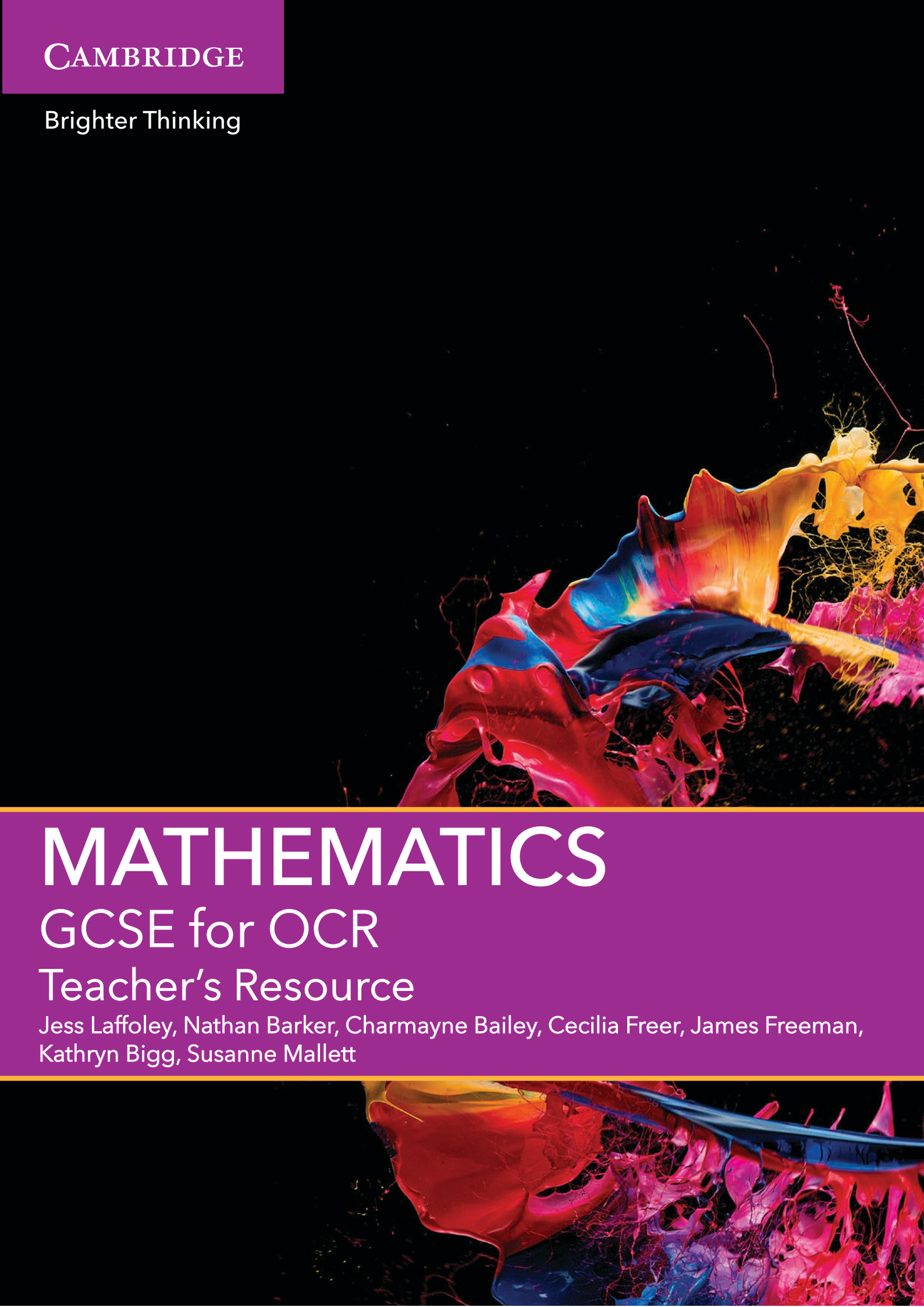


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Brighter Thinking

An abstract, vibrant image of a liquid splash, possibly paint or ink, in shades of red, orange, yellow, and blue, set against a dark background. The splash is dynamic and fluid, with various droplets and streaks visible.

MATHEMATICS

GCSE for OCR

Teacher's Resource

Jess Laffoley, Nathan Barker, Charmayne Bailey, Cecilia Freer, James Freeman,
Kathryn Bigg, Susanne Mallett

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Introduction

This book has been written to support you in delivering the **OCR J560 GCSE Mathematics** specification. It accompanies Cambridge University Press's GCSE Mathematics for OCR student resources, including the Foundation and Higher Student Books, Problem-solving Books and Homework Books. The Teacher's Resource suggests how to use these resources for maximum benefit, as well as providing general advice on teaching the topics in the specification.

STUDENT BOOK SUPPORT

The structure of the Teacher's Resource closely matches the Student Books. The Student Books are divided into chapters and sections, with each section covering a single topic. The Teacher's Resource contains a chapter of teaching guidance for each Student Book chapter, covering both Foundation and Higher tiers, and is divided into corresponding sections.

An introduction details **what your students need to know** before covering the topics in the chapter, **learning outcomes** and **key vocabulary**. The introduction also describes **common misconceptions**, suggesting how to address them, and offers **'hooks'** to introduce the topics in an engaging way.

For each Student Book section, **prompting questions** are provided to promote discussion of the topic and to help students when working through the Student Book exercises. Also provided are suggested activities for **starters**, **plenaries**, **enrichment** and **assessment** – these include links to Web resources as well as classroom activities.

At the end of each chapter, the **Topic links** section describes how the topics in the Student Book chapter link to previous and future learning, and to A Level topics. The **Links to other Cambridge GCSE Mathematics resources** list details the relevant sections in the Homework Books and Problem-solving Books, which you can set students as support, extension or homework tasks.

GENERAL ADVICE AND SUPPORT

This book includes a number of chapters offering general advice for teaching GCSE Mathematics:

- The **Changes to GCSE Mathematics** chapter describes the main changes to the GCSE Mathematics qualification for first teaching from September 2015.
- The **Literacy in mathematics** chapter contains ideas for helping students to become more literate in maths, including advice on extracting information, problem solving and reasoning. It also contains suggestions you can offer parents who are looking to support their child through the course.
- The **Preparing students for exams** chapter contains ideas for supporting students' revision.

REVISION QUIZZES

There are also eight **revision quizzes** (four calculator and four non-calculator) included for each tier, along with accompanying **mark schemes**. Each quiz covers multiple topics and is designed to take a maximum of 40 minutes to complete so you can use it within a single lesson.

The quizzes contain the following icons:



– permitted time



– use of a calculator is permitted



– use of a calculator is not permitted

TIME-SAVING SHEETS

Printable **time-saving sheets** are provided at the end of the Teacher's Resource. These include lists of required formulae, 2 mm graph paper, squared and isometric paper, and other useful handouts.

SCHEMES OF WORK

To help you plan and navigate your way through the course, **one-, two- and three-year schemes of work** are available to download for free from **www.cambridge.org/ukschools/gcsemaths-schemesofwork**.

The schemes of work break the course down by Student Book chapter, with suggested teaching times, key information and details of where to find supporting material in all of the resources in our *GCSE Mathematics for OCR* series.

Each scheme of work is provided as an **editable** Microsoft® Word document, so you can customise it to your own requirements.

GCSE MATHEMATICS ONLINE FOR OCR

Further resources to support you and your students through their GCSE course are available on **GCSE Mathematics Online for OCR**, our brand new interactive teaching and learning subscription service. *GCSE Mathematics Online for OCR* includes lesson notes, games, interactive activities and quizzes, organised in the same chapter structure as the Student Books. It also includes a test generator, reporting and progression tracking.

Please visit **www.cambridge.org/ukschools** for full details of all our GCSE Mathematics resources, including more information on *GCSE Mathematics Online for OCR*.

ANSWERS

Student Book answers

Answers to all Student Book questions are available online as a free PDF. You can download the answer booklet from **www.cambridge.org/ukschools/gcsemaths-studentbookanswers**.

The answer booklet is also included in the Teacher edition of *GCSE Mathematics Online for OCR*.

Problem-solving Book answers

Worked solutions with commentary are included in the back of the Problem-solving Books.

Homework Book answers

A booklet containing answers to all homework exercises is free to download from **www.cambridge.org/ukschools/gcsemaths-homeworkanswers**.

Changes to GCSE Mathematics

This chapter offers an overview of the main changes to the GCSE mathematics qualification for first teaching from September 2015, split to show the effect on both Foundation and Higher tiers.

GRADES

The new GCSE is significantly different from its predecessor. Students will now be graded on a 1–9 scale; students failing to meet the standard for grade 1 will be graded ‘U’. Higher tier will assess grades 4–9 and Foundation tier grades 1–5, hence the overlap grades are now grades 4 and 5, but these new grades do not correspond directly to the A*–G grades previously used. If you wish to read more about grades, the latest information can usually be found on the websites of Ofqual and the awarding bodies.

ASSESSMENT

GCSE mathematics is now double weighted in the league tables and the Department for Education has stated that assessment time must be a minimum of 4.5 hours. The course is linear and assessment is by written papers only, with no controlled assessment or coursework.

Assessment Objectives have been updated for the reformed GCSE, with an increased emphasis on problem solving and mathematical reasoning.

		Weighting	
		Higher	Foundation
A01	Use and apply standard techniques Students should be able to: - accurately recall facts, terminology and definitions - use and interpret notation correctly - accurately carry out routine procedures or set tasks requiring multi-step solutions	40%	50%
A02	Reason, interpret and communicate mathematically Students should be able to: - make deductions, inferences and draw conclusions from mathematical information - construct chains of reasoning to achieve a given result - interpret and communicate information accurately - present arguments and proofs - assess the validity of an argument and critically evaluate a given way of presenting information Where problems require candidates to ‘use and apply standard techniques’ or to independently ‘solve problems’ a proportion of those marks should be attributed to the corresponding Assessment Objective.	30%	25%

AO3	Solve problems within mathematics and in other contexts Students should be able to: • translate problems in mathematical or non-mathematical contexts into a process or a series of mathematical processes • make and use connections between different parts of mathematics • interpret results in the context of the given problem • evaluate methods used and results obtained • evaluate solutions to identify how they may have been affected by assumptions made Where problems require candidates to ‘use and apply standard techniques’ or to ‘reason, interpret and communicate mathematically’ a proportion of those marks should be attributed to the corresponding Assessment Objective.	30%	25%
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FORMULAE

In addition, there have been significant changes to the formulae students receive in the exams and now the only formulae given in the exams are:

- Perimeter, area, surface area and volume formulae
Where r is the radius of the sphere or cone, l is the slant height of a cone and h is the perpendicular height of a cone:
 - curved surface area of a cone = πrl
 - surface area of a sphere = $4\pi r^2$
 - volume of a sphere = $\frac{4}{3}\pi r^3$
 - volume of a cone = $\frac{1}{3}\pi r^2 h$
- Kinematics formulae
Where a is constant acceleration, u is initial velocity, v is final velocity, s is displacement from the position when $t = 0$ and t is time taken:
 - $v = u + at$
 - $s = ut + \frac{1}{2}at^2$
 - $v^2 = u^2 + 2as$

All other formulae will need to be memorised or students will need to learn how to derive it in an exam.

SUBJECT CONTENT

Changes to the subject content on which students will be assessed are detailed below for each tier, alongside references to the Student Book chapters that contain relevant material. Please note that the language of specifications is open to interpretation, so these changes are the authors’ conclusions only and you may wish to conduct your own comparison between the specification for first teaching from 2015 and its predecessor.

Additions to Foundation tier:

The following table lists concepts that are new to the Foundation tier and indicates whether they were previously part of the Higher tier assessment. A reference is given to the DfE’s GCSE subject content and assessment objectives published in November 2013 along with where you will find it in both the Foundation tier Student Book and Higher tier Student Book (where applicable). As you will see, the changes to the material being assessed on Foundation tier are quite substantial and it may be the case that more teaching time is required to cover the content. For comprehensive guidance on topics that may appear in final assessments, centres are directed to the awarding body’s content guidance documents and Specimen Assessment Materials (SAMs), which may be updated during the lifespan of the course.

To indicate the level of challenge each concept presents, the text has been left with the formatting used by the DfE to indicate that despite the expectation that ‘All students will develop confidence and competence with the content

identified by the standard type. All students will be assessed on the content identified by the standard and the underlined type; more highly attaining students will develop competence with all of this content.' (DfE, 2013: p.4) In other words, the underlined type presents the significantly more challenging material Foundation students will now face, whereas there is an expectation that all students will master the normal type.

Concept*	DfE Ref**	Where you can find it	Notes
Calculate with and interpret standard form $A \times 10^n$, where $1 \leq A < 10$ and n is an integer.	N9	F26 / H26 Standard form	Previously Higher only content.
Calculate exactly with ... <u>multiples of π</u> .	N8	F16 / H16 Area, F21 / H21 Volume and surface area	Previously Higher only content.
Round numbers and measures to an appropriate degree of accuracy (e.g. to a specified number of ... significant figures).	N15	F17 / H17 Approximation and estimation	Previously Foundation candidates were only expected to round values to 1 significant figure.
<u>Use inequality notation to specify simple error intervals due to truncation or rounding.</u>	N15	F17 / H17 Approximation and estimation	This is new to both tiers and different to upper and lower bounds which remains in the Higher tier.
Simplify and manipulate algebraic expressions (<u>including those involving surds...</u>)	A4	F3 / H3 Algebraic expressions	There is no expectation that Foundation candidates do anything other than simplify expressions involving surds where they treat the surd as they would any other variable.
<u>Expanding products of two ... binomials.</u>	A4	F7 / H7 Further algebraic expressions	Previously Higher only content.
<u>Factorising quadratic expressions of the form $x^2 + bx + c$, including the difference of two squares.</u>	A4	F7 / H7 Further algebraic expressions	Previously Higher only content. The important thing to note here is that the coefficient of x^2 is always 1.
<u>Argue mathematically to show algebraic expressions are equivalent.</u>	A6	F3 / H3 Algebraic expressions	This extends previous objectives to knowing the difference between an equation and identity to arguing mathematically through solving that two statements are always, rather than sometimes, equal.
<u>Use the form $y = mx + c$ to identify parallel... lines.</u>	A9	F18 / H18 Straight-line graphs, F19 / H19 Graphs of equations and functions.	Previously Higher only content.
Find the equation of the line through two given points, or through one point with a given gradient.	A9	F18 / H18 Straight-line graphs, F19 / H19 Graphs of equations and functions.	This is new to both tiers.
Identify and interpret roots, intercepts, turning points of quadratic functions graphically.	A11	F19 / H19 Graphs of equations and functions.	Previously Higher only content.
Deduce [quadratic] roots algebraically	A11	F8 / H8 Equations	Previously Higher only content.
Recognise, sketch and interpret graphs of linear functions, quadratic functions, simple cubic functions, the reciprocal function $y = \frac{1}{x}$ with $x \neq 0$.	A12	F18 / H18 Straight-line graphs, F19 / H19 Graphs of equations and functions.	Sketching quadratic, cubic and reciprocal functions is new to both tiers.

Plot and interpret graphs (including reciprocal graphs...)	A14	F19 / H19 Graphs of equations and functions.	Previously plotting reciprocal graphs was Higher only content.
Solve quadratic equations ... algebraically by factorising.	A18	F7 / H7 Further algebraic expressions, F8 / H8 Equations	Previously Higher only content.
Find approximate solutions (to quadratics) using a graph.	A18	F8 / H8 Equations	Previously Higher only content.
Solve two simultaneous equations in two variables (linear/linear...) algebraically.	A19	F8 / H8 Equations	Previously Higher only content.
Translate simple situations or procedures into algebraic expressions or formulae; derive an equation (or two simultaneous equations), solve the equation(s) and interpret the solution.	A21	F8 / H8 Equations	The important section of this objective being that which increases the challenge of forming and solving to simultaneous equations.
Recognise and use sequences of triangular, square and cube numbers, simple arithmetic progressions, Fibonacci type sequences, quadratic sequences, and simple geometric progressions (r^n where n is an integer, and r is a rational number > 0).	A24	F4 / H4 Functions and sequences	These different types of sequences have had more emphasis placed on them in the new specification. 'Fibonacci type sequences' implies students now may have to work with sequences defined by a recurrence relation.
Solve problems involving percentage change.	R9	F13 / H13 Percentages	Previously Higher only content.
Original value (percentage) problems.	R9	F12 / H12 Percentages	Previously Higher only content. This content covers problems often referred to as reverse percentage problems.
Simple interest including in financial mathematics.	R9	F33 / H35 Discrete growth and decay	Previously Higher only content.
Solve problems involving direct and inverse proportion, including graphical and algebraic representations.	R10	F34 / H36 Direct and inverse proportion, H39 Interpreting graphs	Previously Higher only content.
Use compound units such as ... density and pressure.	R11	F12 / H12 Units of measurement	Density is previously a compound measure used in Higher tier only, however pressure is a new compound measure to both tiers.
Understand that x is inversely proportional to y is equivalent to x is proportional to $\frac{1}{y}$.	R13	F34 / H36 Direct and inverse proportion	Previously Higher only content.
Interpret equations that describe direct and inverse proportion.	R13	F34 / H36 Direct and inverse proportion	Previously Higher only content.
Interpret the gradient of a straight-line graph as a rate of change.	R14	F18 / H18 Straight-line graphs, F37 / H39 Interpreting graphs	Previously Higher only content.
Recognise and interpret graphs that illustrate direct and inverse proportion.	R14	F34 / H36 Direct and inverse proportion, F37 / H39 Interpreting graphs	
Set up, solve and interpret the answers in growth and decay problems, including compound interest.	R16	F33 / H35 Discrete growth and decay	Previously Higher only content.

Derive and use the sum of angles in a triangle.	G3	F9 / H9 Angles	The important word here being 'derive' as previously there was only an expectation students understood a proof that the sum of angles in a triangle is 180 degrees.
Use the basic congruence criteria for triangles (SSS, SAS, ASA, RHS).	G5	F29 / H30 Congruent triangles	Previously Higher only content.
Apply ... triangle congruence, similarity ... to conjecture and derive results about angles and sides..., and use known results to obtain simple proofs.	G6	F29 / H30 Congruent triangles, F30 / H31 Similarity	Previously Higher only content.
Identify, describe and construct congruent and similar shapes, including on coordinate axes, by considering ... enlargement (including fractional ... scale factors).	G7	F30 / H31 Similarity	Previously Higher only content. The extension to enlargements in the previous specification is that it now includes fractional scale factors. Negative scale factors still only appear in Higher content.
Identify and apply circle definitions and properties, including: ... tangent, arc, sector and segment.	G9	F5 / H5 Properties of shapes and solids	Previously Higher only content.
Know the formulae: ... surface area and volume of spheres, pyramids, cones and composite solids.	G17	F21 / H21 Volume and surface area	Previously Higher only content.
Calculate arc lengths, angles and areas of sectors of circles.	G18	F15 / H15 Perimeter, F16 / H16 Area	Previously Higher only content.
Apply the concepts of congruence and similarity, including the relationships between lengths, ... in similar figures.	G19	F30 / H31 Similarity	Previously Higher only content. Note, Foundation students will only deal with one dimensional scale factors as they apply to lengths of similar shapes.
Know the formulae for: ... the trigonometric ratios, $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ and $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$; apply them to find angles and lengths in right-angled triangles ... in two ... dimensional figures.	G20	F32 / H33 Trigonometry	Previously Higher only content.
Know the exact values of $\sin\theta$ and $\cos\theta$ for $\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ$ and 90° ; know the exact value of $\tan\theta$ for $\theta = 0^\circ, 30^\circ, 45^\circ$ and 60° .	G21	F32 / H33 Trigonometry	This content is new to both tiers.
Apply addition and subtraction of vectors, multiplication of vectors by a scalar, and diagrammatic and column representations of vectors.	G25	F27 / H28 Vectors	Previously Higher only content.
Enumerate sets and combinations of sets systematically, using ... Venn diagrams and tree diagrams.	P6	F24 / H24 Combined events and probability diagrams	Venn diagrams are new material to both tiers.
Calculate the probability of independent and dependent combined events, including using tree diagrams and other representations, and know the underlying assumptions.	P8	F24 / H24 Combined events and probability diagrams	Previously Higher only content.

