Solutions to exercises

Exercise 1.1 A 'stationary' particle in any laboratory on the Earth is actually subject to gravitational forces due to the Earth and the Sun. These help to ensure that the particle moves with the laboratory. If steps were taken to counterbalance these forces so that the particle was really not subject to any net force, then the rotation of the Earth and the Earth's orbital motion around the Sun would carry the laboratory away from the particle, causing the force-free particle to follow a curving path through the laboratory. This would clearly show that the particle did not have constant velocity in the laboratory (i.e. constant speed in a fixed direction) and hence that a frame fixed in the laboratory is not an inertial frame. More realistically, an experiment performed using the kind of long, freely suspended pendulum known as a *Foucault pendulum* could reveal the fact that a frame fixed on the Earth is rotating and therefore cannot be an inertial frame of reference. An even more practical demonstration is provided by the winds, which do not flow directly from areas of high pressure to areas of low pressure because of the Earth's rotation.

Exercise 1.2 The Lorentz factor is $\gamma(V) = 1/\sqrt{1 - V^2/c^2}$.

(a) If $V = 0.1c$, then

$$\gamma = \frac{1}{\sqrt{1-(0.1c)^2/c^2}} = 1.01 \quad \text{(to 3 s.f.)}.$$

(b) If $V = 0.9c$, then

$$\gamma = \frac{1}{\sqrt{1-(0.9c)^2/c^2}} = 2.29 \quad \text{(to 3 s.f.)}.$$

Note that it is often convenient to write speeds in terms of c instead of writing the values in $\mathrm{m\,s^{-1}}$, because of the cancellation between factors of c.

Exercise 1.3 The inverse of a 2×2 matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is

$$M^{-1} = \frac{1}{AD - BC}\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}.$$

Taking $A = \gamma(V)$, $B = -\gamma(V)V/c$, $C = -\gamma(V)V/c$ and $D = \gamma(V)$, and noting that $AD - BC = [\gamma(V)]^2(1 - V^2/c^2) = 1$, we have

$$[\Lambda]^{-1} = \begin{pmatrix} \gamma(V) & +\gamma(V)V/c \\ +\gamma(V)V/c & \gamma(V) \end{pmatrix}.$$

This is the correct form of the inverse Lorentz transformation matrix.

Exercise 1.4 First compute the Lorentz factor:

$$\begin{aligned}\gamma(V) &= 1/\sqrt{1 - V^2/c^2} \\ &= 1/\sqrt{1 - 9/25} = 1/\sqrt{16/25} = 5/4.\end{aligned}$$

Thus the measured lifetime is $\Delta T = 5 \times 2.2/4\,\mu\mathrm{s} = 2.8\,\mu\mathrm{s}$. Note that not all muons live for the same time; rather, they have a range of lifetimes. But a large group of muons travelling with a common speed does have a well-defined *mean lifetime*, and it is the dilation of this quantity that is easily demonstrated experimentally.

Exercise 1.5 The alternative definition of length can't be used in the rest frame of the rod as the rod does not move in its own rest frame. The proper length is therefore defined as before and related to the positions of the two events as observed in the rest frame. (This works, because event 1 and event 2 still occur at the end-points of the rod and the rod never moves in the rest frame S′.)

As before, it is helpful to write down all the intervals that are known in a table.

Event	S (laboratory)	S′ (rest frame)
2	$(t_2, 0)$	(t'_2, x'_2)
1	$(t_1, 0)$	(t'_1, x'_1)
Intervals	$(t_2 - t_1, 0)$ $\equiv (\Delta t, \Delta x)$	$(t'_2 - t'_1, x'_2 - x'_1)$ $\equiv (\Delta t', \Delta x')$
Relation to intervals	$(L/V, 0)$	$(?, L_\mathrm{P})$

By examining the intervals, it can be seen that Δx, Δt and $\Delta x'$ are known. From the interval transformation rules, only Equation 1.33 relates the three known intervals. Substituting the known intervals into that equation gives $L_\mathrm{P} = \gamma(V)(0 - V(L/V))$. In this way, length contraction is predicted as before:

$$L = L_\mathrm{P}/\gamma(V).$$

Exercise 1.6 The received wavelength is less than the emitted wavelength. This means that the jet is approaching. We can therefore use Equation 1.42 provided that we change the sign of V. Combining it with the formula $f\lambda = c$ shows that $\lambda' = \lambda\sqrt{(c - V)/(c + V)}$. Squaring both sides and rearranging gives

$$(\lambda'/\lambda)^2 = (c - V)/(c + V).$$

From this it follows that

$$(\lambda'/\lambda)^2(c + V) = (c - V),$$

so

$$V(1 + (\lambda'/\lambda)^2) = c(1 - (\lambda'/\lambda)^2),$$

thus

$$V = c(1 - (\lambda'/\lambda)^2)/(1 + (\lambda'/\lambda)^2).$$

Substituting $\lambda' = 4483 \times 10^{-10}$ m and $\lambda = 5850 \times 10^{-10}$ m, the speed is found to be $v = 0.26c$ (to 2 s.f.).

Exercise 1.7 Let the spacestation be the origin of frame S, and the nearer of the spacecraft the origin of frame S′, which therefore moves with speed $V = c/2$ as measured in S. Let these two frames be in standard configuration. The velocity of the further of the two spacecraft, as observed in S, is then $\boldsymbol{v} = (3c/4, 0, 0)$. It follows from the velocity transformation that the velocity of the further spacecraft as observed from the nearer will be $\boldsymbol{v}' = (v'_x, 0, 0)$, where

$$v'_x = \frac{v_x - V}{1 - v_x V/c^2} = \frac{3c/4 - c/2}{1 - (3c/4)(c/2)/c^2} = 2c/5.$$

Exercise 1.8 $\Delta x = (5 - 7)\,\mathrm{m} = -2\,\mathrm{m}$ and $c\,\Delta t = (5 - 3)\,\mathrm{m} = 2\,\mathrm{m}$. Since the spacetime separation is $(\Delta s)^2 = (c\,\Delta t)^2 - (\Delta x)^2$ in this case, it follows that

$(\Delta s)^2 = (2\,\mathrm{m})^2 - (2\,\mathrm{m})^2 = 0$. The value $(\Delta s)^2 = 0$ is permitted; it describes situations in which the two events could be linked by a light signal. In fact, any such separation is said to be *light-like*.

Exercise 1.9 Start with $(\Delta s')^2 = (c\,\Delta t')^2 - (\Delta x')^2$. The aim is to show that $(\Delta s')^2 = (\Delta s)^2$.

Substitute $\Delta x' = \gamma(\Delta x - V\,\Delta t)$ and $c\,\Delta t' = \gamma(c\,\Delta t - V\,\Delta x/c)$ so that

$$\begin{aligned}(\Delta s')^2 &= \gamma^2\left(c^2(\Delta t)^2 - 2V\,\Delta x\Delta t + V^2(\Delta x)^2/c^2\right)\\ &\quad - \gamma^2\left((\Delta x)^2 - 2V\,\Delta x\Delta t + V^2(\Delta t)^2\right).\end{aligned}$$

Cross terms involving $\Delta x\,\Delta t$ cancel. Collecting common terms in $c^2(\Delta t)^2$ and $(\Delta x)^2$ gives

$$(\Delta s')^2 = \gamma^2 c^2(\Delta t)^2(1 - V^2/c^2) - \gamma^2(\Delta x)^2(1 - V^2/c^2).$$

Finally, noting that $\gamma^2 = [1 - V^2/c^2]^{-1}$, there is a cancellation of terms, giving

$$(\Delta s')^2 = c^2(\Delta t)^2 - (\Delta x)^2 = (\Delta s)^2,$$

thus showing that $(\Delta s')^2 = (\Delta s)^2$.

Exercise 1.10 Since $(\Delta s)^2 = (c\,\Delta t)^2 - (\Delta l)^2$, and $(\Delta s)^2$ is invariant, it follows that all inertial observers will find $(c\,\Delta t)^2 = (\Delta s)^2 + (\Delta l)^2$, where $(\Delta l)^2$ cannot be negative. Since $(\Delta l)^2 = 0$ in the frame in which the proper time is measured, it follows that no other inertial observer can find a smaller value for the time between the events.

Exercise 1.11 In Terra's frame, Stella's ship has velocity $(v_x, v_y, v_z) = (-V, 0, 0)$. It follows from the velocity transformation that in Astra's frame, the velocity of Stella's ship will be $(v'_x, 0, 0)$, where $v'_x = (v_x - V)/(1 - v_x V/c^2)$. Taking $v_x = -V$ gives

$$v'_x = \frac{(-V - V)}{(1 - (-V)V/c^2)} = \frac{-2V}{1 + V^2/c^2}.$$

Taking the magnitude of this single non-zero velocity component gives the speed of approach, $2V/(1 + V^2/c^2)$, as required.

Exercise 1.12 In Terra's frame, the signals would have an emitted frequency $f_{\mathrm{em}} = 1\,\mathrm{Hz}$. In Astra's frame, the Doppler effect tells us that the signals would be received with a different frequency f_{rec}. On the outward leg of the journey, the signals would be redshifted and the received frequency would be

$$f_{\mathrm{rec}} = f_{\mathrm{em}}\sqrt{(c - V)/(c + V)}.$$

On the return leg of the journey, the signals would be blueshifted and the received frequency would be

$$f_{\mathrm{rec}} = f_{\mathrm{em}}\sqrt{(c + V)/(c - V)}.$$

Exercise 2.1 The Lorentz factor is

$$\gamma = 1/\sqrt{1 - v^2/c^2} = 1/\sqrt{1 - 16c^2/25c^2} = 1/\sqrt{9/25} = 5/3.$$

The electron has mass $m = 9.11 \times 10^{-31}$ kg. Thus the magnitude of the electron's momentum is

$$p = 5/3\times 4c/5\times m = (5/3)\times(4\times 3.00\times 10^{8}\,\mathrm{m\,s^{-1}}/5)\times 9.11\times 10^{-31}\,\mathrm{kg} = 3.6\times 10^{-22}\,\mathrm{kg\,m\,s^{-1}}.$$

Exercise 2.2 The kinetic energy is $E_{\mathrm{K}} = (\gamma - 1)mc^2$. Taking the speed to be $9c/10$, the Lorentz factor is

$$\gamma = 1/\sqrt{1 - v^2/c^2} = 1/\sqrt{1 - (9/10)^2} = 2.29.$$

Noting that $m = 1.88 \times 10^{-28}$ kg, the kinetic energy is

$$E_{\mathrm{K}} = (2.29 - 1) \times 1.88 \times 10^{-28}\,\mathrm{kg} \times (3.00 \times 10^{8}\,\mathrm{m\,s^{-1}})^2 = 2.2 \times 10^{-11}\,\mathrm{J}.$$

Exercise 2.3 $v = 3c/5$ corresponds to a Lorentz factor

$$\gamma(v) = 1/\sqrt{1 - v^2/c^2} = 1/\sqrt{1 - 9/25} = 5/4.$$

The proton has mass $m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg, therefore the total energy is

$$E = \gamma(v)mc^2 = (5/4)\times 1.67\times 10^{-27}\,\mathrm{kg}\times(3.00\times 10^{8}\,\mathrm{m\,s^{-1}})^2 = 1.88\times 10^{-10}\,\mathrm{J}.$$

Exercise 2.4 Since the total energy is $E = \gamma mc^2$, it is clear that the total energy is twice the mass energy when $\gamma = 2$. This means that $2 = 1/\sqrt{1 - v^2/c^2}$. Squaring and inverting both sides, $1/4 = 1 - v^2/c^2$, so $v^2/c^2 = 3/4$. Taking the positive square root, $v/c = \sqrt{3}/2$.

