On Ramanujan zeta function

©Payam Danesh

Department of Biosystems Engineering, University of Tehran, Iran

Email: Payamdanesh71@gmail.com

Abstract

In number theory, the Ramanujan zeta function is a classical object and its special values bring together Fourier coefficients, modular transformations, elliptic integrals, and arithmetic periods. In this paper, we give a clear and rigorous way to study these values and to compute them with guaranteed error bounds. We begin with the discriminant modular form, use its Mellin transform, and then apply the classical elliptic modular parameter to obtain explicit one-dimensional period integrals in the critical range. We also give an exact method for generating the Ramanujan tau coefficients and a certified interval algorithm for numerical computation. The main results are a recurrence for the tau coefficients, a practical tail estimate for the Dirichlet series, and a precision theorem that gives the number of coefficients needed for any prescribed decimal accuracy. The numerical examples show that the method is stable and effective. The main advantage of the approach is that the symbolic formulas and the numerical approximations are kept separate, so each step can be checked directly and rigorously.

Keywords: Ramanujan zeta function; Ramanujan tau function; discriminant modular form; Mellin transform

1. Introduction

Mr. Ramanujan introduced the arithmetic function now denoted by $\tau(n)$ through the Fourier expansion of the discriminant modular form in his work on arithmetical functions [1]. The same Fourier expansion defines one of the most important cusp forms in the theory of modular forms. Its coefficients display deep multiplicative structure, and the associated Dirichlet series gives the Ramanujan zeta function. Mordell proved the Euler product and functional equation for this Dirichlet series [2]. These results placed Ramanujan's observations inside the analytic theory of modular forms. In modern math, the discriminant modular form is the normalized cusp form of weight twelve and level one. Its Mellin transform gives the corresponding modular L-function, and its modular transformation law gives the completed functional equation [3]. The size of the Fourier coefficients is controlled by Deligne's proof of the Ramanujan–Petersson estimate [4]. This estimate is essential for rigorous computation because it gives an explicit bound for the tail of the Dirichlet series. It allows finite sums to be converted into certified numerical intervals rather than unsupported approximations.

The critical special values of modular L-functions also have a period interpretation. For the Ramanujan zeta function, the classical elliptic modular parameter converts the Mellin integral into hypergeometric elliptic integrals in the critical range. The elliptic identities used in this work belong to the standard theory of Ramanujan's modular formulae and complete elliptic integrals [5,6]. This fits naturally with the general period viewpoint of Kontsevich and Zagier [7] and with the deeper motivic framework of Beilinson, Deninger, and Scholl [8,9].

The aim of this study is to give a complete method that connects three layers of the same object: 1-The modular product, 2-The period integral, and 3-The certified numerical value. This paper proves the Mellin representation, derives the critical period formula, gives an exact recurrence for the tau coefficients, proves a usable tail bound, and formulates an interval algorithm with a guaranteed decimal precision. The main contribution is the certified interval theorem. Given an integer $k \geq 8$ and a requested decimal precision, the theorem gives an explicit coefficient cutoff and returns a closed interval containing $L(\Delta, k)$. The interval is produced from exact integer coefficient generation and a rigorous analytic error estimate. The numerical method is therefore reproducible and independent of experimental cancellation.

2. Normalization and classical inputs

For start, let z lie in the complex upper half-plane and define

$$q = \exp(2\pi i z). \qquad (1)$$

The discriminant modular form is

$$\Delta(z) = q \prod_{m=1}^{\infty} (1 - q^m)^{24}. \qquad (2)$$

Ramanujan's tau coefficients are defined by

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)\, q^n. \qquad (3)$$

The Ramanujan zeta function attached to $\Delta$is

$$L(\Delta, s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}. \qquad (4)$$

For large real part of s, the series is absolutely convergent. Its completed form is

$$\Lambda(\Delta, s) = (2\pi)^{-s}\Gamma(s)L(\Delta, s). \qquad (5)$$

The functional equation is centered at $s = 6$, reflecting the weight twelve of Δ.

