, h \in \{1, 2\}$

$$\therefore \text{by parts A \& B} \Rightarrow lS(n) = \{1, 2, 4\}, \forall n \in N_+$$

Final Conclusion:

1 – any loop number r such that $(r/3) \in N_+ \Rightarrow lS(n) = \{1, 2, 4\}, \forall n \in N_+$

2 – any loop number r such that $(r/3) \notin N_+ \Rightarrow lS(n)$ does not exist.

$$\therefore lS(n) = \{1, 2, 4\}, \forall n \in N_+$$

The Equation in Brief

$$\therefore x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + (3^0)2^{2(k-1)}}{2^{r-k}} \dots eq1$$

$$\therefore x = \frac{3^k x + [3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + (3^0)2^{2(k-1)}]}{2^{r-k}}$$

$$\therefore x = \frac{3^k x + [\sum_{i=0}^{k-1} 3^{k-1-i} 2^{2i}]}{2^{r-k}} \dots eq1$$

$$\text{let } S = \sum_{i=0}^{k-1} 3^{k-1-i} 2^{2i}$$

$$\therefore x = \frac{3^k x + S}{2^{r-k}} \dots eq1 (\text{By substitution})$$

$$i) \text{ if } x = \frac{3^k x + S}{2^{r-k} - 3^k} \Rightarrow x \in N_+ \Rightarrow lS(n) = \{1, 2, 4\}, \forall n \in N_+, \forall r \in N_+, \& \left(\frac{r}{3}\right) \in N_+$$

$$ii) \text{ if } x = \frac{3^k x + S}{2^{r-k} - 3^k} \Rightarrow x \notin N_+ \Rightarrow lS(n) \text{ does not exist}, \forall n \in N_+, \forall r \in N_+, \& \left(\frac{r}{3}\right) \notin N_+ \Rightarrow$$

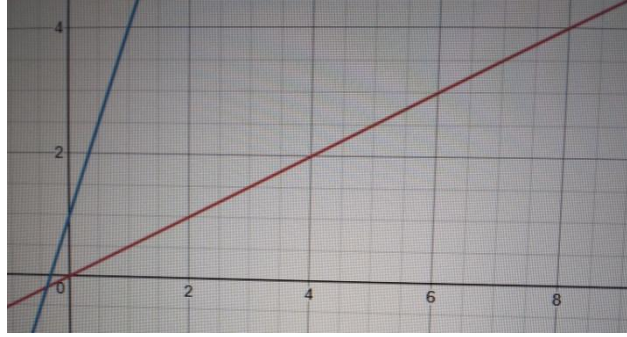
$$\nexists n \in N_+ \text{ makes } lS(n) = \{x\}, \{x, y\}, \{x, y, z, t\}, \dots, \text{ or } \{x, \dots, u\} \text{ when } r \in N_+, \& \left(\frac{r}{3}\right) \notin N_+$$

$$\therefore r \in N_+, \& N_+ = (N_{\frac{r}{3} \in N_+}) \cup (N_{\frac{r}{3} \notin N_+}) = [(N_{\frac{r}{3} \in N_+}) \cup \{\}] = N_{\frac{r}{3} \in N_+}$$

$$\therefore (N_+ = N_{\frac{r}{3} \in N_+}) \dots \text{under Collatz rules only} \Rightarrow n \in N_+ = n \in N_{\frac{r}{3} \in N_+}$$

$$\therefore S(n) = \{\frac{n}{2} \text{ or } 3n + 1, \dots, 4, 2, 1\} \Rightarrow lS(n) = \{1, 2, 4\}, \forall n \in N_+$$

Graph of Collatz Sequence to find $S(n)$ and $LS(n)$, $\forall n \in N_+$



$$y_1 = n/2, n \in N_{\text{even}}, \& n \in x - \text{axis}$$

$$y_2 = 3n + 1, n \in N_{\text{odd}}, \& n \in x - \text{axis}$$

$$4 \text{ by } y_1 \rightarrow 2 \text{ by } y_1 \rightarrow 1 \text{ by } y_2 \rightarrow 4$$

$$\therefore LS(n) = \{4, 2, 1\}, \forall n \in N_+$$

Engineering Collatz Sequence Solution 1st Way By Using Loops

$r = \text{number of elements in } LS(n), n, r \in N_+$

$$S(n) = \left\{ \frac{n}{2} \text{ or } (3n + 1), \dots, x \right\} \Rightarrow LS(n) \neq \{x\}, \forall n \in N_+, r = 1$$