Exercise 2.5 (a) The energy difference is $\Delta E = \Delta m\, c^2$, where $\Delta m = 3.08 \times 10^{-28}$ kg. Thus

$$\Delta E = 3.08 \times 10^{-28}\,\mathrm{kg} \times (3.00 \times 10^{8}\,\mathrm{m\,s^{-1}})^2 = 2.77 \times 10^{-11}\,\mathrm{J}.$$

Converting to electronvolts, this is

$$2.77 \times 10^{-11}\,\mathrm{J}/1.60 \times 10^{-19}\,\mathrm{J\,eV^{-1}} = 1.73 \times 10^{8}\,\mathrm{eV} = 173\,\mathrm{MeV}.$$

(b) From $\Delta E = \Delta m\, c^2$, the mass difference is $\Delta m = \Delta E/c^2$. Now, $\Delta E = 13.6$ eV or, converting to joules,

$$\Delta E = 13.6\,\mathrm{eV} \times 1.60 \times 10^{-19}\,\mathrm{J\,eV^{-1}} = 2.18 \times 10^{-18}\,\mathrm{J}.$$

Therefore

$$\Delta m = 2.18 \times 10^{-18}\,\mathrm{J}/(3.00 \times 10^{8}\,\mathrm{m\,s^{-1}})^2 = 2.42 \times 10^{-35}\,\mathrm{kg}.$$

Note that the masses of the electron and proton are 9.11×10^{-31} kg and 1.67×10^{-27} kg, respectively, so the mass difference from chemical binding is small enough to be negligible in most cases. However, mass–energy equivalence is not unique to nuclear reactions.

Exercise 2.6 The transformations are $E' = \gamma(V)(E - Vp_x)$ and $p'_x = \gamma(V)(p_x - VE/c^2)$. In this case, $E = 3m_{\mathrm{e}}c^2$ and $p_x = \sqrt{8}m_{\mathrm{e}}c^2$. For relative speed $V = 4c/5$ between the two frames, the Lorentz factor is $\gamma = 1/\sqrt{1 - (4/5)^2} = 5/3$. Substituting the values,

$$E' = 5/3(3m_{\mathrm{e}}c^2 - 4c/5 \times \sqrt{8}m_{\mathrm{e}}c) = 1.23m_{\mathrm{e}}c^2$$

and

$$p' = 5/3(\sqrt{8}m_{\rm e}c - 4c/5 \times 3m_{\rm e}c^2/c^2) = 0.714m_{\rm e}c.$$

Exercise 2.7 (a) For a photon $m = 0$, so

$$p = E/c = hf/c = \frac{6.63 \times 10^{-34}\,\mathrm{J\,s} \times 5.00 \times 10^{14}\,\mathrm{s^{-1}}}{3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}} = 1.11 \times 10^{-27}\,\mathrm{kg\,m\,s^{-1}}.$$

(b) Using the Newtonian relation that the force is equal to the rate of change of momentum (we shall have more to say about this later), the magnitude of the force on the sail will be $F = np$, where n is the rate at which photons are absorbed by the sail (number of photons per second). Thus

$$n = F/p = 10\,\mathrm{N}/1.11 \times 10^{-27}\,\mathrm{kg\,m\,s^{-1}} = 9.0 \times 10^{27}\,\mathrm{s^{-1}}.$$

Exercise 2.8 To be a valid energy/momentum combination, the energy–momentum relation must be satisfied, i.e. $E_{\rm f}^2 - p_{\rm f}^2c^2 = m_{\rm f}^2c^4$. For the given values of energy and momentum,

$$E_{\rm f}^2 - p_{\rm f}^2c^2 = 9m_{\rm f}^2c^4 - 49m_{\rm f}^2c^4 = -40m_{\rm f}^2c^4 \neq m_{\rm f}^2c^4.$$

So they are not valid values.

Exercise 2.9 It follows directly from the transformation rules for the last three components of the four-force F^μ that

$$\begin{aligned}
\gamma(v')f'_x &= \gamma(V)\left[\gamma(v)f_x - V\gamma(v)\boldsymbol{f}\cdot\boldsymbol{v}/c^2\right],\\
\gamma(v')f'_y &= \gamma(v)f_y,\\
\gamma(v')f'_z &= \gamma(v)f_z.
\end{aligned}$$

Note that the transformation of f_x involves both the speed of the particle v as measured in frame S and the speed V of frame S$'$ as measured in frame S. Both $\gamma(v)$ and $\gamma(V)$ appear in the transformation.

Exercise 2.10 Since the four-vector is contravariant, it transforms just like the four-displacement. Thus

$$\begin{aligned}
c\rho' &= \gamma(V)(c\rho - VJ_x/c),\\
J'_x &= \gamma(V)(J_x - V(c\rho)/c),\\
J'_y &= J_y,\\
J'_z &= J_z,
\end{aligned}$$

where V is the speed of frame S$'$ as measured in frame S.

The covariant counterpart to $(c\rho, J_x, J_y, J_z)$ is $(c\rho, -J_x, -J_y, -J_z)$.

Exercise 2.11 The components of a contravariant four-vector transform differently from those of a covariant four-vector. The former transform like the components of a displacement, according to the matrix $[\Lambda^\mu{}_\nu]$ that implements the Lorentz transformation. The latter transform like derivatives, according to the inverse of the Lorentz transformation matrix, $[(\Lambda^{-1})_\mu{}^\nu]$. Since one matrix 'undoes' the effect of the other in the sense that their product is the unit matrix, it is to be expected that combinations such as $\sum_{\mu=0}^{3} J_\mu J^\mu$ will transform as invariants, while other combinations, such as $\sum_{\mu=0}^{3} J_\mu J_\mu$ and $\sum_{\mu=0}^{3} J^\mu J^\mu$, will not.

Exercise 2.12 The indices must balance. They do this in both cases, but in the former case the lowering of indices can be achieved by the legitimate process of multiplying by the Minkowski metric and summing over a common index. In the latter case an additional step is required, the replacement of $\mathsf{F}_{\mu\nu}$ by $\mathsf{F}_{\nu\mu}$. This would be allowable if $[\mathsf{F}_{\nu\mu}]$ was symmetric — that is, if $\mathsf{F}_{\mu\nu} = \mathsf{F}_{\nu\mu}$ for all values of μ and ν — but it is not. Making such an additional change will alter some of the signs in an unacceptable way. The general lesson is clear: indices may be raised and lowered in a balanced way, but the order of indices is important and should be preserved. This is why elements of the mixed version of the field tensor may be written as $\mathsf{F}^{\mu}{}_{\nu}$ or $\mathsf{F}_{\mu}{}^{\nu}$ but should not be written as F^{μ}_{ν}.

Exercise 2.13 The field component of interest is given by $c\mathsf{F}'^{10}$, so we need to evaluate

$$\mathsf{F}'^{10} = \sum_{\alpha,\beta} \Lambda^{1}{}_{\alpha} \Lambda^{0}{}_{\beta} \mathsf{F}^{\alpha\beta}.$$

$\Lambda^{1}{}_{\alpha}$ is non-zero only when $\alpha = 0$ and $\alpha = 1$. Similarly, $\Lambda^{0}{}_{\beta}$ is non-zero only when $\beta = 0$ and $\beta = 1$. This makes the sum much shorter, so it can be written out explicitly:

$$\mathsf{F}'^{10} = \Lambda^{1}{}_{0}\Lambda^{0}{}_{0}\mathsf{F}^{00} + \Lambda^{1}{}_{0}\Lambda^{0}{}_{1}\mathsf{F}^{01} + \Lambda^{1}{}_{1}\Lambda^{0}{}_{0}\mathsf{F}^{10} + \Lambda^{1}{}_{1}\Lambda^{0}{}_{1}\mathsf{F}^{11}.$$

Since $\mathsf{F}^{00} = 0$ and $\mathsf{F}^{11} = 0$, the sum reduces to

$$\mathsf{F}'^{10} = \Lambda^{1}{}_{0}\Lambda^{0}{}_{1}\mathsf{F}^{01} + \Lambda^{1}{}_{1}\Lambda^{0}{}_{0}\mathsf{F}^{10}.$$

It is now a matter of substituting known values: $\mathsf{F}^{10} = -\mathsf{F}^{01} = \mathcal{E}_x/c$, $\Lambda^{0}{}_{0} = \Lambda^{1}{}_{1} = \gamma(V)$ and $\Lambda^{0}{}_{1} = \Lambda^{1}{}_{0} = -V\gamma(V)/c$, which leads to

$$\mathcal{E}'_x/c = \gamma^2(1 - V^2/c^2)\mathcal{E}_x/c.$$

Since $1 - V^2/c^2 = \gamma^{-2}$, we have

$$\mathcal{E}'_x = \mathcal{E}_x,$$

as required.

With patience, all the other field transformation rules can be determined in the same way.

Exercise 2.14 $$H'_{\alpha\beta\gamma\delta} = \sum_{\mu,\nu,\rho,\eta=0}^{3} \Lambda_{\alpha}{}^{\mu}\, \Lambda_{\beta}{}^{\nu}\, \Lambda_{\gamma}{}^{\rho}\, \Lambda_{\delta}{}^{\eta}\, H_{\mu\nu\rho\eta}.$$

Exercise 3.1 (a) You could note that $y/x = 4/3$ for all values of u, and also $u = 0$ gives $y = x = 0$, so this is the part of the straight line with positive u values and gradient $4/3$ through the origin. Or you could work out x and y for a few values of u, as shown in the table below.

u	0	1	2	3
x	0	3	12	27
y	0	4	16	36

Either way, your sketch should look like Figure S3.1.

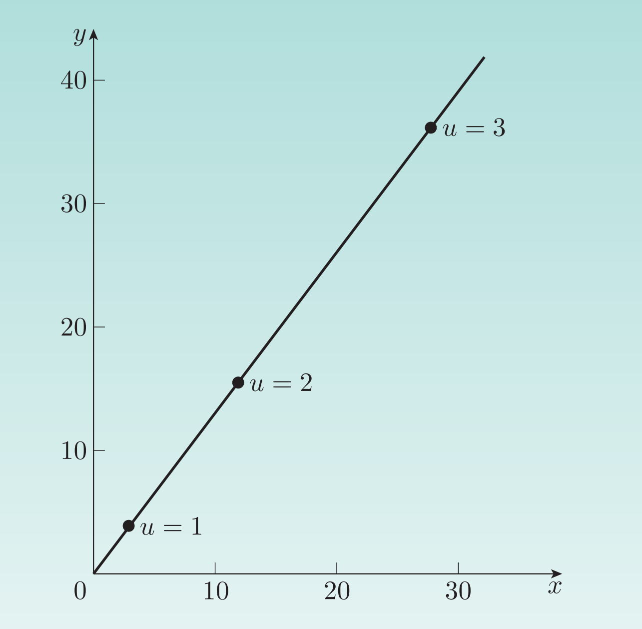

Figure S3.1 Sketch of the line $x = 3u^2$, $y = 4u^2$.