Infer properties of populations or distributions from a sample, whilst knowing the limitations of sampling.	S1	F35 / H37 Collecting and displaying data	Previously Higher only content.
Interpret, analyse and compare the distributions of data sets from univariate empirical distributions through ... appropriate measures of central tendency ... and spread (range, including consideration of outliers...).	S4	F36 / H38 Analysing data	The consideration of outliers when measuring spread is new to both tiers.
Recognise correlation and know that it does not indicate causation.	S6	F36 / H38 Analysing data	New content to both tiers.
Interpolate and extrapolate apparent trends whilst knowing the dangers of so doing.	S6	F36 / H38 Analysing data	Extrapolation was previously Higher only content.

*Concept quoted from DfE GCSE subject content and assessment objectives where appropriate.

**Reference taken from DfE GCSE subject content and assessment objectives.

Additions to Higher tier

The following section details the changes made to the objectives for Higher tier students. There are not as many changes as there are to the Foundation tier and some of the additions are new to both and indicated as such. In the same way that the most challenging content all students will be assessed on was indicated with underlined text, further challenging material for students taking the Higher tier is indicated with bold font. The expectation here is that: 'only the more highly attaining students will be assessed on the content identified by **bold** type. The highest attaining students will develop confidence and competence with the **bold** content.' (DfE, 2013: p.4)

Concept*	DfE Ref**	Where you can find it	Notes
<u>Use inequality notation to specify simple error intervals due to truncation or rounding.</u>	N15	F17 / H17 Approximation and estimation	This is new to both tiers and different from upper and lower bounds which remains in the Higher tier.
<u>Find the equation of the line through two given points, or through one point with a given gradient.</u>	A9	F18 / H18 Straight-line graphs, F19 / H19 Graphs of equations and functions.	This is new to both tiers. Previously students were only expected to identify the gradient of a line in the form $y = mx + c$.
Recognise, sketch and interpret graphs of linear functions, quadratic functions, <u>simple cubic functions, the reciprocal function $y = \frac{1}{x}$ with $x \neq 0$, exponential functions $y = k^x$ for positive values of k, and the trigonometric functions (with arguments in degrees) $y = \sin x$, $y = \cos x$ and $y = \tan x$ for angles of any size.</u>	A12	F19 / H19 Graphs of equations and functions.	Sketching quadratic, cubic and reciprocal functions is new to both tiers. Further challenge for Higher students includes an ability to sketch exponential functions and the three trigonometric functions they learn about at GCSE: sine, cosine and tangent.
Calculate or estimate gradients of graphs and areas under graphs (including quadratic and other non-linear graphs), and interpret results in cases such as distance–time graphs, velocity–time graphs and graphs in financial contexts.	A15	H39 Interpreting graphs	There is an increased emphasis on interpreting the information presented in graphs and in addition to the new requirement for students to calculate the gradient of curves at given points they will also have to interpret this in given situations.

Find the equation of a tangent to a circle at a given point.	A16	H18 Straight line-graphs, H19 Graphs of equations and functions.	This topic does not require calculus as the gradient can be found by considering the negative reciprocal of the gradient of the radius connecting the centre of the circle to a given point. It is a challenging addition and requires strong knowledge of linear equations and an ability to work with equations of circles.
Solve ... quadratic inequalities in one variable; represent the solution set on a number line, using set notation and on a graph.	A22	H40 Algebraic inequalities	Solving quadratic inequalities was previously a topic reserved for A Level. In addition, Higher students are also expected to represent their solutions with set notation and graphically.
Recognise and use sequences of triangular, square and cube numbers, simple arithmetic progressions, <u>Fibonacci type sequences</u> , <u>quadratic sequences</u> , and <u>simple geometric progressions (r^n where n is an integer, and r is a rational number > 0, or a surd)</u> and other sequences.	A24	F3 / H4 Functions and sequences	These different types of sequences have had more emphasis placed on them in the new specification. 'Fibonacci type sequences' implies students now may have to work with sequences defined by a recurrence relation. In addition for Higher students the common ratio of a geometric sequence may also be an irrational number.
Deduce expressions to calculate the n th term of linear and quadratic sequences.	A25	H4 Functions and sequences	The addition here is finding an expression for the n th term of quadratic sequences.
Use compound units such as ... <u>pressure</u> .	R11	F12 / H12 Units and measurement	Pressure is a new compound measure to both tiers.
Interpret the gradient at a point on a curve as the instantaneous rate of change; apply the concepts of average and instantaneous rate of change (gradients of chords and tangents) in numerical, algebraic and graphical contexts.	R15	H39 Interpreting graphs	This addition to the Higher tier introduces students to the idea that a non-linear curve can have a different gradient at different points and that these gradients can be found by investigating the tangent to the curve at that point. It is not the reintroduction of calculus to the GCSE curriculum but does introduce some important ideas that will be studied for students continuing mathematics at A Level.
Derive and use the sum of angles in a triangle.	G3	F9 / H9 Angles	The important word here being 'derive' as previously there was only an expectation students understood a proof that the sum of angles in a triangle is 180 degrees.
Apply and prove the standard circle theorems concerning angles, radii, tangents and chords, and use them to prove related results.	G10	H34 Circle theorems	There is now an expectation that students can prove the standard results for circle theorems.
Know the exact values of $\sin \theta$ and $\cos \theta$ for $\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ$ and 90° ; know the exact value of $\tan \theta$ for $\theta = 0^\circ, 30^\circ, 45^\circ$ and 60° .	G21	F32 / H33 Trigonometry	This content is new to both tiers.

Enumerate sets and combinations of sets systematically, using ... Venn diagrams.	P6	F24 / H24 Combined events and probability diagrams	Venn diagrams are new material to both tiers.
Calculate and interpret conditional probabilities through representation using expected frequencies with two-way tables, tree diagrams and Venn diagrams.	P9	H24 Combined events and probability diagrams	The addition of Venn diagrams extends to working with conditional probability for Higher tier candidates.
Interpret, analyse and compare the distributions of data sets from univariate empirical distributions through: Appropriate measures of central tendency ... and spread (range, including consideration of outliers...).	S4	F36 / H38 Analysing data	The consideration of outliers when measuring spread is new to both tiers.
<u>Recognise correlation and know that it does not indicate causation.</u>	S6	F36 / H38 Analysing data	New content to both tiers.

Topics that have been removed

As well as the large number of additions to the new GCSE there have also been some topics removed that were present in the previous GCSE programme of study. These are:

- conversions between metric and imperial units;
- design and criticism of survey questions;
- identification of sources of bias.

In addition trial and improvement to solve algebraic equations now appears at Higher tier only.

1F / 1H Basic calculation skills

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know the meanings of the words sum, quotient, product and difference;
- be able to use formal and informal methods and algorithms, both mental and written, for the four operations of arithmetic;
- understand the distributive law;
- know that multiplication and addition are commutative.

Learning outcomes

Foundation

Section 1

- To identify the correct operations required and use written calculations to solve worded problems.
- To calculate with all four operations of arithmetic using positive and negative numbers.

Section 2

- To apply the hierarchy of operations to accurately work out calculations involving two or more operations.

Section 3

- To identify and write the inverses for operations and apply these to check the results of calculations and develop the skills required to solve equations.

Higher

Section 1

- To identify the correct operations required and use written calculations to solve worded problems.
- To calculate with all four operations of arithmetic using positive and negative numbers.

Section 2

- To apply the hierarchy of operations to accurately work out calculations involving two or more operations.

Section 3

- To identify and write the inverses for operations and apply these to check the results of calculations and develop the skills required to solve equations.

Vocabulary

integers

Common misconceptions and other issues

- With multiplication, it is important to ensure that students fully understand the distributive law, to avoid a commonly held multiplication misconception creating errors such as $34 \times 23 = 30 \times 20 + 4 \times 3$. (The 'grid' method, to ensure full multiplication of all partitions, can be useful here.)
- Some foundation students may have been confused by the range of possible methods they have been shown and they may try to follow algorithmic procedures with no true understanding. An example of this is with the column method for subtraction, where 'borrowing' would be necessary. Students often ignore this and find the difference between the largest and smallest values in the column, irrespective of order. Use of manipulatives (Dienes blocks, Cuisenaire rods, money) to visually represent the strategies, alongside the formal written methods, can help.
- Students can learn procedural rules that they do not understand, which they then incorrectly apply. For example 'two negatives make a positive' translates to $-n - m = +(n + m)$. Use continuations of patterns (see below) and examples from real life to help. (For example, imagining travelling in a lift that includes sub-ground level floors, considering earning and borrowing money or working out the change in temperature to a liquid after adding hot or cold liquids can all provide useful analogies to help with the concept of adding and subtracting negatives.)

Hooks

1. A good way to introduce key terminology and to develop problem-solving skills is to ask students how many pairs of integers with their sum equal to their product can they find. $2 + 2$ and 2×2 or $0 + 0$ and 0×0 . What relationship must exist between the numbers? $n + m = mn$ or $n = \frac{m}{m-1}$ for $m \neq 1$. What if we remove the restriction of specifying integers? Obviously an infinite set of non-integer solutions can be found; some examples include $\frac{4}{3}$ and 4; 1.25 and 5; -9 and 0.9 ; $\sqrt{2}$ and $2 + \sqrt{2}$.
2. 'Calculation Question' and 'Just Add' at **Mathematical Beginnings**. For the first, ask students to write an explanation to the person who wrote this. For the second, students will inevitably check the addition is correct initially. Then they might begin to consider the sense of what has been done! (mathematicalbeginnings.com)

Q: When I was at school, $2 + 3 \times 5$ (no brackets) equalled 25. This is confirmed by my calculator. However, on my new phone, the answer is given as 17. This is also the answer being taught at our local secondary school. Which is correct?

B. Marrison, Birdwell, Barnsley



SECTION 1F / 1H: BASIC CALCULATIONS

Section 1 introduces the use of basic calculations and choosing the appropriate operation for worded problems. The section then goes on to look at choosing the appropriate operation for word problems and how to calculate with directed numbers.

Prompting questions

Exercise 1A(F) / 1A(H)

While working through the exercise, good prompts for students might be:

- Can you identify which operation is required for each question? Then work through it showing the method fully. Encourage discussion and comparison of solutions. Have students used different methods and can they describe them? Which are more efficient?
- How can you check your solutions? Possibilities include making a sensible estimate before calculating or use inverse operations. For example, in Exercise 1A Question 8 $(8765 - 3087) + (1206 \div 18)$ could be estimated as $(9000 - 3000) + (1200 \div 20) = 6000 + 600 = 6600$.
- What words or clues in the questions can you find to identify which operations and calculations you need to use? For example, in Question 4 the rate of words per minute should indicate that multiplication is required to calculate the number of words typed in 1.5 hours.

Exercise 1B(F) / 1B(H)

While working through the exercise, a good prompt for students might be:

- Why do the 'rules' given in the chapter work? Investigate these patterns to help:

$4 + 2 =$		$4 - 2 =$
$4 + 1 =$		$4 - 1 =$
$4 + 0 =$		$4 - 0 =$
$4 + -1 =$		$4 - -1 =$
$4 + -2 =$		$4 - -2 =$
$4 + -3 =$		$4 - -3 =$

$4 \times 2 =$		$-2 \times 4 =$
$4 \times 1 =$		$-2 \times 3 =$
$4 \times 0 =$		$-2 \times 2 =$
$4 \times -1 =$		$-2 \times 1 =$
$4 \times -2 =$		$-2 \times 0 =$
$4 \times -3 =$		$-2 \times -1 =$
$4 \times -4 =$		$-2 \times -2 =$

$\times$	3	2	1	0	-1	-2	-3
3							
2							
1							
0							
-1							
-2							
-3							

- In 1758 the British mathematician Francis Maseres claimed that negative numbers: '*... darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple.*'
- Maseres and his contemporary, William Friend, took the view that negative numbers did not exist.
 - Why do you think they felt like this? *Perhaps there was a lack of immediate necessity for negative numbers at this time; today we need them to show debt, lower temperature, and so on, which could have been less relevant then. Also, they possibly felt negative numbers over complicated things. Students find multiplying two negative numbers surprising; in Maseres' time such concepts were being thought about and proved for the first time.*
 - Why do we need negative numbers? Think of examples in real life where negative numbers are useful. *For example, temperature, money, numbering floors in buildings which have underground levels.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- **NRICH Product Sudoku.** This provides a twist on traditional Sudoku involving some restrictions in which pairs of numbers must have a specified product. It could be used to develop resilience and students' ability to work to given conditions and follow instructions (nrich.maths.org).
- **(Foundation only)** The **Number Loving** website activity 'Tick or Trash QWC Comparing Prices' encourages students to compare methods and create their own questions and mark scheme for the problem (numberloving.co.uk).
- Cross number (crossums) puzzles are an engaging way to practise basic skills. The **Teachers Resources Online (trol)** website has plenty of these in a variety of formats (cleavebooks.co.uk).
- Number Chains. Give students a starting number and then a string of operations to perform on it to get a final value. For example:
- e.g.

106	$\div 2$	$- 13$	10% of this	$\times 5$	square	(Answer 400)
-----	----------	--------	-------------	------------	--------	--------------

- Target boards. Show students a grid with a range of numbers and ask them to write calculations to make a particular value on the grid, or given them calculations to do to find your target number.
- Magic squares involving negative numbers. A useful source for this is instant maths ideas (Number and Algebra p. 54), available from the **National STEM Centre** (nationalstemcentre.org.uk).
- To give some variation in exercise style, you could create a 'follow-me' loop or treasure hunt using some of the questions from the chapter. Question 4 would be good as a starter question where students could 'race' to complete each calculation.
- **NRICH Connect Three** (nrich.maths.org).
- Search online for the *Negative Numbers Song* video, from the UWE PGCE Mathematics Trio 2011–12, to play for students.
- **Number Loving** has two examples of 'Tick or Trash' for calculations with negative numbers as well as 'Catchphrase' and 'Bingo' games (numberloving.co.uk).

Enrichment activities

- **NRICH Largest Product** is a useful investigation that will ensure students become familiar with the terms 'sum' and 'product', which can be extended to non-integers. Students could be encouraged to try to generalise their findings (nrich.maths.org).
- **NRICH Cinema Problem** is great for developing problem-solving skills and can be approached by exhaustive listings of calculations or introduction of algebra, and therefore offers the possibility of extension for higher tier students (nrich.maths.org).
- **(Foundation only)** Improving Learning in Mathematics **Mostly Number – N8 Using Directed Numbers in Context** available from the National STEM Centre Archive. The activities here work on adding and subtracting negative numbers through the context of temperature (nationalstemcentre.org.uk).
- **(Foundation only)** **NRICH Balloon Game.** This is suitable for students who struggle with the concept of adding and subtracting negative numbers (nrich.maths.org).
- **(Foundation only)** **NRICH Consecutive Negative Numbers.** This investigation will ensure that students have plenty of practice at adding and subtracting negative numbers, while being encouraged to look for patterns and develop their investigative skills (nrich.maths.org).

Assessment ideas

- Ask students to create their own 'cross number' puzzle and a fully worked solution, with the calculation questions having to use at least all four arithmetic operations. Their clues could have to include a number of problems framed in 'real life' scenarios as well as straightforward calculations to perform. This could be extended by using non-integer values.
- Improving Learning in Mathematics **Mostly Number – N9 Evaluating Directed Number Statements** available from the National STEM Centre Archive. This encourages students to evaluate statements using 'always', 'sometimes',

‘never’, as well as explore operations on negative numbers using patterns in a more extended way than the suggestions given previously. This provides plenty of opportunities for in-class assessment (nationalstemcentre.org.uk).

SECTION 2(F) / 2(H): ORDER OF OPERATIONS

This section focuses on the hierarchy of operations and applying them in the appropriate order.

Prompting questions

While working on the material in this section good prompts for promoting discussion may be:

- To have a whole class discussion of the section’s ‘Work it out’ element, identifying which solution is correct and the mistakes made so as to fully draw out any lingering misconceptions.
- To help to pick up any continuing misconceptions, use Question 1 in Exercise C to make a quick multiple choice quiz (along the lines of a ‘Who wants to be a Maths Millionaire?’). Ensure that the incorrect multiple choice options include the answers found simply by working from left to right, or calculating a part of the sum in the incorrect order and discuss why these give the wrong answers.

Exercise 1C(F) / 1C(H)

While working through the exercise, good prompts for students might be:

- Why do you think operations have a hierarchy (i.e. ‘BIDMAS’)? *It would be good to tease out from students that raising a number to a power is a quicker way of increasing a number, compared to repeat multiplication or indeed addition.*
- **Q5(H)** If these calculations are entered into the students’ calculators, do they get the expected answer? Try with a calculation such as $4.2 + 7 + \frac{3 \times \sqrt{4.5^2 + 6^2}}{2}$. How many different solutions have the class achieved? Where should they be entering brackets to ensure their calculator gives the required solution? With the range of calculators students may be using, this is really important to ask.

Starters, plenaries, enrichment and assessment ideas

Starters

- A quick starter is to ask students to write down the numbers 1 to 20. They then roll a dice to randomly generate numbers with which they must make each number from 1 to 20 using the four operations, brackets and powers (from one of the rolled numbers) without using any number more than once. They are not allowed to make numbers by writing the rolled numbers as consecutive digits (e.g. making 16 by writing the rolled numbers 1, 6).
- ‘Four Fours’ is a similar idea to the one above, but here students are asked to make the numbers 1 to 100 using exactly four fours each time. The use of operations can be developed further by involving square roots and factorials and they are allowed to write 44 or 444 with their fours if required. **Cut the Knot** has a comprehensive list of solutions (cut-the-knot.org).

Plenaries

- **(Higher only)** Spot the Mistakes. Give students questions such as those below (there is plenty of opportunity here for differentiation through your choice of questions) to identify the errors that have been made and amend each calculation so it becomes correct.

	$3 + 4 \times 5 = 35$	$8 - 10 \div 2 = -1$	$2 \times 5^2 = 100$	$(9 - 6) \div 3 = 7$
Corrections:	$(3 + 4) \times 5 = 35$	$(8 - 10) \div 2 = -1$	$(2 \times 5)2 = 100$	$9 - (6 \div 3) = 7$

- Make a ‘BIDMAS’ maze (a table with each cell filled with ‘BIDMAS’ appropriate calculations) in which, travelling from left to right, it is possible to find a route of cells with a specified target number as an answer.
- **(Foundation only)** Play the ‘BIDMAS dice’ game. Roll a dice 18 times and ask students to place a number in each box, such as the ones below, with the aim of getting the highest total at the end.

a. $\boxed{} \times \boxed{} + \boxed{} = \text{-----}$ b. $\boxed{} \times \boxed{} - \boxed{} = \text{-----}$ c. $\boxed{} + \boxed{} \times \boxed{} = \text{-----}$

d. $\boxed{} + \boxed{} \times \boxed{} + \boxed{} = \text{-----}$ e. $\boxed{} \times \boxed{} + \boxed{} \times \boxed{} - \boxed{} = \text{-----}$

Grand total _____

Enrichment activities

- **(Foundation only)** The **Number Loving** website has some suitable BIDMAS activities including ‘Collect a joke’ and ‘Top Trumps’ cards activities (numberloving.co.uk).
- For a range of additional BIDMAS activities, this chapter from ‘Instant Maths Ideas’ is available from the **National STEM Centre** (nationalstemcentre.org.uk).
- Develop the previous prompt based on Q5(H) with both higher and foundation students. Try giving students a calculation such as $5.8 + 7 + \frac{4 \times \sqrt{3.5^2 + 7^3}}{3}$. How many different solutions have the class achieved? Where should they be entering brackets to ensure their calculator gives the required solution? With the range of calculators students may be using, this is really important to ask.

