A Level Mathematics for OCR A
Student Book 2 (Year 2) with Cambridge Elevate edition

This Student Book with Cambridge Elevate edition has been specifically written for the OCR A Level Mathematics A specification, for first teaching from 2017. Developed by a highly experienced author team with a wealth of maths teaching expertise, teachers and students can be confident that the content matches the course requirements.

- Overarching themes of problem-solving, proof and modelling are built into every chapter, with additional Focus on pages to further develop students’ proficiency in these key mathematical skills.
- A synoptic approach underpins all the resources. This includes cross-topic review exercises and fast forward/rewind signposts to reinforce concepts and connect Pure and Applied topics.
- An extensive question bank includes drill, discussion, synoptic, past paper and a wealth of practice questions, which are colour-coded for different skill levels.
- Continual assessment is embedded at chapter, strand and course level to help build student and teacher confidence with the new linear assessment.
- Further requirements of the specification are fulfilled through opportunities to practise incorporating technology throughout.
- The unique Work it out feature challenges common misconceptions.

Cambridge Elevate editions are customisable and interactive, allowing students and teachers to annotate text, add audio notes and link out to external resources. Includes worked solutions to all practice questions. Final answers to all exercises are located at the back of the Student Book.

Visit cambridge.org/education for details of all AS/A Level Mathematics resources. Please note, the Student Book has been endorsed by OCR.
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Introduction

You have probably been told that mathematics is very useful, yet it can often seem like a lot of techniques that just have to be learnt to answer examination questions. You are now getting to the point where you will start to see where some of these techniques can be applied in solving real problems. However, as well as seeing how maths can be useful we hope that anyone working through this book will realise that it can also be incredibly frustrating, surprising and ultimately beautiful.

The book is woven around three key themes from the new curriculum:

Proof
Maths is valued because it trains you to think logically and communicate precisely. At a high level, maths is far less concerned about answers and more about the clear communication of ideas. It is not about being neat – although that might help! It is about creating a coherent argument that other people can easily follow but find difficult to refute. Have you ever tried looking at your own work? If you cannot follow it yourself it is unlikely anybody else will be able to understand it. In maths we communicate using a variety of means – feel free to use combinations of diagrams, words and algebra to aid your argument. And once you have attempted a proof, try presenting it to your peers. Look critically (but positively) at some other people’s attempts. It is only through having your own attempts evaluated and trying to find flaws in other proofs that you will develop sophisticated mathematical thinking. This is why we have included lots of common errors in our ‘work it out’ boxes – just in case your friends don’t make any mistakes!

Problem solving
Maths is valued because it trains you to look at situations in unusual, creative ways, to persevere and to evaluate solutions along the way. We have been heavily influenced by a great mathematician and maths educator, George Polya, who believed that students were not just born with problem solving skills – they were developed by seeing problems being solved and reflecting on their solutions before trying similar problems. You may not realise it but good mathematicians spend most of their time being stuck. You need to spend some time on problems you can’t do, trying out different possibilities. If after a while you have not cracked it then look at the solution and try a similar problem. Don’t be disheartened if you cannot get it immediately – in fact, the longer you spend puzzling over a problem the more you will learn from the solution. You may never need to integrate a rational function in future, but we firmly believe that the problem solving skills you will develop by trying it can be applied to many other situations.

Modelling
Maths is valued because it helps us solve real-world problems. However, maths describes ideal situations and the real world is messy! Modelling is about deciding on the important features needed to describe the essence of a situation and turning that into a mathematical form, then using it to make predictions, compare to reality and possibly improve the model. In many situations the technical maths is actually the easy part – especially with modern technology. Deciding which features of reality to include or ignore and anticipating the consequences of these decisions is the hard part. Yet it is amazing how some fairly drastic assumptions – such as pretending a car is a single point or that people’s votes are independent – can result in models that are surprisingly accurate.

More than anything else, this book is about making links. Links between the different chapters, the topics covered and the themes above, links to other subjects and links to the real world. We hope that you will grow to see maths as one great complex but beautiful web of interlinking ideas.

Maths is about so much more than examinations, but we hope that if you take on board these ideas (and do plenty of practice!) you will find maths examinations a much more approachable and possibly even enjoyable experience. However, always remember that the results of what you write down in a few hours by yourself in silence under exam conditions is not the only measure you should consider when judging your mathematical ability – it is only one variable in a much more complicated mathematical model!
How to use this book

Throughout this book you will notice particular features that are designed to aid your learning. This section provides a brief overview of these features.

Learning objectives
A short summary of the content that you will learn in each chapter.

WORKED EXAMPLE
The left-hand side shows you how to set out your working. The right-hand side explains the more difficult steps and helps you understand why a particular method was chosen.

PROOF

WORK IT OUT
Can you identify the correct solution and find the mistakes in the two incorrect solutions?

Before you start...
Points you should know from your previous learning and questions to check that you’re ready to start the chapter.

Key point
A summary of the most important methods, facts and formulae.

Explore
Ideas for activities and investigations to extend your understanding of the topic.

Tip
Useful guidance, including on ways of calculating or checking answers and using technology.

Each chapter ends with a Checklist of learning and understanding and a Mixed practice exercise, which includes past paper questions marked with the icon 📘.

In between chapters, you will find extra sections that bring together topics in a more synoptic way.

Focus on ...
Unique sections relating to the preceding chapters that develop your skills in proof, problem solving and modelling.

CROSS-TOPIC REVIEW EXERCISE
Questions covering topics from across the preceding chapters, testing your ability to apply what you have learnt.

You will find Paper 1, Paper 2 and Paper 3 practice questions towards the end of the book, as well as a glossary of key terms (picked out in colour within the chapters), and answers to all questions. Fully worked solutions can be found on the Cambridge Elevate digital platform, along with other essential resources such as a digital version of this Student Book.
Maths is all about making links, which is why throughout this book you will find signposts emphasising connections between different topics, applications and suggestions for further research.

**how to use this book**

**black** – drill questions. These come in several parts, each with subparts i and ii. You only need attempt subpart i at first; subpart ii is essentially the same question, which you can use for further practice if you got part i wrong, for homework, or when you revisit the exercise during revision.

**green** – practice questions at a basic level.
**blue** – practice questions at an intermediate level.
**red** – practice questions at an advanced level.
**yellow** – designed to encourage reflection and discussion.

Some of the links point to the material available only through the Cambridge Elevate digital platform.

**Elevate**

A support sheet for each chapter contains further worked examples and exercises on the most common question types. Extension sheets provide further challenge for the more ambitious.

**Gateway to A Level**

GCSE transition material that provides a summary of facts and methods you need to know before you start a new topic, with worked examples and practice questions.

**Colour-coding of exercises**

The questions in the exercises are designed to provide careful progression, ranging from basic fluency to practice questions. They are uniquely colour-coded, as shown below.

1. A sequence is defined by \( u_n = 2 \times 3^{-n} \). Use the principle of mathematical induction to prove that \( u_1 + u_2 + \ldots + u_n = 3^1 - 1 \).
2. Show that \( 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \).
3. Show that \( 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \).
4. Prove by induction that \( \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1} \).
5. Prove by induction that \( \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{n^3} = \frac{n}{n+1} \).

6. Prove that \( 1 + 1 \times 3! + 2 \times 3! + 3 \times 3! + \ldots + n \times 3! = (n+1)! - 1 \).
7. Use the principle of mathematical induction to show that \( 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = \frac{n(n+1)}{2} \).
8. Prove that \( (n+1) + (n+2) + (n+3) + \ldots + (2n) = \frac{n(3n+1)}{2} \).
9. Prove using induction that \( \sin \theta + \sin 3\theta + \ldots + \sin (2n-1)\theta = \frac{\sin \theta \cdot n}{\sin \theta / 2} \).
10. Prove that \( \sum_{k=1}^{n} k^2 = (n - 1) 2^{m+1} + 2 \).

**rewind**

Reminders of where to find useful information from earlier in your study.

**fast forward**

Links to topics that you may cover in greater detail later in your study.

**focus on ...**

Links to problem solving, modelling or proof exercises that relate to the topic currently being studied.

**did you know?**

Interesting or historical information and links with other subjects to improve your awareness about how mathematics contributes to society.
Working with the large data set

As part of your course you are expected to work with the large data set covering different methods of transport and age distributions in different parts of the country and in different years. This large data set is an opportunity to explore statistics in real life. As well as supporting the ideas introduced in Chapters 16 and 18 we shall be using the large data set to guide you through four key themes. All of these themes will be explored with examples and questions in the large data set section in the Cambridge Elevate edition. You will not have to work with the full data set in the final examination, but familiarity with it will help you as many examination questions will be set in the context of this data set.

Practical difficulties with data
Unlike most textbook or examination problems, the real world is messy. Often there are difficulties with being overwhelmed by too much data, or perhaps there are errors, missing items or labels which are ambiguous. For example, how do you deal with the fact that in 2001 Cornwall was made up of separate districts that were later combined into a single unitary authority, if you want to compare areas over time? If you are grouping data for a histogram, how big a difference does it make where you choose to put the class boundaries?

Using technology
Modern statistics is heavily based on familiarity with technology. We will be encouraging you to use spreadsheets and graphing packages, looking at the common tools available to help simplify calculations and present data effectively.

One important technique we can employ with modern technology is simulation. We will try to gain a better understanding of hypothesis testing by using the data set to simulate the effect of sampling on making inferences about the population.

Thinking critically about statistics
Why might someone want to use a pie chart rather than a histogram? Whenever statistics are calculated or data sets are represented graphically, some information is lost and some information is highlighted. An important part of modern statistics is to ask critical questions about the way evidence provided by statistics is used to support arguments.

One important part of this is the idea of validating statistics. For example, with the information presented it is not clear which category or categories a person would be included in if they travel to work by bicycle on some days and take the bus on others. We will look at ways in which we can interrogate the data to try to understand it more.

Statistical problem solving
Technology can often do calculations for us. However the art of modern statistics is deciding what calculations to do on what data. One of the big difficulties is that we rarely have exactly the data we want, so we have to make indirect inferences from the data we have. For example, you will probably not see newspaper headlines saying ‘the correlation coefficient between median age and percentage of people cycling to work is 0.64’, but you might see something saying ‘Pensioners promote pedalling!’ Deciding on an appropriate statistical technique to determine whether older people are more likely to use a bicycle and then interpreting results is the type of thing which is hard to examine but very valuable in real-world statistics.

There are lots of decisions to be made. Should you use the total number of cyclists in an area? Or the percentage of people who cycle? Or the percentage of people who travel to work who cycle? We shall see how the answer to your main question depends on decisions like these.
1 Proof and mathematical communication

In this chapter you will:

- review proof by deduction, proof by exhaustion and disproof by counter example
- learn a new method of proof called proof by contradiction
- practise criticising proofs.

Before you start…

| Student Book 1, Chapter 1 | You should be able to use logical connectors. | 1 Insert either $\Rightarrow$, $\Leftarrow$ or $\Leftrightarrow$ in the places marked A and B:  
\[ x^2 - 1 = 8 \]
\[ A \quad x^2 = 9 \]
\[ B \quad x = 3 \]  |
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<td>Student Book 1, Chapter 1</td>
<td>You should be able to use proof by exhaustion.</td>
<td>4 Use proof by exhaustion to prove that 17 is a prime number.</td>
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Developing proof

One of the purposes of this chapter is to act as revision of the material from Student Book 1. It draws on all chapters from that book but, in particular, it builds on the fundamental ideas of proof from Chapter 1. This chapter introduces a new and very powerful method of proof that mathematicians often rely on: proof by contradiction.

Section 1: A reminder of methods of proof

In Student Book 1 you met proof by deduction, proof by exhaustion and disproof by counter example. The following questions show how these methods can be used in topics from throughout Student Book 1.
EXERCISE 1A

1 Use proof by exhaustion to prove that 17 is a prime number.
2 Prove by exhaustion that all square numbers end in 0, 1, 4, 5, 6 or 9.
3 The velocity of a particle after time $t$ is given by $v = t^2 + 3$. Prove that the particle never returns to its original position.
4 a Prove from first principles that if $y = x^2$ then $\frac{dy}{dx} = 2x$.
   b Use a counter example to show that if $\frac{dy}{dx} = 2x$, then it is not necessarily true that $y = x^2$.
5 Use a counter example to show that the following statement is not true:
   $\sin 2x = 1 \Rightarrow x = 45^\circ$
6 Prove that $\binom{n}{1} = n$.
7 A set of data has mean $A$, mode $B$ and median $C$. Consider the following statement: $B < A \Rightarrow C < A$.
   Prove this statement or use a counter example to disprove it.
8 The diagram shows triangle $OAB$, where $AB$ lies on the circle with centre $O$. $M$ is the midpoint of $AB$.
   a Use the cosine rule to prove that $AB = \sqrt{2r^2 - 2r^2 \cos(2\theta)}$.
   b Show that $AM = r \sin \theta$.
   c Hence, prove that $\cos 2\theta = 1 - 2 \sin^2 \theta$.
9 In quadrilateral $OABC$, $O$ is the origin and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the position vectors of points $A, B,$ and $C$. $P$ is the midpoint of $OA$, $Q$ is the midpoint of $AB$, $R$ is the midpoint of $BC$ and $S$ is the midpoint of $OC$.
   a Show that $PQ = \frac{1}{2} \mathbf{b}$.
   b Hence, prove that $PQRS$ is a parallelogram.
   c If $PQRS$ is a rectangle, what can be said about the quadrilateral $OABC$?
10 a Use a counter example to disprove the statement $\ln (x + y) = \ln x + \ln y$.
   b Prove that if $\ln (x + y) = \ln x + \ln y$, then $y = \frac{x}{x-1}$.
   c If $y = \frac{x}{x-1}$, does it follow that $\ln (x + y) = \ln x + \ln y$?
11 a Show that $a^3 + 1 = (a + 1)(a^2 + ka + 1)$, where $k$ is a constant to be determined.
   b Hence, prove that $a^3 + 1$ is prime if and only if $a = 1$.
12 Prove algebraically that if $X \sim \text{B}(n, p)$, then the sum of the probabilities of the different values $X$ can take is 1.
Section 2: Proof by contradiction

Proof by contradiction starts from the opposite of the statement you are trying to prove, and shows that this results in an impossible conclusion.