(b) We have

$$\frac{dx}{du} = 6u \quad \text{and} \quad \frac{dy}{du} = 8u,$$

so

$$L = \int_0^3 \left((6u)^2 + (8u)^2\right)^{1/2} du = \int_0^3 10u\, du = \left[5u^2\right]_0^3 = 45.$$

Exercise 3.2 Since $r = R$ and $\phi = u$, we have $dr = 0$ and $d\phi = du$, so

$$\begin{aligned} C = \int_0^{2\pi} dl &= \int_0^{2\pi} (dr^2 + r^2\, d\phi^2)^{1/2} = \int_0^{2\pi} (0^2 + R^2\, du^2)^{1/2} \\ &= \int_0^{2\pi} R\, du = [Ru]_0^{2\pi} = 2\pi R. \end{aligned}$$

Exercise 3.3 (a) Like the cylinder, the cone can be formed by rolling up a region of the plane. Once again this won't change the geometry; the circles and triangles will have the same properties as they have on the plane. So the cone has flat geometry.

(b) In this case, distances for the bugs are shorter towards the edge of the disc, so the shortest distance from P to Q, as measured by the bugs, will appear to us to curve outwards. The angles of the triangle PQR add up to more than 180°, as shown in Figure 3.12, so for this inverse hotplate the results are qualitatively similar to the geometry of the sphere, and the hotplate again has intrinsically curved geometry despite the lack of any extrinsic curvature.

Exercise 3.4 From Equation 3.10, we have

$$dl^2 = R^2\, d\theta^2 + R^2 \sin^2\theta\, d\phi^2.$$

Again there are only squared coordinate differentials, so $g_{ij} = 0$ for $i \neq j$. We can also see that $g_{11} = R^2$ and $g_{22} = R^2 \sin^2 x^1$, so

$$[g_{ij}] = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 x^1 \end{pmatrix}.$$

Exercise 3.5 In this case we only have squared coordinate differentials, so $g_{ij} = 0$ for $i \neq j$. Also, $g_{11} = 1$, $g_{22} = (x^1)^2$, $g_{33} = (x^1)^2 \sin^2 x^2$, and therefore

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & (x^1)^2 \sin^2 x^2 \end{pmatrix}.$$

Note that the final entry involves the coordinate x^2, *not* x squared.

Exercise 3.6 Defining $x^1 = r$ and $x^2 = \phi$, we have

$$[g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & (x^1)^2 \end{pmatrix}.$$

Exercise 3.7 (a) Since the line element is $\mathrm{d}l^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2$, we have

$$[g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From Equation 3.23, the connection coefficients are defined by

$$\Gamma^i{}_{jk} = \frac{1}{2} \sum_l g^{il} \left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right),$$

and since $\partial g_{ij}/\partial x^k = 0$ for all values of i, j, k, it follows that $\Gamma^i{}_{jk} = 0$ for all i, j, k.

Comment: This argument generalizes to any n-dimensional Euclidean space; consequently, when Cartesian coordinates are used, such spaces have vanishing connection coefficients.

(b) From Exercise 3.4, the metric is

$$[g_{ij}] = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 x^1 \end{pmatrix},$$

and the dual metric is the inverse matrix

$$[g^{ij}] = \begin{pmatrix} 1/R^2 & 0 \\ 0 & 1/R^2 \sin^2 x^1 \end{pmatrix}.$$

But in this case $R = 1$, so

$$[g^{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2 x^1 \end{pmatrix}.$$

Since

$$\Gamma^i{}_{jk} = \frac{1}{2} \sum_l g^{il} \left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right),$$

there are six independent connection coefficients:

$$\Gamma^1{}_{11}, \quad \Gamma^1{}_{12} (= \Gamma^1{}_{21}), \quad \Gamma^1{}_{22},$$
$$\Gamma^2{}_{11}, \quad \Gamma^2{}_{12} (= \Gamma^2{}_{21}), \quad \Gamma^2{}_{22}.$$

However,

$$\frac{\partial g_{22}}{\partial x^1} = 2\sin x^1 \cos x^1, \qquad \text{while} \qquad \frac{\partial g_{ij}}{\partial x^k} = 0$$

for all other values of i, j, k. Also, $g^{il} = 0$ for $i \neq l$, from which we can see that

$$\Gamma^1{}_{11} = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right) = 0,$$

$$\Gamma^1{}_{12} = \frac{1}{2} g^{11} \left(\frac{\partial g_{12}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^1} \right) = 0,$$

$$\Gamma^1{}_{22} = \frac{1}{2} g^{11} \left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1} \right) = -\frac{1}{2} g^{11} \frac{\partial g_{22}}{\partial x^1},$$

$$\Gamma^2{}_{11} = \frac{1}{2} g^{22} \left(\frac{\partial g_{21}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2} \right) = 0,$$

$$\Gamma^2{}_{12} = \frac{1}{2} g^{22} \left(\frac{\partial g_{22}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^2} \right) = \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial x^1},$$

$$\Gamma^2{}_{22} = \frac{1}{2} g^{22} \left(\frac{\partial g_{22}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^2} \right) = 0.$$

Consequently, the only non-zero values of the six independent connection coefficients listed above are

$$\Gamma^1{}_{22} = -\frac{1}{2} g^{11} \frac{\partial g_{22}}{\partial x^1} = -\sin x^1 \cos x^1 \qquad \text{and} \qquad \Gamma^2{}_{12} = \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial x^1} = \frac{\cos x^1}{\sin x^1} = \cot x^1.$$

(The only other non-zero connection coefficient is $\Gamma^2{}_{21} = \Gamma^2{}_{12}$.)

Exercise 3.8 From Exercise 3.7(a), $\Gamma^i{}_{jk} = 0$ for all i, j, k in this metric, so Equation 3.27 reduces to

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} = 0,$$

giving the solutions $x^i = a_i \lambda + b_i$ for constants a_i, b_i. Writing this as $x(\lambda) = a\lambda + b$ and $y(\lambda) = c\lambda + d$, we see that these equations parameterize the straight line through (b, d) with gradient c/a.

Exercise 3.9 Using our usual coordinates for the surface of a sphere, $x^1 = \theta$, $x^2 = \phi$, and the results of Exercise 3.7(b) for the connection coefficients, Equation 3.27 becomes

$$\frac{\mathrm{d}^2 \theta}{\mathrm{d}\lambda^2} - \sin\theta \cos\theta \left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} \right)^2 = 0 \tag{3.69}$$

and

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}\lambda^2} + 2 \frac{\cos\theta}{\sin\theta} \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = 0. \tag{3.70}$$

(a) The portion of a meridian A can be parameterized by

$$\theta(\lambda) = \lambda, \quad 0 \leq \lambda \leq \tfrac{\pi}{2},$$
$$\phi(\lambda) = 0,$$

so we have

$$\frac{d\theta}{d\lambda} = 1, \quad \frac{d^2\theta}{d\lambda^2} = \frac{d^2\phi}{d\lambda^2} = \frac{d\phi}{d\lambda} = 0,$$
$$\sin\theta = \sin(\lambda), \quad \cos\theta = \cos(\lambda).$$

Equation 3.69 becomes

$$0 - \sin(\lambda)\cos(\lambda) \times 0 = 0,$$

and Equation 3.70 becomes

$$0 + 2\cot(\lambda) \times 1 \times 0 = 0.$$

So A satisfies the geodesic equations and is a geodesic.

Comment: This is what we would expect, because A is part of a great circle.

(b) B can be parameterized by

$$\theta(\lambda) = \tfrac{\pi}{2},$$
$$\phi(\lambda) = \lambda, \quad 0 \leq \lambda < 2\pi.$$

So we have

$$\frac{d\phi}{d\lambda} = 1, \quad \frac{d^2\phi}{d\lambda^2} = \frac{d^2\theta}{d\lambda^2} = \frac{d\theta}{d\lambda} = 0,$$
$$\sin\theta = 1, \quad \cos\theta = 0.$$

Equation 3.69 becomes $0 - 1 \times 0 \times 1 = 0$, and Equation 3.70 becomes $0 + 2 \times 0 \times 1 \times 0 = 0$. So B satisfies the geodesic equations and is a geodesic.

(c) C can be parameterized by

$$\theta(\lambda) = \tfrac{\pi}{4},$$
$$\phi(\lambda) = \lambda, \quad 0 \leq \lambda < 2\pi.$$

So we have

$$\frac{d\phi}{d\lambda} = 1, \quad \frac{d^2\phi}{d\lambda^2} = \frac{d^2\theta}{d\lambda^2} = \frac{d\theta}{d\lambda} = 0,$$
$$\sin\theta = \cos\theta = \sqrt{2}.$$

Equation 3.69 becomes $0 - \sqrt{2} \times \sqrt{2} \times 1 = -2 \neq 0$, and Equation 3.70 becomes $0 + 2 \times 1 \times 0 \times 1 = 0$. So C is not a geodesic because it doesn't satisfy both geodesic equations.

Exercise 3.10 (a) Since k is constant at every point on the curve and $k = 1/R$, we have

$$R = \frac{1}{k} = \frac{1}{0.2\,\text{cm}^{-1}} = 5\,\text{cm}.$$

So the best approximating circle at every point on the curve is a circle of radius 5 cm, and the curve itself is a circle of radius 5 cm.

(b) Here again k will be constant, as the straight line has constant 'curvature'. However big we draw the circle, a larger circle will approximate the straight line better, so the curvature of a straight line must be smaller than $1/R$ for all possible R. Hence k must be zero. In other words,

$$k = \lim_{R\to\infty} \frac{1}{R} = 0.$$

Exercise 3.11 The parabola can be parameterized by $x(\lambda) = \lambda$ and $y(\lambda) = \lambda^2$. Consequently,

$$\dot{x} = 1, \quad \ddot{x} = 0, \quad \dot{y} = 2\lambda, \quad \ddot{y} = 2,$$

and for $\lambda = 0$ we have

$$\dot{x} = 1, \quad \ddot{x} = 0, \quad \dot{y} = 0, \quad \ddot{y} = 2.$$

So the curvature at $\lambda = 0$ is

$$k = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|1 \times 2 - 0 \times 0|}{(1^2 + 0^2)^{3/2}} = 2,$$

and the approximating circle has the radius

$$R = \frac{1}{k} = \frac{1}{2}.$$

The centre of the circle is at $x = 0$, $y = 0.5$.

Exercise 3.12 The derivatives of x and y are given by

$$\dot{x} = -a\sin\lambda, \quad \ddot{x} = -a\cos\lambda, \quad \dot{y} = b\cos\lambda, \quad \ddot{y} = -b\sin\lambda,$$

so the curvature is given by

$$k = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{ab\sin^2\lambda + ab\cos^2\lambda}{(a^2\sin^2\lambda + b^2\cos^2\lambda)^{3/2}} = \frac{ab}{(a^2\sin^2\lambda + b^2\cos^2\lambda)^{3/2}}.$$

For the circle of radius R we have $a = R$ and $b = R$, so

$$k = \frac{ab}{(a^2\sin^2\lambda + b^2\cos^2\lambda)^{3/2}} = \frac{R^2}{(R^2\sin^2\lambda + R^2\cos^2\lambda)^{3/2}} = \frac{1}{R},$$

which is as expected.