The elliptic reduction uses the hypergeometric function

$$F(\alpha) = F_{2.1}\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right), 0 < \alpha < 1. \qquad (6)$$

We define

$$u(\alpha) = \frac{F(1-\alpha)}{2F(\alpha)}. \qquad (7)$$

The classical derivative formula is

$$\frac{du}{d\alpha} = -\frac{1}{2\pi\alpha(1-\alpha)F(\alpha)^2}. \qquad (8)$$

The corresponding discriminant identity is

$$\Delta(iu(\alpha)) = \frac{1}{16}\alpha(1-\alpha)^4 F(\alpha)^{12}. \qquad (9)$$

Equations (7)–(9) are standard consequences of the elliptic modular lambda parameter and the theta-function expression for the discriminant form. They are used as classical inputs, with the normalization matching the choice $q = \exp(2\pi iz)$.

3. Mellin transform and critical period formula

The Mellin transform gives the analytic bridge between the modular form and its Dirichlet series.

Theorem 1.
For every positive integer k,

$$L(\Delta,k) = \frac{(2\pi)^k}{\Gamma(k)}\int_0^\infty u^{k-1}\,\Delta(iu)\,du. \qquad (10)$$

Proof of Theorem 1.
For sufficiently large real part of s, substitute the Fourier expansion of Δ into the Mellin integral along the positive imaginary axis. Since $q = \exp(-2\pi u)$, absolute convergence permits termwise integration. The elementary identity

$$\int_0^\infty u^{s-1}\,e^{-2\pi nu}\,du = \frac{\Gamma(s)}{(2\pi n)^s} \qquad (11)$$

gives the claimed formula in the region of absolute convergence.

The modular transformation law of Δ gives exponential decay at both endpoints of the positive real axis after the standard change $u \mapsto 1/u$ near zero. Hence the Mellin integral defines the analytic continuation of the same completed L-function. Evaluating at positive integers gives the result.

The next theorem gives the critical period formula.

Theorem 2.
For $1 \le k \le 11$,

$$L(\Delta,k) = \frac{\pi^{k-1}}{16\Gamma(k)}\int_0^1 (1-\alpha)^3 F(\alpha)^{11-k} F(1-\alpha)^{k-1}\,d\alpha. \qquad (12)$$

Proof of Theorem 2.
Insert the discriminant identity (9) into the Mellin formula (10). Use the substitution $u = u(\alpha)$. The derivative is given by (8), and the negative sign is removed by reversing the integration limits. The power

u^{k-1}contributes $F(1-\alpha)^{k-1}$and $2^{1-k}F(\alpha)^{1-k}$. The factor $\Delta(iu(\alpha))$contributes $\alpha(1-\alpha)^4F(\alpha)^{12}/16$. The Jacobian contributes $1/(2\pi\alpha(1-\alpha)F(\alpha)^2)$. After cancellation, the remaining integrand and constant are exactly those in (12).

The condition $1 \le k \le 11$ensures that the power of $F(\alpha)$in the integrand is nonnegative. This is the direct critical range of the Mellin-period method.

4. Exact generation of the tau coefficients

Certified computation requires exact coefficient generation. The product defining Δgives such a recurrence.

Let

$$\sigma_1(n) = \sum_{d|n} d. \qquad (13)$$

Theorem 3.
The coefficients $\tau(n)$satisfy

$$\tau(1) = 1, \tau(n) = -\frac{24}{n-1}\sum_{m=1}^{n-1} \sigma_1(m)\tau(n-m), n \ge 2. \qquad (14)$$

Proof of Theorem 3.
Take the logarithmic derivative of the product in (2) with respect to q. This gives

$$\frac{q\Delta'(q)}{\Delta(q)} = 1 - 24\sum_{m=1}^{\infty} \frac{mq^m}{1-q^m}. \qquad (15)$$

The divisor-sum expansion is

$$\sum_{m=1}^{\infty} \frac{mq^m}{1-q^m} = \sum_{n=1}^{\infty} \sigma_1(n)q^n. \qquad (16)$$

Now multiply the right-hand side of (15) by the Fourier expansion in (3). The coefficient of q^non the left is $n\tau(n)$. The coefficient of q^non the right is

$$\tau(n) - 24\sum_{m=1}^{n-1} \sigma_1(m)\tau(n-m). \qquad (17)$$

Equating the two coefficients and solving for $\tau(n)$proves (14).