WORKED EXAMPLE 1.1

Use proof by contradiction to prove that there are an infinite number of prime numbers.

Assume that there is a largest prime number, \( P \).

Construct another number, \( N \), that is the product of all the prime numbers up to and including \( P \).

Consider \( N + 1 \). This is 1 greater than a number divisible by all the primes up to and including \( P \), so it cannot be divisible by any of the primes up to and including \( P \).

Therefore, \( N + 1 \) is either itself prime, or is divisible by primes larger than \( P \).

Either way, you have shown that there must be a prime larger than \( P \). This contradicts the premise that there is a largest prime number.

Therefore, there are an infinite number of prime numbers.

Proof by contradiction always starts by assuming the opposite of what you want to prove.

Now set about trying to find a larger prime than \( P \).

Here, the contradiction to the original assumption (that there is a largest prime) occurs.

Did you know?

A variation on the previous proof can be found in Euclid’s masterpiece *The Elements*, a textbook written in approximately 300 BCE but still in use in many schools in the first half of the twentieth century!
Prove that $\sqrt{2}$ is irrational. You may use the fact that if $a^2$ is even, so is $a$.

**Assume that** $\sqrt{2} = \frac{p}{q}$  

where $p$ and $q$ are integers with no common factors.

**Squaring both sides gives:**  

$$2 = \frac{p^2}{q^2}$$  

$\iff p^2 = 2q^2$ (\*)

This means that $p^2$ is even so $p$ must also be even.

You can then write that $p = 2k$, for some integer $k$, so $p^2 = 4k^2$.

Substituting this into the equation marked (\*) gives:  

$$4k^2 = 2q^2$$  

$\iff 2k^2 = q^2$  

This means that $q^2$ is even, so $q$ must be even.

But you have shown that both $p$ and $q$ are even, so they share a factor of 2. This contradicts the original assertion (that $p$ and $q$ are integers with no common factors), so it must be incorrect.

Therefore, $\sqrt{2}$ is irrational and cannot be written as $\frac{p}{q}$.

---

**EXERCISE 1B**

1. Prove that if $n^2$ is even, then $n$ is also even.
2. Prove that $\sqrt{3}$ is irrational.
3. Prove that there are infinitely many even numbers.
4. Prove that the sum of a rational and irrational number is irrational.
5. Prove that if $ab$ is even with $a$, $b$ integers, then at least one of them is even.
6. Prove that $\sqrt{3}$ is irrational.
7. Prove that $\log_2 3$ is irrational.

---

**Elevate**

See Support sheet 1 for an example of the same type as Question 7 and further practice questions on proof by contradiction.
8 Suppose that \( n \) is a composite integer. Prove that \( n \) has a prime factor less than or equal to \( \sqrt{n} \).

9 Prove that if any 25 different dates are chosen, some 3 must be within the same month.

10 Prove that the value of \( a^2 - 4b^2 \) is never 2 if \( a \) and \( b \) are whole numbers.

11 a Show that if \( x = \frac{p}{q} \) is a solution to the equation \( x^3 + x + 1 = 0 \), then \( p^3 + pq^2 + q^3 = 0 \).

b Explain why there is no solution to this equation if \( p \) is odd or \( q \) is odd.

c Prove that there are no rational solutions to \( x^3 + x + 1 = 0 \).

12 Prove that if a triangle has sides \( a, b \) and \( c \) such that \( a^2 + b^2 = c^2 \), then it is a right-angled triangle.

Section 3: Criticising proofs

In Student Book 1 you were introduced to the notation used in logic:

\( A \iff B \) means that statements \( A \) and \( B \) are equivalent.

\( A \implies B \) means if \( A \) is true, then so is \( B \).

\( A \impliedby B \) means if \( B \) is true, then so is \( A \).

When checking a proof (including solving equations, which is a type of proof) you have probably looked out for errors in things like arithmetic or algebra. You now need to also look out for errors in logic.

**WORKED EXAMPLE 1.3**

Yas was solving the equation \( 2 \log_{10} x = 4 \). Find the errors in her working.

1 \( 2 \log_{10} x = 4 \)

2 \( \iff \log_{10} (x^2) = 4 \)

3 \( \implies x^2 = 10^4 = 10000 \)

4 \( \iff x = \pm 1000 \)

**On line 2, the symbol should be \( \implies \): if \( x \) is negative, line 2 could be correct but line 1 is not possible.**

**In line 3 the symbol should be \( \iff \): if \( x^2 = 10^4 \), then \( \log_{10} (x^2) = 4 \).**

**In line 4 the positive square root of 10 000 should be 100.**

**Because one of the implications goes only one way, the final solutions might not work in the original equation. They should be checked.**

This is an error in logic.

This is an error in logic.

This is an arithmetic error.

This is an error in logic. Even if Yas had not made the arithmetic error, she would still need to state that \( x = -100 \) is not a valid solution, because a negative number cannot be substituted into the original equation.
EXERCISE 1C

1. Lambert was asked to solve the equation \( x = \sqrt{3x+4} \). Here is his working:

1. \( x = \sqrt{3x+4} \)
2. \( \iff x^2 = 3x + 4 \)
3. \( \iff x^2 - 3x - 4 = 0 \)
4. \( \iff (x - 4)(x + 1) = 0 \)
5. \( \iff x = 4 \text{ or } x = -1 \)

a. By checking his working, find the correct solution.

b. In which line of working is his mistake?

2. Craig was asked to solve the equation \( x^2 = 3x \). Here is his solution:

1. \( x^2 = 3x \)
2. \( \iff x = 3 \)

a. Show that \( x = 0 \) is also a solution to the original equation.

b. What logical symbol should Craig have used in the second line?

3. Freja was asked to solve the equation \( x - \frac{1}{x} = 1 + \frac{5-2x}{x-3} \). Here is her working:

1. \( \iff x - \frac{1}{x} = 1 + \frac{5-2x}{x-3} \)
2. \( \iff x - \frac{6-2x}{x-3} \)
3. \( \iff (x-1)(x-3) = 6-2x \)
4. \( \iff x^2 - 4x + 3 = 6 - 2x \)
5. \( \iff x^2 - 2x - 3 = 0 \)
6. \( \iff (x-3)(x+1) = 0 \)
7. \( \iff x = 3 \text{ or } x = -1 \)

a. By checking her working, find the correct solution.

b. In which line of working is her mistake?

4. Jamie was asked to solve \( \log_2(-x) + \log_2(2 - x) = 3 \). Here is her working:

1. \( \log_2(-x) + \log_2(2 - x) = 3 \)
2. \( \iff \log_2(-x(2 - x)) = 3 \)
3. \( \iff \log_2(x^2 - 2x) = 3 \)
4. \( \iff x^2 - 2x = 2^3 \)
5. \( \iff x^2 - 2x - 8 = 0 \)
6. \( \iff (x - 4)(x + 2) = 0 \)
7. \( \iff x = 4 \text{ or } x = -2 \)

In which line of working did Jamie make a mistake?
Andrew was asked to prove the following statement:
The function \( y = x^3 - 3x \) has a minimum at \( x = 1 \). His working is shown below.

1. \( \frac{dy}{dx} = 3x^2 - 3 = 0 \)
2. \( x^2 = 1 \)
3. \( x = 1 \)
4. \( \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) \)
5. \( = \frac{d}{dx} (0) \)
6. \( = 0 \)
7. So it is a minimum.

Describe the errors in this proof.

Criticise the following proof of the statement:
If \( x + q \) is a factor of \( x^3 + px + q \), then the other factor is \( x^2 - qx + 1 \).

If \( x + q \) is a factor of \( x^3 + px + q \), then you can write:

1. \( x^3 + px + q = (x + q)(x^2 + bx + 1) \)
2. \( \equiv x^3 + x^2(b + q) + x(bq + 1) + q \)
3. Comparing coefficients of \( x^2 \): \( 0 = b + q \)
4. Therefore, the remaining factor is \( x^2 - qx + 1 \).

Find the error in the following proof that \( \sqrt{16} \) is irrational.

1. Assume that \( \sqrt{16} = \frac{p}{q} \), where \( p \) and \( q \) are integers with no common factors.
2. Squaring both sides gives \( 16 = \frac{p^2}{q^2} \).
3. So \( p^2 = 16q^2 \) (*).
4. This means that \( p^2 \) is even, so \( p \) must also be even.
5. You can then write that \( p = 2k \), so \( p^2 = 4k^2 \).
6. Substituting this into the equation marked (*) gives \( 4k^2 = 16q^2 \).
7. So \( k^2 = 4q^2 \).
8. This means that \( q^2 \) is even, so \( q \) must be even.
9. But you have shown that both \( p \) and \( q \) are even, so they share a factor of 2.
10. This contradicts the original assertion, so it must be incorrect. This means that \( \sqrt{16} \) cannot be written as \( \frac{p}{q} \).

Checklist of learning and understanding

- You should be able to apply counter examples, proof by exhaustion and proof by deduction to material from Student Book 1.
- Proof by contradiction is a method of proof that works by showing that assuming the opposite of the required statement leads to an impossible situation.
- When criticising proofs, look out for flaws in logic as well as mistakes in algebra or arithmetic.
Mixed practice 1

1. Prove that \( n^2 - n \) is always even.
2. Prove that \( \sqrt{5} \) is irrational.
3. Prove that there are infinitely many square numbers.
4. Find the error(s) in the following working to solve \( \tan x = 2 \sin x \) for \( 0^\circ \leq x < 360^\circ \):
   1. \( \frac{\sin x}{\cos x} = 2 \sin x \)
   2. \( \Leftrightarrow \sin x = 2 \sin x \cos x \)
   3. \( \Leftrightarrow 1 = 2 \cos x \)
   4. \( \Leftrightarrow 0.5 = \cos x \)
   5. \( \Leftrightarrow x = 60^\circ \)
5. Choosing from options A to D, which symbol should be used to replace \( \? \) in the working below?
   A \( \Rightarrow \) B \( \Leftrightarrow \) C \( \Leftarrow \) D \( \equiv \)
   1. \( x^2 = 8x \)
   2. \( \? \ x = 8 \)
6. Prove that \( \log_2 5 \) is irrational.
7. \( OABC \) is a parallelogram with \( O \) at the origin, and \( a, b, c \) are the position vectors of points \( A, B, \) and \( C \). \( P \) is the midpoint of \( BC \) and \( Q \) is the point on \( OB \) such that \( OQ : QB = 2 : 1 \).
   Prove that \( AQP \) is a straight line.
8. Prove that if \( a \) and \( b \) are whole numbers, then \( a^2 - b^2 \) is either odd or a multiple of 4.
9. a. Prove that if \( f(x) \) is a polynomial of finite order with integer coefficients and \( n \) is an integer, then \( f(n) \) is an integer.
   b. Use a counter example to show that the following statement is not always correct:
      If \( f(x) \) is a polynomial, where \( f(n) \) is an integer whenever \( n \) is an integer, then \( f(x) \) must have integer coefficients.
10. Consider the following working to solve \( x + \frac{4x}{x-2} = \frac{8}{x-2} \):
    1. \( x + \frac{4x}{x-2} = \frac{8}{x-2} \)
    2. \( \Leftrightarrow x(x-2) + 4x = 8 \)
    3. \( \Leftrightarrow x^2 - 2x + 4x = 8 \)
    4. \( \Leftrightarrow x^2 + 2x = 8 \)
    5. \( \Leftrightarrow (x + 1)^2 = 9 \)
    6. \( \Leftrightarrow x + 1 = 3 \)
    7. \( \Leftrightarrow x = 2 \)
   a. In which lines are there mistakes?
   b. Rewrite the solution correctly, making appropriate use of logical connectors.
Fermat said that if \( x \) is prime then \( 2x + 1 \) is prime. Find the smallest value of \( x \) that provides a counter example to this statement.

The proof below is trying to demonstrate that there are an arbitrary number of consecutive composite (non-prime) numbers.