Exercise 3.13 Interchanging the j, k indices in Equation 3.35, we get

$$R^l{}_{ikj} = \frac{\partial\Gamma^l{}_{ij}}{\partial x^k} - \frac{\partial\Gamma^l{}_{ik}}{\partial x^j} + \sum_m \Gamma^m{}_{ij}\,\Gamma^l{}_{mk} - \sum_m \Gamma^m{}_{ik}\,\Gamma^l{}_{mj}.$$

Swapping the first and second terms, and the third and fourth terms, leads to

$$R^l{}_{ikj} = -\frac{\partial\Gamma^l{}_{ik}}{\partial x^j} + \frac{\partial\Gamma^l{}_{ij}}{\partial x^k} - \sum_m \Gamma^m{}_{ik}\,\Gamma^l{}_{mj} + \sum_m \Gamma^m{}_{ij}\,\Gamma^l{}_{mk}.$$

Comparison with Equation 3.35 shows that the expression on the right-hand side of this equation is $-R^l{}_{ijk}$, hence proving that $R^l{}_{ijk} = -R^l{}_{ikj}$.

Exercise 3.14 From Exercise 3.7(a), all connection coefficients for this space are zero, and hence from Equation 3.35, we have

$$R^l{}_{ijk} = 0.$$

Since the connection coefficients also vanish for an n-dimensional Euclidean space, it follows that the Riemann tensor is zero for such spaces.

Exercise 3.15 From Equation 3.35 and Exercise 3.7(b), we have

$$\begin{aligned} R^1{}_{212} &= \frac{\partial\Gamma^1{}_{22}}{\partial x^1} - \frac{\partial\Gamma^1{}_{21}}{\partial x^2} + \sum_\lambda \Gamma^\lambda{}_{22}\,\Gamma^1{}_{\lambda 1} - \sum_\lambda \Gamma^\lambda{}_{21}\,\Gamma^1{}_{\lambda 2} \\ &= \frac{\partial\Gamma^1{}_{22}}{\partial x^1} - \frac{\partial\Gamma^1{}_{21}}{\partial x^2} + \Gamma^1{}_{22}\,\Gamma^1{}_{11} + \Gamma^2{}_{22}\,\Gamma^1{}_{21} - \Gamma^1{}_{21}\,\Gamma^1{}_{12} - \Gamma^2{}_{21}\,\Gamma^1{}_{22}. \end{aligned}$$

But from Exercise 3.7(b),

$$\Gamma^1{}_{11} = \Gamma^1{}_{12} = \Gamma^1{}_{21} = \Gamma^2{}_{11} = \Gamma^2{}_{22} = 0,$$

so

$$\begin{aligned} R^1{}_{212} &= \frac{\partial \Gamma^1{}_{22}}{\partial x^1} - \Gamma^2{}_{21}\,\Gamma^1{}_{22} \\ &= \frac{\partial}{\partial x^1}(-\sin x^1 \cos x^1) - \frac{\cos x^1}{\sin x^1}(-\sin x^1 \cos x^1) \\ &= -\cos^2(x^1) + \sin^2(x^1) + \cos^2(x^1) \\ &= \sin^2 x^1. \end{aligned}$$

Exercise 3.16 From the earlier in-text question, we know that $K = a^{-2}$, and from Exercise 3.15,

$$R^1{}_{212} = \sin^2 x^1.$$

However, from Exercise 3.7(b),

$$[g_{ij}] = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2 x^1 \end{pmatrix},$$

so

$$g = \det[g_{ij}] = a^4 \sin^2 x^1.$$

Also, from Chapter 2 we know that lowering the first index on $R^1{}_{212}$ gives

$$R_{1212} = \sum_{i=1}^{2} g_{1i} R^i{}_{212} = g_{11} R^1{}_{212} + g_{12} R^2{}_{212}.$$

However, $g_{12} = 0$, hence

$$\frac{R_{1212}}{g} = \frac{a^2 \times \sin^2 x^1}{a^4 \sin^2 x^1} = \frac{1}{a^2},$$

which is the same as K.

Exercise 3.17 (a) Just as in Exercise 3.7(a), the connection coefficients are zero since the metric is constant.

(b) Since the connection coefficients for a Minkowski spacetime are zero, as shown in part (a), and each term in the Riemann tensor defined by Equation 3.35 involves at least one connection coefficient, it follows that all components of the Riemann tensor are zero.

Exercise 3.18 (a) The metric is

$$[g_{ij}] = \begin{pmatrix} c^2 & 0 \\ 0 & -f^2(t) \end{pmatrix}$$

and the dual metric is

$$[g^{ij}] = \begin{pmatrix} 1/c^2 & 0 \\ 0 & -1/f^2(t) \end{pmatrix}.$$

As in Exercise 3.7(b), there are only six independent connection coefficients:

$$\begin{aligned} &\Gamma^0{}_{00}, \quad \Gamma^0{}_{01}(=\Gamma^0{}_{10}), \quad \Gamma^0{}_{11}, \\ &\Gamma^1{}_{00}, \quad \Gamma^1{}_{01}(=\Gamma^1{}_{10}), \quad \Gamma^1{}_{11}. \end{aligned}$$

Moreover,

$$\frac{\partial g_{11}}{\partial x^0} = -2f\dot{f}, \quad \text{where } \dot{f} \equiv \frac{\mathrm{d}f(t)}{\mathrm{d}t},$$

and

$$\frac{\partial g_{ij}}{\partial x^k} = 0$$

for all other values of i, j, k. Also, $g^{il} = 0$ for $i \neq l$, from which we can see that

$$\Gamma^0{}_{00} = \frac{1}{2} g^{00} \left(\frac{\partial g_{00}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^0} \right) = 0,$$

$$\Gamma^0{}_{01} = \frac{1}{2} g^{00} \left(\frac{\partial g_{01}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^1} - \frac{\partial g_{01}}{\partial x^0} \right) = 0,$$

$$\Gamma^0{}_{11} = \frac{1}{2} g^{00} \left(\frac{\partial g_{01}}{\partial x^1} + \frac{\partial g_{10}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^0} \right) = -\frac{1}{2} g^{00} \frac{\partial g_{11}}{\partial x^0},$$

$$\Gamma^1{}_{00} = \frac{1}{2} g^{11} \left(\frac{\partial g_{10}}{\partial x^0} + \frac{\partial g_{01}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^1} \right) = 0,$$

$$\Gamma^1{}_{01} = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial x^0} + \frac{\partial g_{01}}{\partial x^1} - \frac{\partial g_{01}}{\partial x^1} \right) = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^0},$$

$$\Gamma^1{}_{11} = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right) = 0.$$

Consequently, the only non-zero values of the six independent connection coefficients listed above are

$$\Gamma^0{}_{11} = -\frac{1}{2} g^{00} \frac{\partial g_{11}}{\partial x^0} = -\frac{1}{2} \times \frac{1}{c^2} \times (-2f\dot{f}) = \frac{f\dot{f}}{c^2}$$

and

$$\Gamma^1{}_{01} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^0} = \frac{1}{2} \times \frac{-1}{f^2} \times (-2f\dot{f}) = \frac{\dot{f}}{f}.$$

The only other non-zero connection coefficient is $\Gamma^1{}_{10} = \Gamma^1{}_{01}$.

(b) As in Exercise 3.15,

$$\begin{aligned} R^0{}_{101} &= \frac{\partial \Gamma^0{}_{11}}{\partial x^0} - \frac{\partial \Gamma^0{}_{10}}{\partial x^1} + \sum_\lambda \Gamma^\lambda{}_{11} \Gamma^0{}_{\lambda 0} - \sum_\lambda \Gamma^\lambda{}_{10} \Gamma^0{}_{\lambda 1} \\ &= \frac{\partial \Gamma^0{}_{11}}{\partial x^0} - \frac{\partial \Gamma^0{}_{10}}{\partial x^1} + \Gamma^0{}_{11} \Gamma^0{}_{00} + \Gamma^1{}_{11} \Gamma^0{}_{10} - \Gamma^0{}_{10} \Gamma^0{}_{01} - \Gamma^1{}_{10} \Gamma^0{}_{11}. \end{aligned}$$

Since $\Gamma^0{}_{00} = \Gamma^0{}_{01} = \Gamma^0{}_{10} = \Gamma^1{}_{00} = \Gamma^1{}_{11} = 0$, we have

$$\begin{aligned} R^0{}_{101} &= \frac{\partial \Gamma^0{}_{11}}{\partial x^0} - \Gamma^1{}_{10} \Gamma^0{}_{11} = \frac{\partial}{\partial x^0} \left[\frac{f\dot{f}}{c^2} \right] - \frac{\dot{f}}{f} \times \frac{f\dot{f}}{c^2} \\ &= \frac{1}{c^2} \frac{\partial}{\partial t} \left[f\dot{f} \right] - \frac{\dot{f}^2}{c^2} = \frac{1}{c^2} \left[\dot{f}\dot{f} + f\ddot{f} \right] - \frac{\dot{f}^2}{c^2} \\ &= \frac{f\ddot{f}}{c^2}. \end{aligned}$$

Exercise 4.1 (a) Suppose that the separation is l and the distance from the centre of the Earth is R, as shown in Figure S4.1.

Then the magnitude of the horizontal acceleration of each object is $g\sin\theta \approx g\theta$, so the total (relative) acceleration is $g2\theta$. However, $2\theta = l/R$, so the magnitude of the total acceleration, a, is given by

$$a = \frac{gl}{R} = \frac{9.81 \times 2.00}{6.38 \times 10^6}\,\mathrm{m\,s^{-2}} = 3.08 \times 10^{-6}\,\mathrm{m\,s^{-2}}.$$

(b) Suppose that one object is a distance l vertically above the other object. Since Newtonian gravity is an inverse square law, the magnitudes of acceleration at R and $R+l$ are related by

$$\frac{g(R)}{g(R+l)} = \frac{(R+l)^2}{R^2} = \left(1 + \frac{l}{R}\right)^2 \approx 1 + \frac{2l}{R}.$$

Hence Δg, the difference between the magnitudes of acceleration at R and $R+l$, is given by

$$\Delta g = \frac{2gl}{R} = \frac{2 \times 9.81 \times 2.00}{6.38 \times 10^6}\,\mathrm{m\,s^{-2}} = 6.15 \times 10^{-6}\,\mathrm{m\,s^{-2}}.$$

Exercise 4.2

(a) As indicated by Figure S4.2, the coordinates are related by $x = r\cos\theta,\ y = r\sin\theta$.

Setting $(x'^1, x'^2) = (x, y)$ and $(x^1, x^2) = (r, \theta)$, we have

$$\frac{\partial x'^1}{\partial x^1} = \frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial x'^1}{\partial x^2} = \frac{\partial x}{\partial \theta} = -r\sin\theta$$

and

$$\frac{\partial x'^2}{\partial x^1} = \frac{\partial y}{\partial r} = \sin\theta, \qquad \frac{\partial x'^2}{\partial x^2} = \frac{\partial y}{\partial \theta} = r\cos\theta.$$

In this case, the general tensor transformation law reduces to

$$A'^1 = \sum_\nu \frac{\partial x'^1}{\partial x^\nu} A^\nu, \quad \text{and} \quad A'^2 = \sum_\nu \frac{\partial x'^2}{\partial x^\nu} A^\nu.$$

This means that A'^μ and A^μ must be related by

$$A'^1 = \cos\theta\, A^1 - r\sin\theta\, A^2, \quad \text{and} \quad A'^2 = \sin\theta\, A^1 + r\cos\theta\, A^2.$$

(b) In the case of the infinitesimal displacement, this general transformation rule implies that

$$\mathrm{d}x = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta, \quad \text{and} \quad \mathrm{d}y = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta.$$

But this is exactly the relationship between these different sets of coordinates given by the chain rule, so the infinitesimal displacement does transform as a contravariant rank 1 tensor.