Assessment ideas

- To help to pick up any continuing misconceptions, use Question 1 in Exercise C to make a quick multiple choice quiz (along the lines of a ‘Who wants to be a Maths Millionaire?’). Ensure that the incorrect multiple choice options include the answers found simply by working from left to right, or calculating a part of the sum in the incorrect order.

SECTION F3 / H3: INVERSE OPERATIONS

This section introduces inverse operations to ‘undo’ a previous calculation.

Prompting questions

Exercise 1D(F)/1D(H)

While working through the exercise, good prompts for students might be:

- In addition to Question 3(F) and Question 2(H), which asks students to use inverse operations to check solutions, go back through some of the students’ own solutions to previous exercises and get them to use inverse operations to double check their solutions so as to reinforce the use of inverses as a method of checking work.
- Can you give examples where use of inverse operations is helpful? *Double checking calculations, solving equations, rearranging formulae...*
- What would the inverse of squaring or cubing a number be? *Square root or cube root respectively.*
- Can you link the idea of inverses to other areas of maths? For example, what would the inverse of a 90° clockwise rotation about a particular point be? *A 90° anticlockwise or 270° clockwise rotation about the same point.*
- Investigate the definition of a reciprocal and consider how this links to inverse operations. *A number multiplied by its reciprocal = 1, therefore $1 \div \text{the original number} = \text{the reciprocal}$.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- **NRICH Alphabetti Sudoku.** In addition to a traditional Sudoku this requires students to solve nine additional problems with the use of inverse operations (nrich.maths.org).

- Give students some ‘Think of a number’ problems to solve. You can differentiate this activity by varying the types and numbers of operations you used. At this stage there is no need for any formal expression of this using algebra, although those students wishing to do so could be encouraged. For some quick examples of this see the **Transum Starter of the Day** (transum.org).

Enrichment activities

- Extend the ‘Think of a Number’ by asking students to write their own problems and give them to another student to work through.
- **NRICH Twisting and Turning**. Although it is marked as a Key Stage 3 activity, this is a useful way to help students think about ‘undoing’, thus developing concepts such as inverse operations, inverse functions and in particular reciprocals, since the ‘turn’ operation is to use a reciprocal action (nrich.maths.org).
- Twisting and Turning has two follow on activities, **More Twisting and Turning** and **All Tangled Up**, which further develop the use of reciprocals (nrich.maths.org).

Topic links

Previous learning

This topic provides a good opportunity to revisit and practise arithmetic skills using both mental and written methods, including formal algorithms. For foundation students, it may be useful to review the use of number lines for working with directed numbers. For an additional game to practise basic use of ‘BIDMAS’, try **NRICH The 24 Game**. For students struggling with the concept of negative numbers, revisiting a couple of KS3 problems on the NRICH website could be useful: for example, **Strange Bank Account** and **Strange Bank Account Part 2** (nrich.maths.org).

Future learning

Fluency with the calculation skills from this chapter will be required for successful problem solving and calculations within most GCSE topics. They are also important when working with algebraic problems; for example, solving equations and rearranging formulae require the use of inverse operations.

Gateway to A Level

This is a straightforward topic at GCSE, but to move successfully to KS5 it is important that students are confident when performing calculations involving integers, order of operations and inverse operations, as these will be required for algebraic manipulation and equation solving. At A Level this will be developed further to include other types of calculation, for example involving complex numbers, or modular arithmetic which in turn links to group theory and congruence classes.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 2 Question 1
- Chapter 4 Question 11
- Chapter 9 Questions 1, 2, 8, 9, 10
- Chapter 10 Questions 1, 5

Higher

- Chapter 2 Question 1
- Chapter 4 Questions 4, 10
- Chapter 9 Questions 1, 2
- Chapter 10 Question 1

Homework Book

Foundation

- Chapter 1

Higher

- Chapter 1

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 11 worksheets (+ solutions)
- 13 animated widgets
- 13 interactive walkthroughs
- 5 auto-marked quickfire quizzes
- 5 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

2F / 2H Whole number theory

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to list the factors and multiples of a number;
- recognise square, cube and prime numbers;
- be able to express a number as a product of its prime factors;
- know how to write a series of numbers that have been multiplied together in index form (**Higher only**).

Learning outcomes

Foundation

Section 1

- To recall and understand key definitions of types of number.
- To consolidate understanding of basic place value

Section 2

- To apply knowledge of factors and primes to express a number as a product of its prime factors
- To simplify a collection of numbers that have been multiplied together by writing them in index form

Section 3

- To use the 'listing method' to find the highest common factor and lowest common multiple of a set of numbers
- To use prime factors to find the highest common factor and lowest common multiple of a set of numbers

Higher

Section 1

- To recall and understand key definitions of types of number.
- To consolidate understanding of basic place value

Section 2

- To apply knowledge of factors and primes to express a number as a product of its prime factors
- To simplify a collection of numbers that have been multiplied together by writing them in index form

Section 3

- To use the 'listing method' to find the highest common factor and lowest common multiple of a set of numbers
- To use prime factors to find the highest common factor and lowest common multiple of a set of numbers

Vocabulary

consecutive, prime factor

Common misconceptions and other issues

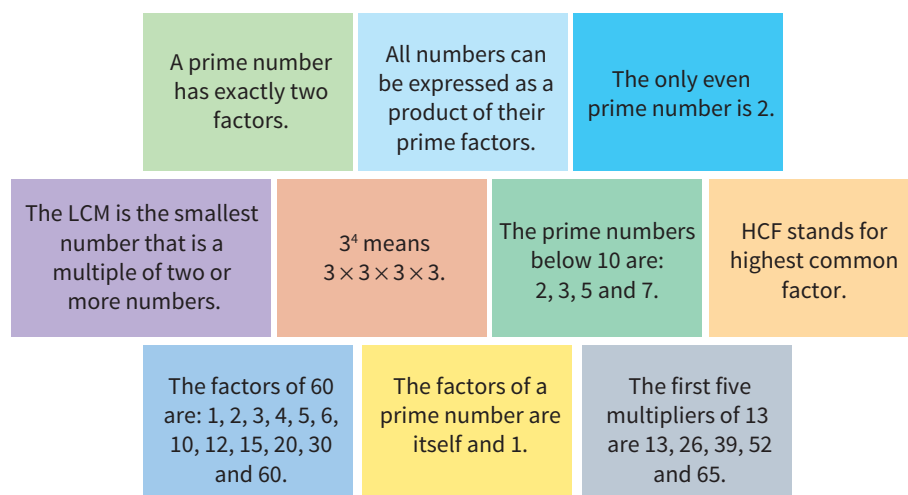
- That 1 is a prime number.
- That there is no even prime number.
- Students use other operations instead of the correct multiplication symbol.
- Students may confuse 3^4 ($3 \times 3 \times 3 \times 3$) with 3×4 .

Hooks

Kim's game

This topic requires students to recognise and understand various types of number. A great way to introduce or recap this knowledge is through a quick activity called Kim's game. Ten mathematical facts are displayed on the screen. Students get the opportunity to study the facts but cannot write anything down. Every 30 seconds or so, a fact is

removed. Students are then asked to remember and write down the fact that was removed. An example of suitable facts can be seen below:



SECTION 1F / 1H: REVIEW OF NUMBER PROPERTIES

This section recaps basic number properties.

Prompting questions

Exercise 2A(F) / 2A(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** What is the difference between a factor and a multiple? *Factors divide into a number without leaving a remainder.*
- **Q2(F) / Q2(H)** What is the only even prime number and why? *Two: all other even numbers are multiples of two.*
- As an extension, can you use algebra to prove why the sum of two odd numbers is even (and so on with other questions)? *Let $n = a$ number, therefore we can say the first odd number is $2n - 1$ and the second $2n + 1$. If we find the sum of these two numbers, $(2n - 1) + (2n + 1) = 4n$. Since $4n = 2(2n)$ then all values of n are even.*

Exercise 2B(F) / 2B(H)

While working through the exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** Which place value column is 100 times bigger than the units column? *Hundreds, for example $4 \times 100 = 400$.*
- **Q2(F) / Q2(H)** Explain why you cannot place the number 11 into the units column. *The number 11 consists of one ten and one unit.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Give each student a hundred square. Identify a number on the grid and create a series of clues based on its properties (this needs to be unique). Read the clues aloud. The students can use their grid to identify the number. After this, students can create their own clues for a selected number.

Starters or plenaries

- **NRICH Factors and multiples game.** A game that can be played as a class vs a teacher or boy vs girl and so on. Each player has to take it in turn to choose a number between 1 and 100 that is a multiple or factor of the previous number. This can also be used as a challenge in which students attempt to get the longest string of numbers (nrich.maths.org).

Enrichment activities

- **NRICH Factors and multiples puzzle.** A challenging puzzle that asks students to arrange numbers in a table based on different headings, for example multiple of 3. To differentiate this activity, the teacher could place the headings on the table before giving the numbers to the students (nrich.maths.org).
- **NRICH Sieve of Eratosthenes.** This is an excellent way of helping students to identify prime numbers and understand what a prime number is (nrich.maths.org).
- **NRICH How much can we spend?** An activity based on lower common multiples (nrich.maths.org).
- **National Stem Centre.** A selection of number based activities (nationalstemcentre.org.uk).
- **National Stem Centre.** Factors, multiples and primes ‘who dunnit’ (nationalstemcentre.org.uk).
- **NRICH Take three from five** (nrich.maths.org).

SECTION 2F / 2H: PRIME NUMBERS AND PRIME FACTORS

Section 2 focuses on prime numbers and prime factors.

Prompting questions

Exercise 2C(F) / 2C(H)

While working through the exercise, good prompts for students might be:

- Why would it not be a good idea to divide the number by one when creating a factor tree? *This would just duplicate the previous branch of the tree.*
- When creating a factor tree, does it matter which factor pair you start with as the first ‘branches’? *No.*
- **Q2(F) / Q2(H)** Can every number be made using products of primes? *Yes, any integer greater than one.*
- **Q2(F) / Q2(H)** Could any two numbers have the same prime number decomposition? *No.*
- **Q3(F) / Q3(H)** What is the difference between 3^3 and 3×3 ? *3^3 is equal to $3 \times 3 \times 3$.*
- **Q3(F) / Q3(H)** How many unique prime factor trees can you create for the number you identified in part a? Do they all result in the same product of primes? *Yes, they do. It doesn't matter which factors are chosen at first.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Search on the internet for the ‘Prime Factor Trees’ song to play to students.

Enrichment activities

- Give students pieces of incorrect homework (not from actual students). Ask them to identify the mistakes and correct the homework.
- Show students how to apply prime decomposition to be able to easily simplify fractions and hence use this as a ‘trick’ for division. Ask them to create ‘difficult’ division questions for each other to solve (such as $1512 \div 54$) where, thanks to prime decomposition, they know there must be an integer solution.
- Investigate how many ways can the number 1 000 000 be expressed as the product of three positive integers. **NRICH Factoring a Million** (nrich.maths.org).
- **NRICH Gaxinta** (nrich.maths.org).

SECTION 3F / 3H: MULTIPLES AND FACTORS

Section 3 focuses on lowest common multiple and highest common factors.

Prompting questions

Exercise 2D(F) / 2D(H)

While working through the exercise, good prompts or questions for students might be:

- **Q5(F) / Q4(H)** Why is it useful to find the LCM for this question? *This is the most efficient way of recording the information.*

Starters, plenaries, enrichment and assessment ideas

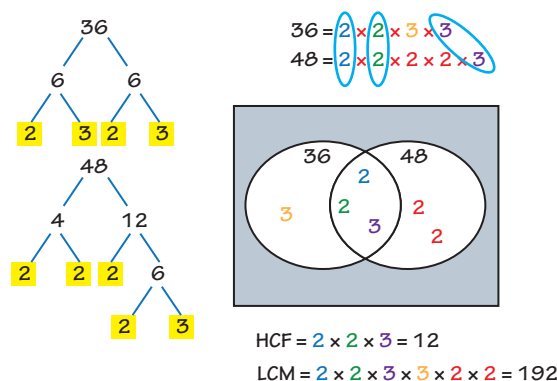
Starters or plenaries

- **Visualisation of prime numbers** (datapointed.net).

Enrichment activities

- **NRICH Got it** (nrich.maths.org).
- **NRICH Counting Cogs** (nrich.maths.org).
- Students may like to use a Venn diagram to find the HCF and LCM of a pair of numbers. For example:

HCF and LCM of 36 and 48.



- Students may like to investigate the fact that the HCF of two numbers must be a factor of the difference between them and also try to explain why this is the case.

Topic links

Previous learning

- Students should be able to multiply and divide integers.
- Indices: students should understand index notation and be able to express a series of repeated multiplications in this way.

Future learning

- Indices: Whole number theory is used in indices, see *Chapter 25 Powers and roots* for further information.

Gateway to A Level

- Indices;
- The Fundamental Theorem of Arithmetic.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 9 Questions 3, 11
- Chapter 10 Question 6

Higher

- Chapter 9 Question 9
- Chapter 10 Question 2

Homework Book

Foundation

- Chapter 2

Higher

- Chapter 2

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 6 worksheets (+ solutions)
- 9 animated widgets
- 15 interactive walkthroughs
- 5 auto-marked quickfire quizzes
- 5 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- Crossword puzzle (for use with Chapter review Question 1)

3F / 3H Algebraic expressions

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to add, subtract, multiply and divide directed numbers;
- understand what a power is and does, for example squaring and cubing;
- be able to use the laws of indices (**Higher only**);
- know how to add, subtract, multiply and divide fractions (**Higher only**).

Additional useful prior knowledge

- How the commutative, associative and distributive laws apply to numerical operations, for example, that addition is commutative but that subtraction isn't so, while $3 + 5 = 5 + 3$, the same does not hold for subtraction. The language of these laws is used in the chapter and while it would be beneficial for students to know the formal language of these laws, understanding what they mean for each operation will be enough for foundation students.

Learning outcomes

Foundation

Section 1

- To understand and work with correct, formal algebraic language and notation.
- To form algebraic expressions from worded instructions and geometric problems.
- To substitute given values into algebraic expressions and evaluate the result.

Section 2

- To simplify algebraic expressions by collecting like terms.
- To simplify products and quotients.

Section 3

- To expand the product of a single term and binomial.

Section 4

- To factorise expressions by taking out common factors and recognise that the HCF must be used for an expression to be fully factorised.

Section 5

- To form expressions from word problems in a variety of contexts, including number problems, and use algebra to solve them.

Higher

Section 1

- To understand and work with correct, formal algebraic language and notation.
- To simplify products and quotients and apply the index laws to simplify.
- To form algebraic expressions from worded instructions and geometric problems.

Section 2

- To simplify algebraic expressions by collecting like terms.
- To simplify products and quotients.

Section 3

- To expand the product of a single term and binomial.

Section 4

- To factorise expressions by taking out common factors and recognise that the HCF must be used for an expression to be fully factorised.

Section 5

- To form expressions from word problems in a variety of contexts, including number problems, and use algebra to solve them.

Vocabulary

variable, expression, term, product, evaluate, expand, factorise

Common misconceptions and other issues

There are a huge number of misconceptions that students encounter in understanding why letters are used in mathematics. Research into this issue has been published extensively and it is worth reading Küchermann's chapter in detail for background knowledge of students' use of letters in mathematics (Küchermann, D (1981) *Algebra*, in K. Hart, *Children's Understanding of Mathematics: 11-16*. London: Alden Press). The most common misconceptions are listed below:

- Using a letter as shorthand in place of an object rather than to represent the number of those objects. For example, $3b = 3$ bananas rather than 3 times the number of bananas. Avoiding introducing simplifying as shorthand for collecting objects, for example $3 \text{ apples} + 2 \text{ bananas} + 2 \text{ apples} = 5 \text{ apples and } 2 \text{ bananas}$ will help with this.
- Confusing like terms with terms with the same letter so wanting to simplify terms that contain the same letter regardless of its power. Asking students to substitute in a number for the letter, for example 3, can help them identify why x and x^2 are not like terms and consequently their simplified expression is not equivalent to their starting point.
- Students don't realise that $3b$ represents the product of 3 and the value of b so when substituting, for example when $b = 2$, students replace b with 2 and get 32 rather than 6. Avoid discussing substitution as replacing the letter with the value given and instead introduce it as an extension to work on function machines where discussions are connected to the work on forming expressions and the meaning of the algebra in the expressions is explicit.

Hooks

1. A member of one maths department likes to start by telling students that she uses algebra to cook a chicken, moving on to showing them a cooking label and the weight of the chicken and deriving a formula from the image. Foundation year 11 students are often heard explaining how algebra is needed to cook a chicken.
2. Play games where points are awarded for being as quick as possible when adding the same value several times (e.g. $3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3$) and identify that those answering the problem quickest are counting how many of the value they have and multiplying (e.g. $10 \times 3 = 30$). You may find that a useful way to introduce simplifying and the use of a letter as a variable is by repeating the same type of sum, for example $5 + 5 + 5 + 5 + 5 + 5 = 6 \times 5$ or $7 + 7 + 7 + 7 + 7 + 7 = 6 \times 7$, to conclude that any number summed 6 times is the same as 6 times that number (i.e. $n + n + n + n + n + n = 6n$).

SECTION 1F / 1H: ALGEBRAIC NOTATION

Section 1 introduces the basics of the language and conventions of algebra. Students are introduced to a lot of key vocabulary in this section and learn to formulate expressions using formal conventions. In Exercise A they are required to simplify products and quotients using formal methods. Higher students are also expected to apply their knowledge of adding, subtracting, multiplying and dividing fractions to algebraic fractions.

Prompting questions

While working on the material in this section good prompts for promoting discussion may be:

- Why do we use letters in mathematics? *To represent an unknown or any number. Students may suggest that we use it like a code, for example $n = \text{number}$, $a = \text{apple}$, and so on, or that it stands for an unknown number.*
- Why do we need conventions in how we write things down? *Like spellings and grammar in English, conventions and rules exist so that mathematical statements can be understood universally.*

Exercise 3A(F) / 3A(H)

While working through the exercise, good prompts for students might be:

- **Q2(F) / Q1(H)** Are we trying to find the value of the letters in these questions? *No, the value is unimportant.*

- **Q5(F) / Q4(H)** Why can we use fractions to represent divisions? *One explanation is: because a fraction is a 'lazy way' of writing a division. We are sharing the numerator into the denominator but because we won't have an integer solution we prefer it written in this form.*
- **Q5(F) / Q4(H)** If we multiply by 4 and then divide by 2 what have we done to the original number? *Multiplied by 2 only. Students may struggle to identify what is happening here. Suggest they try it with more than one value and perhaps then suggest they look at multiplying by 5 followed by multiplying by 2.*
- **Q4-6(F) / Q3-4(H)** Why might we want to write these multiplications and divisions more concisely and uniformly? *To reduce the number of operations applied to a number and the amount of work when simplifying.*
- **Q4(F) / Q3(H)** What will happen when we multiply a negative by a negative (or any other combination involving directed numbers)? Can you demonstrate the rules using a diagram? *Giving students a multiplication grid from -5 to 5 multiplied by -5 to 5 and asking them to fill out what they know can be useful for highlighting these rules.*

$\times$	+ve	-ve
+ve	+	-
-ve	-	+

- **Q5(F) / Q4(H)** Repeat the above question for division. Why is it the same rules for multiplying and dividing? *Because division is multiplying by the reciprocal of the multiplier, in other words they are the same thing.*
- **Q5(H)** Why can we add the powers when multiplying? Can you give me an algebraic example of why this is true? *For example $a \times a \times a \times a \times a = a^5$*
- **Q5(H)** Why are these laws useful when simplifying? *To reduce the number of operations we have to apply to the value of the letter.*
- **Q7(F) / Q6(H)** How do we find the area of a rectangle? How can we represent this given we don't know the length of the rectangle? *Multiply the base by height.*
- **Q7(F) / Q6(H)** How can we write the length of the rectangle in part a using the unknown? *Add the two lengths to get $x + 6$. How else could you find the area of this rectangle? Find the area of the two individual rectangles and add them. What does this tell you about $2(x + 6)$ and $2x + 12$? They are equal, note these last few questions can be replicated for parts b and c.*
- **Q7(F) / Q6(H)** What makes the shape in part d different? Can we form two equal expressions for this shape? *No, it is compound so we can't just find base $\times$ height.*
- **Q7(F) / Q6(H)** Are there any squares in these diagrams? How do you know? *In parts b and d we have $x \times x = x^2$; the side lengths are equal so it is a square.*

Exercise 3B(F) / 3B(H)

While working through the exercise, good prompts for students might be:

- **Q1-2(F) / Q1-2(H)** What word do we use to remember the order in which operations should be completed? *BIDMAS/BODMAS; importantly, all indices should be dealt with first, and all products/quotients should be found before sums/differences are calculated.*
- **Q1(F) / Q1(H)** What do we mean by evaluate? *I am interested in the value of the letter(s) and, using them, I want to know the numerical value of the expression.*

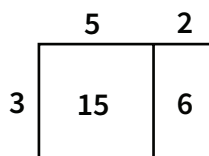
Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

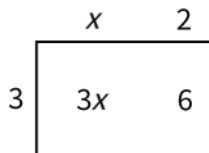
- The matching task (Exercise 3A(F) Question 3) could be made into cards to create an appropriate starter or plenary. Removing one or more of the cards could increase the challenge of this problem and also provide appropriate differentiation with the addition or subtraction of pairs.
- Another 'quick' problem that encourages the use of forming expressions to solve a problem is **How many eggs**, which may be faster than students' trial and improvement methods (nrich.maths.org).