A  Consider \( n! \) for \( n \geq r \geq 1 \).

B  \( n! + r \) is divisible by \( r \).

C  So the numbers \( n! + 1, n! + 2 \ldots n! + n \) are not prime.

D  This is a list of \( n \) consecutive composite numbers.

Which is the first line to contain an error?

Prove that if \( a, b \) and \( c \) are integers such that \( a^2 + b^2 = c^2 \), then either \( a \) or \( b \) is even.

a  By considering a right-angled triangle, prove that if \( A \) is an acute angle, then \( \tan(90^\circ - A) = \frac{1}{\tan A} \).

b  Hence, prove that \( \tan 10^\circ \times \tan 20^\circ \times \tan 30^\circ \ldots \times \tan 80^\circ \) is a rational number.

Prove that for \( x \) between \( 90^\circ \) and \( 180^\circ \), \( \sin x - \cos x \geq 1 \).

See Extension sheet 1 to complete the details of a couple of famous proofs.
Before you start...

1. Given that \( f(x) = 2 - x \), evaluate:
   a. \( f(3) \)
   b. \( f(-4) \)

2. Write the following sets using the interval notation.
   a. \( \{ x : x > 3 \text{ and } x \leq 6 \} \)
   b. \( \{ x : x < 3 \text{ or } x \geq 6 \} \)

3. a. Express \( f(x) = x^2 + 5x + 3 \) in the form \( (x + a)^2 + b \).
    b. Hence, state the coordinates of the turning point of \( f(x) \).

4. Solve the inequality \( x^2 - 4x - 5 > 0 \).

5. Make \( x \) the subject of the following.
   a. \( y = e^{2x-1} \)
   b. \( y = \ln(3x + 4) \)

6. Find the range of \( x \) values for which \( f(x) = x^3 - 2x \) is an increasing function.

Why study functions?

Doubling, adding 5, finding the largest prime factor – these are all instructions that can be applied to numbers to produce a numerical result. This idea comes up a lot in mathematics. The formal study of it leads to the concept of a function.
Functions can be used whenever you need to express how one quantity changes with another, whether it is how the strength of the gravitational force varies with distance, or how the amount of paint needed depends on the area of a wall.

In this chapter we will focus on developing the theory of functions. You have already seen many examples of modelling with linear, quadratic and exponential functions in Student Book 1. In Chapter 7 of this book you will meet further models using trigonometric functions.

**Section 1: Mappings and functions**

A **mapping** takes numbers from a given set (inputs) and assigns to each of them one or more output values, using a mapping rule. For example:

- \( x \mapsto x^2 + 3, x \in \mathbb{R} \) maps 4 to 19, 0 to 3, and \(-2.1\) to 7.41.
- \( x \mapsto \pm 3x, x \in \mathbb{Z} \) maps 2 to 6 and \(-6\), \(-4\) to 12 and \(-12\), and 0 to 0.
- \( x \mapsto \) a factor of \( x, x \in \{1, 2, 3, 4, 5, 6\} \) maps 1 to 1, 5 to 1 and 5, and 6 to 1, 2, 3 and 6.

We can represent a mapping using the arrow notation, as shown above, or we can give a name to the output value and write, for example, \( y = x^2 + 3, x \in \mathbb{R} \). We can also use a mapping diagram:

![Mapping Diagram](image)

Another useful representation is on a graph, where the input values are shown on the horizontal axis and the output values on the vertical axis.

You should remember that, to fully define a mapping, you need to state the set of the input values as well as the mapping rule. This set of all possible input values is called the **domain** of the mapping. For example, the mapping \( x \mapsto \pm 3x, x \in \{-1, 0, 1, 2\} \) is different from the mapping \( x \mapsto \pm 3x, x \in \mathbb{R} \), as can be seen from their graphs:

![Graphs](image)

**Did you know?**

There is a more general way to describe a relationship between two sets, called a **relation**, where the output does not need to be written explicitly in terms of the input. For example, a relation could be given by \( \{(x, y): x^2 + y^2 = 4\} \); you already know that the graph of this relation is a circle. Although you will not study relations in much detail, you will learn how to find gradients of some curves defined by relations (see Chapter 10, Section 4).

**Fast forward**

You will learn more about domains in Section 2.
A mapping rule can assign more than one output to each input. A special type of mapping, where each input value has only one output, is called a **function**. Of the three mappings given previously, only \( x \mapsto x^2 + 3, \ x \in \mathbb{R} \) is a function.

When a mapping is given by its graph, the easiest way to decide whether or not it is a function is to carry out a **vertical line test**:

![Graphs showing function and not a function](image)

**Key point 2.1**

- A mapping is a function if every input value maps to a single output value.
- **Vertical line test**: if a mapping is a function, any vertical line will meet its graph at most once.

Functions are often named with letters, such as \( f \) or \( g \). For example, \( f : x \mapsto x^2 + 3, \ x \in \mathbb{R} \). You can also use the function notation \( f(x) = x^2 + 3, \ x \in \mathbb{R} \). Then \( f(1) = 4, f(0) = 3 \) and \( f(-2.1) = 7.41 \); we say that 7.41 is the **image** of \(-2.1\).

Having decided that a mapping is a function, you can ask whether each output comes from just one input. To check this, you can apply the **horizontal line test**:

![Graphs showing many-one and one-one](image)
A mapping where a single input corresponds to more than one output (so is not a function) is called one–many.

A function is:
- **one–one** if every $y$ value corresponds to only one $x$ value.
- **many–one** if there is at least one $y$ value that comes from more than one $x$ value.

Horizontal line test: if a function is one–one, any horizontal line will meet the graph at most once.

**Key point 2.2**

**WORKED EXAMPLE 2.1**

Which of the graphs below could represent functions? For those that could be functions, classify them as one–one or many–one.

Any vertical line meets the graph at most once; therefore it could be a function.
At least one of the horizontal lines meets the graph at more than one point; therefore it could be a many-one function.

Draw several horizontal lines and see how many times they cross the graph.

At least one of the vertical lines meets the graph more than once; therefore it is not a function.

Draw several vertical lines and see how many times they cross the graph.

Any vertical line meets the graph at most once; therefore it could be a function.

Draw several vertical lines and see how many times they cross the graph. Remember that a vertical line through an open circle does not count as an intersection.

Tip
An open circle on a graph means that that point is not a part of the graph, and a closed circle means that it is.
Any horizontal line meets the graph at most once; therefore it could be a one–one function.

When you are given a rule for a function, rather than its graph, it can be more difficult to decide whether the function is one–one. If a function is not one–one, you need to find only one example of an output value with two corresponding input values. To prove that a function is one–one, you need to rearrange the equation \( y = f(x) \) and show that each \( y \) comes from only one \( x \).

You should remember that whether a function is one–one depends on its domain, as well as the function rule.

**WORKED EXAMPLE 2.2**

- a The function \( f \) is defined for all real numbers \( x \) and has the rule \( f(x) = x^2 + 3 \). Show that \( f \) is not a one–one function.

- b The function \( g \) also has the rule \( g(x) = x^2 + 3 \) but its domain is \( x > 0 \). Show that \( g \) is a one–one function.

- a \( f(-1) = f(1) = 4 \)
  
  Two inputs have the same output, hence \( f \) is not a one–one function.

- b Let \( y = g(x) \). Then:
  
  \[
  y = x^2 + 3 \\
  \iff x^2 = y - 3 \\
  \iff x = \pm \sqrt{y - 3}
  
  However, \( x > 0 \) so \( x = \sqrt{y - 3} \).

  Each \( y \) value comes from just one \( x \) value, so \( g \) is a one–one function.

- Look for two input values that have the same output. In this case, you know that squaring a number gives the same result as squaring its negative.

- You need to show that each possible \( y \) comes from only one \( x \). One way to do this is to find \( x \) in terms of \( y \).

- This equation appears to give two \( x \) values for each \( y \) value. However, the domain of \( g \) is \( x > 0 \), so \( x \) must be positive.
EXERCISE 2A

1 Which of these graphs could not represent functions? For each one that could represent a function, state if it is one–one or many–one.

   a
   b
   c
   d
   e
   f

2 Determine whether each of the following functions is one–one.
   a i \( f(x) = x^2 - 2, x < 3 \)    ii \( f(x) = x^2 + 5, x > -2 \)
   b i \( g(x) = x^2 - 3, x \leq -2 \)    ii \( g(x) = x^3 + 1, x \geq 3 \)
   c i \( h(x) = x^3 + 5, x > 1 \)    ii \( h(x) = x^3 - 2, x \leq 0 \)

Section 2: Domain and range

The previous section mentioned that, in order to fully define a function, you must specify both the set of allowed inputs and the rule that tells you what to do with each input.

Key point 2.3

The set of allowed input values is called the domain of the function.

WORKED EXAMPLE 2.3

Sketch the graph of \( f(x) = x + 1 \) over the domain:
   a \( x \in \mathbb{R}, x > 2 \)    b \( x \in \mathbb{Z} \)

Tip

Remember that \( \mathbb{R} \) stands for the set of all real numbers and \( \mathbb{Z} \) for the set of whole numbers.
Sketch the graph over the domain $x \in \mathbb{R}$.

Discard the part of the graph that is outside the required domain. Since the end point is not included you must label it with an open circle.

Use the same original graph as in part a, but this time include only integer values.

If no domain is explicitly mentioned, you can assume that the domain is the largest possible set of real numbers for which the values are defined. You may wonder why you would ever need any other domain. There are two main reasons.

1. You may be modelling a physical situation where the variables can take only particular values; for example, if the variable is age of humans, you would not want it to be negative or much beyond 120.

2. The mathematical operation you are using may not be able to handle certain types of numbers. For example, if you are looking for the largest prime factor of a number, you would normally only be looking at positive integers. When working with real numbers, the four most common reasons to restrict the domain are:
   - You cannot divide by zero.
   - You cannot take a square root of a negative number.
   - You cannot take the logarithm of a non-positive number.
   - You cannot find the tan or cot of certain angles (for example, tan 90 or cot $\theta$).

Tip

Unless told otherwise, you can assume that the domain is a subset of real numbers. For example, $x > 2$ in fact means $x \in \mathbb{R}$ and $x > 2$. 

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WORKED EXAMPLE 2.4

Find the largest possible domain of a function with the rule \( f(x) = \ln(4-2x) \).

You need \( 4-2x > 0 \).

So the largest possible domain is \( x < 2 \).

WORKED EXAMPLE 2.5

A function, \( g \), is defined on the domain \([0, k) \cup (k, 180] \) and given by the rule \( g: x \mapsto \tan((x-30)^\circ) \). Find the value of \( k \).

\[
\tan((x-30)^\circ) = \frac{\sin((x-30)^\circ)}{\cos((x-30)^\circ)}
\]

Look for any values of \( x \) for which \( \tan((x-30)^\circ) \) is not defined.

Not defined when \( \cos((x-30)^\circ) = 0 \).

This is when \( x-30 = 90 \),

so \( x = 120 \).

Hence, \( k = 120 \).

Tip

Remember that \( f: x \mapsto x + 3 \) is just an alternative notation for \( f(x) = x + 3 \).

WORKED EXAMPLE 2.6

What is the largest possible domain of a function with the rule \( h: x \mapsto \frac{1}{x-2} + \sqrt{x+3} \)? Write your answer using interval notation.

There will be division by zero when \( x - 2 = 0 \).

Look for division by zero.

There will be a square root of a negative number when \( x + 3 < 0 \).

Look for a square root of a negative number.

The largest possible domain is: \( x \geq -3 \) and \( x \neq 2 \)

Decide what can therefore be allowed into the function.

Hence, \( x \in [-3,2) \cup (2,\infty) \)

This describes two intervals: from \(-3\) to \(2\) and from \(2\) to infinity, excluding \(2\).
As well as knowing the set of inputs for a function (the domain), it is useful to identify the set of possible outputs.

**Key point 2.4**

The set of all possible outputs of a function is called the **range**.

The easiest way of finding the range is to sketch the graph.

**WORKED EXAMPLE 2.7**

A function, \( f \), is given by the rule \( f(x) = x^2 + 3 \). Find its range if the domain is:

a. \( x \in \mathbb{R} \)

b. \( x > 2 \)

**a**

Sketch the graph \( y = f(x) \).

**Range:** \( f(x) \geq 3 \)

Use the graph to state which \( y \) values can occur.

**b**

Sketch the graph \( y = f(x) \).

**Range:** \( f(x) > 7 \)

Use the graph to state which \( y \) values can occur.
EXERCISE 2B

In this exercise, unless otherwise stated, \( x \) is a real number.