The recurrence is exact over the integers. It is therefore suitable for rigorous interval computation.

5. Dirichlet-series tail estimate

The next result supplies the analytic error bound for truncating the Ramanujan zeta function.

Theorem 4.
Let

$$S_N(k) = \sum_{n=1}^{N} \frac{\tau(n)}{n^k}. \qquad (18)$$

For every integer $k \geq 8$,

$$| L(\Delta, k) - S_N(k) | \leq \frac{2}{k-7} N^{7-k}. \qquad (19)$$

Proof of Theorem 4.
Deligne's estimate gives

$$| \tau(n) | \leq d(n) n^{11/2}, \qquad (20)$$

where $d(n)$is the divisor-counting function. The elementary bound $d(n) \leq 2\sqrt{n}$gives

$$| \tau(n) | \leq 2n^6. \qquad (21)$$

Thus the absolute value of the tail after N is bounded by

$$2 \sum_{n=N+1}^{\infty} n^{6-k}. \qquad (22)$$

For $k \geq 8$, the function $x^{6-k}$is positive and decreasing on the positive real axis. The standard integral comparison therefore gives

$$\sum_{n=N+1}^{\infty} n^{6-k} \leq \int_N^{\infty} x^{6-k} \, dx.$$

Since $k > 7$, the integral converges, and direct evaluation gives

$$\int_N^{\infty} x^{6-k} \, dx = \frac{N^{7-k}}{k-7}.$$

It follows that

$$| L(\Delta, k) - S_N(k) | \leq \frac{2}{k-7} N^{7-k}.$$

This proves the claimed tail estimate. **Figure 1** gives a concrete view of the coefficient estimate used in the proof. The argument only needs the uniform bound

$$|\tau(n)| \leq 2n^6,$$

because this bound is enough to control the entire remaining Dirichlet tail. The figure plots the normalized coefficients

$$\frac{\tau(n)}{n^6}, 1 \leq n \leq 200,$$

together with the two horizontal lines corresponding to the envelope

$$-2 \leq \frac{\tau(n)}{n^6} \leq 2.$$

The plotted values remain well inside this certified range. For $2 \leq n \leq 200$, the largest observed absolute normalized value is