Enrichment activities

- The area problems in Exercise 3A Q7(F) / Q6(H) make an excellent introduction to expanding brackets and forming identities (Section 3 and 4). Investigate these shapes and start with some where the length isn't a variable, for example:



We can deduce that $3(5 + 2) = 15 + 6$ and hence $3(x + 2) = 3x + 6$. We can then change the problem to give one of the lengths as an unknown or variable x , for example:



Students could work on several problems with the same structure, where the value of x is the only change, to support the idea of a letter representing a variable.

- There are many investigations that can support students' development in forming appropriate expressions to generalise their findings. **Crossed Ends** is effective because it uses only a 100-square grid (rich.maths.org). When students have worked on expanding two brackets they can also look at a similar problem using a 100-square grid where they compare the product of the opposite corners of a square drawn on the grid. There are many shapes that make interesting number problems on a 100-square grid.

SECTION 2F / 2H: SIMPLIFYING EXPRESSIONS

Section 2 focuses on like terms and simplifying expressions by collecting these like terms. For higher students there is more key vocabulary introduced in this section and a further emphasis is placed on simplifying products and quotients.

Prompting questions

Exercise 3C(F) / 3C(H)

While working through the exercise, good prompts for students might be:

- Q1(F) / Q1(H)** What do we mean by *like terms*? Can you give me any examples of like terms? *For example, $2a$ and $5a$ whereas $3x$ and x^2 are not like terms.*
- Q2(F) / Q2(H)** Why are x and x^2 not like terms? Can you give any examples where they would be equal? *Because $2x$ is 2 lots of x whereas $2x^2$ is 2 lots of x^2 and x and x^2 are different values.*
- Q2(F) / Q2(H)** What do we mean by simplify? *I like the informal definition of writing things in the shortest way possible to make it easier to read.* Are we interested in the value of the letter when simplifying? *No, it can take any value.*
- Q2(F) / Q2(H)** Which term does the positive or negative sign apply to in the expression? *The one immediately after it.*
- Q2(F) / Q2(H)** How could you identify different terms when simplifying expressions? *Using different colours or symbols to categorise them in the original expression. Please note that the use of colours should not be employed in an exam due to scanning reasons. You could get students to draw different shapes around the like terms instead, for example, squares and triangles.*

$$3a + 2b + 2a - b = 5a + b$$

- Q6(F) / Q6(H)** How do I simplify fractions? *I cancel common factors.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- The first question in Exercise 3C could be turned into a card sort and used as a starter or plenary categorising task in which students sort the cards into like terms and non-like terms.

Enrichment activities

- Toy train sets can be used to introduce the idea of simplifying where the variable is the length of the track piece. Given that there are only a few different types of track pieces, a track can be made up using two or more pieces and its total length represented by an algebraic expression. (Note: you only need pictures of toy train tracks to do this task; having an actual set only enhances this task for weaker students.)
- Perimeter Expressions** is a similar, far more challenging NRICH task based on lengths of proportional sizes of paper (having only used cuts that cut a piece exactly in half). It is very useful for developing students' awareness of a variable as all paper sizes are proportional and students completing this task with different sizes of A paper can bring out some generality (nrich.maths.org).

SECTION 3F / 3H: MULTIPLYING OUT BRACKETS

This section focuses on expanding single brackets and applying the laws of BIDMAS to collect like terms. An effective method for teaching students to expand brackets is using the 'box method', which demonstrates the expansion as finding an area (see examples of this in Exercise 3A). This method creates a solid foundation from which students can extend these ideas to the expansion of two binomials.

Prompting questions

While working on the material in this section good prompts for promoting discussion may be:

- Why is the identity sign useful in algebra? Why might we have less need for it in number? *Because in algebra we use letters to generalise number. In number we can tell if something is always true, for example $3 + 7 = 10$, because it is one case. However, we would have to test an infinite set of numbers to be sure with algebraic expressions, so we use our knowledge of associativity, commutativity and distributivity to form identities.*

Exercise 3D(F) / 3D(H)

While working through the exercise, good prompts for students might be:

- Q2-3(F) / Q2-3(H)** Can you use diagrams to represent the bracket being expanded? *Use the area problems in Exercise 3A to demonstrate this.*
- Q5(F) / Q5(H)** Why can substitution help show that equations are not identities? *Because identities are true for all values so we only need to find one it doesn't work for.*

Starters, plenaries, enrichment and assessment ideas

Starters

- There are many tasks, such as **Your Number Is...**, which are good for giving students a purpose for algebra and make an appropriate starter (nrich.maths.org). If you want to really challenge your students, asking them to invent their own 'think of a number' problems will stretch their manipulation skills and their understanding of inverse operations.

Starters or plenaries

- The questions in Exercise 3D Question 1 could be produced as cards and used as a starter or plenary in which students sort them into two piles (correct and not) or turned into a 'tick or trash' PowerPoint.

Assessment ideas

- Expanding brackets offers many opportunities for matching exercises and there are many examples of Tarsia puzzles (and other similar tasks) available on the internet.

SECTION 4F / 4H: FACTORISING EXPRESSIONS

Section 4 introduces students to factorising the product of a term and a binomial by finding the HCF of the terms in the expression.

Prompting questions

Exercise 3E(F) / 3E(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** What is the HCF of both terms in the expression? *For example, for part e the HCF is $5xy$.*
- **Q2(F) / Q2(H)** What is the HCF of all the terms of the expression? Is $x - 2$ a factor of the expression in part d? *Yes. We do not need to factorise first and this leads us into further ideas of factorising in Chapter 7 Further expressions.*

Starters, plenaries, enrichment and assessment ideas

Enrichment ideas

- Set up a worksheet consisting of three columns where the first column contains a series of single bracket expansions to be worked out and the other two are blank. Students fold over the paper to hide the right-hand column and they write the expansion of the factorised expressions in the middle column. The students unfold the last column and fold over the left hand column. They are then required to factorise the expanded form in the centre column. This task is very good for helping students identify the difference between factorised and fully factorised forms and they definitely remember less of the original questions than you would think.

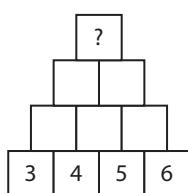
SECTION 5F / 5H: USING ALGEBRA TO SOLVE PROBLEMS

Section 5 is a consolidation of all the material covered in the chapter and offers a selection of ‘typical’ algebraic problems, including magic squares and addition pyramids, as well as links to geometry with area and perimeter problems.

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- A nice extension to the pyramid task is to tell students that you can work out the top number in their pyramid, given their starting number in the bottom left box, when the base of the pyramid is four consecutive numbers:



- Ask students to find the rule that allows you to calculate the solution so quickly.

Topic links

Previous learning

There are a few prior connections for this topic:

- The order in which operations are completed (BIDMAS). Opportunities for revising these rules come through substitution and remembering that for $2x^2$, x is squared before it is multiplied by 2.
- (Higher only) Knowledge of how to add, subtract, multiply and divide fractions. If students previously used informal methods to add and subtract fractions, based on identifying the lowest common multiple of both denominators and finding appropriate equivalent fractions first, this chapter provides opportunities for them to establish formal methods that will increase speed and accuracy, for example. $\frac{a}{b} + \frac{c}{d} \equiv \frac{ad+bc}{bd}$ However, while this may be appropriate for weaker Higher candidates, those looking to continue with further study at A Level will be better prepared for

advanced A2 rational functions work if they continue to combine fractions by manipulating them so they have a common denominator. Applying the rule above can make the final stages of simplifying more demanding than necessary when polynomials of degree 3 or more are involved and need factorising. It is better for students to use the following method: $\frac{3x}{(x+4)^2} - \frac{1}{x+4} = \frac{3x}{(x+4)^2} - \frac{1}{x+4} \times \frac{x+4}{x+4} = \frac{3x}{(x+4)^2} - \frac{1 \times (x+4)}{(x+4)^2} = \frac{3x - (x+4)}{(x+4)^2} = \frac{3x - x - 4}{(x+4)^2} = \frac{2x - 4}{(x+4)^2}$ rather than using the rule above, which necessitates additional factorising and cancelling in the final step.

Future learning

The obvious extensions to this topic are the later algebraic manipulation chapters (Chapter 7 Further expressions, Chapter 8 Equations and Chapter 14 Algebraic formulae). Students will continue to simplify and substitute into more complex expressions as well as expand more than one bracket.

Students will also need strong foundations in this chapter in order to successfully solve algebraic equations and understand how many (if any) solutions exist.

In addition to the above, understanding the place of a letter as a variable will be instrumental in later sequence and functions chapters.

Gateway to A Level

At A Level the demand on students' algebraic manipulation skills is high, particularly when working with fractions and solving equations. Their knowledge is extended beyond laws of indices to include laws of logarithms and students are expected to be able to manipulate expressions with ease. You may observe that students with weak conceptual understanding of the use of algebra and functions in mathematics struggle greatly with the A Level course and will

happily cancel incorrectly a variety of things when simplifying or expanding brackets, for example $\frac{\sin \theta}{\cos \theta} = \frac{\sin}{\cos}$ or $\frac{x+3}{x-5} = \frac{3}{-5}$ or $2x(5-x) = 10x - x$ and so on. This prevents them working with a variety of new material as they can't start the problem correctly and limits their ability to be successful in the course. Having a strong conceptual understanding regarding variables and the difference between an identity and an equation will be important when students prove identities in trigonometry.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 4 Question 1
- Chapter 7 Question 9

Higher

- Chapter 5 Question 6
- Chapter 8 Questions 1, 12, 13
- Chapter 9 Question 3
- Chapter 10 Question 9

Homework Book

Foundation

- Chapter 3

Higher

- Chapter 3

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 11 worksheets (+ solutions)
- 12 animated widgets
- 16 **(F)** / 18 **(H)** interactive walkthroughs
- 5 auto-marked quickfire quizzes
- 5 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

4F / 4H Functions and sequences

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- be able to recall the multiplication tables;
- recognise square and cube numbers;
- be able to spot numerical patterns;
- be able to substitute into formulae, particularly with negative numbers.

Learning outcomes

Foundation

Section 1

- To generate terms of a sequence from a term-to-term rule.

Section 2

- To generate terms of a sequence from a position-to-term rule and find the n th term of a linear sequence.

Section 3

- To generate terms of a sequence using a function.

Section 4

- To identify special sequences.

Higher

Section 1

- To generate terms of a sequence from a term-to-term rule.

Section 2

- To generate terms of a sequence from a position-to-term rule and find the n th term of a linear sequence.

Section 3

- To generate terms of a sequence using a function.
- To use and find composite and inverse functions.

Section 4

- To identify special sequences.
- To find the n th term of a quadratic sequence.

Vocabulary

sequence, term, consecutive terms, first difference, term-to-term rule, arithmetic sequence, geometric sequence, position-to-term rule, function

(Higher only) composite function, inverse function

Common misconceptions and other issues

- Confusing the term-to-term and position-to-term rules.
- Incorrectly writing $n + 4$ for a position-to-term rule when the common difference is $+4$.
- Applying operations in a function incorrectly. For example with $3(n + 4)$, incorrectly multiplying by 3 before adding 4.
- Applying composite functions in the wrong order.
- Forgetting the second difference must be halved to give the coefficient of n^2 .

Hooks

Write two numbers on the board (e.g. 3, 6) and ask students what the next number could be. They could suggest 9 (from 'add 3') or 12 (from 'double') or possibly other more obscure answers. Write up another two numbers (e.g. 5, 9) and ask for further suggestions. This opens up for discussion what rules govern different sequences and how we can write them.

SECTION 1F / 1H: SEQUENCES AND PATTERNS

This section introduces the idea and notation of term-to-term rules. Most students will be familiar with the language required for describing term-to-term rules (e.g. add, subtract, multiply, divide) and the concepts in this section are straightforward.

Prompting questions

Exercise 4A(F) / 4A(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** What is the next term in the sequence? Why? *a) 16, adding 3 each time.*
- **Q2(F) / Q2(H)** How do you get from one number to the next in the given sequence? *a) add 7, b) subtract 4 and so on.*
- **Q4(F) / Q4(H)** How can we work out the height after the first bounce? *Halve 96 (48).*

Starters, plenaries, enrichment and assessment ideas

Starters

- Assign each student a different term-to-term rule. Read out the terms in a sequence and ask students to stand up as soon as they know they have the rule for it.
- Give students two numbers and ask them what the third number in the sequence could be.

Starters or plenaries

- Distribute numbered cards. Project/write up the first three terms in a sequence and ask students to hold up their number if it is in the sequence.
- Project different sequences and ask students to write the term-to-term rule on mini-whiteboards.
- **NRICH Coordinate patterns** (nrich.maths.org).

SECTION 2F / 2H FINDING THE n TH TERM

Section 2 begins by introducing the position-to-term rule and how it can be used to generate terms of a sequence. It may be necessary to remind students of the order of operations, as they need to be understand this when substituting values into their position-to-term rule. Students may also need reminding to check their working carefully if negative numbers are involved. Their greatest difficulty is often determining which number to substitute into a position-to-term rule (e.g. for the 4th term we must substitute in 4).

The section then goes on to finding the n th term of a sequence. The six-step problem-solving grid provides a thorough and logical approach to finding the n th term rule for a linear sequence.

Students typically remember to find the common difference between terms, but often then struggle to remember how it is linked to the n th term rule. For example, for 4, 7, 10, 13, ... students will recognise the common difference is + 3, but will then give the answer $n + 3$. To combat this, drawing out the link with the times tables can help reinforce the idea that any linear sequence is essentially a shifted times table.

Prompting questions

Exercise 4B(F) / 4B(H)

While working through the exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** What does n represent? *The position in the sequence, in other words for the 3rd number we substitute $n = 3$. Are there any numbers n cannot be? Zero or the negatives.*
- **Q3(F) / Q3(H)** How do you get from one number to the next in the sequence? Which times table is this related to? How is the given sequence different? *Add 4, the four times table, it is one more than the four times table (i.e. $4n + 1$).*
- **Q4(F) / Q4(H) (questions for part b, but can be adapted for all)** How do you get from one number to the next in the sequence (e.g. 3, 7, 11, 15,...)? *Add 4.*

- **Q4(F) / Q4(H) (questions for part b, but can be adapted for all)** What other set of numbers can you think of where you add (e.g. 4) each time? *The four times table.*
- **Q4(F) / Q4(H) (questions for part b, but can be adapted for all)** How is this sequence different from the four times table? *It is one less, so the n th term is $4n - 1$.*
- **Q4(F) / Q4(H) (questions for part b, but can be adapted for all)** Will the number 1000 be in this sequence? How do you know? *No. 1000 is in the four times table, whereas this sequence only has numbers that are one less than the four times table, so 1000 cannot be in the sequence.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

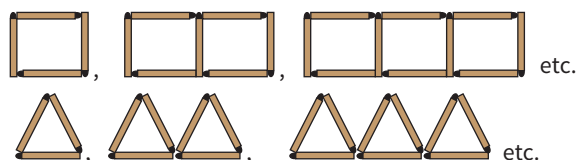
- Display a sequence only showing the 1st, 2nd and 5th terms. Ask students to fill in the 3rd and 4th terms.
- The **Sequence Generator** spreadsheet is excellent as a whole-class teaching tool. It shows ten terms of a sequence and the n th term rule, but also allows you to hide/reveal particular terms or the rule (maths-it.org.uk).
- **NRICH Shifting times tables** task (nrich.maths.org).

Plenaries

- Consequences game. Each student writes a position-to-term rule at the top of a piece of paper and passes it to the student on their left. This student writes the first five terms of the sequence underneath, folds over the top line of writing (so that their writing, but not the original is visible) and passes to their left. The new student writes down the position-to-term rule, folds over the line of writing with the five terms so that only the new writing is visible and passes to their left.

Enrichment activities

- Create a treasure hunt around the classroom. A card holds a question (e.g. what is the 5th term in the sequence $2n + 1$). Another card holds the answer to this in a bubble in the top left corner, and has another question of the same format. Produce a set of such cards (which loop back with the final answer being on the top of the first card), place them around the room and ask students to complete the treasure hunt.
- Use matchsticks to create physical sequences and ask students to work out the position-to-term rule. Students can then create their own patterns with matchsticks and they can work out the rules for other students' sequences.



- **NRICH Seven Squares** task (nrich.maths.org). The teacher notes with this resource suggest how to use this in the classroom, enabling the formulation of the position-to-term rule to naturally emerge from discussions on the construction of the sequence.

SECTION 3F / 3H: FUNCTIONS

Section 3 introduces the idea of a function. Most students will be familiar with the concept of a function, especially if direct links are made with computer science or spreadsheets. This link can help with the language of 'inputs' and 'outputs'. Note that knowledge of function notation is not required by the specification.

The higher version of Section 3 then gives a brief introduction to composite functions. The section finishes by introducing the idea of the inverse of a function. Students will already be familiar with the idea of an inverse ('division undoes multiplication').

It may be worth pointing out that functions always have an inverse if they are 'one-one', i.e. if each value of y has just one corresponding x value. A 'many-to-one' function (like any quadratic or trigonometric function) doesn't necessarily have an inverse. If you draw a horizontal line across the graph of the function it intercepts the graph at more than one point, and that means it has no inverse. However, if you only plot part of the function (i.e. restrict the input values) so that the horizontal line only intercepts at a single point, then it does have an inverse. Constant value functions (for example $y = 2$) do not have an inverse function.

Prompting questions

While working on the material in this section good prompts for promoting discussion might be:

- Do we have to use the letter x or n for the input? (Note that n is commonly used when we expect the domain to be the integers.)

Exercise 4C(F) / 4C(H)

While working through the exercise, good prompts or questions for students might be:

- **Q1(F) / Q1(H)** What answer do you get if you input 1? 2? 3? How could you describe this sequence? a) 4, 5, 6, add 1.
- **Q5(H)** How can you undo the function and get back to x ? a) add 7, b) divide by 4 and so on.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- **NRICH Alison's Mapping** task uses the **NRICH Number Plumber** which provides an interactive function machine that you can use to create your own functions, and view the results on a graph. Students could create their own functions on the Number Plumber and challenge their classmates to discover what they are (nrich.maths.org).