1. Find the largest possible domain, and the corresponding range, of the functions with the following rules.
   a i \( f(x) = 2^x \) ii \( f(x) = a^x, \ a > 0 \)
   b i \( f(x) = \log_{10} x \) ii \( f(x) = \log_a x, \ b > 1 \)

2. Find the largest possible domain of the functions with the following rules.
   a i \( f(x) = \frac{1}{x+2} \) ii \( f(x) = \frac{5}{x-7} \)
   b i \( f(x) = \frac{3}{x-2)(x+4) \) ii \( g(x) = \frac{x}{x^2 - 9} \)
   c i \( r(y) = \sqrt{y^2 - 1} \) ii \( h(x) = \sqrt{x+3} \)
   d i \( f(a) = \frac{1}{\sqrt{a-1}} \) ii \( f(x) = \frac{5x}{\sqrt{2-5x}} \)
   e i \( a(x) = \frac{1+x}{x+1} \) ii \( f(x) = \sqrt{x+1} + \frac{1}{x+2} \)
   f i \( f(x) = \sqrt{x+1} + \frac{1}{x+3} - x^3 + 5 \) ii \( f(x) = e^x + \sqrt{2x+3} - \frac{1}{x^2 + 4} - 2 \)

3. Find the range of the following functions.
   a i \( f(x) = 7 - x^2, \ x \in \mathbb{R} \) ii \( f(x) = x^2 + 3, \ x \in \mathbb{R} \)
   b i \( g(x) = x^2 + 3, \ x \geq 3 \) ii \( g(x) = x^2 - 1, x < -3 \)
   c i \( h(x) = x - 2, x < 5, x \in \mathbb{Z} \) ii \( h(x) = x + 1, x > 3, x \in \mathbb{Z} \)
   d i \( d(x) = \frac{1}{x}, x \geq -1, x \neq 0 \) ii \( q(x) = 3\sqrt{x}, x > 0 \)

4. Find the largest possible domain and the corresponding range of the functions with the following rules.
   a i \( f(x) = x^2 - 4x - 1 \) ii \( f(x) = x^2 + 2x + 5 \)
   b i \( g: x \mapsto 5 - x^2 \) ii \( g: x \mapsto 3 - 2x^2 \)
   c i \( f(x) = \sqrt{x^2 - 5} \) ii \( f(x) = \sqrt{9 - x^2} \)
   d i \( f: x \mapsto 2\sqrt{x^2 - 6x + 8} \) ii \( f: x \mapsto 4\sqrt{x^2 + 2x - 3} \)

5. a Write \( 2x^2 + 6x - 3 \) in the form \( a(x + p)^2 + q \).
   b Hence, state the range of the function \( f: x \mapsto 2x^2 + 6x - 3, x \in \mathbb{R} \).

6. Find the largest possible domain and the corresponding range of the function with the rule \( g(x) = \ln(6 + 4x) \).

7. The function \( f \) has the rule \( f(x) = \sqrt{\ln(x - 4)} \). Find the largest possible domain of this function.

8. Find the largest possible domain of the function with the rule \( f(x) = \frac{4\sqrt{x-1}}{x+2} - \frac{1}{x^2 - 5x + 6} + x^2 + 1 \).
9. a Sketch the graph of \( y = 6 - x - 2x^2 \).

   b Hence, find the largest possible domain of the function with the rule \( f: x \to \sqrt{6 - x - 2x^2} \).

10. Find the largest possible domain of the function with the rule \( g(x) = \ln(x^2 + 3x + 2) \).

11. Find the largest set of real values of \( x \) such that the function \( f \) with the rule \( f(x) = \sqrt{\frac{8x - 4}{x - 12}} \) takes real values.

12. a State the largest possible domain of the function with the rule \( f(x) = \sqrt{x - a + \ln(b - x)} \) if:

   i \( a < b \)

   ii \( a \geq b \)

   b Find \( f(a) \) in each of the two cases.

Section 3: Composite functions

After applying a function to a number it is possible to apply another function to the image. The resulting rule is called a composite function.

Key point 2.5

Applying the function \( g \) to \( x \) and then the function \( f \) to the result is written:

\[ f(g(x)) \] or \[ f \circ g \] or \[ f \circ g \]

It can be useful to refer to \( g(x) \) as the inner function and \( f(x) \) as the outer function.

WORKED EXAMPLE 2.8

If \( f(x) = x^2 \) and \( g(x) = x - 3 \), find:

a \( f \circ g(1) \)  

b \( f \circ g(x) \)  

c \( g \circ f(x) \)

\[ g(1) = 1 - 3 = -2 \]

\[ f(-2) = (-2)^2 = 4 \]

\[ \therefore f(g(1)) = 4 \]

\[ f(g(x)) = f(x - 3) \]

\[ = (x - 3)^2 = x^2 - 6x + 9 \]

\[ g(f(x)) = g(x^2) = x^2 - 3 \]

Tip

Evaluate \( g(1) \) and then apply \( f \) to the result.

Note that you don’t need to work out the general expression for \( f \circ g(x) \).

Replace \( x \) in \( f(x) \) with the expression for \( g(x) \).

Replace \( x \) in \( g(x) \) with the expression for \( f(x) \).

Tip

Be careful: \( f(g(x)) \) and \( g(f(x)) \) are not the same function.
Two functions are defined for all real numbers by \( f(x) = 3x - 2 \) and \( g(x) = x^2 - 1 \). Find \( f \circ g (5) \).

Which of the following solutions is correct? Identify the mistake in the other two.

<table>
<thead>
<tr>
<th>Solution 1</th>
<th>Solution 2</th>
<th>Solution 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(5) = 15 - 2 = 13 )</td>
<td>( g(5) = 25 - 1 = 24 )</td>
<td>( f(5) = 15 - 2 = 13 )</td>
</tr>
<tr>
<td>( g(5) = 25 - 1 = 24 )</td>
<td>( f(24) = 72 - 2 = 70 )</td>
<td>( f \circ g (5) = 13^2 - 1 = 168 )</td>
</tr>
<tr>
<td>( f(5) g(5) = 312 )</td>
<td>( g(5) = 25 - 1 = 24 )</td>
<td>( f \circ g (5) = 13^2 - 1 = 168 )</td>
</tr>
</tbody>
</table>

You can also compose a function with itself.

**Key point 2.6**

\( f \circ f (x) \) can also be written as \( f^2 (x) \).

**WORKED EXAMPLE 2.9**

Given that \( f(x) = \frac{2}{x - 3} \), find and simplify an expression for \( f^2(x) \).

\[
f^2(x) = f(f(x))
\]

\[
= \frac{2}{x - 3}
\]

\[
= \frac{2(x - 3)}{2 - 3(x - 3)}
\]

\[
= \frac{2x - 6}{11 - 3x}
\]

For the composite function \( fg \) to exist, the range of \( g \) must lie entirely within the domain of \( f \), otherwise you would be trying to put values into \( f \) that it cannot take.

In the diagram above, the large blue rectangle represents the domain of \( g \). Its image (which is the range of \( g \)) is represented by the smaller blue rectangle. The larger red rectangle represents the domain of \( f \).
WORKED EXAMPLE 2.10

The functions \( f \) and \( g \) are defined by \( f: x \mapsto x^2 - 5, \ x \in \mathbb{R} \) and \( g: x \mapsto \sqrt{x + 3}, \ x \geq -3 \).

**a** Explain why the composite function \( g f \) is not defined.

**b** Find the largest possible domain for which \( g f \) is defined. In this case, state the range of \( g f \).

**a** You need \( f(x) \geq -3 \).

Check whether the output from \( f \) is within the domain of \( g \).

But, for example, \( f(0) = -5 \), and this is not in the domain of \( g \).

**b** \( x^2 - 5 \geq -3 \)

\[ \Leftrightarrow x^2 \geq 2 \]

\[ \Leftrightarrow x \leq -\sqrt{2} \text{ or } x \geq \sqrt{2} \]

This is a quadratic inequality; the solution consists of two separate intervals.

The range of \( g f \) is \( g f(x) \geq 0 \).

With the domain above, \( f(x) \) takes all real values \( \geq -3 \), so \( g(f(x)) = \sqrt{(f(x) + 3)} \) can be any non-negative real number.

A more complex problem is to recover one of the original functions when you have a composite function. A good way to do this is to use a substitution.

WORKED EXAMPLE 2.11

If \( f(x + 1) = 4x^2 + x \), find the expression for \( f(x) \) in the form \( ax^2 + bx + c \).

\[ y = x + 1 \]

Substitute \( y \) for the inner function.

\[ x = y - 1 \]

Rearrange to get \( x = \ldots \).

\[ f(y) = 4(y - 1)^2 + (y - 1) \]

Replace all instances of \( x \).

\[ = 4y^2 - 8y + 4 + y - 1 \]

\[ = 4y^2 - 7y + 3 \]

\[ f(x) = 4x^2 - 7x + 3 \]

Write the answer in terms of \( x \).
In this exercise, functions are defined with their largest possible real domain, unless specified otherwise.

1. Given that \( f(x) = x^2 + 1 \) and \( g(x) = 3x + 2 \), find:
   a) \( g(f(0)) \)
   b) \( f \circ g(-2) \)

2. Given that \( f(x) = x^2 + 1 \) and \( g(x) = 3x + 2 \), find:
   a) \( g(f(x)) \)
   b) \( f \circ g(x) \)

3. Given that \( f(x) = x^2 + 1 \) and \( g(x) = 3x + 2 \), find:
   a) \( g^2(\sqrt{a} + 1) \)
   b) \( f^2(y - 1) \)

4. Given that \( f(x) = x^2 + 1 \) and \( g(x) = 3x + 2 \), find:
   a) \( g(f(x - 2)) \)
   b) \( g(f(3 - x)) \)

5. Find \( f(x) \), given the following conditions.
   a) \( f(2a) = 4a^2 \)
   b) \( f(x + 1) = 3x - 2 \)
   c) \( f(1 - y) = 5 - y \)
   d) \( f(e^x) = \ln k \)
   b) \( f(x) = 3x + 2 \), solve the equation \( f(g(x)) = g(f(x)) \).

6. Given that \( f(x) = \frac{x}{x^2 + 25} \), solve the equation \( g(f(x)) = 0 \).

7. The function \( f \) is defined by \( f: x \rightarrow x^3 \). Find an expression for \( g(x) \) in terms of \( x \) in each of the following cases.
   a) \( (f \circ g)(x) = 2x + 3 \)
   b) \( (g \circ f)(x) = 2x + 3 \)

8. Let \( f \) and \( g \) be two functions. Given that \( (f \circ g)(x) = \frac{x + 2}{3} \) and \( g(x) = 2x + 5 \), find an expression for \( f(x - 1) \).
**Section 4: Inverse functions**

Functions transform an input into an output, but sometimes you want to reverse this process: to be able to say which input produced a given output. When this is possible, it is done by finding the **inverse function**, usually labelled $f^{-1}$.

For example, if $f(x) = 3x$, then $f^{-1}(12)$ is a number that, when put into $f$, produces output 12. In other words, you are looking for a number $x$ such that $f(x) = 12$. Hence, $f^{-1}(12) = 4$.

**Tip**

Make sure you don’t get confused about this notation. With numbers, the superscript ‘$-1$’ denotes reciprocal; for example, $x^{-1} = \frac{1}{x}$, $3^{-1} = \frac{1}{3}$.

With functions, $f^{-1}$ denotes the inverse function of $f$.

**Finding the inverse function**

To find the inverse function you must rearrange the formula to find the input ($x$) in terms of the output ($y$).

**Key point 2.7**

To find the expression for inverse function $f^{-1}(x)$, given an expression for $f(x)$:

1. Start with $y = f(x)$.
2. Rearrange to get $x$ (the input) in terms of $y$ (the output).
3. Give $f^{-1}(x)$ by replacing every instance of $y$ with $x$.

**WORKED EXAMPLE 2.12**

The function $f$ is defined for $x > -4$ by the rule $f(x) = 3 \ln (x + 4)$. Find an expression for $f^{-1}(x)$.

$y = 3 \ln (x + 4) \\
\iff \frac{y}{3} = \ln (x + 4) \\
\iff x + 4 = e^{\frac{y}{3}} \\
\iff x = e^{\frac{y}{3}} - 4 \\
$ $f^{-1}(x) = e^{\frac{y}{3}} - 4$

Start with $y = f(x)$.

Rearrange to make $x$ the subject.

Taking $e$ to the power of both sides removes the logarithm.

Write the resulting function in terms of $x$. 

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WORK IT OUT 2.2

Find the rule for the inverse function of \( f(x) = \frac{3x - 1}{x + 4}, \ x \neq 4 \).

Which of the following solutions is correct? Identify the mistake in the other two.

<table>
<thead>
<tr>
<th>Solution 1</th>
<th>Solution 2</th>
<th>Solution 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \frac{3x - 1}{x + 4} )</td>
<td>( f(x) = \frac{3x - 1}{x + 4} )</td>
<td>( y = \frac{3x - 1}{x + 4} )</td>
</tr>
<tr>
<td>( \Rightarrow xy + 4y = 3x - 1 )</td>
<td>( \Rightarrow f^{-1}(x) = \frac{x + 4}{3x - 1} )</td>
<td>( \Rightarrow xy + 4y = 3x - 1 )</td>
</tr>
<tr>
<td>( \Rightarrow x( y-3) = -1 - 4y )</td>
<td></td>
<td>( \Rightarrow xy = 3x - 1 - 4y )</td>
</tr>
<tr>
<td></td>
<td>( \Rightarrow x = \frac{4y + 1}{3y} )</td>
<td>( \Rightarrow x = 3x - 1 - 4y )</td>
</tr>
<tr>
<td>( \therefore f^{-1}(x) = \frac{4x + 1}{3 - x} )</td>
<td>( \therefore f^{-1}(x) = \frac{x + 4}{3x - 1} )</td>
<td>( \therefore f^{-1}(x) = 3x - 5 )</td>
</tr>
</tbody>
</table>

The relationship between \( f \) and \( f^{-1} \)

Once you know how to find inverse functions, there are a couple of very important facts you need to know about them.