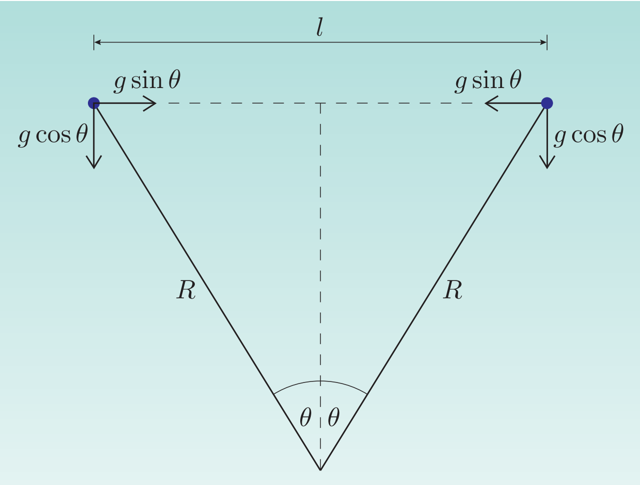

Figure S4.1 Accelerations of horizontally separated masses in a freely falling lift.

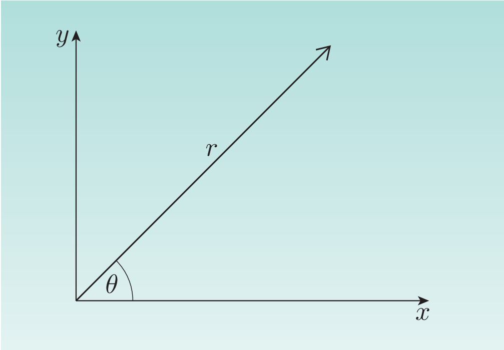

Figure S4.2 Polar coordinates.

Exercise 4.3 We know that

$$A_\mu = \sum_{\alpha=0}^{3} g_{\mu\alpha}\, A^\alpha.$$

Multiplying by $g^{\nu\mu}$ and summing over μ, we have

$$\sum_{\mu=0}^{3} g^{\nu\mu}\, A_\mu = \sum_{\mu=0}^{3}\sum_{\alpha=0}^{3} g^{\nu\mu}\, g_{\mu\alpha}\, A^\alpha.$$

Reversing the order in which we do the summation on the right-hand side of this equation enables us to write it as

$$\sum_{\mu=0}^{3} g^{\nu\mu}\, A_\mu = \sum_{\alpha=0}^{3} A^\alpha \sum_{\mu=0}^{3} g^{\nu\mu}\, g_{\mu\alpha}.$$

However,

$$\sum_{\mu=0}^{3} g^{\nu\mu}\, g_{\mu\alpha} = \delta^\nu{}_\alpha.$$

Since $\delta^\nu{}_\alpha = 1$ when $\nu = \alpha$ and $\delta^\nu{}_\alpha = 0$ when $\nu \neq \alpha$, we have

$$\sum_{\mu=0}^{3} g^{\nu\mu}\, A_\mu = A^\nu.$$

Exercise 4.4 (a) There are two reasons. The μ index is up on A^μ but down on B_μ. The K term has no μ index.

(b) The ν index cannot be up on both $Y^{\mu\nu}$ and Z^ν; it must be up on one term and down on the other.

(c) There cannot be three instances of the ν index on the right-hand side of this equation.

Exercise 4.5 Being a scalar, this quantity has no contravariant or covariant indices. So in this particular case, covariant differentiation simply gives

$$\nabla_\lambda S = \frac{\partial S}{\partial x^\lambda}.$$

Exercise 4.6 We know that

$$[\eta_{\mu\nu}] = [\eta^{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$[U^\mu] = \gamma(v)(c, \boldsymbol{v}) = \gamma(v)\left(c, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right).$$

Since $U^0 = c$ in the instantaneous rest frame, we have $T^{00} = \rho c^2$. Also, $T^{0i} = 0$ since $\eta^{0i} = 0$ and $U^i = 0$ in this frame. Likewise,

$$T^{ii} = \left(\rho + \frac{p}{c^2}\right) U^i U^i + p = p.$$

Finally, for $i \neq j$,

$$T^{ij} = \left(\rho + \frac{p}{c^2}\right) U^i U^j - p\eta^{ij} = 0$$

since $\eta^{ij} = 0$ for $i \neq j$ and $U^i = 0$ in the instantaneous rest frame.

Exercise 4.7 Multiplying Equation 4.34 by $g^{\mu\nu}$ and summing over both indices, we obtain

$$\sum_{\mu,\nu} g^{\mu\nu} R_{\mu\nu} - \sum_{\mu,\nu} \tfrac{1}{2} R\, g_{\mu\nu}\, g^{\mu\nu} = \sum_{\mu,\nu} -\kappa\, g^{\mu\nu}\, T_{\mu\nu}.$$

Now using the fact that

$$\sum_{\mu,\nu} g^{\mu\nu}\, g_{\mu\nu} = \sum_{\nu} \delta^\nu{}_\nu = 4,$$

this becomes

$$R - 2R = -\kappa T.$$

Hence $R = \kappa T$, which we can substitute in Equation 4.34 to obtain Equation 4.35:

$$R_{\mu\nu} - \tfrac{1}{2}\kappa T\, g_{\mu\nu} = -\kappa\, T_{\mu\nu},$$

so

$$R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\, T\right).$$

Exercise 5.1 From the definition of the Einstein tensor,

$$G_{00} = R_{00} - \tfrac{1}{2} g_{00} R$$

and we have

$$R_{00} = -\mathrm{e}^{2(A-B)}\left(A'' + (A')^2 - A'B' + \frac{2A'}{r}\right),$$

$$g_{00} = \mathrm{e}^{2A}$$

and

$$R = -2e^{-2B}\left(A'' + (A')^2 - A'B' + \frac{2}{r}(A' - B') + \frac{1}{r^2}\right) + \frac{2}{r^2}.$$

So

$$\begin{aligned}
G_{00} &= R_{00} - \tfrac{1}{2}g_{00}R \\
&= -e^{2(A-B)}\left(A'' + (A')^2 - A'B' + \frac{2A'}{r}\right) \\
&\quad + e^{2(A-B)}\left(A'' + (A')^2 - A'B' + \frac{2}{r}(A' - B') + \frac{1}{r^2}\right) - \frac{e^{2A}}{r^2} \\
&= -e^{2(A-B)}\left(\frac{2B'}{r} - \frac{1}{r^2}\right) - \frac{e^{2A}}{r^2},
\end{aligned}$$

as required.

Exercise 5.2 (a) The only place where the coordinate ϕ appears in the Schwarzschild line element is in the term $r^2 \sin^2\theta\,(d\phi)^2$. But since $\phi' = \phi + \phi_0$, the difference in the ϕ-coordinates of any two events will be equal to the difference in the ϕ'-coordinates of those events, and in the limit, for infinitesimally separated events, $d\phi' = d(\phi + \phi_0) = d\phi$. So the Schwarzschild line element is unaffected by the change of coordinates apart from the replacement of ϕ by ϕ'. This establishes the form-invariance of the metric under the change of coordinates.

(b) In a system of spherical coordinates, a given value of the coordinate ϕ corresponds to a meridian of the kind shown in Figure S5.1.

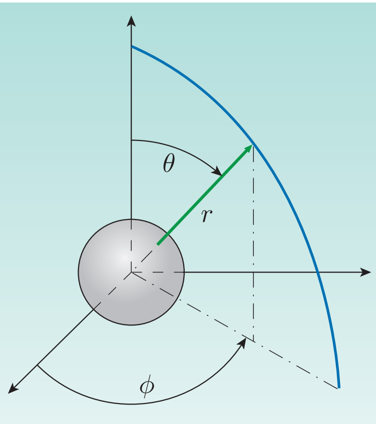

Figure S5.1 Radial coordinates with a (meridian) line of constant ϕ.

The replacement of ϕ by ϕ' effectively shifts every such meridian by the same angle ϕ_0. Since the body that determines the Schwarzschild metric is spherically symmetric, the displacement of the meridians will have no physical significance. Moreover, since each meridian is replaced by another, all that really happens in this case is that each meridian is relabelled, and this will not even change the form of the metric.

Exercise 5.3 We require

$$\frac{d\tau}{dt} \leq 1 - 10^{-8}.$$

With $\mathrm{d}r = \mathrm{d}\theta = \mathrm{d}\phi = 0$ the metric reduces to

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} \approx 1 - \frac{GM}{c^2 r}, \qquad \text{so} \qquad \frac{GM}{c^2 r} \leq 10^{-8}.$$

Rearranging gives

$$r \geq \frac{GM}{c^2 \times 10^{-8}} = 1.5 \times 10^{11} \text{ metres.}$$

We have not yet found the relationship between the Schwarzschild coordinate r and physical (proper) distance — that is the subject of the next section. Nonetheless it is interesting to note that a proper distance of 1.5×10^{11} metres is about the distance from the Earth to the Sun.

Exercise 5.4 The proper distance $\mathrm{d}\sigma$ between two neighbouring events that happen at the same time ($\mathrm{d}t = 0$) is given by the metric via the relationship $(\mathrm{d}s)^2 = -(\mathrm{d}\sigma)^2$. Thus

$$(\mathrm{d}\sigma)^2 = \frac{(\mathrm{d}r)^2}{1 - \frac{2GM}{c^2 r}} + r^2(\mathrm{d}\theta)^2 + r^2 \sin^2\theta\,(\mathrm{d}\phi)^2.$$

For the circumference at a given r-coordinate in the $\theta = \pi/2$ plane, $\mathrm{d}r = \mathrm{d}\theta = 0$, hence

$$(\mathrm{d}\sigma)^2 = r^2(\mathrm{d}\phi)^2.$$

So

$$\mathrm{d}\sigma = r\,\mathrm{d}\phi \qquad \text{and therefore} \qquad C = \int_0^{2\pi} r\,\mathrm{d}\phi = 2\pi r,$$

as required.

Exercise 5.5 It follows from the general equation for an affinely parameterized geodesic that

$$\frac{\mathrm{d}^2 x^0}{\mathrm{d}\lambda^2} + \sum_{\nu,\rho} \Gamma^0{}_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} = 0.$$

Since the only non-zero connection coefficients with a raised index 0 are $\Gamma^0{}_{01} = \Gamma^0{}_{10}$, the sum may be expanded to give

$$\frac{\mathrm{d}^2 x^0}{\mathrm{d}\lambda^2} + 2\Gamma^0{}_{01} \frac{\mathrm{d}x^0}{\mathrm{d}\lambda} \frac{\mathrm{d}x^1}{\mathrm{d}\lambda} = 0.$$

Identifying $x^0 = ct$, $x^1 = r$ and $\Gamma^0{}_{01} = \frac{GM}{r^2 c^2 \left(1 - \frac{2GM}{c^2 r}\right)}$, we see that

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} + \frac{2GM}{c^2 r^2 \left(1 - \frac{2GM}{c^2 r}\right)} \frac{\mathrm{d}r}{\mathrm{d}\lambda} \frac{\mathrm{d}t}{\mathrm{d}\lambda} = 0,$$

as required.