SECTION 4F / 4H: SPECIAL SEQUENCES

This section introduces some common and historically significant sequences. It is probably worth tackling head-on the fact that we have very many ways of describing the same thing in mathematics: sequences that are linear could also be called arithmetic; we sometimes use the word 'progression' to mean the same as sequence.

Students are generally good at spotting whether sequences are arithmetic or geometric, especially for well-chosen numbers where the difference or ratio is easy to spot.

Most students will be familiar with square and cube numbers, but reinforcing the idea that '5 squared' is the area of a square with side length 5, and '5 cubed' is the volume of a cube with side length 5 can help justify the slightly unintuitive names ('Miss, why is squared a little 2 when a square has four sides?').

Following Exercise 4D, the higher version of Section 4 goes on to finding the n th term of a quadratic sequence. This method will be new to many teachers and is described using two worked examples.

Students will be most efficient if they can remember that the second difference is double the coefficient of x^2 , but it is important that students establish this through considering the second differences of x^2 , $2x^2$, $3x^2$, and so on. Keeping the coefficient of x^2 constant and finding the second difference of the family of sequences $3x^2 + 2x$, $3x^2 + 7$, $3x^2 - 4x + 6$ should help consolidate this idea.

The method essentially partitions a quadratic sequence into an x^2 term and a linear expression. Students will have practised finding expressions for linear sequences in Section 2, so once the x^2 term is identified, it is a case of determining what linear 'adjustment' is made to give the desired sequence.

Students will continue to need encouragement to consistently remember to test their rules!

Prompting questions

While working on the material in this section good prompts for promoting discussion might be:

- Why do we use the word 'linear' to describe a sequence like 5, 7, 9, ...?
- Is 0, 0, 0, ... a linear sequence? Is it geometric?
- What happens if I start on 1 and keep multiplying by -2? What would it look like drawn on a graph?
- What if we start with 360 and keep dividing by 2? What about dividing by -2? How long until we get to zero?
- Is zero a square/triangular/and so on, number?
- Is -1 a square/ triangular /cube number?
- Are any numbers both square and triangular numbers? Yes! 0, 1, 36, 1225, ...
- **(Higher only)** Why are sequences with a squared sign called quadratic? *This is linked to areas of quadrilaterals, for example $x(x+1) = x^2 + x$ is the area of a rectangle with a side one unit longer than the other.*

- **(Higher only)** If the second difference is 5, what is the coefficient of x^2 ? If the second difference is $\frac{1}{2}$, ...? If the second difference is $-\left(\frac{1}{3}\right)$, ...? $\frac{5}{2}, \frac{1}{4}, -\left(\frac{1}{6}\right)$. *Always halve the second difference to find the coefficient of x^2 .*
- **(Higher only)** If the coefficient of x^2 is 6, what will the second difference be? If the coefficient of x^2 is 5, ...? 12, 10. *We double the coefficient of x^2 to find the second difference.*
- **(Higher only)** If the second difference is zero, what will the coefficient of x^2 be? Zero; there will not be an x^2 term. *This means the sequence is linear.*

Exercise 4D(F) / 4D (H)

While working through the exercise, good prompts for students might be:

- **Q3(F) / Q3(H)** How many parents does each bee have? How many grandparents? What is the pattern for calculating the next generation back? 2, 4, *double the current generation.*
- **Q7(F) / Q7(H)** What does n^3 mean? 'n cubed', or $n \times n \times n$.

Exercise 4E(H)

While working through the exercise, good prompts for students might be:

- **Q2(H)** How can you get from 1 to $\sqrt{2}$? *Multiply by $\sqrt{2}$, we can't add a number to get exactly to $\sqrt{2}$.*
- **Q4(H)** How can you work out u_1 ? *Substitute $n = 1$ into the formula.*

Starters, plenaries, enrichment and assessment ideas

Plenaries

- Students can use the 'ANS' key on their calculator (which substitutes the previous answer) to quickly generate terms of a sequence. For example typing 1 and pressing equals; then type $2 \times \text{ANS}$ and repeatedly pressing equals will generate a geometric sequence that doubles the previous value.
- When given a series of calculator key presses, students could fully describe the sequence in words (e.g. 'geometric progression, starting on 5 and doubling').

Starters or plenaries

- **OEIS** allows you to type terms of a sequence and matches them against a comprehensive database of named and described sequences (oeis.org). It is useful to highlight that many sequences are possible if just a few terms are provided (e.g. 154 known sequences contain the numbers 1, 4, 9, 16, 25 in order).
 - **(Higher only)** The **Sequence Generator** spreadsheet displays first and second differences for a configurable quadratic sequence (maths-it.org.uk).
 - **(Higher only)** The method can be used to provide another justification for the $\frac{1}{2}n(n+1)$ formula for triangular numbers described earlier.

Enrichment activities

- Search on the internet for Zeno's paradox.
- No discussion of triangular numbers would be complete without reference to the legend of Gauss calculating the sum of the integers 1 to 100. Writing out pairs that sum to 101 (1 and 100, 2 and 99, 3 and 98, etc.) and realising there are 50 pairs is generally a convincing argument for the $\frac{1}{2}n(n+1)$ formula.
- **NRICH Fibs** task (nrich.maths.org).
- Fibonacci sequences make many appearances in nature and are a rich topic for students to research.
 - **(Higher only)** **NRICH Steel Cables** task (nrich.maths.org).
 - **(Higher only)** **NRICH Handshakes** task, or a different form of the same problem, **Mystic Rose** (nrich.maths.org).

Topic links

Previous learning

This topic provides a good opportunity to return to the work on basic calculations covered in *Chapter 1 Basic calculation skills* and *Chapter 2 Whole number theory*. There are opportunities to consider how square and cube numbers are related to physical shapes and how a numerical sequence can be linked to geometrical or physical patterns.

Future learning

- Functions and sequences are used in *Chapter 18 Straight-line graphs* for describing the relationship between the x and y coordinates.
- Quadratic sequences are used in *Chapter 19 Graphs of functions and equations* for describing the relationship between the x and y coordinates when plotting graphs of quadratic functions.

Gateway to A Level

- This topic will be built upon in KS5. Having a strong understanding of this concept will be necessary for students to extend their knowledge in A Level modules. In addition to the notation learnt at GCSE, students will also learn to describe sequences as arithmetic progressions. They will calculate terms in a sequence and the sum to n terms.
- In addition to arithmetic progressions, geometric progressions are a further extension of this topic, which builds upon sequences such as 2, 4, 8, 16, 32. Students will learn to calculate terms in a sequence and the sum to n terms, including the sum to infinity for converging sequences.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 8 Question 9
- Chapter 9 Question 4

Higher

- Chapter 1 Question 1
- Chapter 5 Question 1
- Chapter 6 Question 11
- Chapter 8 Questions 2, 14

Homework Book

Foundation

- Chapter 4

Higher

- Chapter 4

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 7 worksheets (+ solutions)
- 4 animated widgets
- 8 interactive walkthroughs
- 3 auto-marked quickfire quizzes
- 3 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

5F / 5H Properties of shapes and solids

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- understand the use of angle as a measure of turn, including the use of degrees;
- be able to recall and use angle sums (straight line, around a point).

Learning outcomes

Foundation

Section 1

- To know the names and features of common polygons and polyhedra.
- To know how to describe and label common features (congruent shapes, parallel sides, etc.) of plane figures.

Section 2

- To identify and describe line and rotational symmetry in plane figures.

Section 3

- To know and use properties of triangles, including their interior angle sum.

Section 4

- To know and use properties of quadrilaterals, including their interior angle sum.

Section 5

- To know and use properties of three-dimensional solids.

Higher

Section 1

- To know the names and features of common polygons and polyhedra.
- To know how to describe and label common features (congruent shapes, parallel sides, etc.) of plane figures.

Section 2

- To identify and describe line and rotational symmetry in plane figures.

Section 3

- To know and use properties of triangles, including their interior angle sum.

Section 4

- To know and use properties of quadrilaterals, including their interior angle sum.

Section 5

- To know and use properties of three-dimensional solids.

Vocabulary

plane shape, polygon, regular polygon, irregular polygon, circumference, diameter, radius, arc of a circle, sector, semicircle, chord, segment, tangent, polyhedron, equidistant, reflection, line of symmetry, rotational symmetry, order of rotational symmetry, adjacent, bisect, congruent

Common misconceptions and other issues

- Arithmetic errors, for example 42 and 58 being complements to 90. Students should be encouraged to check their answers, for example by adding them together again.
- Not identifying shapes because they are not in 'standard position' (e.g. an isosceles triangle where the base angles are not at the 'bottom'). It is useful, when drawing figures on the board, to frequently draw them in non-standard positions.
- Confusion between 'parallel' and 'perpendicular'. The two letter 'l's in parallel (which ARE parallel) can be a useful mnemonic for which is which.
- A lot of the content of this chapter will have been met by students in previous years, which can lead to complacency and/or arrogance in some. However, it is vital that students realise the importance of revisiting this content and learning all of the facts, as this will allow them to access a multitude of exam questions.

Hooks

1. Use photographs, such as those in Section 4, as a starting point for discussion. What shapes can they see in the photographs? It will be even more of a hook if the photographs are of somewhere they recognise easily, for example somewhere in the school or local area.
2. Show students some exam-style questions that rely on understanding/recalling properties of shapes to get started. For example, a question on setting up and solving an equation based on a diagram of an isosceles triangle. This will establish a need to revisit this topic.

SECTION 1F / 1H: TYPES OF SHAPES

This section should be familiar ground to most students, but it is important that their use of terminology is precise and accurate. In particular the correct usage of the $\parallel$ and $\perp$ symbols, and unambiguous description of angles should be a focus. Many students struggle with the ‘three letter’ notation for angles and forget that the middle letter is the one at the vertex. It is important to stress, too, that angles should be written with upper case letters to prepare the way for standard notation for non-right angled triangles in trigonometry.

Prompting questions

Exercise 5A(F) / 5A(H)

While working through the exercise, good prompts for students might be:

- I take a regular hexagon and divide it into two equal parts with a single cut. Describe for me what shape(s) might be created. *Two trapeziums.*

Exercise 5B(F) / 5B(H)

While working through the exercise, good prompts for students might be:

- Bring up a street map of midtown Manhattan in New York (e.g. search for ‘Empire State Building’ on the internet). Tell me a street that is perpendicular to 5th Avenue... that is parallel to 35th Street... a road that is neither perpendicular nor parallel to 31st Street.
- When is it ok for me to refer to angle B rather than ABC ? *When it is opposite side b .*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- Dan Meyer’s **Pick a Point** provides students with justification of the need to label geometrical objects. In particular extending the final slide (with letters on points) to draw on some angles shows easily that angle B may be ambiguous but PBD is less so (see below) (blog.mrmeyer.com).
- While naming angles, take the opportunity to ask students to estimate their size. It is also a good idea to highlight that there are two angles at PBD (i.e. one acute, one reflex).

Assessment ideas

- The Bowland **Three of a Kind** assessment task is a good activity for getting students to think about properties of similar shapes that form a pattern (bowland.org.uk).

SECTION 2F / 2H: SYMMETRY

This section introduces the concepts of line symmetry and rotational symmetry. When drawing lines of symmetry students frequently either stop at one line of symmetry, or add in too many. Changing the orientation of some shapes into non-standard positions (e.g. a square without any of its sides horizontal or vertical) can cause students further difficulties.

The book’s idea of folding paper can be very helpful to many students. With good quality tracing paper and a forgiving photocopier, you should be able to copy shapes onto tracing paper, which can be folded and held up to the light to confirm the two ‘halves’ are the ‘same’. In pairs, with two pieces of tracing paper, students can rotate the shapes to count how many times they look the same during one complete rotation. Adding an ‘up’ arrow can be helpful for keeping track of the starting place.

Prompting questions

Exercise 5C(F) / 5C(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** How many lines of symmetry does this rectangle have (many students will include the diagonals)? When are the diagonals of a rectangle lines of symmetry? *When the rectangle is square.*
- **Q1(F) / Q1(H)** How many lines of symmetry does this parallelogram have? *Probably none.* When does a parallelogram have lines of symmetry? *When it's a square.*
- **Q1(F) / Q1(H)** How many lines of symmetry does a circle have? *An infinite number.*
- **Q2(F) / Q2(H)** How many lines of symmetry does a regular pentagon have? *Five.* A regular octagon? *Eight.* A regular dodecagon? *12.* A regular icosagon? *20, and so on.*
- **Q2(F) / Q2(H)** What is the smallest/biggest order of line symmetry a shape can have? $0/\infty$ How is this different to rotational symmetry? *Minimum is 1.*
- **Q3(F) / Q3(H)** Why do all shapes have rotational symmetry of at least order 1? *Because they can all be rotated a full 360° . It is impossible to have a shape with rotational symmetry of order 0.*

Starters, plenaries, enrichment and assessment ideas

Starters

- The mirror scene from the Marx Brothers' film *Duck Soup* (1933) can be a good stimulus for discussing reflection, including the situation found in mathematics that is impossible in the real world of crossing through the mirror.

Enrichment activities

- Once students are comfortable with the two described types of symmetries, combining both within the same question can provide good practice. For example, students could be given a selection of traffic signs that need to be classified. These could be placed on an enlarged two-way table on A3 paper (orders of rotational symmetry as the rows; line symmetry as the columns).
- Kim Scott's ambigrams, for example his **Half Words**, have some great suggestions for classroom activities (scottkim.com).

SECTION 3F / 3H: TRIANGLES

This section introduces two ways of classifying triangles: (a) by the number of sides of the same length and (b) by the size of the largest angle. Many students will find it difficult to appreciate that a particular triangle can be classified in (at least) two different ways, for example right-angled and isosceles.

Prompting questions

Exercise 5D(F) / 5D(H)

While working through the exercise, good prompts for students might be:

- **Q3(F) / Q3(H)** Two angles of a triangle add up to 75 degrees. What is the size of the third angle? 105° .
- **Q3(F) / Q3(H)** Dante claims that he can construct a triangle with angles 45° , 70° and 80° . Can he do this? *No, the angles must add up to 180° .* What about if the triangle was drawn on a globe with one vertex on the North Pole and two on the Equator? *This 'triangle' could, for example, have three 90° angles.*
- **Q3(F) / Q3(H)** Can a triangle have more than one right angle? *No.* What about more than one obtuse angle? *No.* Can a triangle have a reflex angle? *No.*
- **Q4(F) / Q4(H)** If one of the angles in a right-angled triangle is 31° , what is the other (non-RA)? 59° .
- **Q4(F) / Q4(H)** What kind of triangle is one with two 45° angles? *A right-angled isosceles triangle.*
- **Q5(F) / Q5(H)** My triangle has its three angles being consecutive numbers. What are its angles? 59° , 60° , 61° .
- **Q6(F) / Q6(H)** My isosceles triangle has two vertices at (0, 0) and (10, 0). What might the third vertex be? $(5, x)$ where x is not 0.

Starters, plenaries, enrichment and assessment ideas

Starters

- A starter activity could involve students renumbering a dart board. The three darts need to hit three numbers that could be the angles in a triangle. Alternatively, project a target board grid and students can write down possible combinations of angles on a mini-whiteboard. As well as asking students which three numbers could be the angles in a triangle, you can ask which three angles could be in an equilateral triangle, an isosceles triangle or a right-angled triangle. An example target board is below.

20	110	60	15	70
35	85	90	20	60
140	45	35	15	5
45	30	60	150	90

Enrichment activities

- Students could investigate the various ‘centres’ of a triangle (circumcentre, incentre, etc). The film **Journey to the Center of a Triangle** (1977) demonstrates these nicely without the exact procedures being revealed (archive.org). Students could then work out how to find these centres using compass and pencil.
- Triangle Properties** activity. Ask students to arrange some given triangles into a Carroll diagram or Venn diagram to illustrate that triangles can be classified using their angles or their sides. Alternatively, present the students with a blank table and ask students to draw an example of a different type of triangle in each box. This can be used as a homework activity (tes.co.uk).
- True or false? Present students with a series of diagrams of triangles and ask them to sort the triangles into those that are true triangles (angles that add up to 180°) and false triangles (angles that don’t add to 180°). An extension is to classify the true triangles. This activity could be done as a card sort or as a quick-fire starter/plenary by projecting the pictures.
- There are many NRICH problems that you could use to support this topic (nrich.maths.org). A few are listed below:
 - Notes on a triangle** uses a beautiful René Jodoin film as a prompt for students to think about a whole host of properties of triangles and other shapes.
 - Terminology** – this problem involves using angle properties of an equilateral and isosceles triangle, as well as angles along a straight line, to form some algebra.
 - The first part of each of the questions in **Cyclic Quadrilaterals** requires students to draw isosceles triangles on different dotted circles, and work out the angles.

SECTION 4F / 4H: QUADRILATERALS

This section focuses on properties of named quadrilaterals, in particular the nature of the sides, interior angles and diagonals. Manipulatives (example.g. Geo Strips) can be helpful when looking at these properties. Students are often uncomfortable with the idea that a given shape may have multiple names; for example that a square is also a rectangle.

Prompting questions

Exercise 5E(F) / 5E(H)

While working through the exercise, good prompts for students might be:

- Q2(F) / Q2(H)** With the definitions of various special quadrilaterals, get students to test a square to see whether it is also a trapezium, a kite, a parallelogram, a rhombus or a rectangle. *Yes to all!*
- Q3(F) / Q3(H)** Draw a square shape but with no side lengths or indications that its sides are congruent. Ask what information needs to be added to be confident it actually is a square (and not a rhombus or a rectangle). *All sides*

and diagonals equal, angles 90° , opposite sides parallel, diagonals bisect angles and each other, and are at right angles.

- **Q3(F) / Q3(H)** Draw a square, rotated by 45° so that its diagonals are horizontal and vertical. What is the mathematical name for this shape? *It is still a square (but is also a rhombus and a rectangle and a parallelogram!). Diamond would not be an acceptable response.*
- Draw an arrowhead (chevron/dart). Which (if any) of our definitions of quadrilateral does this match? *A kite, if one of the diagonals is allowed to be extended.*

Starters, plenaries, enrichment and assessment ideas

Starters

- The target board starter, as described above in Section 3, can be adapted for use with quadrilaterals or other polygons.

Enrichment activities

- Quadrilateral Rummy, in the ATM book **Geometry Games** (atm.org.uk). A variation is available on NRICH called **Quadrilaterals Game** (nrich.maths.org).
- The NRICH problem **Quadrilaterals** asks students to find as many different quadrilaterals as possible by joining dots on the circumference of a circle. The problem initially starts with a circle with eight evenly spaced dots, but is easily adapted for other numbers of dots. This problem can be used with students of all levels to revisit vocabulary, then using the angles of isosceles triangles to work out the angles in each quadrilateral and start to explore the properties of cyclic quadrilaterals (**Higher only**). A slightly more structured version of this activity is **Cyclic Quadrilaterals** (nrich.maths.org).
- The ‘true or false’ activity, as described above in Section 3, can also be adapted for use with angles in quadrilaterals or any other angle facts.
- Devise an activity in which students match up the properties of special quadrilaterals with their names/pictures and create **revision cards**. Another variation is a card sort from Mr Barton Maths called **Properties of Quadrilaterals** group activity (tes.co.uk).

SECTION 5F / 5H: PROPERTIES OF 3D OBJECTS

This section reviews the basic terminology associated with 3D objects, especially polyhedra. In addition, it considers some properties and naming of prisms.