When you are finding the inverse function you switch the inputs and the outputs, so on the graph you switch the \( x \)- and \( y \)-axes:

Key point 2.8

The graph of \( y = f^{-1}(x) \) is a reflection of the graph of \( y = f(x) \) in the line \( y = x \).

When you do and undo a function, you get back to where you started.

Key point 2.9

\[ f\left[f^{-1}(x)\right] = f^{-1}(f(x)) = x \]
The fact that the graphs of \( f \) and \( f^{-1} \) are reflections of each other gives you a very useful trick to solve some equations that would otherwise involve complicated (or impossible) algebra.
In this question you must show detailed reasoning.

This diagram shows a part of the graph of function \( f(x) = \frac{1}{27}x^3 + x - 8, x \in \mathbb{R} \).

**Tip**

‘You must show detailed reasoning’ means that you must solve equations algebraically rather than, for example, using intersections of the graph on your calculator.

---

**a** On the same axes, sketch the graph of \( y = f^{-1}(x) \).

**b** Solve the equation \( f(x) = f^{-1}(x) \).

---

The graph of \( f^{-1}(x) \) is the reflection of the graph of \( y = f(x) \) in the line \( y = x \).

Finding the equation for the inverse function would involve solving a cubic equation, and you don’t know how to do that.

Luckily, you can see from the graph that the graphs of \( f \) and \( f^{-1} \) intersect on the line \( y = x \). This means that solving the equation \( f = f^{-1} \) is equivalent to solving the equation \( f(x) = x \).

The reflection in the line \( y = x \) swaps the domain and the range of a function (because it swaps \( x \) and \( y \) coordinates).

---

**Key point 2.10**

- The domain of \( f^{-1}(x) \) is the same as the range of \( f(x) \).
- The range of \( f^{-1}(x) \) is the same as the domain of \( f(x) \).
The function \( f \) is defined by \( f(x) = \frac{1+x}{3-x} \) for \( x \neq 3 \).

a Find an expression for \( f^{-1}(x) \) and state its domain and range.

b State the range of \( f \).

\[ a \]
\[
\begin{align*}
y &= \frac{1+x}{3-x} \\
y(3-x) &= 1+x \\
3y-yx &= 1+x \\
3y-1 &= x+xy \\
3y-1 &= x(1+y) \\
x &= \frac{3y-1}{1+y} \\
\end{align*}
\]

\[ f^{-1}(x) = \frac{3x-1}{1+x} \]

The domain of \( f^{-1} \) is \( x \neq -1 \).

The range of \( f^{-1} \) is \( f^{-1}(x) \neq 3 \).

\[ b \]
\[ \text{The range of } f \text{ is } f(x) \neq -1. \]

EXERCISE 2D

Find an expression for \( f^{-1}(x) \) if:

1. \[ a \]
   \[ i \] \( f(x) = 3x+1 \)
   \[ ii \] \( f(x) = 7x-3 \)

   \[ b \]
   \[ i \] \( f(x) = \frac{2x}{3x-2}, x \neq \frac{2}{3} \)
   \[ ii \] \( f(x) = \frac{x}{2x+1} \)

   \[ c \]
   \[ i \] \( f(x) = \frac{x-a}{x-b}, x \neq b \)
   \[ ii \] \( f(x) = \frac{ax-1}{bx-1}, x \neq \frac{1}{b} \)

   \[ d \]
   \[ i \] \( f(a) = 1-a \)
   \[ ii \] \( f(y) = 3y+2 \)

   \[ e \]
   \[ i \] \( f(x) = \sqrt{3x-2}, x \geq \frac{2}{3} \)
   \[ ii \] \( f(x) = \sqrt{2-5x}, x \leq \frac{2}{5} \)

   \[ f \]
   \[ i \] \( f(x) = \ln(1-5x), x < 0.2 \)
   \[ ii \] \( f(x) = \ln(2x+2), x > -1 \)

   \[ g \]
   \[ i \] \( f(x) = 7e^x \)
   \[ ii \] \( f(x) = 9e^{10x} \)

   \[ h \]
   \[ i \] \( f(x) = x^2 - 10x + 6, x > 5 \)
   \[ ii \] \( f(x) = x^2 + 6x - 1, x > 0 \)
2 Sketch the inverses of the following functions.

\[ a \]

\[ y \]

\[ b \]

\[ y \]

\[ c \]

\[ y \]

\[ d \]

\[ y \]

3 Each of the following functions is defined on the largest possible real domain such that the inverse function exists. Find the expression for \( f^{-1} \) in each case. State the domain and range for both \( f \) and \( f^{-1} \).

\[ a \]

\[ i \] \( f(x) = \frac{3x - 1}{x + 2} \)

\[ ii \] \( f(x) = \frac{2x + 3}{x - 2} \)

\[ b \]

\[ i \] \( f(x) = 2 - 3\sqrt{3x} \)

\[ ii \] \( f(x) = \frac{1}{2} \sqrt{4 - 3x} + 1 \)

\[ c \]

\[ i \] \( f(x) = 3 + \ln(4x - 3) \)

\[ ii \] \( f(x) = 2\ln(x + 3) - 1 \)

\[ d \]

\[ i \] \( f(x) = 3 - 2e^{x/2} \)

\[ ii \] \( f(x) = 3e^{x/2} + 1 \)

4 Below is a table giving selected values of the one–one function \( f(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

a Evaluate \( f(2) \). 

b Evaluate \( f^{-1}(3) \).

5 In this question you must show detailed reasoning.

The function \( f \) is defined by \( f: x \mapsto \sqrt{3 - 2x} \) for \( x \leq \frac{3}{2} \).

Evaluate \( f^{-1}(7) \).

6 Given that \( f(x) = 3e^{2x} \):

a Find the inverse function \( f^{-1}(x) \).

b State the domain and range of \( f^{-1} \).

7 Given functions \( f: x \mapsto 2x + 3 \) and \( g: x \mapsto x^3 \), find the function \((f \circ g)^{-1}\).
8 Let \( f \) and \( g \) be two functions such that \( f \circ g \) is defined, and suppose that both \( f^{-1} \) and \( g^{-1} \) exist. Let \( h(x) = f \circ g (x) \). Prove that \( h^{-1}(x) = g^{-1} \circ f^{-1}(x) \).

9 The diagram shows the graph of \( y = f(x) \). The lines \( y = -9 \) and \( y = 9 \) are the asymptotes of the graph.
   a. Copy the graph and, on the same axes, sketch the graph of \( y = f^{-1}(x) \).
   b. State the domain and range of \( f^{-1} \).
   c. Write down the solutions to the equation \( f(x) = f^{-1}(x) \).

10 The functions \( f \) and \( g \) are defined by \( f: x \mapsto e^{2x} \) and \( g: x \mapsto x + 1 \).
   a. Calculate \( f^{-1}(3) \times g^{-1}(3) \).
   b. Show that \( (f \circ g)^{-1}(3) = \ln \sqrt{2} - 1 \).

11 The function \( f \) is defined for \( x \leq 0 \) by \( f(x) = \frac{x^2 - 4}{x^2 + 9} \). Find an expression for \( f^{-1}(x) \).

12 Let \( f(x) = \ln(x - 1) + \ln 3 \), for \( x > 1 \).
   a. Find \( f^{-1}(x) \).
      Let \( g(x) = e^x \) for \( x \in \mathbb{R} \).
   b. Find \( (g \circ f)(x) \), giving your answer in the form \( ax + b \), where \( a, b \in \mathbb{Z} \). Find the domain and range of \( g \circ f \).

13 In this question you must show detailed reasoning.
   The function \( f \) is defined by \( f(x) = \frac{\sqrt{3}x^3 + 30x - 45}{x} \) for \( x \in \mathbb{R} \). The graph of \( y = f(x) \) is shown in the diagram.
   a. On the same set of axes, sketch the graph of \( y = f^{-1}(x) \).
   b. Solve the equation \( f(x) = f^{-1}(x) \).

When does the inverse function exist?

All functions have inverse mappings, but these inverse mappings are not necessarily themselves functions. Since an inverse function is a reflection in the line \( y = x \), for the result to pass the vertical line test the original function must pass the horizontal line test. But, as you saw in Section 1, this means it must be a one–one function.
Only one–one functions have inverse functions.

This leads to one of the most important uses of domains. By restricting the domain you can turn any function into a one–one function, which allows you to find its inverse function.

**WORKED EXAMPLE 2.16**

a. Find the largest value of $k$ such that the function $f(x) = (x - 3)^2$, $x \leq k$ is one–one.

b. For this value of $k$, find $f^{-1}(x)$ and state its range.

---

**Sketch the graph of $y = (x - 3)^2$, $x \in \mathbb{R}$.**

**Eliminate the points towards the right of the graph which cause the horizontal line test to fail.**

**Decide which section remains.**

**Follow standard procedure for finding inverse functions.**

**Use the fact that $x \leq 3$ to decide which root to take.**

**Write $f^{-1}(x)$.**

**The range of $f^{-1}$ is $f^{-1}(x) \leq 3$.**

The range of $f^{-1}$ is the domain of $f$. 

---

**Since $x \leq 3$, $x = 3 - \sqrt{y}$.**

**$f^{-1}(x) = 3 - \sqrt{x}$.**
WORK IT OUT 2.3

What is the inverse function of \( f(x) = x^2 - 3, x \in \mathbb{R} \)?

Which of the following solutions is correct? Identify the mistake in the other two.

<table>
<thead>
<tr>
<th>Solution 1</th>
<th>Solution 2</th>
<th>Solution 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = x^2 - 3 )</td>
<td>( y = x^2 - 3 )</td>
<td>It doesn’t exist.</td>
</tr>
<tr>
<td>( \Rightarrow x^2 = y + 3 )</td>
<td>( \Rightarrow x^2 = y + 3 )</td>
<td>( \Rightarrow x = \sqrt{y + 3} )</td>
</tr>
<tr>
<td>( \Rightarrow x = \pm \sqrt{y + 3} )</td>
<td>( \Rightarrow x = \sqrt{y + 3} )</td>
<td>( \therefore f^{-1}(x) = \pm \sqrt{x + 3} )</td>
</tr>
<tr>
<td>( \therefore f^{-1}(x) = \pm \sqrt{x + 3} )</td>
<td>( \therefore f^{-1}(x) = \sqrt{x + 3} )</td>
<td></td>
</tr>
</tbody>
</table>

It should be clear from the horizontal line test that if a function, either increases or decreases throughout its domain, then it is one-one. As soon as there is a turning point, the function is no longer one-one (and therefore has no inverse).

WORKED EXAMPLE 2.17

In this question you must show detailed reasoning.

If \( f(x) = x^2 - 8x + 18 \), for \( x \geq 0 \), prove that \( f \) has an inverse function.

\[
\begin{align*}
f'(x) &= 2x - 8x + 18 \\
&= 2(x - 6x + 9) \\
&= 2(x^2 - 3)^2 \geq 0 \\
\text{for all } x \geq 0. \\
\therefore \text{ } &f \text{ is an increasing function} \\
\text{for all } x \geq 0.
\end{align*}
\]

Hence, \( f \) is one-one and so has an inverse function.

There are two things to notice in problems about one-one functions. First of all, in Worked Example 2.17, the fact that the gradient is \( \geq 0 \) allowed you to conclude that the function is always increasing. This may not be true if its graph has asymptotes or other breaks, as illustrated by the two diagrams below. The graphs have positive gradients but are not always increasing.
Secondly, although a function increasing (or decreasing) implies that it is one–one, the converse is not true: a one–one function can have both increasing and decreasing sections. The diagram below shows the graph of the function
\[ f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 3 - x, & 1 \leq x \leq 2 \end{cases} \]
You can use the horizontal line test to show that this function is one–one, and therefore has an inverse function.

Most functions you will meet in this course will not require you to consider these issues, but it is useful to be aware of them.

**EXERCISE 2E**

1. In the following situations, find the value of \( k \) that gives the largest possible domain such that the inverse function exists. For this domain, find the inverse function.
   a. \( y = x^2, x \leq k \)
   b. \( y = (x + 1)^2 + 2, x > k \)
   c. \( y = 5 + 2x - x^3, x \leq k \)
   d. \( y = x^3 + 4x + 3, x > k \)

2. For each function shown in the diagrams below, determine a possible domain of the given form for which the inverse function exists.
   a. Domain: \( x \leq k \)
b  Domain: \( x \in [a, b] \)

\[
\begin{align*}
\text{Graph of } f(x) &= e^x - 3x \\
&\text{for } x \in \mathbb{R}.
\end{align*}
\]

c  Domain: \( a \leq x < b \)

\[
\begin{align*}
\text{Graph of } f(x) &= e^x - 3x \\
&\text{for } x \in \mathbb{R}.
\end{align*}
\]

3  Given that \( f(x) = e^x - 3x \) for \( x \in \mathbb{R} \):
   a  Find \( f'(x) \).
   b  Explain why \( f^{-1}(x) \) does not exist.