Exercise 5.6 For circular motion at a given r-coordinate in the equatorial plane, u is constant, so

$$\frac{\mathrm{d}u}{\mathrm{d}\phi} = \frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} = 0 \quad \text{and also} \quad \frac{\mathrm{d}r}{\mathrm{d}\tau} = 0.$$

(a) It follows from the orbital shape equation (Equation 5.36) that for a circular orbit with $J^2/m^2 = 12G^2M^2/c^2$,

$$\frac{3GMu^2}{c^2} - u + GM\left(\frac{12G^2M^2}{c^2}\right)^{-1} = 0,$$

that is

$$\frac{3GMu^2}{c^2} - u + \frac{c^2}{12GM} = 0.$$

Solving this quadratic equation in u gives $u = c^2/6GM$, so $r = 6GM/c^2$ is the minimum radius of a stable circular orbit.

(b) The corresponding value of E may be determined from the radial motion equation (Equation 5.32), remembering that $\mathrm{d}r/\mathrm{d}\tau = 0$:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{J^2}{m^2r^2}\left(1 - \frac{2GM}{c^2r}\right) - \frac{2GM}{r} = c^2\left[\left(\frac{E}{mc^2}\right)^2 - 1\right].$$

So

$$0 + \frac{12G^2M^2}{c^2}\left(\frac{c^2}{6GM}\right)^2\left(1 - \frac{2GM}{c^2}\frac{c^2}{6GM}\right) - 2GM\frac{c^2}{6GM}$$
$$= c^2\left[\left(\frac{E}{mc^2}\right)^2 - 1\right].$$

Simplifying this, we have

$$\frac{c^2}{3}\left(1 - \frac{2}{6}\right) - \frac{c^2}{3} = c^2\left[\left(\frac{E}{mc^2}\right)^2 - 1\right]$$

that is

$$-\frac{c^2}{9} = c^2\left[\left(\frac{E}{mc^2}\right)^2 - 1\right],$$

which can be rearranged to give $E = \sqrt{8}mc^2/3$.

Exercise 6.1 (a) For the Sun, $R_\mathrm{S} = 3\,\mathrm{km}$. So for a black hole with three times the Sun's mass, the Schwarzschild radius is $9\,\mathrm{km}$. Substituting this value into Equation 6.10, we find that the proper time required for the fall is just

$$\tau_\mathrm{fall} = 6 \times 10^3/(3 \times 10^8)\,\mathrm{s} = 2 \times 10^{-5}\,\mathrm{s}.$$

(b) For a $10^9\,\mathrm{M}_\odot$ galactic-centre black hole, the Schwarzschild radius and the in-fall time are both greater by a factor of $10^9/3$. A calculation similar to that in part (a) therefore gives a free fall time of 6700 s, or about 112 minutes. (Note that these results apply to a body that starts its fall from far away, not from the horizon.)

Exercise 6.2 According to Equation 6.12, for events on the world-line of an outward radially travelling photon,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c(1 - R_\mathrm{S}/r).$$

For a stationary local observer, i.e. an observer at rest at r, we saw in Chapter 5 that intervals of proper time are related to intervals of coordinate time by $\mathrm{d}\tau = \mathrm{d}t\,(1 - R_\mathrm{S}/r)^{1/2}$, while intervals of proper distance are related to intervals of coordinate distance by $\mathrm{d}\sigma = \mathrm{d}r\,(1 - R_\mathrm{S}/r)^{-1/2}$. It follows that the speed of light as measured by a local observer, irrespective of their location, will always be

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}t}\,\frac{1}{1 - R_\mathrm{S}/r}.$$

So, in the case that the intervals being referred to are those between events on the world-line of a radially travelling photon, we see that the locally observed speed of the photon is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = c(1 - R_\mathrm{S}/r)\frac{1}{1 - R_\mathrm{S}/r} = c.$$

Exercise 6.3 According to the reciprocal of Equation 6.17, for events on the world-line of a freely falling body,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -cR_\mathrm{S}^{1/2}\frac{1 - R_\mathrm{S}/r}{(1 - R_\mathrm{S}/r_0)^{1/2}}\left(\frac{r_0 - r}{rr_0}\right)^{1/2}.$$

We already know from the previous exercise that for a stationary local observer,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}t}\,\frac{1}{1 - R_\mathrm{S}/r}.$$

So, in the case of a freely falling body, the measured inward radial velocity will be

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} &= -cR_\mathrm{S}^{1/2}\frac{1 - R_\mathrm{S}/r}{(1 - R_\mathrm{S}/r_0)^{1/2}}\left(\frac{r_0 - r}{rr_0}\right)^{1/2}\frac{1}{1 - R_\mathrm{S}/r} = -cR_\mathrm{S}^{1/2}\frac{1}{(1 - R_\mathrm{S}/r_0)^{1/2}}\left(\frac{r_0 - r}{rr_0}\right)^{1/2}\\ &= -c\left(\frac{R_\mathrm{S}}{(r_0 - R_\mathrm{S})} \times \frac{r_0 - r}{r}\right)^{1/2}.\end{aligned}$$

In the limit as $r \to R_\mathrm{S}$, the locally observed speed is given by $|\mathrm{d}\sigma/\mathrm{d}\tau| \to c$.

Exercise 6.4 Initially, the fall would look fairly normal with the astronaut apparently getting smaller and picking up speed as the distance from the observer increased. At first the frequency of the astronaut's waves would also look normal, though detailed measurements would reveal a small decrease due to the Doppler effect. As the distance increased, the astronaut's speed of fall would continue to increase and the frequency of waving would decrease. This would be accompanied by a similar change in the frequency of light received from the falling astronaut, so the astronaut would appear to become redder as well as more distant. The reddening would be increased due to gravitational redshift, though the astronaut's motion would continue to contribute. As the astronaut approached the event horizon, the effect of spacetime distortion would become dominant. The astronaut's rate of fall would be seen to decrease, but the image would become very red and would rapidly dim, causing the departing astronaut to fade away.

Though something along these lines is the expected answer, there is another effect to take into account, that depends on the mass of the black hole. This is a consequence of tidal forces and will be discussed in the next section.

Exercise 6.5 The increasing narrowness and gradual tipping of the lightcones as they approach the event horizon indicates the difficulty of outward escape for photons and, by implication, for any particles that travel slower than light. This effect reaches a critical stage at the event horizon, where the outgoing edge of the lightcone becomes vertical, indicating that even photons emitted in the outward direction are unable to make progress in that direction. A diagrammatic study of lightcones alone is unable to prove the impossibility of escape from within the event horizon, but the progressive narrowing and tipping of lightcones in that region is at least suggestive of the impossibility of escape, and it is indeed a fact that all affinely parameterized geodesics that enter the event horizon of a non-rotating black hole reach the central singularity at some finite value of the affine parameter.

Exercise 6.6 The time-like geodesic for the Schwarzschild case has already been given in Figure 6.11. The nature of the lightcones is also represented in that figure, so the expected answer is shown in Figure S6.1a. In the case of Eddington–Finkelstein coordinates, Figure 6.13 plays a similar role, suggesting (rather than showing) the form of the time-like geodesic and indicating the form of the lightcones. The expected answer is shown in Figure S6.1b.

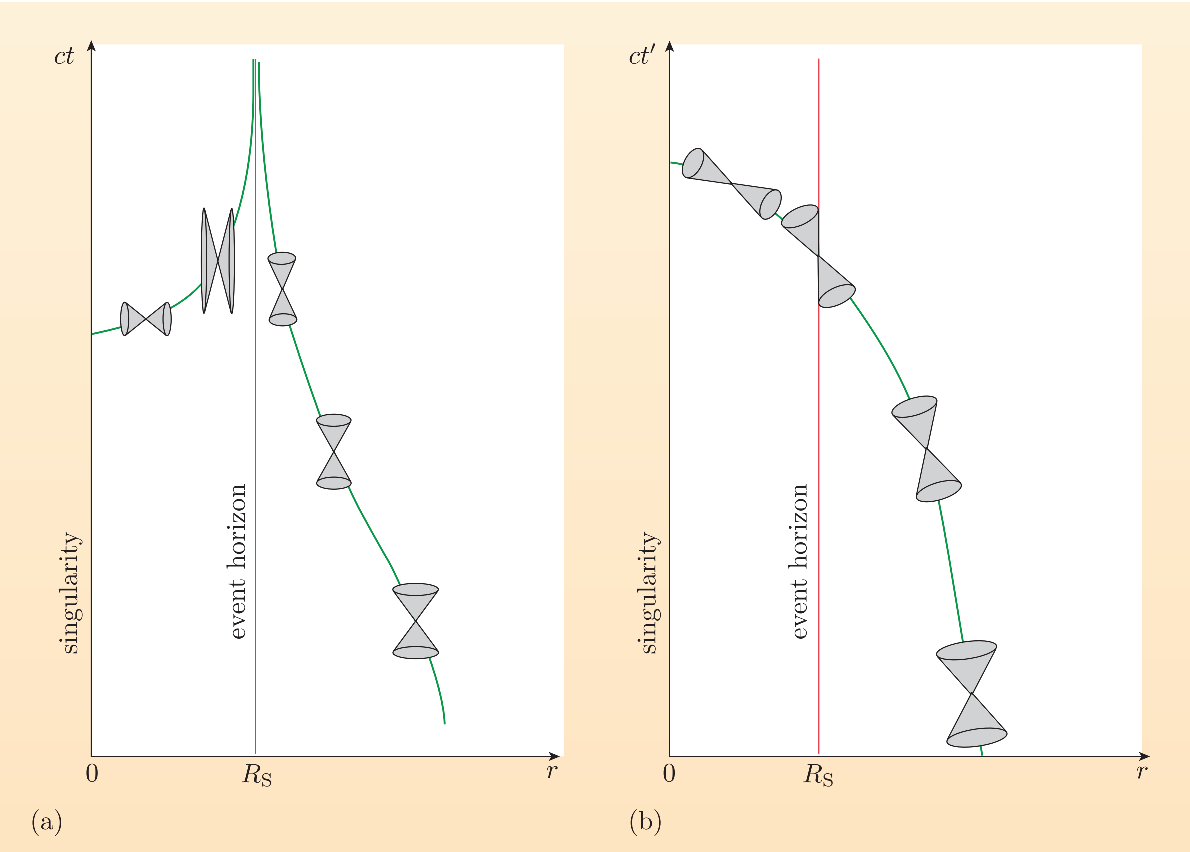

Figure S6.1 Lightcones along a time-like geodesic in (a) Schwarzschild and (b) advanced Eddington–Finkelstein coordinates.

Exercise 6.7 (a) When $J = GM^2/c$, we have $a = J/Mc = GM/c^2 = R_S/2$. Inserting this into Equations 6.32 and 6.33, the second term vanishes and we find

$r_\pm = R_S/2$.

(b) When $J = 0$, we have $a = 0$ and we obtain $r_+ = R_S$, $r_- = 0$.

In both cases (a) and (b), there is only one event horizon as the inner horizon vanishes.

Exercise 6.8 (a) The path indicated by the dashed line in Figure 6.20 shows no change in angle as it approaches the static limit. Space outside the static limit is also dragged around, even though rotation is no longer compulsory. However, a particle in free fall must be affected by this dragging, and so a particle in free fall could not fall in on the dashed line. The path of free fall would have to curve in the direction of rotation of the black hole.

(b) It is possible to follow the dashed path, but the spacecraft would have to exert thrust to counteract the effects of the spacetime curvature of the rotating black hole that make the paths of free fall have a decreasing angular coordinate.

(c) The dotted path represents an impossible trip for the spacecraft. Inside the ergosphere, no amount of thrust in the anticlockwise direction can make the spacecraft maintain a constant angular coordinate while decreasing the radial coordinate.