Prompting questions

Exercise 5F(F) / 5F(H)

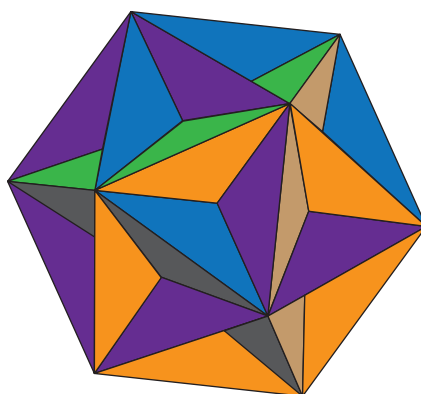
While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Tell me the name of a solid that has four faces. *Triangular pyramid (tetrahedron).* Five faces? *Square pyramid.*
- **Q1(F) / Q1(H)** Tell me the name of a solid that has eight vertices. *Cube.* Another. *Cuboid.* Ten vertices. *Pentagonal prism.* Fewer than eight. *Hexagonal pyramid.*
- **Q1(F) / Q1(H)** Tell me a mathematical name for the container of a tube of Smarties®. *Hexagonal prism.* Toblerone®. *Triangular prism.* Rolos®. *Cylinder.*
- **Q2(F)** If I held up a sphere in front of a light source, what shape would its shadow be? *A circle.* How about a cube? *A square.* How could I hold a cube to give me a hexagonal shadow? *Hold it so one of the vertices is towards you.*
- **Q3(F)** Consider a cone. How many faces does it have? *One, the (flat) circle.* Is the curved part a face? *No.* Is the apex a vertex, even if it doesn't connect edges? *Yes, this shows the issue of defining an edge as connecting two vertices.* Does it have any edges? (Note that Euler's polyhedron formula doesn't help clear this up since a cone isn't a polyhedron.)

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- Having physical polyhedra available is invaluable. As an enrichment activity, students could make these for display in the classroom. They hang nicely from the ceiling, making an effective (and useful) display.
- **(Higher only) Euler's polyhedron formula** provides an opportunity to consider the relationship between faces, edges and vertices of a solid (plus.maths.org). It is considered in Question 2 of Exercise 5F (Higher only). Some students could also consider which solids do not have a Euler characteristic of 2.
- Students could justify why only five Platonic solids are possible by considering the **possible polygons that might meet at each vertex** (mathsisfun.com).
- Classrooms could be decorated during an end of term activity, for example constructing **plaited origami polyhedra** (nrich.maths.org).
- A carefully coloured great dodecahedron can provide a wonderful exercise in visualisation and description. Students need to carefully phrase the shapes/solids that they see to other students. For example, some will see a star on a blue background, others on a purple, and so on.



- The **NRICH Which Solid?** activity encourages students to identify mystery solids using as few direct questions as possible. This is good for encouraging students' precision with mathematical language (nrich.maths.org).

Topic links

Previous learning

- This topic provides ample practice for mental methods of addition and subtraction.
- Students could also practise measuring lengths and angles by checking the precision of some hand-drawn shapes: is a shape with four sides varying between 6.9 cm and 7.1 cm and angles varying between 89° and 91° really a square?

Future learning

- In *Chapter 9 Angles*, students will develop their understanding of angles, particularly with parallel lines and exterior angles of polygons.
- Isosceles triangles are vital in future work with circle theorems covered in *Chapter 34H Circle theorems*.
- Equations of parallel and perpendicular lines are considered in *Chapter 18 Straight-line graphs*.
- Calculating angles and lengths of sides of triangles is developed using Pythagoras' theorem in *Chapter 31F/32H Pythagoras' theorem*.
- In *Chapter 32F/33H Trigonometry*, students will learn to find missing sides and angles in right-angled triangles and in *Chapter 33H Trigonometry* this is developed into use of the sine and cosine rules for finding sides and angles in triangles that do not have right angles.
- Isometric drawing of 3D shapes and drawing plans and elevations are covered in *Chapter 20 Three-dimensional shapes*.
- Volume of 3D solids is covered in *Chapter 21 Volume and surface area*.
- *Chapter 28F/29H Isometric plane transformations* extends the symmetry work in this chapter.

- Students will need to set up and solve equations involving properties of shapes. Linear examples are considered in *Chapter 8 Equations*.

Gateway to A Level

- In calculus and coordinate geometry students will learn to use perpendicular and parallel lines when working out the equations of tangents and normals to curves. Later students will learn how to use calculus to calculate the volumes of solids created by revolving a curve through 360° around an axis.
- Students will learn to classify functions as 'odd' or 'even' based on the symmetry of their graphs.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 3 Question 1
- Chapter 8 Question 10
- Chapter 9 Question 12

Higher

- Chapter 6 Question 12
- Chapter 8 Question 3
- Chapter 9 Questions 4, 10

Homework Book

Foundation

- Chapter 5

Higher

- Chapter 5

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- 5 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- Polygon diagrams

6F / 6H Construction and loci

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know the properties of angles;
- know the properties of shapes and associated language, including circles;
- be able to read and write angles and lines using formal notation conventions;
- be able to measure and construct angles using a protractor;
- understand notation such as $\perp$ for perpendicular to and $\parallel$ for parallel to;
- have a basic knowledge of ratio and scale.

Additional useful prior knowledge

- To use knowledge of complements of angles around a point to accurately construct reflex angles using a 180° protractor.
- To convert metric units of length.

Learning outcomes

Foundation

Section 1

- To use ruler, protractor and pair of compasses to accurately construct angles and shapes.
- To accurately copy diagrams using rulers and a pair of compasses only.

Section 2

- To construct the perpendicular bisector of a line.
- To construct the perpendicular at a given point on a line.
- To construct a perpendicular from a given point to a line.
- To bisect an angle.

Section 3

- To use constructions to solve loci problems.

Section 4

- To apply appropriate constructions and loci knowledge to a variety of problems including those set in context.

Higher

Section 1

- To use ruler, protractor and pair of compasses to accurately construct angles and shapes.
- To accurately copy diagrams using rulers and a pair of compasses only.

Section 2

- To construct the perpendicular bisector of a line.
- To construct the perpendicular at a given point on a line.
- To construct a perpendicular from a given point to a line.
- To bisect an angle.

Section 3

- To use constructions to solve loci problems.

Section 4

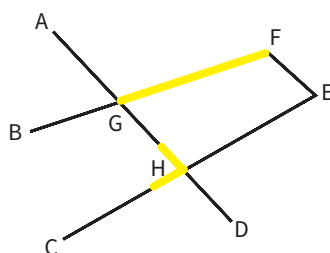
- To apply appropriate constructions and loci knowledge to a variety of problems including those set in context.

Vocabulary

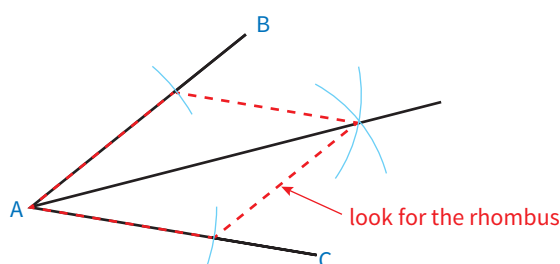
bisect, midpoint, perpendicular bisector, locus (loci)

Common misconceptions and other issues

- Students struggle with formal language and notations in geometry. For example when a vertex is labelled B, students refer to the angle as B forgetting that there are two possible angles this could be, one less than 180° and the other greater than 180° as a minimum. Having starters based on notation, and highlighting angles and lines given their labels, can help identify and recognise these errors at the beginning. For example:



- Many students have problems measuring and drawing angles using a protractor. These problems arise mainly from their misuse of a protractor. Students either fail to line up the centre of the protractor with the vertex of the angle or read from the wrong scale of the protractor. Labelling the vertex before they draw the rest of the angle to have something to line up to and reminding students to always start from 0 on the scale can be helpful in preventing these problems. In addition to this, reinforcing estimation skills relating to angles, including using games such as **Estimating Angles** can also support students in recognising their errors in reading from the wrong scale (nrich.maths.org). The tip in Section 1, which suggests that students extend the lines of an angle if struggling to read the size from their protractor, can also assist them in to accurately read off the size of an angle.
- Students' work may not be accurate enough. It could be that they are struggling to work with their pair of compasses and pencil. Having a set of Christmas cracker screwdrivers to hand for loose compasses and reminding students to sharpen their pencils helps with this. Students who struggle to use their pair of compasses because they put too much pressure on it may find it better to spin the paper not the pair of compasses.
- Some students struggle to remember how to draw basic constructions. Time spent working on ways to prompt students' memories can be useful, such as considering the constructions as all linked by finding/drawing the rhombus. If you join all construction points in any construction problem they form a rhombus (see example in diagram). You could ask students to add these lines to their construction notes so that you can prompt 'Where is the rhombus?'



You could also encourage students to make their own 'How to...' guides for each construction as an assessment task.

- Students may want to rub out their construction lines or they may be too faint in the scan of their answer. Reinforcing that construction lines are mandatory to get the marks and spending time marking exam questions with students, or scanning in their work to show what problems may occur, can help to prevent this.
- Students who have problems drawing arcs (because they misjudge where they are going to intersect) could draw the full circle when constructing rather than just the arcs required.

Hooks

Introducing loci problems with analogies relating to animals tethered to poles and the space they can move within helps students gain a conceptual understanding of what is going on in loci problems. You could 'act out' loci problems with the students themselves: using rope to attach them to a fixed point, or, given two students standing on opposite sides of the classroom, ask students one by one to stand an equal distance between them.

SECTION 1F / 1H: GEOMETRICAL INSTRUMENTS

Section 1 revises the basic construction skills students need in this chapter. It focuses on drawing and measuring angles using a protractor and then teaches students a method for accurately constructing a triangle when the three sides are known, using a pair of compasses and a ruler.

For less-able students, having diagrams part drawn for Exercise 6A or providing supporting sketches would be advisable.

Prompting questions

While working on the material in this section good prompts for promoting discussion may be:

- What is the difference between a sketch and a drawing? *A sketch is not accurate; it only indicates the general diagram.*
- Can you sketch what you are being asked to draw first to give an idea of what you could do? *General advice that could help students who are struggling to know what needs to be done without worrying about accuracy.*
- What equipment does the question allow you to use? *Make sure students read each line of the question and use the equipment stated.*

Exercise 6A(F) / 6A(H)

While working through the exercise, good prompts for students might be:

- **Q1, 3, 7-8(F) / Q1, 3, 7-9(H)** Have you placed the centre of the protractor on the vertex of your angle? *General advice that could help students.*
- **Q1, 3, 7-8(F) / Q1, 3, 7-9(H)** Which scale are you going to use on your protractor? *The one where the line is on 0.*
- **Q2(F) / Q2(H)** What do angles around a point sum to? How can we use this information to draw reflex angles? *360°.*
- **Q2(F) / Q2(H)** What else could we do to draw a reflex angle? *Split our angle into a straight line and the additional part of the angle.*
- **Q4-6(F) / Q4-6(H)** If we have the diameter of a circle how can we find the centre of the circle? Hint: what is the connection between the diameter and the radius of a circle? *The diameter is twice the length of the radius.*
- **Q8(F) / Q8(H)** Are there any angle or shape facts you could use to identify what you have to do? *For example, what do we know about an equilateral triangle? What do we know about angles around a point?*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- For students who need some additional practice with a pair of compasses and those who can gain some additional conceptual knowledge regarding properties of triangles, **Constructing Triangles** is an ideal investigation to look at (nrich.maths.org). True/false triangle constructions based on AAS, SAS and SSS triangle facts are also useful.

Assessment ideas

- Provide solutions to Exercise A that have been photocopied onto OHP acetate (or similar) to allow students to then lay them over their own work and self-assess their own accuracy. For simple angle constructions give students $\pm 2^\circ$ accuracy guidelines to check their working and give them an idea of how their work will be marked in the exam. (Note: this also makes a good peer-assessment tool for classes.)

SECTION 2F / 2H: BISECTORS AND PERPENDICULARS

Section 2 starts with bisections of a line and then continues this construction to include constructing a perpendicular at a given point on a line and constructing a perpendicular from a point to a line. It then covers how to bisect an angle. Throughout this chapter, you can suggest that students look for the rhombus (see previous misconceptions) and when first working on these constructions students can draw in the rhombus in a different coloured pen.

Prompting questions

While working on the material in this section good prompts for promoting discussion may be:

- Have you opened your compasses wide enough so that your arcs will intersect? *Students may wish to use a general rule of thumb to help here such as ‘open your compasses to $\frac{3}{4}$ of the length of the line’.*
- Have you kept your pair of compasses at the same width when working on your construction? *The answer hopefully will be ‘yes’.*
- Have you labelled all the parts of the shape as required? *Make sure all line segments are appropriately labelled to prevent errors in more complex problems.*

Exercise 6B(F) / 6B(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H):** Which construction finds the midpoint of a line? *Perpendicular bisector.*
- **Q4(F) / Q4,8(H):** If you have not spotted any patterns or rules yet, what could you do next? *Repeat the investigation.*
- **Q6(F) / Q6(H):** What does $\perp$ or $\parallel$ mean? *Perpendicular to or parallel to.*

Starters, plenaries, enrichment and assessment ideas

Assessment ideas

- Use exam questions and photocopy solutions on OHP acetate to give students an idea of how accurate they need to be and what construction lines will be looked for in the exam.
- Having questions and solutions pre-printed saves time and maximises the number of constructions students complete in a lesson.
- Having one or two heavily annotated examples for students to copy up as step-by-step guides for each construction is often very helpful for future revision of this work. Get students to write them up on A5 card and laminate them as revision cards to store in their books/folders.

SECTION 3F / 3H: LOCI

Section 3 introduces the idea that there may be more than one point that solves a problem and focuses on using constructions to find the paths formed by loci problems. The locus of points may be a line or a region and may have to satisfy more than one condition.

The images at the end of the section in the chapter offer ample opportunity for students to discuss the type of shapes and constructions they can see.

Prompting questions

While working on the material in this section good prompts for promoting discussion might be:

- What language can we use to describe what will happen? What shapes can you see? Is the solution an area or a line?

Exercise 6C(F) / 6C(H)

While working through the exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** Can you draw a picture of the problem being described to help you? *For example, what does a tennis court look like?*
- **Q7(F) / Q7(H)** How do I find the centre of a rectangle? *Find the point where the diagonals intersect.*
- **Q9(H)** Where is a valve on a bike wheel? What is going to happen to this valve as the wheel rotates? *It is attached to the edge of the wheel by the tyre so will go up and down as the wheel rotates (creating a cycloid).*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- **NRICH How Far Does It Move?** ties into the ideas being looked at in Question 8, based on rotating regular polygons (nrich.maths.org).
- **NRICH Like a Circle in a Spiral** makes a nice end of term activity (nrich.maths.org).

SECTION 4F / 4H: MORE COMPLEX PROBLEMS

Section 4 connects the ideas of the previous section to contextual problems. This adds a layer of complexity related to drawing and solving problems where a scale is involved.

Prompting questions

While working on the material in this section good prompts for promoting discussion might be:

- Can you sketch the problem to work out what shapes and constructions will be involved? *General advice that could help students who are struggling to know what needs to be done without worrying about accuracy.*
- Have you answered the question by clearly stating which line or area is the locus of points required? *Some students might need a general reminder to clearly show their solution.*

Exercise 6D(F) / 6D(H)

While working through the exercise, good prompts for students might be:

- **Q2-4(F) / Q2-4(H)** Are there any properties of shapes that would be useful in this question? *For example in Question 2, which diagonal will be longer?*
- **Q5-6(F) / Q5-9(H)** What is the scale of the problem? Do you need to convert any lengths to fit your scale diagram? *Yes, or they won't fit on the page. Recommend students use 1 cm : 100 cm (1 m) for their drawings (or 1 cm : 10 m for Question 9).*
- **Q6(F) / Q6(H)** Is there any additional information you need to consider to solve this problem? *The length of an average arm so that people can't reach over the rails to touch the monkey either.*
- **Q8(H)** What does 'equidistant' mean? What constructions can I use to find the line of points equidistant? *Angle or perpendicular bisector.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- You could use a range of activities, such as treasure hunting or deciding where to build a new shop, based on maps of the local area. These tasks take a while to create but are much better when personal to the area in which students live. They can also be differentiated for different classes to include a different number of clues, which can be printed on card to also promote problem-solving skills.

Topic links

Previous learning

A lot of the work in this chapter relies on strong geometric foundations in language, notation conventions and properties of shapes that were introduced in *Chapter 5 Properties of shapes and solids*.

Future learning

A basic knowledge of scale and proportion is required in *Chapter 22 Calculations with ratio*.

(Higher only) When working on *Chapter 34H Circle theorems* there is a nice opportunity to recap on constructions involving circles by looking at the **Curvy Areas** investigation (nrich.maths.org). In addition to this, the ideas presented in this chapter about multiple solutions being valid, and hence a line or area solution, are also important for graphing inequalities later in *Chapter 38F/40H Algebraic inequalities*.

Gateway to A Level

Parametric equations, which are covered in the second year of A Level, describe the loci of points on the plane for a given parameter. Students will also learn about cycloid curves, similar to the lines formed in the problems related to rolling shapes in Section 3.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 1 Questions 7, 8
- Chapter 10 Question 13

Higher

- Chapter 1 Questions 2, 3, 12

Homework Book

Foundation

- Chapter 6

Higher

- Chapter 6

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 10 worksheets (+ solutions)
- 5 animated widgets
- 3 interactive walkthroughs
- 3 auto-marked quickfire quizzes
- 3 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

7F / 7H Further algebraic expressions

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to simplify expressions by collecting like terms including those with different powers;
- know how to simplify products of expressions;
- **(Higher only)** know how to find the HCF of one or more expressions;
- **(Higher only)** know how to simplify fractions by cancelling common factors.

Additional useful prior knowledge

- Area formulae.
- **(Higher only)** Volume formulae.
- Pythagoras' theorem.

Learning outcomes

Foundation

Section 1

- To know what a quadratic expression is.
- To be able to expand the product of two binomials.

Section 2

- To be able to factorise expressions of the form $x^2 + bx + c$.

Section 3

- To form algebraic expressions to solve problems.

Higher

Section 1

- To know what a quadratic expression is.
- To expand the product of two binomials.

Section 2

- To factorise expressions of the form $ax^2 + bx + c$.

Section 3

- To complete the square on a quadratic expression.

Section 4

- To simplify and manipulate algebraic fractions.

Section 5

- To form algebraic expressions to solve problems.

Vocabulary

binomial, binomial product, quadratic expression, trinomial, perfect square, coefficient, constant

Common misconceptions and other issues

- Common algebraic misconceptions related to finding the product of expressions follow through to this chapter, for example $x \times x = 2x$. In cases like this substitution to offer a counter example may work best to help students identify this mistake.
- Students may miss out the product of one or more pairs of terms when expanding the product of two or more polynomials. The use of arcs to track their product pairs (e.g. 'smiley face method' or 'moon method') can help to prevent this. Alternatively, working with products as areas of rectangles and filling in the area of each sub-rectangle, although more time consuming to use, can help students who often miss product pairs in their calculations. This is often a more pressing issue when the product of two polynomials is being found, one of which has more than two terms (Higher only). However, for students who are more comfortable with the arcs method, recording the products on each arc is a good way of making sure no product pair is missed (Foundation and Higher).