The domain of \( f \) is changed to \( x \leq k \) so that \( f^{-1} \) now exists.

c  Find the largest possible value of \( k \).

4  A function is called \textit{self-inverse} if \( f(x) = f^{-1}(x) \) for all \( x \) in the domain. Find the value of the constant \( k \) so that \( g(x) = \frac{3x - 5}{x + k} \) is a self-inverse function.

5  A function is defined by \( f(x) = x^3 + 3ax^2 + 3ax + 1 \), where \( a \) is a constant. Find the set of values of \( a \) for which \( f^{-1}(x) \) exists for all \( x \).
Checklist of learning and understanding

- A mapping takes input values from a given set and maps each one of them to one or more output values.
- A mapping is a function if every \( x \) value maps to a single \( y \) value.
- The output value corresponding to the input \( x \) is called the image of \( x \).
- The set of all allowed input values of a mapping is called the domain.
- The set of all possible outputs of a mapping is called the range.
- A function is:
  - one–one if every \( y \) value corresponds to only one \( x \) value.
  - many–one if at least one \( y \) value that come from more than one \( x \) value.
- The vertical and horizontal line tests can be applied to graphs of mappings.
  - Vertical line test: if a mapping is a function, any vertical line will meet its graph at most once.
    - If a mapping fails a vertical line test, it is called one–many and it is not a function.
  - Horizontal line test: if a function is one–one, any horizontal line will meet the graph at most once.
- The composite function formed by applying \( g \) to \( x \) and then \( f \) to the result is written as:
  \[ f \circ g(x) \]
- The inverse, \( f^{-1} \), of a function \( f \) is such that:
  \[ f(f^{-1}(x)) = f^{-1}(f(x)) = x \]
- To find the expression for the inverse function \( f^{-1}(x) \), given an expression for \( f(x) \):
  - Start with \( y = f(x) \).
  - Rearrange to get \( x \) (the input) in terms of \( y \) (the output).
  - Obtain \( f^{-1}(x) \) by replacing every instance of \( y \) with \( x \).
  - Only one–one functions have inverse functions.
  - The graph of \( y = f^{-1}(x) \) is a reflection of the graph of \( y = f(x) \) in the line \( y = x \).
    - The domain of \( f^{-1} \) is the same as the range of \( f \).
    - The range of \( f^{-1} \) is the same as the domain of \( f \).
Mixed practice 2

In this exercise, if a domain of a function is not stated, you may assume that it is all real numbers.

1. Find the inverse of the following functions.
   a. \( f(x) = \log_5(x + 3), \ x > -3 \)
   b. \( f(x) = 3e^{x-1}, \ x \in \mathbb{R} \)

2. The diagram shows three graphs.

   A is part of the graph of \( y = x \).
   B is part of the graph of \( y = 2^x \).
   C is the reflection of graph B in line A.

Write down:
   a. the equation of C in the form \( y = f(x) \)
   b. the coordinates of the point where C cuts the x-axis.

3. Let \( f(x) = \sqrt{x} \) and \( g(x) = 2^x \), where both functions are defined on their largest possible real domain. Solve the equation \( (f^{-1} \circ g)(x) = 0.25 \).

4. Let and \( f(x) = x^2 + 1, \ x \geq 3 \) and \( g(x) = 5 - x \).
   a. Evaluate \( f(3) \).
   b. Find and simplify an expression for \( g(f(x)) \).
   c. State the geometric relationship between the graphs of \( y = f(x) \) and \( y = f^{-1}(x) \).
   d. i. Find an expression for \( f^{-1}(x) \).
      ii. Find the range of \( f^{-1}(x) \).
      iii. Find the domain of \( f^{-1}(x) \).
   e. Explain why the equation \( f(x) = g(3x) \) has no solutions.

5. Functions f and g are defined for all real values of x by
   \( f(x) = x^3 + 4 \) and \( g(x) = 2x - 5 \).
   Evaluate,
   i. \( fg(1) \),
   ii. \( f^{-1}(12) \)
6 The function \( f \) is given by \( f(x) = x^2 - 6x + 10 \), for \( x \geq 3 \).
   a Write \( f(x) \) in the form \((x - p)^2 + q\).
   b Find the inverse function \( f^{-1}(x) \), stating its domain.

7 Let \( f(x) = 2x + 1 \), \( x \in \mathbb{R} \) and \( g(x) = \frac{x + 3}{x - 1} \), \( x \neq 1 \).
   a Find and simplify:
      i \( f(7) \)
      ii the range of \( f(x) \)
      iii \( f(z) \)
      iv \( f \circ g(x) \)
      v \( f \circ f(x) \)
   b Explain why \( g \circ f(x) \) does not exist.
   c i Find the form of \( g^{-1}(x) \).
      ii State the geometric relationship between the graphs of \( y = g(x) \) and \( y = g^{-1}(x) \).
      iii State the domain of \( g^{-1}(x) \).
      iv State the range of \( g^{-1}(x) \).

8 In this question you must show detailed reasoning.

The functions \( f \) and \( g \) are defined over the domain of all real numbers by \( f(x) = x^2 + 4x + 9 \) and \( g(x) = e^x \).
   a Write \( f(x) = x^2 + 4x + 9 \), \( x \in \mathbb{R} \) in the form \( f(x) = (x + p)^2 + q \).
   b Hence, sketch the graph of \( y = x^2 + 4x + 9 \), labelling all axes intercepts and the coordinates of the turning point.
   c State the range of \( f(x) \) and \( g(x) \).
   d Hence, or otherwise, find the range of \( h(x) = e^{2x} + 4e^x + 9 \).
Let \( h(x) = x^2 - 6x + 2 \).

a  Write \( h(x) \) in the form \((x - p)^2 + q\).

b  Hence, or otherwise, find the range of \( h \) when its domain is all real numbers.

c  Another function \( g \) is now defined with the same rule as \( h \), but with the largest possible domain of the form \( x \geq k \), for which it has an inverse function. Find an expression for \( g^{-1}(x) \).

a  Show that if \( g(x) = \frac{1}{x} \), then \( gg(x) = x \).

b  A function satisfies the identity \( f(x) + 2f\left(\frac{1}{x}\right) = 2x + 1 \).

By replacing all instances of \( x \) with \( \frac{1}{x} \), find another identity satisfied by \( f(x) \).

c  By solving these two identities simultaneously, express \( f(x) \) in terms of \( x \).

In this question you must show detailed reasoning.

The functions \( f(x) \) and \( g(x) \) are given by \( f(x) = \sqrt{x + 2} \) and \( g(x) = x^2 + x \). The function \( (f \circ g)(x) \) is defined for \( x \in \mathbb{R} \), except for the interval \( a < x < b \).

a  Calculate the value of \( a \) and of \( b \).

b  Find the range of \( f \circ g \).

An odd function is any function \( f(x) \) that satisfies \( f(x) = -f(-x) \).

a  Show that \( f(x) = x^3 \) is an odd function.

b  What type of symmetry must the graph of any odd function have?

c  Given any function \( g(x) \), show that \( g(x) - g(-x) \) is an odd function.

An even function is any function that satisfies \( f(x) = f(-x) \).

d  Show that \( f(x) = x^4 \) is an even function.

e  What type of symmetry must the graph of any even function have?

f  Given any function \( g(x) \), show that \( g(x) + g(-x) \) is an even function.

g  Hence, or otherwise, show that any function can be written as the sum of an even function and an odd function.
Before you start...

| Student Book 1, Chapter 5 | You should be able to recognise a graph transformation from the equation. | 1 | The graph of \( y = f(x) \) is shown in the diagram. Sketch the graph of: 

a. \( y = f(x+2) \) 

b. \( y = -f(x) \) |
|---------------------------|---------------------------------------------------------------------------|---|--------------------------------------------------------------------------|
| Student Book 1, Chapter 5 | You should be able to change the equation of a graph to achieve a given transformation. | 2 | A graph has equation \( y = x^2 - 3x \). Find the equation of the graph after: 

a. a translation of 5 units in the positive \( y \) direction 

b. a horizontal stretch with scale factor 2 |
| Student Book 1, Chapter 1 | You should be able to use interval notation to express solution of inequalities. | 3 | In this question, write the solution using interval notation. 

a. Solve the inequality \( 3x - 2 \geq 7 \). 

b. Solve the system of inequalities \( 2x + 5 > 1 \) and \( 3 - 2x > 1 \). |

Combining transformations

In this chapter you’ll take the transformations you met in Student Book 1, Chapter 5, and combine them to produce a sequence of transformations of the original graph.

You’ll also meet the modulus function, which is used in many contexts where a quantity needs to be positive. For example, the total distance travelled by a particle must be the sum of positive quantities even though
it may change direction. You’ve also seen this idea used to find the area between a curve and the $x$-axis when the enclosed region is below the axis.

**Section 1: Combined transformations**

In this section we look at what happens when you apply two transformations to a graph. An important question to consider is: does the order in which the two transformations are done affect the outcome? To investigate this question, let us first consider transformations of a single point.

The point $(1, 3)$ is translated 2 units up and then reflected in the $x$-axis.

The new point is $(1, -5)$.

The point $(1, 3)$ is reflected in the $x$-axis first and then translated 2 units up. The new point is $(1, -1)$.

The point $(1, 3)$ is translated 2 units up and then reflected in the $y$-axis.

The new point is $(-1, 5)$.

The point $(1, 3)$ is reflected in the $y$-axis first and then translated 2 units up. The new point is $(-1, 5)$.

These examples suggest the following rules for combining transformations.

**Key point 3.1**

When two vertical transformations or two horizontal transformations are combined, changing the order may affect the outcome.

When one vertical and one horizontal transformation are combined, the outcome does not depend on the order.

In advanced mathematics, algebra is much more than using letters to represent numbers. Unknowns can include transformations, as well as many other things.

As demonstrated in this section, the rules for transformations are different from the rules for numbers, but there are certain similarities, too. The study of this more general form of algebra includes group theory, which has many applications, from particle physics to painting polyhedra.
Combining one vertical and one horizontal transformation

The diagram shows the graph of the function \( y = f(x) \).

On separate diagrams, draw the graphs of:

\[ a \quad y = 2f(x+3) \]

\[ b \quad y = -f\left(\frac{x}{2}\right) \]

Both these questions involve one horizontal and one vertical transformation.
Since the order doesn’t matter, you can carry out the transformation in brackets first.

\[ a \quad y = f(x+3): \text{translate} \]
\[ 3 \text{ units to the left.} \]

\[ y = 2f(x+3): \text{vertical stretch with} \]
\[ \text{scale factor } 2. \]
Combining two vertical transformations

To transform the graph of \( y = f(x) \) into the graph of:

\[
y = p f(x) + c
\]

you first multiply \( f(x) \) by \( p \) and then add \( c \). The flow chart shows the order of operations and the corresponding transformations:

Notice that this follows the normal order of operations: multiplication is done before addition.
Combining two horizontal transformations

If we combine two horizontal transformations, we can transform the graph of \( y = f(x) \) into the graph of

\[
y = f(qx + d)
\]

We can achieve this by first replacing \( x \) with \( x + d \) and then replacing all occurrences of \( x \) by \( qx \). The flow chart shows the resulting order of transformations.

Notice that the transformations are in the ‘wrong’ order: the translation, which corresponds to addition, is done first.
WORKED EXAMPLE 3.3

The following graph shows \( y = f(x) \). Sketch the graph of:

a. \( y = f(2x + 1) \)  

b. \( y = f(2(x + 1)) \)

**a. \( y = f(2x + 1) \)**

There are two transformations:

1. \( x \) is replaced by \( x + 1 \).
2. \( x \) is replaced by \( 2x \).
3. \( x \) is replaced by \( x + 1 \) → horizontal translation \( \begin{pmatrix} -1 \\ 0 \end{pmatrix} \).

This is a combination of two horizontal transformations, so you deal with the addition first.

Replace \( x \) with \( x + 1 \).

Change \( y = f(x) \) to \( y = f(x + 1) \).

**b. \( y = f(2(x + 1)) \)**

Replace \( x \) with \( 2x \).

Change \( y = f(x + 1) \) to \( y = f(2x + 1) \).

The red graph shows the final answer.
\[ y = f(2(x + 1)) \]

The two transformations are:

1. \( x \) is replaced by \( 2x \).
2. \( x \) is replaced by \( x + 1 \).
3. \( x \) is replaced by \( 2x \rightarrow \text{stretch, scale factor } \frac{1}{2} \)

The presence of brackets means that now we deal with the multiplication first.

Replace \( x \) with \( 2x \).

Change \( y = f(x) \) to \( y = f(2x) \).