$$S(n) = \left\{ \frac{n}{2} \text{ or } (3n + 1), \dots, x, y \right\} \Rightarrow LS(n) \neq \{x, y\}, \forall n \in N_+, r = 2$$

$$S(n) = \left\{ \frac{n}{2} \text{ or } (3n + 1), \dots, x, y, z \right\} \Rightarrow LS(n) = \{x, y, z\} = \{4, 2, 1\} \forall n \in N_+, r = 3$$

$$S(n) = \left\{ \frac{n}{2} \text{ or } (3n + 1), \dots, x, y, z, t \right\} \Rightarrow LS(n) \neq \{x, y, z, t\}, \forall n \in N_+, r = 4$$

$$S(n) = \left\{ \frac{n}{2} \text{ or } (3n + 1), \dots, x, y, z, t, u \right\} \Rightarrow LS(n) \neq \{x, y, z, t, u\}, \forall n \in N_+, r = 5$$

$$S(n) = \left\{ \frac{n}{2} \text{ or } (3n + 1), \dots, x, y, z, t, u, v \right\} \Rightarrow LS(n) = \{4, 2, 1, 4, 2, 1\} = \{4, 2, 1\}, \forall n \in N_+, r = 6$$

$$S(n) = \left\{ \frac{n}{2} \text{ or } (3n + 1), \dots, x, y, z, t, u, v, g \right\} \Rightarrow LS(n) \neq \{x, y, z, t, u, v, g\}, \forall n \in N_+, r = 7$$

$$\left(\frac{r}{3} \right) \in N_+ \Rightarrow (r = 3k) \Rightarrow LS(n) = \{4, 2, 1\}, \forall n \in N_+, \& r \in N_{3k}, k \in N_+$$

$$\left(\frac{r}{3} \right) \notin N_+ \Rightarrow (r = 3k + h, \& h \in \{1, 2\}) \Rightarrow LS(n) \text{ does not exist}, \forall n \in N_+, \& r \in N_{3k+h}$$

$$\therefore N_+ = N_{r=3k} \cup N_{r=3k+h}$$

$$\therefore S(n) = \left\{ \frac{n}{2} \text{ or } (3n + 1), \dots, 4, 2, 1 \right\} \Rightarrow LS(n) = \{4, 2, 1\} \text{ the only loop } \forall n \in N_+$$

$$S = \sum_{i=0}^{k-1} 3^{k-1-i} 2^{2i}, \text{ \& } x = \frac{3^k x + S}{2^{r-k}}$$

Proof

1 – *Geometric Series Approach:*

$$\therefore 2^{2i} = 4^i$$

$$\therefore S = \sum_{i=0}^{k-1} 3^{k-1-i} 2^{2i} = \sum_{i=0}^{k-1} 3^{k-1-i} 4^i$$

$$S = \sum_{i=0}^{k-1} 3^{k-1} \left(\frac{4}{3}\right)^i = 3^{k-1} \sum_{i=0}^{k-1} \left(\frac{4}{3}\right)^i \dots (\text{by common factor})$$

$$S = 3^{k-1} \left[\frac{\left(\frac{4}{3}\right)^k - 1}{\frac{4}{3} - 1} \right] = 3^{k-1} \left[\frac{\left(\frac{4}{3}\right)^k - 1}{\frac{1}{3}} \right] = 3^k \left[\left(\frac{4}{3}\right)^k - 1 \right] = 4^k - 3^k = 2^{2k} - 3^k$$

$$\therefore S = 2^{2k} - 3^k$$

2 – *Induction Approach:*

$$\therefore S = \sum_{i=0}^{k-1} 3^{k-1-i} 2^{2i} \Rightarrow$$

$$S = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \dots + 3^{k-k+1}(2^{2(k-2)}) + (3^0)2^{2(k-1)}$$

$$\therefore S = 3^{k-1} + 3^{k-2}(4) + 3^{k-3}(4^2) + \dots + 3(4^{k-2}) + 4^{(k-1)}$$

$$\therefore 3S = 3^k + 3^{k-1}(4) + 3^{k-2}(4^2) + \dots + 3^2(4^{k-2}) + 3(4)^{(k-1)}$$

$$\text{ \& } 4S = 3^{k-1}(4) + 3^{k-2}(4^2) + 3^{k-3}(4^3) + \dots + 3(4^{k-1}) + 4^k]$$

$$\text{Now } 4S - 3S = 4^k - 3^k \Rightarrow S = 4^k - 3^k = 2^{2k} - 3^k$$