- Some students confuse product with sum, particularly when negatives are involved. This is a misconception that may come through using the box method to expand products. You could advise students to write down the polarity of the second term when setting up their grids, for example:

$\times$	x	$+4$
x		
-2		

- This can sometimes result in students writing 2 in the bottom right hand corner because they have found the sum of 4 and -2 rather than the product. To combat this encourage the use of a multiplication cross in the top left corner; as well as the use of smaller superscript polarity signs, for example $^{-}2$ and $^{+}4$.
- Squaring both terms in a perfect square rather than expanding the binomial product. Encouraging students not to work mentally and write down all steps of their working helps prevent this and students should always be encouraged to start with the following: $(x + 3)^2 = (x + 3)(x + 3)$.
- Factorising quadratics is a concept more procedural students may struggle with. This is because there is not a formula or procedure that provides the factorisation without conscious thought. These students are often reluctant to write down any working other than that which leads them to the correct answer so encourage the use of a factor pairs sum table alongside their working, for example:

Factors of 12	Sum
1, 12	13
2, 6	8
3, 4	7

- The polarities of the constant terms of the linear factors also cause students problems. Rather than encouraging students to learn a series of rules, you could focus on the polarity of the factors so that when students have identified a negative coefficient in the given quadratic they then test more factor pairs, for example:

Factors of -12	Sum	or when the coefficient of x is negative:	Factors of 12	Sum
-1, 12	11		1, 12	13
1, -12	-11		-1, -12	-13
-2, 6	4		2, 6	8
2, -6	-4		-2, -6	-8
-3, 4	1		3, 4	7
3, -4	-1		-3, -4	-7

- (Higher only)** Students often forget to factorise out the coefficient of x^2 (a) when completing the square. There are different approaches to this, for example factorising a out of all 3 terms or just factorising it out of the x^2 and x terms and dealing with the constant in the final stages of manipulation. So students either do $a(x^2 + \frac{b}{a}x + \frac{c}{a})$ or $a(x^2 + \frac{b}{a}x) + c$. You may find that the second method is better for students who are less successful at dealing with fractions as it often reduces the amount of manipulation they have to do. However, neither method particularly increases the success rate at dealing with factorising out a . The most successful method may be to encourage students to check their work by expanding their expressions once done.

Hooks

Introduce the expansion of brackets by connecting the ideas to multiplying two 2-digit numbers. Start with the example 14×27 and connect it to long multiplication methods that make use of a grid to connect the product to an area. Generalising the problem to a variable length can help students make the connection between these ideas and identify how binomial products can be expanded.

SECTION 1F / 1H: MULTIPLYING TWO BINOMIALS

Section 1 introduces and focuses on expansion of double brackets. Students are introduced to some new terminology relating to polynomials and opportunities for creating conceptual links to long multiplication and areas of rectangles are offered. Students are also introduced to special cases of quadratics: perfect squares and the difference of two squares.

(Higher only) This conceptual idea extends to the product of three or more binomials and consequently an ability to expand the product of a binomial and trinomial is required.

As well as defining binomials to students it is also worth at this point considering what a polynomial is, as later expansions will produce quadratics and, for higher tier students, cubics.

(Higher only) It is important that students understand that a quadratic is not just an expression in which the highest power is 2 but also one in which all powers of x are positive integers.

Prompting questions

While working on the material in this section good prompts for promoting discussion might be:

- Why do I have to find the product of all possible pairs of terms? *Use a diagram and compare to finding the total area of the rectangle.*
- What do I mean by expand the binomial product? *Multiply out the brackets to find the product of both binomials.*
- What do we mean by a perfect square? *A number or expression found by multiplying the same expression by itself.*
- Do we need to remember the generalised patterns for perfect squares? *No, because we can always expand the binomial product, but it is useful to recognise the expanded form of a perfect square in the next section.*

Exercise 7A(F) / 7A(H)

While working through the exercise, good prompts for students might be:

- **Q1-2(F) / Q1-2(H)** Why does it not matter what letter is used? *Because we can use any letter to represent a variable.*
- **Q1-2(F) / Q1-2(H)** What will happen when I multiply a negative by a negative (or any other combination involving directed numbers)? Can you demonstrate the rules using a diagram? *Giving students a multiplication grid from -5 to 5 multiplied by -5 to 5 and asking them to fill in what they know can be useful for highlighting these rules.*

X	+ve	-ve
+ve	+	−
-ve	−	+

- **Q4-5(F) / Q4(H)** What do you have to remember when you are given a binomial squared, for example $(x + 3)^2$? *That it is the product of the binomial multiplied by itself. By remembering BIDMAS we know that brackets come first so we can't just bring the power in to act on every term within the bracket.*

Exercise 7B(F) / 7B(H)

While working through the exercise, good prompts for students might be:

- **Q4(F) / Q2(H)** How have you arrived at the shortcut rule for the difference of two squares? *Hopefully by identifying patterns in what has been expanded.*
- **Q4(F) / Q2(H)** Can you draw a diagram to explain what is going on? *(For Foundation students you may wish to suggest an example case first.)*
- **Q6(F) / Q3(H)** What is squared when you apply the rule? What if I had the binomial product $(2x - 4y)(2x + 4y)$? *Everything, not just the variable so $4x^2$ not $2x^2$.*

Exercise 7C(H)

While working through the exercise, good prompts for students might be:

- **Q1(H)** What could the product of three binomials represent? *The volume of a cuboid.*

- **Q1(H)** Could you use a grid in 3D to show the product pairs required for this expansion? Why might this not be a sensible way of expanding the polynomial product? *Because it is hard to fill in a 3D grid, multiplying a quadratic by a linear expression is far easier.*
- **Q1(H)** Why is the order I expand in unimportant? *Because multiplication is commutative.*
- **Q2(H)** Why would it be easiest to start with expansions that form perfect squares of the difference of two squares? *Because, if you can spot patterns, it could make your work quicker and reduce the number of product pairs in the next part of the expansion.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- By considering the problems in Exercise A as areas of rectangles, students can spend time working on forming equivalent expressions to represent the total area both as a product and a sum. Structured support sheets can be used for weaker students with the rectangles pre-drawn and gaps to fill in to guide them. Use of the identity symbol to form appropriate statements about both the area represented as a product and as a sum will further consolidate these concepts from Chapter 3.
- The diagram given to support the explanation of an expansion relating to the difference of two squares could be assigned as an investigation in which students find the area of the shape formed between two perfect square numbers and attempt to generalise their findings. Asking them to form an appropriate diagram to explain a perfect square would be appropriate useful extension and plenary to this task.

Starters

- The parts of Q6(F) and Q4(H) could be used as a starter to a lesson that is moving on to factorising by giving students different statements in which to fill in the gaps and then share their findings with the class.

Plenaries

- The 'work it out' problem in Section 1 is a lovely plenary for students and you could get the class to make a list of common errors students make when expanding (**Foundation only**).

Starters or plenaries

- Having a series of binomials or binomial products on cards and asking students to categorise them can help students identify their own patterns and rules regarding positives and negatives and assist them in making the correct choices regarding negatives when factorising in the next section.

Assessment ideas

- Matching tasks are good assessment and investigation tools for this topic. By removing some pairs you can encourage students to start factorising through logical guesswork and pattern spotting and you can differentiate appropriately for students in your class. The scale of the task (how many pairs are involved) could make this an appropriate starter or plenary or even the main task in a lesson.

SECTION 2F / 2H: FACTORISING QUADRATIC EXPRESSIONS

Section 2 introduces and focuses on factorising quadratics after a brief recap of the factorising covered in Chapter 3. The section starts with worked examples for factorising quadratics in the form $ax^2 + bx + c$ where the value of a is 1. When introducing factorising to students it is worth offering them some practice with just positive coefficients initially to prevent confusion with the added complexity of negative numbers. What follows then is an explanation of the short cut for factorising the difference of two squares by spotting that the identity can be used.

(Higher only) For higher students, the section continues to looking at factorising quadratics where the coefficient of x^2 is not 1 or 0 first through factorising out the coefficient where it is a common factor of all terms and then by considering the extension to the step by step process introduced earlier in the chapter by considering this time what the product of the two constant terms in the binomials must now be. In addition to this, students are also encouraged to look for the special case where they have a perfect square, in some cases disguised by an additional factor.

It is also worth considering the use of the multiplication grid when factorising. A useful visual tool is using the 'Battenberg grid'. For example, for $x^2 + 8x + 12$ the yellow parts of the grid would be filled in and the challenge is about finding the values for the pink cells and consequently the two factors. This helps support students who struggle to

understand why we are looking at the product and sum. It also offers additional support for those who struggle with negatives and which way around they go, as well as how to deal with a coefficient of x^2 that is not 1, through trial and improvement methods.

	x	
x	x^2	
		12

When students start looking at factorising quadratics where $a \neq 1$ (but is still positive in this section) it is worth considering the general case with students and why it is more complex this time. In the textbook students are merely told that the product is now ac which, although a quick adequate method, does rely on them remembering which two letters are involved and is not necessary for success if students just learn to play with factors of the numbers (and most quadratics they work with have a prime coefficient of x^2 or one with very few factors, e.g. 4). An explanation of why follows:

(Higher only) When we consider the expansion of $(x+d)(x+e) \equiv x^2 + (d+e)x + de$ we can conclude that by comparing to the general form $ax^2 + bx + c$ that $a = 1$ and b is the sum and c is the product of two unknowns. These two clues are enough for us to work out logically what those two numbers must be. When we have the following expansion things are much more complex: $(fx+d)(gx+e) \equiv fgx^2 + (dg+ef)x + de$. This is because we now have four unknowns: d, e, f, g , which makes it harder to find the unknowns now $f, g \neq 1$. In this case, to form two useful statements about our four unknowns, we consider that the sum of the coefficients of x gives us $dg + ef = b$ but that the product of these two numbers is no longer the constant term c . If we find the product of the two values we are summing to find the coefficient b we get $fgde = ac$ the product of the coefficient of x^2 and the constant term of the quadratic. By considering the general case and comparing to when $f, g = 1$ more able students should be able to identify that actually they are always doing this but ac in the first case is just $1 \times c = c$.

Prompting questions

While working on the material in this section good prompts for promoting discussion might be:

- What is a quadratic expression? *A polynomial where the largest power is 2.*
- What do we mean by factorise? *Undo the expansion of brackets to write the quadratic as a product of two linear expressions.*
- What happens when you expand the product? $(x+\alpha)(x+\beta)$ If we always get a quadratic in the form $x^2 + bx + c$ what can you tell me about b and c ? $b = \alpha + \beta$ $b =$ and $c = \alpha\beta$.
- What are possible pairs of numbers you could multiply to make -12 ? *Confirm to students that they need to consider two pairs instead of 1 now, for example 3×-4 and -3×4 .*

Exercise 7C(F) / 7D(H)

While working through the exercise, good prompts for students might be:

- **Q2-4(F) / Q1-3(H)** Can you write down what the sum and product is for each of the quadratics in the exercise?

Exercise 7D(F) / 7E(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1-2(H)** Is an expression in the form $x^2 - c$ still a quadratic? *Yes, it just means the coefficient of x is 0.*
- **Q1(F) / Q1-2(H)** What do both numbers have to sum to when we have a quadratic in the form $x^2 - c$? $b = 0$. What does that tell you about what both numbers must be? *The same number but one positive and one negative.*
- **Q2e(H)** Which two numbers can I multiply together to make -7 ? $\sqrt{7}$ and $-\sqrt{7}$.
- **Q2(F) / Q3(H)** Will factorising the difference of two squares always make your Pythagoras calculation easier? *No, it depends on how large the values are.*

Exercise 7F(H)

While working through the exercise, good prompts for students might be:

- **Q1(H)** What is the product and sum of the two values you are looking for?
- **Q4(H)** What part of the expression is making the factorising hard? (*The repeating linear expression.*) Replace $x + y$ and so on with another letter to make the quadratic easier to identify. Question 4 in Exercise F is a type of problem many A Level students struggle with. You could call them quadratics in disguise and advise students to use a substitution to make the expression more recognisable as a quadratic before replacing their substitution in the last line. This is a concept that often requires a lot of practice and exposure to applying similar techniques, even for the most able students.

Starters, plenaries, enrichment and assessment ideas**Starters**

- Product/sum diamonds (**10 ticks** has a series of them) in which students fill in various gaps of the diamonds, finishing by identifying the pair of numbers required to form the correct product and sum.

Product		
5		8
Sum		

Product	60	
6		
Sum		

Product	12	
Sum	7	

Plenaries

- A suitable plenary might be to ask students whether they think all quadratics can be factorised. Ask them then to provide you with their own combination of coefficients that cannot be factorised so as to facilitate further discussion of rules and patterns, ready for future work on this concept. This can be easily differentiated for Higher and Foundation students: Foundation students may suggest ones where the coefficient of x^2 is not 1 whereas Higher students may suggest ones with fractional or irrational roots or even ones with imaginary solutions.
- A possible plenary could be: Can you give me three different quadratics to factorise? One where the constants of both linear factors are positive, one where both are negative and one where they are of different polarity.

Enrichment ideas

- An investigation in which students are given four linear factors and asked to find all possible quadratics from them, for example $(x+3)$, $(x-2)$, $(x-3)$, $(x+5)$ encourages them to find their own rules about the values of a , b and c for $ax^2 + bx + c$.

Assessment ideas

- A series of matching task competitions in which students identify common patterns (product and sum) can help improve their speed.
- An assessment task that you could use to both assess students understanding and create appropriate revision material is to set students the task of writing a 'how to' guide similar to the one given in the textbook.

SECTION 3F: APPLY YOUR SKILLS

Section 3 offers a series of problems for Foundation students to apply their new manipulation skills to.

Prompting questions

Exercise 7E(F)

While working through the exercise, good prompts for students might be:

Q3(F) Can you draw a diagram to support your workings?

Starters, plenaries, enrichment and assessment ideas

Starters

- Exercise 7E, Question 1 makes a suitable starter for assessing what students have learned in this chapter.

SECTION 3H: COMPLETING THE SQUARE

Section 3 focuses on manipulating quadratics by using the completed square form. Although solving quadratics is mentioned briefly it is not covered in detail until *Chapter 8 Equations*.

Prompting questions

Exercise 7G(H)

While working through the exercise, good prompts for students might be:

- Q1(H)** What does a quadratic need to look like to be a complete square? Talk about general form of square rooting the constant and seeing if the coefficient of x is twice this value.
- Q2(H)** How do we square a fraction? *Square both the numerator and denominator; think of multiplying fractions.*
- Q3(H)** What is the closest square you can find? How can we adapt our expression to include it? *The x^2 and x term are the most important so we match our square to them.*
- Q3(H)** What can I add to an expression without changing it? *Add values that sum to 0.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- Draw the perfect squares and show the addition/subtraction we need to form the expressions. These areas can then be manipulated to give the factorised form (for those that can be factorised simply or as an extension using the difference of two squares method found above).
- Matching exercises can be used for students who lack confidence in completing the square. Removing cards so that not all cards have a pair to match to can help challenge students to move between both expanding and completing the square.
- A supportive task for weaker students is to give them a card sort that requires them to order the steps of writing a quadratic in completed square form for example:

$$5x^2 + 13x + 3$$

$$5\left(x + \frac{13}{10}\right)^2 - \frac{109}{20}$$

$$5\left[\left(x + \frac{13}{10}\right)^2 - \frac{169}{100}\right] + 3$$

$$5\left(x^2 + \frac{13}{5}x\right) + 3$$

$$5\left(x + \frac{13}{10}\right)^2 - \frac{169}{20} + 3$$

Additional steps could also be added for students to go from completed square form to factorised.

An interesting extension for more able students is to consider why $(x - y)^2$ is the same as $(y - x)^2$.

SECTION 4H: ALGEBRAIC FRACTIONS

Section 4 revisits some of the manipulation skills students worked on in *Chapter 3 Algebraic expressions* for working with and simplifying fractions and adds problems involving quadratics. Many conceptual problems students face have been covered in the teacher notes for *Chapter 3*.

Prompting questions

Exercise 7H(H)

While working through the exercise, good prompts for students might be:

- **Q1(H)** What can we cancel when simplifying? *Factors not terms.*
- **Q1(H)** How can you find the factors of a quadratic expression? *Factorise.*

Starters, plenaries, enrichment and assessment ideas

Assessment ideas

- Asking students to write incorrect solutions (or using their incorrect solutions scanned from a homework task) and peer marking is good for getting students to identify common errors such as the ones highlighted in the textbook.

SECTION 5H: APPLY YOUR SKILLS

Section 5 offers a series of problems to which Higher tier students can apply their new manipulation skills. Encourage students to write down all working and use diagrams to support their method where possible, as mistakes are often related to problems with negatives and lost terms because they try to do too much in their heads.

Prompting questions

Exercise 7I(H)

While working through the exercise, good prompts for students might be:

- **Q5(H)** Can you draw a diagram to support your workings?
- **Q7(H)** Can you add or subtract something to make the values nicer to work with? Hint: think about what is needed to round it to the nearest whole number.
- **Q8(H)** What can you tell me about any number that is squared? *It is positive.* What is therefore the smallest value that the expression can have?

Topic links

Previous learning

The connections back to *Chapter 3 Algebraic expressions* are obvious as these ideas are just extended to quadratics in this chapter. However, more opportunities for generalising students' long multiplication methods are offered here and they are better equipped to consider more challenging 'Think of a number...' problems.

There are several problems in this chapter that require knowledge of Pythagoras' theorem and these exercises could be used to revisit this topic with a focus on how non-calculator skills, including difference of two squares and surds, can be employed to solve problems

Future learning

Students will need to use the manipulation skills they have learnt in this chapter to solve quadratics in *Chapter 8 Algebraic equations*. They will learn how the completed square and factorised forms can be used to solve quadratics. In addition, for the strongest students, an ability to understand how the completed square form helps to derive the quadratic formula would be appropriate.

Gateway to A Level

At A Level these manipulation skills are essential. Students will be required to manipulate quadratics with ease between various forms and understand how these different forms connect to the features of a quadratic graph and its

transformations. They will also learn long division of polynomials in order to factorise expressions of a higher degree than 2.

Students who work with grid expansion methods can adapt their grids to divide. Those that do not have a conceptual understanding of how the grids support multiplication, and hence division, often struggle to learn a new, long algorithm, both in traditional format and by means of a grid. An ability to expand fluently two or more binomials, trinomials and larger expressions will be required.

A Level students often struggle with simple expansions, concluding for example that $(x+5)^2 = x^2 + 25$ or $(x+3)(x^2+4x+2) = x^3 + 3x^2 + 12x + 6$. The second example is the reason that students should use grids to multiply rather than arcs to show the products when expanding brackets. Once students have mastered expansions and have grasped that they first find the product of all possible pairs of terms and then add them, the use of grids can be stopped.

Students who use a method of tracking their products through the use of arcs often miss one or more products, particularly when there are more than two terms in one of the brackets, as in the example above. Many students, particularly weaker candidates, continue to use this method for the duration of the A Level course.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 1 Question 9
- Chapter 2 Question 7

Higher

- Chapter 1 Question 4
- Chapter 2 Question 2
- Chapter 8 Questions 15, 28

Homework Book

Foundation

- Chapter 7

Higher

- Chapter 7

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 9 (F) / 13 (H) worksheets (+ solutions)
- 3 (F) / 6 (H) animated widgets
- 11 (F) / 19 (H) interactive walkthroughs
- 4 (F) / 7 (H) auto-marked quickfire quizzes
- 4 (F) / 7 (H) auto-marked question sets, each with four levels
- Auto-marked chapter quiz

8F / 8H Equations

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- be able to use variables correctly to form algebraic expressions;
- know how to factorise quadratics in the form: $x^2 + bx + c$ (**Foundation only**);
- know how to factorise quadratics in the form: $ax^2 + bx + c$ (**Higher only**);
- know how to complete the square of a quadratic in the form: $ax^2 + bx + c$ (**Higher only**);
- understand inverse operations;
- be able to draw graphs of linear functions;
- be able to draw graphs of quadratic functions.

Additional useful prior knowledge

- Angle facts including sums of angles in polygons.
- Properties of shapes.
- Area and perimeter formulae.
- Speed, distance, time formulae.

Learning outcomes

Foundation

- To form equations from a variety of problems including geometric scenarios.

Section 1

- Solve linear equations.
- To understand that identities are equations for which there are an infinite number of solutions as they are true for all values x can take.

Section 2

- To solve quadratic equations.
- To understand that different types of equations have a different possible number of solutions.

Section 3

- To solve linear simultaneous equations.

Section 4

- To know how to read and interpret graphs in various contexts.
- To be able to use graphs to find approximate solutions to equations.