Replace \( x \) with \( x + 1 \).

Change \( y = f(2x) \) to \( y = f(2(x + 1)) \).

The green graph shows the final answer.

Note that this is a different answer from the one in part a.

As illustrated in Worked Example 3.3, if you want to perform a horizontal stretch before a translation you need to use brackets correctly in the equation.

**WORKED EXAMPLE 3.4**

The graph of \( y = \sin(x) \) is transformed using a horizontal stretch with scale factor 2 and then translated 3 units to the right. Find the equation of the resulting graph.

1. Replace \( x \) by \( \frac{x}{2} \):
   
   \[ y = \sin(x) \] is changed to \( y = \sin\left(\frac{x}{2}\right) \).

   A horizontal stretch with scale factor \( q \) is achieved by replacing \( x \) with \( \frac{x}{q} \).
WORK IT OUT 3.1

Describe the sequence of two transformations that transform the graph of \( y = f(x) \) to the graph of \( y = f(2x + 4) \).

Which of the following solutions is correct? Identify any mistakes in the other two.

<table>
<thead>
<tr>
<th>Solution 1</th>
<th>Solution 2</th>
<th>Solution 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Add 4 to ( x ); this is a horizontal translation 4 units to the left.</td>
<td>1 Replace ( x ) by 2x; this is a horizontal stretch with scale factor ( \frac{1}{2} ).</td>
<td>1 Add 4 to ( x ); this is a horizontal translation 4 units to the left.</td>
</tr>
<tr>
<td>2 Replace ( x ) by 2x; this is a horizontal stretch with scale factor ( \frac{1}{2} ).</td>
<td>2 Add 4 to ( x ); this is a horizontal stretch with scale factor 2.</td>
<td></td>
</tr>
</tbody>
</table>

EXERCISE 3A

1 The following graphs show \( y = f(x) \) and \( y = g(x) \).

Sketch the graphs of:

a i \( y = 2f(x) - 1 \)  
ii \( y = \frac{1}{2}g(x) + 3 \)

b i \( y = 4 - f(x) \)  
ii \( y = 2 - 2g(x) \)
Given that \( f(x) = x^2 \), express each of the following functions as \( af(x) + b \) and, hence, describe the sequence of transformations mapping the graph of \( f(x) \) to the graph of the given function.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</thead>
</table>
| a | i  | \( k(x) = 2x^2 - 6 \)  
|   | ii | \( k(x) = 5x^2 + 4 \) |
| b | i  | \( h(x) = 5 - 3x^2 \)  
|   | ii | \( h(x) = 4 - 8x^2 \) |

Given that \( f(x) = 2x^3 - 4 \), give the function \( g(x) \) that represents the graph of \( f(x) \) after the following transformations.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
</table>
| a | i  | Translation \( \begin{pmatrix} 0 \\ 2 \end{pmatrix} \) followed by a vertical stretch of scale factor 3.
|   | ii | Translation \( \begin{pmatrix} 0 \\ 6 \end{pmatrix} \) followed by a vertical stretch of scale factor \( \frac{1}{2} \).
| b | i  | Vertical stretch of scale factor \( \frac{1}{2} \) followed by a translation \( \begin{pmatrix} 0 \\ 6 \end{pmatrix} \).
|   | ii | Vertical stretch of scale factor \( \frac{7}{2} \) followed by a translation \( \begin{pmatrix} 0 \\ 10 \end{pmatrix} \).
| c | i  | Reflection in the horizontal axis followed by a translation \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).
|   | ii | Reflection in the horizontal axis followed by a translation \( \begin{pmatrix} 0 \\ 2 \end{pmatrix} \).
| d | i  | Reflection in the horizontal axis, followed by a vertical stretch of scale factor \( \frac{1}{2} \), followed by a translation \( \begin{pmatrix} 0 \\ 3 \end{pmatrix} \).
|   | ii | Reflection in the horizontal axis, followed by a translation \( \begin{pmatrix} 0 \\ -6 \end{pmatrix} \), followed by a vertical stretch of, scale factor \( \frac{3}{2} \).

Given that \( f(x) = x^3 \), express each of the following functions as \( f(ax + b) \) and, hence, describe the transformation mapping the graph of \( f(x) \) to the graph of the given function.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
</table>
| a | i  | \( g(x) = x^3 + 2x + 1 \)  
|   | ii | \( g(x) = x^3 - 5x + 9 \) |
| b | i  | \( k(x) = 4x^3 + 8x + 4 \)  
|   | ii | \( k(x) = 9x^3 - 6x + 1 \) |

Given that \( f(x) = 2x^2 - 4 \), give the function \( g(x) \) that represents the graph of \( f(x) \) after the following transformations.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
</table>
| a | i  | Translation \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) followed by a horizontal stretch of scale factor \( \frac{1}{4} \).
|   | ii | Translation \( \begin{pmatrix} -2 \\ 0 \end{pmatrix} \) followed by a horizontal stretch of scale factor \( \frac{1}{2} \).
b i Horizontal stretch of scale factor \( \frac{1}{2} \) followed by a translation \( \begin{pmatrix} 4 \\ 0 \end{pmatrix} \).

ii Horizontal stretch of scale factor \( \frac{2}{3} \) followed by a translation \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

c i Translation \( \begin{pmatrix} -3 \\ 0 \end{pmatrix} \) followed by a reflection in the \( y \)-axis.

ii Reflection in the vertical axis followed by a translation \( \begin{pmatrix} -3 \\ 0 \end{pmatrix} \).

Find the resulting equation after the graph of \( y = \sin(x) \) is transformed using each sequence of transformations.

a A vertical translation \( c \) units up, then a vertical stretch with scale factor \( p \).

b A vertical stretch with scale factor \( p \) followed by a vertical translation \( c \) units up.

c A horizontal stretch with scale factor \( q \), then a horizontal translation \( d \) units to the left.

d A horizontal translation \( d \) units to the left followed by a horizontal stretch with scale factor \( q \).

The graph of \( y = x^2 \) is transformed using a horizontal stretch with scale factor 2 followed by a vertical stretch with scale factor 4. Find the equation of the resulting graph. Can you explain why this is the case?

b The graph of \( y = e^x \) is transformed using a horizontal translation 2 units to the left followed by a vertical stretch with scale factor \( q \). The equation of the resulting graph is again \( y = e^x \). Find the value of \( q \).

c The graph of \( y = \ln(x) \) is translated 2 units up. What transformation (other than a translation 2 units down) will return the graph to its original position?

Use technology to sketch the graphs and see why this is the case.

The diagram shows the graph of \( y = f(x) \).

On separate axes, sketch the graphs of:

a \( y = f(2x) - 3 \)  
b \( y = 1 - 3f(x) \)

Sketch the following graphs.

a \( y = \ln x \)  
b \( y = 3\ln(x + 2) \)  
c \( y = \ln(2x - 1) \)

In each case, indicate clearly the positions of the vertical asymptote and the \( x \)-intercept.
10 The graph of \( y = x^2 - 3x \) is translated 2 units to the right and then reflected in the \( x \)-axis. Find the equation of the resulting graph, in the form \( y = ax^2 + bx + c \).

11 The graph of \( y = ax + b \) is transformed by the following sequence:
- translation by \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \)
- reflection in \( y = 0 \)
- horizontal stretch with scale factor \( \frac{1}{3} \).

The resulting graph has equation \( y = 1 - 15x \). Find the values of \( a \) and \( b \).

12 The graph of \( y = ax^2 + bx + c \) is transformed by the following sequence:
- reflection in \( x = 0 \)
- translation by \( \begin{pmatrix} -1 \\ 3 \end{pmatrix} \)
- horizontal stretch with scale factor 2.

The resulting graph has equation \( y = \frac{1}{2}x^2 + \frac{1}{2}x - 3 \). Find the values of \( a, b \) and \( c \).

13 Given that \( f(x) = 2^x + x \), give in simplest terms the formula for \( h(x) \), which is obtained by transforming \( f(x) \) by the following sequence of transformations:
- vertical stretch, scale factor 8
- translation by \( \begin{pmatrix} 1 \\ 4 \end{pmatrix} \)
- horizontal stretch, scale factor \( \frac{1}{2} \).

Section 2: Modulus function

The modulus (or absolute value) is a function that leaves non-negative numbers alone but reverses the sign of negative numbers. We use \( |x| \) to denote the modulus of number \( x \); for example, \( |5| = 5 \) and \( |-3| = 3 \). Note that \( |0| = 0 \).

You can define the modulus function by this equation:

\[
|x| = \begin{cases} 
  x & x \geq 0 \\
  -x & x < 0 
\end{cases}
\]

Key point 3.2

Did you know?

You can think of the modulus function as giving the distance of a number from zero on the number line. You applied a similar idea to vectors in Student Book 1. But the distance between two objects is not always easy to define. For example, what length gives the ‘distance’ between two points on the surface of the Earth? This question leads to the idea of a metric. You may want to find out about the Minkowski Metric for finding the distance between two points in space–time in the theory of relativity.
The graph of $y = |x|$ is shown. It will be useful to call the red branch the \textit{reflected branch} (as it is the reflection of a part of the graph $y = x$ in the $x$-axis) and the blue branch the \textit{unreflected branch}.

The domain of $|x|$ is all real numbers, whereas the range is all positive numbers and zero.

You can combine this with the rules for transforming graphs to sketch some other functions involving the modulus.

**WORKED EXAMPLE 3.5**

Sketch the graphs of:

a. $y = |x - 3|$

b. $y = |x + 2| - 5$

In each case, find the $y$-intercept of the graph.

a. $y$-intercept: $|0 - 3| = |-3| = 3$

$x$ has been replaced by $x - 3$, so the graph is shifted 3 units to the right.

The $y$-intercept is when $x = 0$.

b. $y$-intercept: $|0 + 2| - 5 = 2 - 5 = -3$

$x$ has been replaced by $x + 2$, so the graph is shifted 2 units to the left.

Subtracting 5 shifts the graph 5 units down.
You can also think about applying the modulus function to the graph of the function ‘inside’ the modulus. Any parts of the ‘original’ graph that are below the x-axis will have their y values changed from negative to positive; so those parts of the graph will be reflected in the x-axis.

WORKED EXAMPLE 3.6

Sketch the graph of \( y = |5 - 2x| \), indicating the intercepts with the coordinate axes.

Start by drawing the graph of \( y = 5 - 2x \).
This crosses the x-axis at (2.5, 0) and the y-axis at (0, 5).

The part of the graph to the right of the x-intercept has negative y values; taking the modulus will make those positive, so this part of the graph needs to be reflected in the x-axis.

Key point 3.3

To sketch the graph of \( y = |f(x)| \), start with the graph of \( y = f(x) \) and reflect any parts that are below the x-axis.

In particular, the graph of \( y = |ax + b| \) has a V-shape with the vertex at \( \left(-\frac{b}{a}, 0\right) \) and y-intercept \((0, b)\).

Using modulus notation in inequalities

You often meet inequalities where the variable is between a number and its negative; for example, if \( x^2 < 9 \) then \(-3 < x < 3\). You can write this more concisely using modulus notation: \( |x| < 3 \).

It is possible to extend this notation to write other inequalities that represent a single interval. For example, what does \( |x - 5| < 3 \) mean? If we replace \( x - 5 \) by \( y \) then, as in the previous example, \( |y| < 3 \) means that \(-3 < y < 3\). So \(-3 < x - 5 < 3\), which can be rearranged into \( 2 < x < 8 \).
There is a nice interpretation of this inequality on the number line:

![Number Line Diagram]

The number 5 is in the middle of the interval (2, 8), 3 units away from each end. So the inequality $|x - 5| < 3$ can be read as: ‘the distance between $x$ and 5 is less than 3’.

**Key point 3.4**

The modulus inequality $|x - a| < b$ is equivalent to $a - b < x < a + b$.

**Fast forward**

If you study Further Mathematics, in Student Book 1, you will extend the modulus notation to measure distances between points in the complex plane.

**Rewind**

You already use the modulus of a vector to represent the distance between two points.

**WORKED EXAMPLE 3.7**

a Write the inequality $|x + 4| < 6$ in the form $p < x < q$.

b Write the interval $[-3, 7]$ using an inequality of the form $|x - a| \leq b$.

\[
\begin{align*}
\text{a } & \quad -4 - 6 < x < -4 + 6 \\
& \quad \text{So } -10 < x < 2.
\end{align*}
\]

Use the result from Key point 3.4:

\[
|x - a| < b \iff a - b < x < a + b
\]

In this example, $a = -4$ and $b = 6$.