Exercise 6.9 The discovery of a mini black hole would imply (contrary to most expectations) that conditions during the Big Bang were such as to lead to the production of mini black holes. This would be an important development for cosmology.

Such a discovery would also open up the possibility of confirming the existence of Hawking radiation, thus giving some experimental support to attempts to weld together quantum theory and general relativity, such as string theory.

Exercise 7.1 We first need to decide how many days make up a century. This is not entirely straightforward because leap years don't simply occur every 4 years in the Gregorian calendar. However, it is the Julian year that is used in astronomy and this is defined so that one year is precisely 365.25 days. Consequently we have 36 525 days per century, which we denote by d. If we use T to denote the period of the orbit in (Julian) days, then the number of orbits per century is d/T. Equation 7.1 gives the angle in radians, but it is more usual to express the observations in seconds of arc so we need to use the fact that π radians equals 180×3600 seconds of arc. Putting all this together, we find that the general relativistic contribution to the mean rate of precession of the perihelion in seconds of arc per century is given by

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}t} &= \frac{d}{T} \times \frac{6\pi G\mathrm{M}_\odot}{a(1-e^2)c^2} \times \frac{648\,000}{\pi}\ \text{seconds of arc} = \frac{dG\mathrm{M}_\odot}{Ta(1-e^2)c^2} \times 3\,888\,000\ \text{seconds of arc}\\ &= \frac{36\,525 \times 6.673 \times 10^{-11} \times 1.989 \times 10^{30} \times 3\,888\,000}{87.969 \times 5.791 \times 10^{10} \times (1-(0.2067)^2) \times (2.998 \times 10^8)^2}\ \text{seconds of arc per century}\\ &= 42''.99\ \text{per century.}\end{aligned}$$

Exercise 7.2 For rays just grazing the Sun, b is the radius of the Sun, which is $\mathrm{R}_\odot = 6.96 \times 10^8$ m, and M is $\mathrm{M}_\odot = 1.989 \times 10^{30}$ kg. Hence the deflection in seconds of arc is given by

$$\Delta\theta = \frac{4GM_\odot}{c^2 b} \times \frac{648\,000}{\pi} \text{ seconds of arc} = \frac{6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{(2.998 \times 10^8)^2 \times 6.96 \times 10^8} \times \frac{2\,592\,000}{\pi} \text{ seconds of arc}$$
$$= 1''.75.$$

Exercise 7.3 (a) Let $R_\oplus = 6371.0\,\text{km}$ be the mean radius of the Earth, $M_\oplus = 5.9736 \times 10^{24}\,\text{kg}$ be the mass of the Earth, and $h = 20\,200\,\text{km}$ be the height of the satellite above the Earth. From Equation 5.14, the coordinate time interval at $R_\oplus$ and the coordinate time interval at $R_\oplus + h$ are related by

$$\frac{\Delta t_{R_\oplus+h}}{\Delta t_{R_\oplus}} = \left(\frac{1 - \frac{2M_\oplus G}{c^2(R_\oplus+h)}}{1 - \frac{2M_\oplus G}{c^2 R_\oplus}} \right)^{-1/2}.$$

Since the time dilation is small, we can use the first few terms of a Taylor expansion to evaluate this. Putting $2M_\oplus G/c^2(R_\oplus + h) = x$ and $2M_\oplus G/c^2 R_\oplus = y$, the right-hand side above becomes $(1 - x)^{-1/2} \times (1 - y)^{1/2}$. By a Taylor expansion, this is approximately $(1 + \frac{x}{2})(1 - \frac{y}{2}) \approx 1 + \frac{x}{2} - \frac{y}{2}$. So we have

$$\Delta t_{R_\oplus+h} \approx \left(1 + \frac{M_\oplus G}{c^2(R_\oplus + h)} - \frac{M_\oplus G}{c^2 R_\oplus}\right)\Delta t_{R_\oplus} = \Delta t_{R_\oplus} - \frac{M_\oplus G h}{c^2 R_\oplus (R_\oplus + h)}\Delta t_{R_\oplus}.$$

The discrepancy over 24 hours is given by

$$\Delta t_{R_\oplus+h} - \Delta t_{R_\oplus} = -\frac{5.9736 \times 10^{24} \times 6.673 \times 10^{-11} \times 2.02 \times 10^7}{(2.998 \times 10^8)^2 \times 6.371 \times 10^6 \times (6.371 + 20.2) \times 10^6} \times 24 \times 3600\,\text{s}$$
$$= -45.7\,\mu\text{s}.$$

The negative sign indicates that the effect of general relativity is that the satellite clock runs more rapidly than a ground-based one.

(b) Special relativity relates a time interval Δt for a clock moving at speed v with the time interval Δt_0 for one at rest by

$$\Delta t = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \Delta t_0.$$

For a satellite of mass m orbiting the Earth at a distance h from the Earth's surface, its speed v is given by

$$\frac{GM_\oplus m}{(R_\oplus + h)^2} = \frac{mv^2}{R_\oplus + h} \qquad \text{therefore} \qquad v^2 = \frac{GM_\oplus}{R_\oplus + h}$$

and hence

$$\Delta t = \left(1 - \frac{GM_\oplus}{c^2(R_\oplus + h)}\right)^{-1/2} \Delta t_0 \approx \left(1 + \frac{GM_\oplus}{2c^2(R_\oplus + h)}\right)\Delta t_0.$$

Hence the discrepancy over 24 hours between satellite- and ground-based clocks is

$$\Delta t - \Delta t_0 \approx \frac{GM_\oplus}{2c^2(R_\oplus + h)}\Delta t_0 = \frac{6.673 \times 10^{-11} \times 5.9736 \times 10^{24}}{2 \times (2.998 \times 10^8)^2 \times (6.371 + 20.2) \times 10^6} \times 24 \times 3600\,\text{s}$$
$$= 7.2\,\mu\text{s}.$$

The positive result indicates that the effect of special relativity is that the satellite clock runs slower than a ground-based one.

(c) The total effect of the results obtained in parts (a) and (b) is a discrepancy between ground-based and satellite-based clocks of $(-45.7 + 7.2) = -38.5\,\mu$s per day. Since the basis of the GPS is the accurate timing of radio pulses, over 24 hours this could lead to an error in distance of up to

$$c(\Delta t - \Delta t_0) = 2.998 \times 10^8 \times 38.5 \times 10^{-6}\,\text{m} = 11.5\,\text{km}.$$

Exercise 7.4 We can approximate the radius of the satellite's orbit by the Earth's radius. Hence the period of the orbit, T, is given by

$$T = 2\pi\sqrt{\frac{R_\oplus^3}{GM_\oplus}}.$$

Since

$$\frac{GM_\oplus}{c^2 R_\oplus} \approx 10^{-9} \ll 1,$$

Equation 7.13 can be approximated by

$$\alpha \approx 2\pi\left[1 - \left(1 - \frac{3GM_\oplus}{2c^2 R_\oplus}\right)\right] \approx 3\pi\frac{GM_\oplus}{c^2 R_\oplus}.$$

After a time Y, the number of orbits is Y/T and the total precession is given by

$$\alpha_{\text{total}} = \frac{Y}{T} \times 3\pi\frac{GM_\oplus}{c^2 R_\oplus} = \frac{Y}{2\pi}\left(\frac{GM_\oplus}{R_\oplus^3}\right)^{1/2} \times 3\pi\frac{GM_\oplus}{c^2 R_\oplus} = \frac{3Y}{2c^2}\sqrt{\frac{G^3 M_\oplus^3}{R_\oplus^5}}.$$

Converting from radians to seconds of arc, we find that the total precession angle for one year is

$$\alpha_{\text{total}} = \frac{3 \times 365.25 \times 24 \times 3600}{2 \times (2.998 \times 10^8)^2} \times \sqrt{\frac{(6.673 \times 10^{-11})^3 \times (5.974 \times 10^{24})^3}{(6.371 \times 10^6)^5}} \times \frac{180 \times 3600}{\pi} = 8''.44.$$

Exercise 7.5 We have previously carried out a similar calculation for low Earth orbit, the only difference here being that the radius of the orbit is now $R = (6.371 \times 10^6\,\text{m}) + (642 \times 10^3\,\text{m})$ instead of 6.371×10^6 m. Consequently, the expected precession is

$$8''.44 \times \left(\frac{6.371}{7.013}\right)^{5/2} = 6''.64.$$

Exercise 7.6 When considering light rays travelling from a distant source to a detector, it is not just one ray that travels from the source to the detector, but a cone of rays. Gravitational lensing effectively increases the size of the cone of rays that reach the detector. The light is *not* concentrated in the same way as in Figure 7.15, but it is concentrated.

Exercise 8.1 (i) On size scales significantly greater than 100 Mly, the large-scale structure of voids and superclusters (i.e. clusters of clusters of galaxies) does indeed appear to be homogeneous and isotropic.

(ii) After removing distortions due to local motions, the mean intensity of the cosmic microwave background radiation differs by less than one part in ten thousand in different directions. This too is evidence of isotropy and homogeneity.

(iii) The uniformity of the motion of galaxies on large scales, known as the Hubble flow, is a third piece of evidence in favour of a homogeneous and isotropic Universe.

Exercise 8.2 Geodesics are found using the geodesic equation. The first step is to identify the covariant metric coefficients of the relevant space-like hypersurface (only g_{11}, g_{22} and g_{33} will be non-zero). The contravariant form of the metric coefficients will follow immediately from the requirement that $[g_{ij}]$ is the matrix inverse of $[g^{ij}]$. The covariant and contravariant components can then be used to determine the connection coefficients $\Gamma^{i}{}_{jk}$. Once the connection coefficients for the hypersurface have been determined, the spatial geodesics may be found by solving the geodesic equation for the hypersurface. At that stage it would be sufficient to demonstrate that a parameterized path of the form $r = r(\lambda)$, $\theta =$ constant, $\phi =$ constant does indeed satisfy the geodesic equation for the hypersurface.

Exercise 8.3 The Minkowski metric differs in that it does not feature the scale factor $R(t)$. It is true that $k = 0$ for both cases, and this means that space is flat. But the presence of the scale factor in the Robertson–Walker metric allows spacetime to be non-flat.