$$\left|\frac{\tau(2)}{2^6}\right| = \frac{24}{64} = 0.375,$$

and the root-mean-square normalized size over the same range is approximately

$$9.60 \times 10^{-2}.$$

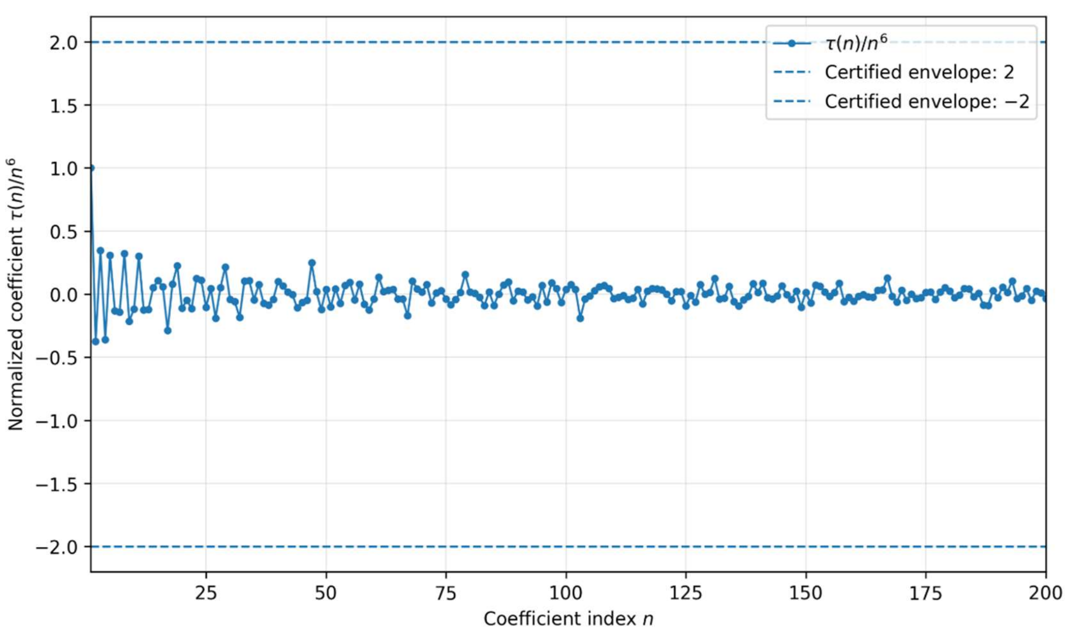

Figure 1. Normalized Ramanujan tau coefficients

https://doi.org/10.33774/coe-2026-cvm8q Content not peer-reviewed by Cambridge University Press.

This comparison explains why the estimate in Theorem 4 is deliberately conservative. It is not meant to describe the typical size of the tau coefficients. Its purpose gives a clean unconditional certificate for the full tail of the Ramanujan zeta series. The oscillation visible in the graph reflects the arithmetic variation of Ramanujan's tau function, while the width of the envelope shows why the proof does not rely on cancellation among the coefficients.

6. Interval algorithm

In this section, we give the main computational theorem.

For integers $k \geq 8$ and $D \geq 1$, define

$$N_D(k) = \left\lceil \left(\frac{4 \cdot 10^D}{k-7} \right)^{1/(k-7)} \right\rceil. \qquad (23)$$

Algorithm 1. Certified interval evaluation of $L(\Delta, k)$**.**

First input: integers $k \geq 8$and $D \geq 1$.

1. And compute $\sigma_1(n)$for $1 \leq n \leq N_D(k)$by a divisor sieve.
2. Then compute $\tau(n)$for $1 \leq n \leq N_D(k)$using Theorem 3.
3. At the end, compute $S_{N_D(k)}(k)$ using exact rational arithmetic or outward-rounded interval arithmetic.
4. Set

$$R_D(k) = \frac{2}{k-7} N_D(k)^{7-k}. \qquad (24)$$

5. Return the interval

$$\mathcal{I}_D(k) = \left[S_{N_D(k)}(k) - R_D(k), S_{N_D(k)}(k) + R_D(k) \right], \qquad (25)$$

enlarged by the rounding interval from Step 3 when floating-point interval arithmetic is used.

Theorem 5.
Algorithm 1 returns a closed interval containing $L(\Delta, k)$. With exact rational summation, the interval length is at most 10^{-D}. With outward-rounded interval summation of radius at most $10^{-D}/4$, the interval length is at most $3 \cdot 10^{-D}/2$.

Proof of Theorem 5.
Theorem 3 gives exact coefficients. Theorem 4 gives the analytic tail bound $R_D(k)$. By the definition of $N_D(k)$,

$$N_D(k)^{k-7} \geq \frac{4 \cdot 10^D}{k-7}. \qquad (26)$$

Therefore,

$$R_D(k) \le \frac{1}{2 \cdot 10^D}. \qquad (27)$$

The interval centered at the exact finite sum with radius $R_D(k)$ contains the true value and has length at most 10^{-D}. If the finite sum is itself computed inside an interval of radius at most $10^{-D}/4$, the full interval length increases by at most $10^{-D}/2$ which gives the stated bound.