Alternatively, you can think of $|x + 4| < 6$ as saying that $x$ is at most 6 units on either side of $-4$.

\[
\begin{align*}
\text{b } & \quad a = \frac{(-3) + 7}{2} = 2 \\
& \quad b = 7 - 2 = 5 \\
& \quad \text{So } |x - 2| \leq 5.
\end{align*}
\]

\[
\begin{align*}
& \quad a \text{ is the number in the middle of the interval.} \\
& \quad b \text{ is the distance from the middle of the interval to one end.} \\
& \quad \text{The end points of the interval are included.}
\end{align*}
\]

**EXERCISE 3B**

1 Sketch the following graphs, showing the axes intercepts.

\[
\begin{align*}
\text{a } & \quad \text{i } y = |x - 3| & \quad \text{ii } y = |x + 5| \\
\text{b } & \quad \text{i } y = |3x + 5| & \quad \text{ii } y = \frac{1}{2}x - 1 \\
\text{c } & \quad \text{i } y = |4 - 2x| & \quad \text{ii } y = |-3x + 2| \\
\text{d } & \quad \text{i } y = |x| + 1 & \quad \text{ii } y = |x| - 2
\end{align*}
\]
2. Write the equation of each graph in the form \( y = |ax + b| \).
   
   a. i. \[ y = |x - 2| \]
      ii. \[ y = |x - 2| + 3 \]
   
   b. i. \[ y = |x + 1| \]
      ii. \[ y = |x + 6| \]
   
   c. i. \[ y = |x| \]
      ii. \[ y = |x| \]

3. Write the following inequalities in the form \( a < x < b \).
   
   a. i. \( |x - 5| < 8 \)
      ii. \( |x - 2| < 9 \)
   
   b. ii. \( |x + 1| < 5 \)
      ii. \( |x + 6| < 3 \)

4. Write the following statements in the form \( |x - a| \leq b \).
   
   a. i. \( x \in [5, 13] \)
      ii. \( x \in [11, 25] \)
   
   b. ii. \( x \in [-14, 10] \)
      ii. \( x \in [-16, 2] \)
   
   c. i. \( x \in [3, 10] \)
      ii. \( x \in [5, 20] \)

5. Sketch the graphs of the following, showing the coordinates of the \( y \)-intercept and the vertex.
   
   a. \( y = |x - 2| \)
   b. \( y = |x - 2| + 3 \)
6 Sketch the graph of \( y = |2x + 3| \), labelling the intercepts with the coordinate axes.

7 Sketch the graph of \( y = 5 - |x - 2| \). Give the coordinates of the points where the graph crosses the y-axis.

8 Sketch the graph of \( y = x|x| \).

Section 3: Modulus equations and inequalities

You can use graphs to solve equations and inequalities involving the modulus function.

**Key point 3.5**

When solving an equation involving a modulus function, sketch the graph.

You need to use the graph to decide whether the intersection is on the reflected or the unreflected part of the graph. If it is on the unreflected part, you can rewrite the equation without the modulus sign. If it is on the reflected part, you need to replace the modulus sign by a minus sign.

**WORKED EXAMPLE 3.8**

Solve the equation \( \frac{x}{2} - |x - 1| \).

Sketch the graphs of \( y = \frac{x}{2} \) and \( y = |x - 1| \).

There are two intersection points:

A the blue graph intersects the reflected part of the red graph.

B the blue line intersects the unreflected part of the red graph.

You need to write a separate equation for each.

For the reflected part, replace the modulus sign with brackets and a minus sign.

For the unreflected part, just remove the modulus sign.

So the solution is:

\[ x = \frac{2}{3} \text{ or } 2 \]
You can also intersect two modulus graphs.

**WORKED EXAMPLE 3.9**

Solve the equation $|x + 1| = |2x - 1|$.

Sketch the graphs of $y = |x + 1|$ and $y = |2x - 1|$.

There are two intersections:

- **A** the unreflected blue line and the reflected red line.
- **B** the unreflected blue line and the unreflected red line.

**A**

$x + 1 = -(2x - 1)$

$x + 1 = -2x + 1$

$3x = 0$

$x = 0$

**B**

$x + 1 = 2x - 1$

$x = 2$

The solution is: $x = 0$ or 2.

You need the reflected part of the red graph, so replace the modulus sign with brackets and a minus sign.

You need the unreflected parts for both graphs, so remove the modulus signs.

There is an alternative method that can be used to solve an equation where both sides are inside a modulus. You then know that both sides of the equation are non-negative, so you can square both sides.

**Key point 3.6**

$|a| = |b| \iff a^2 = b^2$

We illustrate this with the equation from Worked Example 3.9. See which method you prefer.

**WORKED EXAMPLE 3.10**

Solve the equation $|x + 1| = |2x - 1|$.

Both sides are positive so can be squared.

$(x + 1)^2 = (2x - 1)^2$

$\iff x^2 + 2x + 1 = 4x^2 - 4x + 1$

$\iff 3x^2 - 6x = 0$

$\iff 3(x - 2) = 0$

$\iff x = 0$ or 2

Write as a quadratic equation with the $x^2$ terms positive.
Solve the equation $|2x - 1| = |3 - x|$. Which of the following solutions is correct? Identify the mistake in the other two.

<table>
<thead>
<tr>
<th>Solution 1</th>
<th>Solution 2</th>
<th>Solution 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x - 1 = 3 - x$ or $2x - 1 = -3 + x$</td>
<td>$2x - 1 = 3 - x$ or $(2x - 1) = -(3 - x)$</td>
<td>$2x - 1 = 3 - x$ or $2x - 1 = 3 + x$</td>
</tr>
<tr>
<td>$3x = 4$ or $x = -2$</td>
<td>$3x = 4$ or $-3x = -4$</td>
<td>$x = -2$ or $x = 4$</td>
</tr>
<tr>
<td>Solution: $x = \frac{4}{3}$ or $-2$</td>
<td>Solution: $x = \frac{4}{3}$</td>
<td>Solution: $x = -2$ or $4$</td>
</tr>
</tbody>
</table>

To solve inequalities, first sketch the graphs and find their intersections. You can then decide on which parts of the graph the inequality is satisfied.

**WORKED EXAMPLE 3.11**

Solve $|2x - 3| > x + 4$.

Sketch the graphs of $y = |2x - 3|$ and $y = x + 4$, highlighting where the inequality is satisfied.

**A**

$x + 4 = -(2x - 3)$

$3x = -1$

$x = -\frac{1}{3}$

**B**

$x + 4 = 2x - 3$

$x = 7$

$x \in \left(-\infty, -\frac{1}{3}\right) \cup (7, \infty)$

Find the intersection points:

A is on the reflected part of the red line.

B is on the unreflected part of the red line.

Describe the highlighted region in terms of $x$. Remember that you can use either the inequality notation or the interval notation to do this.
EXERCISE 3C

1 Solve the following equations.

a i \( |x| = 4 \)  
ii \( |x| = 18 \)

b i \( |2x - 4| = 4 \)  
ii \( |3x + 1| = 2 \)

c i \( |2x + 4| = 4 - x \)  
ii \( |5 - 2x| = x + 3 \)

d i \( |3x - 4| = 8 - x \)  
ii \( |5 + 2x| = 3 - 2x \)

e i \( |3 - 2x| - |x + 1| \)  
ii \( |4 + x| = |b - 3x| \)

f i \( |6 - x| = |5 + x| \)  
ii \( |4 + 3x| = |x| \)

2 Solve the following inequalities. Write your answer using the stated notation.

a Use interval notation.
   i \( |x| > 5 \)  
ii \( |x| > 2 \)

b Use inequality notation.
   i \( |x| < 3 \)  
ii \( |x| < 10 \)

c Use interval notation.
   i \( |2x + 4| > 4 \)  
ii \( |3x - 2| < 3 \)

d Use inequality notation.
   i \( |2x - 5| < 3 \)  
ii \( |5 - 3x| > 2x \)

e Use interval notation.
   i \( |2x + 1| > |x + 4| \)  
ii \( |3x - 4| > |2x + 1| \)

f Use inequality notation.
   i \( |x + 4| > |2x| \)  
ii \( |1 + 3x| < |x + 3| \)

3 a Solve the equation \( |4x + 1| = x + 3 \).

b Solve the inequality \( |4x + 1| < x + 3 \).

4 a Solve the equation \( |2x - 5| = |x + 2| \).

b Solve the inequality \( |2x - 5| \leq |x + 2| \). Write your answer using the interval notation.

5 a Sketch the graph of \( y = |3x - 7| \).

b Hence, solve the inequality \( |3x - 7| < 1 - x \).

6 Solve the inequality \( |3x + 1| > 2x \).

7 Given that \( k > 0 \), find in terms of \( k \) the solution of the inequality \( |x - k| \leq |2x - k| \).

8 Solve the equation \( |x + k| = |x| + k \), where \( k > 0 \).
You can combine any two (or more) transformations of graphs: translations, stretches and reflections (both horizontal and vertical).

The order in which transformations occur may affect the outcome.

One horizontal and one vertical transformation can be done in either order.

Changing the order of two horizontal or two vertical transformations may affect the outcome.

For a function of the form \( y = p f(x) + c \), the stretch is performed before the translation.

For a function of the form \( y = f(qx + d) \), the translation is performed before the stretch.

The modulus function can be used to reflect the part of the graph below the \( x \)-axis so that the whole graph is on or above it.

The graph of \( y = |mx + c| \) has a V-shape with the vertex at \( \left( -\frac{c}{m}, 0 \right) \).

To solve equations and inequalities involving the modulus function, always use graphs. You need to decide whether the solutions are on the reflected or unreflected part of the graph.

You can solve some modulus equations by squaring both sides, using \(|a| = |b| \iff a^2 = b^2\).

You can use modulus notation to express some inequalities. In particular, \( |x - a| < b \iff a - b < x < a + b \).
Mixed practice 3

1 The graph of \( y = f(x) \) is shown.

![Graph of \( y = f(x) \)](image)

On separate diagrams, sketch the graphs of:

\[ a \quad y = 3f(x - 2) \quad b \quad y = 3 - f(2x) \]

2 The graph of \( y = x^3 - 1 \) is transformed by applying a translation with vector \( \begin{pmatrix} 2 \\ 0 \end{pmatrix} \) followed by a vertical stretch with scale factor 2. Find the equation of the resulting graph in the form \( y = ax^3 + bx^2 + cx + d \).

3 a On the same set of axes, sketch the graphs of \( y = x \) and \( y = |2x - 1| \).

b Hence, solve the inequality \( |2x - 1| < x \).

4 a Describe two transformations that transform the graph of \( y = x^2 \) to the graph of \( y = 3x^2 - 12x + 12 \).

b Describe two transformations that transform the graph of \( y = x^3 + 6x - 1 \) to the graph of \( y = x^2 \).

c Hence, describe a sequence of transformations that transform the graph of \( y = x^3 + 6x - 1 \) to the graph of \( y = 3x^2 - 12x + 12 \).

5 The transformations \( R, S \) and \( T \) are defined as follows:

- \( R \) : reflection in the \( x \)-axis
- \( S \) : stretch in the \( x \) direction with scale factor 3
- \( T \) : translation in the positive \( x \) direction by 4 units.

i The curve \( y = \ln x \) is transformed by \( R \) followed by \( T \). Find the equation of the resulting curve.

ii Find, in terms of \( S \) and \( T \), a sequence of transformations that transforms the curve \( y = x^3 \) to the curve \( y = \left( \frac{1}{9}x - 4 \right)^3 \). You should make clear the order of the transformations.

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6 Find two transformations whose composition transforms the graph of \( y - (x - 1)^2 \) to the graph of \( y = 3(x + 2)^2 \).

7 Solve the inequality \( |2x - 1| > |x - 6| \).

8 a Describe two transformations whose composition transforms the graph of \( y = f(x) \) to the graph of \( y = 3f\left(\frac{x}{2}\right) \).
   b Sketch the graph of \( y = 3 \ln \left(\frac{x}{2}\right) \).
   c Sketch the graph of \( y = 3 \ln \left(\frac{x}{2} + 1\right) \) marking clearly the positions of any asymptotes and \( x \)-intercepts.

9 a State the sequence of three transformations that transform the graph of \( y = |x| \) to the graph of \( y = 5 - 3|x| \). Hence, sketch the graph of \( y = 5 - 3|x| \).
   b Solve the equation \( |2x - 1| = 5 - 3|x| \).
   c Write down the solution of the inequality \( |2x - 1| \leq 5 - 3|x| \).

10 a Describe a transformation that transforms the graph of \( y = f(x) \) to the graph of \( y = f(x + 2) \).
   b On the same diagram, sketch the graphs of:
      i \( y = \ln(x + 2) \)
      ii \( y = 3 - \ln(x + 2) \)
      Mark clearly any asymptotes and \( x \)-intercepts on your sketches.
   c The graph of the function \( y = g(x) \) has been translated and then reflected in the \( x \)-axis to produce the graph of \( y = h(x) \).
i State the translation vector.

ii If \( g(x) = x^3 - 3x + 5 \), find constants \( a, b, c, d \) such that
\[
h(x) = ax^4 + bx^3 + cx^2 + d.
\]

The function \( f \) is defined by \( f(x) = 2 - \sqrt{x} \) for \( x \geq 0 \). The graph of \( y = f(x) \) is shown above.

i State the range of \( f \).

ii Find the value of \( f(4) \).

iii Given that the equation \( |f(x)| = k \) has two distinct roots, determine the possible values of the constant \( k \).

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12 Solve the equation \( x|x| = x^2 \).

13 Sketch the graph of \( y = |x| + x \).