Exercise 8.4 We start with the energy equation

$$\frac{1}{R^2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2}, \qquad \text{(Eqn 8.27)}$$

and differentiate it with respect to time t. We use the product rule on the left-hand side and obtain

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{R^2}\right) + \frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{8\pi G}{3}\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right) - kc^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{R^2}\right).$$

We then use the chain rule to replace $\frac{\mathrm{d}}{\mathrm{d}t}$ with $\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)\frac{\mathrm{d}}{\mathrm{d}R}$, which gives

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{1}{R^2}\right)+\frac{2}{R^2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right) = \frac{8\pi G}{3}\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)-kc^2\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{1}{R^2}\right).$$

Then carrying out the various differentiations with respect to R and t, we get

$$-\frac{2}{R^3}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right) + \frac{2}{R^2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^2R}{\mathrm{d}t^2}\right) = \frac{8\pi G}{3}\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right) + \frac{2kc^2}{R^3}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right).$$

We then substitute back in for $\frac{1}{R^2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2$ in the first term on the left-hand side, using the energy equation again, to get

$$-\frac{2}{R}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)\left(\frac{8\pi G\rho}{3} - \frac{kc^2}{R^2}\right)+\frac{2}{R^2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^2R}{\mathrm{d}t^2}\right) = \frac{8\pi G}{3}\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)+\frac{2kc^2}{R^3}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right).$$

We now substitute for $\frac{1}{R}\left(\frac{\mathrm{d}^2R}{\mathrm{d}t^2}\right)$ in the second term on the left-hand side, using the acceleration equation (Equation 8.28), to get

$$-\frac{2}{R}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)\left(\frac{8\pi G\rho}{3} - \frac{kc^2}{R^2}\right)+\frac{2}{R}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)\left[-\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)\right] = \frac{8\pi G}{3}\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)+\frac{2kc^2}{R^3}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right).$$

Now we collect all terms with $\frac{1}{R}\left(\frac{dR}{dt}\right)$ as a common factor, to get

$$\frac{8\pi G}{3}\left(\frac{d\rho}{dt}\right)+\frac{1}{R}\left(\frac{dR}{dt}\right)\left[\frac{2kc^2}{R^2}+\frac{16\pi G\rho}{3}-\frac{2kc^2}{R^2}+\frac{8\pi G\rho}{3}+\frac{8\pi Gp}{c^2}\right]=0.$$

The terms in $2kc^2/R^2$ cancel out, and dividing through by $\frac{8\pi G}{3}$ gives

$$\left(\frac{d\rho}{dt}\right)+\frac{1}{R}\left(\frac{dR}{dt}\right)\left[2\rho+\rho+\frac{3p}{c^2}\right]=0,$$

which clearly yields the fluid equation as required:

$$\frac{d\rho}{dt}+\left(\rho+\frac{p}{c^2}\right)\frac{3}{R}\frac{dR}{dt}=0. \qquad \text{(Eqn 8.31)}$$

Exercise 8.5 The density and pressure term in the original version of the second of the Friedmann equations (Equation 8.28) may be written as

$$\rho+\frac{3p}{c^2}=\rho_\mathrm{m}+\rho_\mathrm{r}+\rho_\Lambda+\frac{3}{c^2}\left(p_\mathrm{m}+p_\mathrm{r}+p_\Lambda\right).$$

The dark energy density term is constant (ρ_Λ), and the other density terms may be written as

$$\rho_\mathrm{m}=\rho_\mathrm{m,0}\left[\frac{R_0}{R(t)}\right]^3, \quad \rho_\mathrm{r}=\rho_\mathrm{r,0}\left[\frac{R_0}{R(t)}\right]^4.$$

The pressure due to matter is assumed to be zero (i.e. dust), the pressure due to radiation is $p_\mathrm{r}=\rho_\mathrm{r}\,c^2/3$, and the pressure due to dark energy is $p_\Lambda=-\rho_\Lambda/c^2$. Putting all this together, we have

$$\begin{aligned}\rho+\frac{3p}{c^2}&=\rho_\mathrm{m,0}\left[\frac{R_0}{R(t)}\right]^3+\rho_\mathrm{r,0}\left[\frac{R_0}{R(t)}\right]^4+\rho_\Lambda+\frac{3}{c^2}\left(0+\frac{\rho_\mathrm{r}c^2}{3}-\frac{\rho_\Lambda}{c^2}\right)\\&=\rho_\mathrm{m,0}\left[\frac{R_0}{R(t)}\right]^3+\rho_\mathrm{r,0}\left[\frac{R_0}{R(t)}\right]^4+\rho_\Lambda+\frac{3}{c^2}\left(\frac{\rho_\mathrm{r,0}c^2}{3}\left[\frac{R_0}{R(t)}\right]^4-\frac{\rho_\Lambda}{c^2}\right)\\&=\rho_\mathrm{m,0}\left[\frac{R_0}{R(t)}\right]^3+2\rho_\mathrm{r,0}\left[\frac{R_0}{R(t)}\right]^4-2\rho_\Lambda, \quad \text{as required.}\end{aligned}$$

Exercise 8.6 (a) Substituting the proposed solution into the differential equation, we have

$$\frac{d}{dt}\left(R_0(2H_0t)^{1/2}\right)=\sqrt{\frac{8\pi G}{3}\rho_\mathrm{r,0}}\,\frac{R_0^2}{R_0(2H_0t)^{1/2}}.$$

Evaluating the derivative, we get

$$R_0(2H_0)^{1/2}\frac{1}{2t^{1/2}}=\sqrt{\frac{8\pi G}{3}\rho_\mathrm{r,0}}\,\frac{R_0}{(2H_0)^{1/2}\,t^{1/2}}.$$

Cancelling the factor $R_0/t^{1/2}$ on both sides and collecting terms in H_0, this yields

$$H_0=\sqrt{\frac{8\pi G}{3}\rho_\mathrm{r,0}}, \quad \text{as required.}$$

(b) Using the definition of the Hubble parameter,

$$H(t)=\frac{1}{R}\frac{dR}{dt},$$

we substitute in for $R(t)$ from the proposed solution to get

$$H(t) = \left(\frac{1}{R_0(2H_0t)^{1/2}}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(R_0(2H_0t)^{1/2}\right) = \left(\frac{1}{R_0(2H_0t)^{1/2}}\right)\frac{R_0(2H_0)^{1/2}}{2t^{1/2}} = \frac{1}{2t},$$

as required. Hence $H_0 = 1/2t_0$, and substituting this into the proposed solution gives

$$R(t_0) = R_0(2H_0t_0)^{1/2} = R_0\left(\frac{2t_0}{2t_0}\right)^{1/2} = R_0,$$

again as required.

Exercise 8.7 Setting $\mathrm{d}R/\mathrm{d}t = 0$ and $\rho_{\mathrm{m},0} = 0$ in the first Friedmann equation implies that

$$0 = \frac{8\pi G}{3}\left[\rho_{\mathrm{m},0}\left[\frac{R_0}{R(t)}\right]^3 + \rho_\Lambda\right] - \frac{kc^2}{R^2}.$$

But we already know from Equation 8.50 that ρ_Λ and $\rho_{\mathrm{m},0}$ must have the same sign in this case. Consequently, k must be positive and hence equal to $+1$. Using Equation 8.50, and the first Friedmann equation at $t = t_0$, we can therefore write

$$\frac{8\pi G}{3}\left[\frac{3\rho_{\mathrm{m},0}}{2}\right] = \frac{c^2}{R_0^2},$$

leading immediately to the required result

$$R_0 = \left(\frac{c^2}{4\pi G\rho_{\mathrm{m},0}}\right)^{1/2}.$$

Inserting values for G and c, along with the quoted approximate value for the current mean cosmic density of matter, gives $R_0 = 1.8 \times 10^{26}$ m. Since $1\,\mathrm{ly} = 9.46 \times 10^{15}$ m, it follows that, in round figures, $R_0 = 20\,000$ Mly in this static model. Recalling that a parsec is 3.26 light-years, we can also say, roughly speaking, that in the Einstein model, for the given matter density, R_0 is about 6000 Mpc.

Exercise 8.8 The condition for an expanding FRW model to be accelerating at time t_0 is that $\frac{1}{R}\frac{\mathrm{d}^2R}{\mathrm{d}t^2}$ should be positive at that time. We already know from Equation 8.50 that the condition for it to vanish is that

$$\Omega_{\Lambda,0} = \frac{\Omega_{\mathrm{m},0}}{2}.$$

Examining the equation, it is clear that the condition that we now seek is

$$\Omega_{\Lambda,0} \geq \frac{\Omega_{\mathrm{m},0}}{2}.$$

Exercise 8.9 In the $\Omega_{\Lambda,0}$–$\Omega_{\mathrm{m},0}$ plane, the dividing line between the $k = +1$ and $k = -1$ models corresponds to the condition for $k = 0$. This is the condition that the density should have the critical value $\rho_\mathrm{c}(t) = 3H^2(t)/8\pi G$, and may be expressed in terms of $\Omega_{\Lambda,0}$ and $\Omega_{\mathrm{m},0}$ as

$$\Omega_{\mathrm{m},0} + \Omega_{\Lambda,0} = 1.$$

(i) The de Sitter model is at the point $\Omega_{\mathrm{m},0} = 0$, $\Omega_{\Lambda,0} = 1$.

(ii) The Einstein–de Sitter model is at the point $\Omega_{\mathrm{m},0} = 1$, $\Omega_{\Lambda,0} = 0$.

(iii) The Einstein model has a location that depends on the value of $\Omega_{\mathrm{m},0}$, so in the $\Omega_{\Lambda,0}$–$\Omega_{\mathrm{m},0}$ plane it is represented by the line $\Omega_{\Lambda,0} = \Omega_{\mathrm{m},0}/2$, which coincides with the dividing line between accelerating and decelerating models.

Exercise 8.10 The scale change $R(t_{\mathrm{ob}})/R(t_{\mathrm{em}})$ shows up in extragalactic redshift measurements because the light has been 'in transit' for a long time as space has expanded. To measure changes in $R(t)$ locally requires our measuring equipment to be in free fall, far from any non-gravitational forces that would mask the effects of general relativity. However, the large aggregates of matter within our galaxy distort spacetime locally and create a gravitational redshift that would almost certainly mask the effects of cosmic expansion on the wavelength of light. Nearby stars simply will not participate in the cosmic expansion due to these local effects. Thus a local measurement would not be expected to reveal the changing scale factor — any more than a survey of the irregularities on your kitchen floor would reveal the curvature of the Earth.

Exercise 8.11 The figure of 5 billion light-years relates to the proper distances of sources at the time of emission. For sources at redshifts of 2 or 3, as in the case of Figure 8.2, the current proper distances of the sources are between about 16 and 25 billion light-years. The distances quoted in Figure 8.2 indicate that, in a field such as relativistic cosmology where there are many different kinds of distance, there is a problem of converting measured quantities such as redshifts into 'deduced' quantities such as distances. When such deduced quantities are used, it is always necessary to provide clear information about their precise meaning if they are to be properly interpreted.

Exercise 8.12 Historically, the discovery of the Friedmann–Robertson–Walker models was a rather tortuous process. Einstein initiated relativistic cosmology with his 1917 proposal of a static cosmological model. Einstein's model featured a positively curved space ($k = +1$) and used the repulsive effect of a positive cosmological constant Λ to balance the gravitational effect of a homogeneous distribution of matter of density ρ_{m}. Later in the same year, Willem de Sitter introduced the first model of an expanding Universe, effectively introducing the scale factor $R(t)$, though he did not present his model in that way. De Sitter's model included flat space ($k = 0$), and a cosmological constant but no matter, so there was nothing to oppose a continuously accelerating expansion of space. In 1922, Alexander Friedmann, a mathematician from St Petersburg, published a general analysis of cosmological models with $k = +1$ and $k = 0$, showing that the models of Einstein and de Sitter were special cases of a broad family of models. He published a similar analysis of $k = -1$ models in 1924. Together, these two publications introduced all the basic features of the Robertson–Walker spacetime but they were based on some specific assumptions that detracted from their appeal. In 1927 Lemaître introduced a model that was supported by Eddington, in which expansion could start from a pre-existing Einstein model. Lemaître later (1933) proposed a model that would be categorized nowadays as a variant of Big Bang theory and he became interested in models that started from $R = 0$.
By 1936 Robertson and Walker had completed their essentially mathematical investigations of homogeneous relativistic spacetimes, giving Friedmann's ideas a

more rigorous basis and associating their names with the metric. This set the scene for the naming of the Friedmann–Robertson–Walker models. (Sometimes they are referred to as Lemaître–Friedmann–Robertson–Walker models.)