Proposition 6.
Let $N = N_D(k)$. The divisor-sum table in Algorithm 1 requires $O(N\log\, N)$integer additions. The coefficient recurrence requires $O(N^2)$integer arithmetic operations.

Proof of Proposition 6.
The divisor sieve adds each positive integer $d \le N$to its multiples. The number of additions is

$$\sum_{d=1}^{N} \left\lfloor \frac{N}{d} \right\rfloor, \qquad (28)$$

which is $O(N\log\, N)$. The recurrence for $\tau(n)$uses a sum of length $n-1$. Summing these lengths from $n = 2$ to Ngives $N(N-1)/2$. This proves the stated complexity. Theorem 5 is the certification result which gives a precise cutoff and a guaranteed decimal enclosure.

7. Discussion

In this work, we established a certified Mellin-period method for the Ramanujan zeta function. The Mellin transform gives the direct bridge from the discriminant modular form to its Dirichlet series. In the critical range, the classical elliptic parameter converts that integral into a 1-dimensional hypergeometric period formula. For computation, the tau coefficients are generated by an exact recurrence, and the tail of the Dirichlet series is bounded by an explicit theorem. For every $k \ge 8$ and every requested decimal precision, the algorithm gives a concrete coefficient cutoff and returns a closed interval containing $L(\Delta, k)$. The proof uses exact coefficient generation, Deligne's coefficient estimate, and elementary integral comparison. No unverified cancellation is used. The resulting framework gives both a clean period interpretation in the critical range and a rigorous numerical method for certified evaluation.

Dedication

I have long been personally amazed by the works of **Srinivasa Ramanujan** and this paper is dedicated to him, with deep admiration for the extraordinary insight that continues to shape modern number theory. His work on the tau function revealed a hidden depth in arithmetic that still guides mathematicians more than a century later. May his memory remain blessed, and may his mathematics continue to inspire those who seek beauty, truth, and structure in numbers.

References

[1] S. Ramanujan, “On certain arithmetical functions,” *Transactions of the Cambridge Philosophical Society*, vol. 22, no. 9, pp. 159–184, 1916.

[2] L. J. Mordell, “On Mr. Ramanujan’s empirical expansions of modular functions,” *Proceedings of the Cambridge Philosophical Society*, vol. 19, pp. 117–124, 1917.

[3] J.-P. Serre, *A Course in Arithmetic*, Graduate Texts in Mathematics, vol. 7, Springer, New York, 1973. DOI: 10.1007/978-1-4684-9884-4.

[4] P. Deligne, “La conjecture de Weil. I,” *Publications Mathématiques de l’Institut des Hautes Études Scientifiques*, vol. 43, pp. 273–307, 1974. DOI: 10.1007/BF02684373.

[5] B. C. Berndt and K. Ono, “Ramanujan’s unpublished manuscript on the partition and tau functions with proofs and commentary,” *Séminaire Lotharingien de Combinatoire*, vol. 42, Article B42c, 63 pp., 1999.

[6] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 5th ed., edited by V. H. Moll, Cambridge University Press, Cambridge, 2021. DOI: 10.1017/9781009004091.

[7] M. Kontsevich and D. Zagier, “Periods,” in *Mathematics Unlimited — 2001 and Beyond*, B. Engquist and W. Schmid, Eds., Springer, Berlin, pp. 771–808, 2001. DOI: 10.1007/978-3-642-56478-9_39.

[8] A. A. Beilinson, “Higher regulators of modular curves,” in *Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory*, Contemporary Mathematics, vol. 55, American Mathematical Society, Providence, RI, pp. 1–34, 1986. DOI: 10.1090/conm/055.1/862627.

[9] C. Deninger and A. J. Scholl, “The Beilinson conjectures,” in *L-Functions and Arithmetic*, J. Coates and M. J. Taylor, Eds., London Mathematical Society Lecture Note Series, vol. 153, Cambridge University Press, Cambridge, pp. 173–210, 1991. DOI: 10.1017/CBO9780511526053.007.

[10] R. E. Moore, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1966.

With best regards,