Higher

- To form equations from a variety of problems including geometric scenarios.

Section 1

- To solve linear equations.
- To understand that identities are equations for which there are an infinite number of solutions as they are true for all values x can take.

Section 2

- To solve quadratic equations.
- To understand that different types of equations have a different possible number of solutions.

Section 3

- To solve linear simultaneous equations.
- To solve linear and quadratic simultaneous equations.

Section 4

- To know how to read and interpret graphs in various contexts.
- To be able to use graphs to find approximate solutions to equations.

Section 5

- To use iterative methods to find approximate solutions to equations.

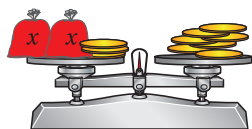
Vocabulary

unknown, variable, linear equation, roots, solution, simultaneous equations

Common misconceptions and other issues

2 bags and 3 coins = 7 coins

Algebra

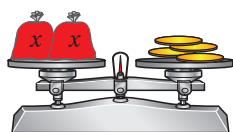


$$2x + 3 = 7$$

-3 coins -3 coins

$$-3 \quad -3$$

2 bags = 4 coins

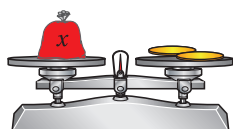


$$2x = 4$$

÷ 2 ÷ 2

$$\div 2 \quad \div 2$$

1 bag = 2 coins



$$x = 2$$

$$\begin{aligned} \text{Check by substitution} &= 2 \times 2 + 3 \\ &= 4 + 3 = 7 \end{aligned}$$

- There is often incorrect use of the equality sign so that when solving $2x + 3 = 7$ students write $7 - 3 = 4/2 = 2$. While this may not seem problematic because students are still attaining the correct answer, students who are showing their working in this way tend to be working on a reverse operation principle without recognising that they are applying the same operation to both sides of the equation. These students tend to have problems later when faced with unknowns on both sides. Visualising the problem as a balancing scale and even using diagrams to support students' working alongside the algebra, where x represents a bag of unknown value, can help here (example in diagram above). In addition to using the analogy of balancing scales, it is worth revisiting some problems involving the use of an equality sign for example:
 $5 + 13 = + 12$, to assess whether students are using the equality to show the next step and hence write 18 in the box or recognise it as showing an equivalence and hence write 6 in the box.
- There is often a lack of understanding that adding a negative is equal to subtraction and so on. This may be evident in the 'before you start' section in which students are asked to fill in the gaps to identify their knowledge of inverses. An example to highlight this is $7 + = 0$ in which students have to recognise that adding -7 is the same as subtracting 7. Revisiting the rules of addition and subtraction of negatives will help support this.
- The following common misconceptions and issues can be prevented by encouraging students to always check their solutions by substitution. They are often unwilling to do this. There is no simple answer to this other than constant reminders and modelling good practice (even if it does take an additional couple of minutes to do so).
 - Not applying an operation to every term on both sides of the equation but to selected terms, for example $3x + 15 = 18 \Rightarrow x + 15 = 6$. Remind students that they need to multiply or divide **every** term in the whole equation to keep the 'scales'/equation balanced so as to avoid this error.
 - When solving by factorising, omitting to set each bracket to zero before writing the solutions and so stating the wrong sign for them. For example, for $(x + a)(x + b)$ stating that the roots are a and b rather than $-a$ and $-b$.
 - Leaving their solutions to simultaneous equations incomplete by forgetting to find the value of the second variable.

- Many students want to give exact or decimalised answers rather than working out their solutions in fractional form. For example, given $3x = 10$ students are reluctant or unsure of how to divide 10 by 3. Few students are confident that $10/3$ is an acceptable form for their solution and many want to round to 3.3 (which of course causes problems when using substitution to check their values of x). Exposure to examples involving fractional answers and discussions about fractions as 'lazy' divisions can be very helpful.
- For issues based on factorising and completing the square (**Higher only**) see the notes for *Chapter 7 Further expressions*.
- Students often fail to find all solutions, particularly when solving a quadratic that is the difference of two squares or when a factor is x . Encouraging students to always factorise and apply the zero factor principle often helps prevent this issue.
- When students apply the iterative formula method they do not always use technology to support them. They need to be taught to use the 'ANS' key in their calculator to prevent issues with rounding and reduce the amount of work they have to do.
- Students confuse the decimal search and bisection method. The meaning of the word bisection can be used to help students remember this.

Hooks

1. Diagrammatic problems in which the game is to find the value of a symbol can often help draw students in. Using problems where the scales are set up as a mobile where each strand is equal allows students to realise they can use knowledge of equivalence (in this case removing the same object) to solve the problem.
2. Introduce the need for simultaneous equations by considering two different shopping receipts in which the value of the products is unknown but the total value of the shopping is known. The addition of refunded items makes for an interesting investigation in which strategies for solving can be formed by the students.

SALE Price Bread Roll 32 Strawberries 32 Total: £4.40	SALE Price Bread Roll Strawberries Strawberries Strawberries Total: £5.00
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SECTION 1F / 1H: LINEAR EQUATIONS

Section 1 explores different types of linear equation including: solving simple linear with one unknown on one side (**Foundation only**), solving equations with the unknown on both sides, solving equations that involve manipulation of fractions (**Higher only**), and setting up equations from information given before going on to solve them. There are also some identities to solve in order to reinforce the idea that these are true for all values of x in contrast to equations that are only true for certain values.

Prompting questions

Exercise 8A(F)

While working through the exercise, good prompts for students might be:

- **Q3(F)** What other options are available to you other than expanding the bracket? *Dividing both sides by the constant factor.*
- **Q4(F)** Why might you want to expand the brackets first rather than divide by a constant factor? *If there is more than one bracket, both with different constant factors.*
- **Q4(F)** Why is $-x = 4$ not an acceptable solution? *Because we want to know what x is, not what -1 times x is.*

Exercise 8B(F) / 8A(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** What should you try to do first when there are unknowns on both sides? *Use the expression ‘pay your debt’; in other words, add any negative x terms when needed first to reduce problems with finding negative x later. This may require expansion of brackets first.*
- **Q1(F) / Q1(H)** What can you add or subtract to both sides of the equation to ‘undo’ a term? *The same thing that will cancel with a term on one side.*
- **Q1(F) / Q1(H)** Can you represent your problem with balancing scales? *See example in misconceptions question for using diagrams.*
- **Q2(F) / Q2(H)** Why might you want to expand the brackets first rather than divide by a constant factor? *If there is more than one bracket, both with different constant factors.*
- **Q3(F) / Q3(H)** What don’t you like about your solution? Does your equation even have a solution? What could this mean? *Identities work for all values for the unknown so we can always reduce the equation to $0 = 0$.*

Exercise 8B(H)

While working through the exercise, good prompts for students might be:

- **Q1(H)** What has to be the same for you to be able to add fractions? *The denominator.*
- **Q1(H)** How can you find a common multiple of both denominators? *Multiply the denominators together.*

Exercise 8C(F) / 8C(H)

While working through the exercise, good prompts for students might be:

- What is your unknown? What does your letter represent?
- Could you draw a diagram to represent your working?
- **Q1e(F) / Q1e(H)** How can you represent any two consecutive numbers? (x , $x + 1$ or $x - 1$, x , etc.) Does it matter which option you choose? *No, the algebra will still give the same solution. Suggest students attempt both methods to see.*
- **Q3(F) / Q3(H)** What formulas are useful to you in this problem? What equations can you form from the information you are given?

Starters, plenaries, enrichment and assessment ideas

Starters

- A simple introduction to solving linear equations by applying inverse operations to undo. **NRICH Your Number Was...** is a standard ‘I think of a number...’ problem(nrich.maths.org).

Starters or plenaries

- ‘What is wrong with the answer?’ starter or plenary short task. This will help to reinforce the idea of the necessity for constantly checking their work for errors. Giving students their teacher’s ‘homework’ to mark is a simple task that takes very little time to write.

Enrichment activities

- **NRICH Good Work If You Can Get It** is a challenging problem in terms of forming and solving. There is a lot of information to get through and students will need to define their own variables for the amount given to each man and the total amount. The trick to solving comes from knowing the fraction of the total amount that each man is due based on the number of days he worked (nrich.maths.org).

Assessment ideas

- **NRICH Addition Equation Sudoku** is a nice homework/assessment task for students who enjoy Sudoku puzzles as it includes the need for defining the unknowns before solving (nrich.maths.org).
- Compare solutions of students who have opted to define their initial unknown in different ways, for example in Example 1 defining x as the largest angle rather than the smallest to promote the idea that the initial decisions aren’t as important as seeing their ideas through and that they may have different working but the same answer as others.

SECTION 2F / 2H: QUADRATIC EQUATIONS

Section 2 focuses on solving quadratic equations by drawing on all the methods for factorising and manipulating quadratic expressions from *Chapter 7 Algebraic expressions*. Students again work on their skills at forming and setting up equations but this time, due to the number of solutions, must recognise that not all their solutions are valid in the context of the question. It is important that students do not rely too much on using the square root function as an inverse, thinking for example that $\sqrt{9} = 3$ or -3 , as it is defined for a positive range of outputs only. It is instead better to talk about what values could be squared to give a required value and using $\pm\sqrt{9}$ as is used in the quadratic formula. This will be very important for later work in functions covered at A Level.

(Higher only) The method for completing the square is used to deduce the quadratic formula and students are asked to consider the number of roots based on the graphical representation of the equation.

Prompting questions

Exercise 8D(F) / 8D(H)

While working through the exercise, good prompts for students might be:

- **Q1-3(F) / Q1(H)** What should I do first when I have a quadratic equation equal to 0? *Factorise.*
- What does one of the factors/brackets have to be if the product is 0? *0. The only way to get 0 with multiplication is to multiply by 0.*
- What must you always do at the end of your solution? *State the values x can take.*
- What does a quadratic look like when plotted on a graph? How does this help us to know how many solutions there are? *The number of solutions is the number of intersections with the x-axis. Showing and discussing various examples could help students understand that not all equations have solutions.*
- How many solutions are possible when solving a quadratic equation? *One, two or none depending on how many different factors the solution has or whether a graph of the function cuts the x-axis at two places (two solutions), just 'sits' on it at one place (one solution) or doesn't meet it at all (no solutions).*
- **Q4(F) / Q3(H)** What should I do if I do not have a quadratic equal to 0? *Rearrange so that it is in the form $ax^2 + bx + c = 0$.*
- **Q3(H)** What if my equation doesn't look like a quadratic? What should I do then? *Rearrange so that it is equal to 0 and then you will know what type of equation you are solving.*

Exercise 8E(H)

While working through the exercise, good prompts for students might be:

- What are the values for a , b and c for this quadratic? *Encouraging students to start by writing down $a =$, $b =$ and $c =$ will reduce the number of errors and prevent problems with negatives being missed.*

Exercise 8E(F) / 8F(H)

While working through the exercise, good prompts for students might be:

- What is your unknown? What does your letter represent?
- Could you draw a diagram to represent your working?
- **Q2-3(F) / Q3-6(H)** What formulas are useful to you in this problem? What equations can you form from the information you know? *General area formulae and (Higher only) Pythagoras' theorem are required.*
- **Q2(H)** What does Pythagoras' theorem state? *The sum of the squares of the two shorter sides is equal to the square of the hypotenuse in a right-angled triangle.*
- **Q2-3(F) / Q3-6(H)** If you are dealing with a value squared, what type of units or problem are you dealing with? *Area.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- **NRICH How Old Am I?** is a challenging problem that requires students to have the ability to set up equations.

Assessment ideas

- There are numerous past exam questions that have involved forming and solving quadratics based on information about an area of a shape. Combining these problems with those based solely on perimeter, and hence linear equations, is good for getting students to recognise what the different dimensions can mean in a contextual question.

SECTION 3F / 3H: SIMULTANEOUS EQUATIONS

Section 3 introduces students to the idea of having two unknowns and using two pieces of information to solve them simultaneously. The hook given above is a nice way to start this topic and can be used to varying degrees as an investigation if needed. Students learn two methods for solving: substitution and elimination. Once students know they can eliminate they are often reluctant to substitute, but it is often the better method, particularly if multiple solutions are involved, so the order in which the substitution and elimination methods are introduced should be considered. The idea of using graphs to find solutions is first raised in this chapter and students should think about what it means graphically to solve equations such as $3x + 9 = 0$.

(Higher only) Work on solving simultaneously is extended to include quadratic and linear simultaneous equations tying together the work from the rest of this chapter.

Prompting questions

Exercise 8F(F) / 8G(H)

While working through the exercise, good prompts for students might be:

- Q2(F) / Q2(H)** Do you need to do any rearranging first to change the subject?
- Have you got a value for both x and y ?
- How can you check your solutions? *Using substitution.*

Exercise 8G(F) / 8H(H)

While working through the exercise, good prompts for students might be:

- Line the statements up, with the variables aligned vertically, like you would a long addition or subtraction. Does this help you see whether you should be adding or subtracting?
- Repeat the question using the substitution method from the previous exercise. Do you get the same answer? *You should!*
- Q2(F) / Q2-3(H)** Why might the elimination method be the better option? *Should be quicker and prevent rearranging if the subject of the formula can form a direct substitution.*
- Q2(F) / Q3(H)** Multiply through by a factor to have the same number of x or y so that elimination is possible.
- Q2(F) / Q3(H)** Why can I multiply the whole equation by a variable? *At this point you may wish to give the analogy of buying shampoos and conditioners (or any other products). If s is the price of shampoo and c the price of conditioner then $s + c = £2.50$. If I buy twice as much then I have $2s + 2c = £5$.*

Exercise 8H(F) / 8I(H)

While working through the exercise, good prompts for students might be:

- What are your two unknowns? Can you define the variables?
- What do you know about your two unknowns? Can you write the sentences out in full English and then adapt by replacing 'the number of...' or 'the unknown number' with your variable?

This section is very challenging for Foundation students and it requires strong forming and solving skills.

Exercise 8J(H)

While working through the exercise, good prompts for students might be:

- Can you sketch your functions to identify how many solutions you will have?
- How can you tell how many solutions you have from your algebra? *Whether the quadratic gives two roots, one or none.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- **NRICH What's It Worth?** has a huge number of possibilities for challenging students of different levels with an attached randomly generated worksheet file. The different solutions that are included offer probes for group discussion about working systematically and elegantly. If you wish, you could ignore the last solution as it is a unique method to this problem (but great fun for a plenary!) (nrich.maths.org).
- Once students are confident with ideas relating to solving simultaneous equations, **NRICH Matchless** is a nice problem for them to attempt. The challenge here is that students will need to set up suitable equations to begin with so as to form their own simultaneous equations. They will then solve their equations and check their answers. You might wish to remove some of the cards to promote a class discussion about the amount of information required to complete the task (nrich.maths.org).
- **(Higher only)** For the most able students, **NRICH Intersections** will force them to consider what is actually going on to make the following equations so different: It also requires strong manipulation skills and attention to detail to prevent errors (nrich.maths.org).
- **(Higher only)** **NRICH Surds** is a good activity for the most able students who are fluent with surds and can recognise that the first statement forms an equation in one variable so that they have enough information to tackle the simultaneous equations (nrich.maths.org).
- Similar to the Sudoku puzzle in Section 1, simultaneous equations are required to solve the **NRICH All-variables Sudoku** (nrich.maths.org).

Starters or plenaries

- Give students a series of equations to be solved and ask them to sort them into 'always', 'sometimes' and 'never true'. Within 'sometimes', they can be further sorted into the number of solutions to help them understand that the order of the polynomial does not necessarily dictate this. The activity also helps consolidate material on identities and works better when a mixture of linear and quadratic equations is used.
- To update the chickens and goats examples, you may wish to turn the problem into a 'Harry Potter potions' class with spiders and lizards. You can produce a similar problem based on heads and legs and possibly show a YouTube clip of Professor Snape telling them off first.

SECTION 4F / 4H: USING GRAPHS TO SOLVE EQUATIONS

Section 4 recaps prior learning on graphical representations of functions and extends this to estimating solutions to equations by reading values off the graph. The example has no context so students may need some support when working on the first couple of questions in Exercise 8I(F) / 8K(H). These are based on a distance–time graph and you may wish to give them an example of this before they tackle the exercise.

Prompting questions

Exercise 8I(F) / 8K(H)

While working through the exercise, good prompts for students might be:

- What does each axis of your graph represent?
- So is your question about a value on the x -axis or y -axis or neither?
- **Q1-2(F) / Q1-2(H)** How do we find the average speed of a moving object? *Total distance divided by total time.*
- How do we find the equation of a straight line from the graph? *We use $y = mx + c$ and find the y -intercept (c) and the gradient (m).*
- **Q6(F) / Q6(H)** What is special about the coordinates of the point at which two straight lines meet? *It is the point of intersection and the solution to the pair of simultaneous equations formed by the two lines.*
- **Q7-8(F) / Q7-10(H)** How many times can a quadratic curve cross the x -axis? *Maximum of two but also one or none.* What equation are you solving when you consider the x -axis intercepts? $y = 0$.

- **Q7-8(F) / Q7-10(H)** At how many different points can a straight line and a quadratic curve meet? *Twice, once or not at all.*
- Why might we prefer algebraic solutions over a graphical solution? *Because algebra is more accurate and can give an exact answer if the solution is irrational.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- **NRICH What's That Graph?** goes beyond the graphs and problems covered in this section as does Standards Unit: A6 – Interpreting Distance–Time Graphs. A mixed matching task could be designed to cover all the functions in this section in which students are asked to match scenarios to graphs as well as read information off them (nrich.maths.org).

SECTION 5(F): USING EQUATIONS AND THEIR GRAPHS

This final section contains a mixed exercise to consolidate and test all the material covered in the chapter.

SECTION 5(H): FINDING APPROXIMATE SOLUTIONS BY ITERATION

This section considers three different methods for approximating solutions of equations: the iterative formula, the decimal search method and the interval bisection method. It is worth considering the order in which you introduce these methods. That is, that some students may find it easier to start with the graphs and change of sign (decimal search and interval bisection) methods first. You may also wish to start the section with an exploration of trial and improvement methods to solve non-linear and non-quadratic equations (e.g. cubic) to help students understand why iterative methods are required.

Exercise 8L(H)

- What is the first value of x you are going to substitute into the iterative formula? *The one given as x_1 .*

Exercise 8M(H)

- **Q2(H)** What is the difference between the decimal search and the interval bisection method? *Decimal search goes up in equal intervals whereas the interval bisection method cuts the interval between the approximate roots in half and checks the median value each time. Students have come across bisect before and the meaning of the word should be explored as a way of helping them remember.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Identify the interval for one/both roots of a quadratic equations using change of sign/trial and improvement methods. Students can then use the quadratic formula to verify that the solutions lie in the intervals given.

Enrichment ideas

- The student book does not consider why the iterative formula can be used and how the given formula is connected to the equation being solved through rearrangement. This is not something students need to know but an appropriate extension for the most able candidates is to explore graphically what is happening when x is made the subject of the equation and the iterative method is applied.
- The use of technology can be used to reduce working and allow students to spot patterns more quickly. Setting up 'speedsheets' that apply iterative formulae can help students identify appropriate values with which to start the substitution and why looking for an initial change of sign is required.

SECTION 6(H): USING EQUATIONS AND GRAPHS TO SOLVE PROBLEMS

This final section contains a mixed exercise to test all the material covered in the chapter.

Topic links

Previous learning

This chapter uses all the manipulation skills, including factorising and simplifying expressions, that were worked on in *Chapter 3 Algebraic expressions* and *Chapter 7 Further expressions* to help to solve a variety of equations.

Future learning

Students will revisit solving to explore ranges of solutions to inequalities in *Chapter 38(F) 40(H) Algebraic Inequalities*. In addition, in *Chapter 14 Algebraic formulae* they will apply the techniques learned here to ‘solve’ formulae in order to change their subject.

Gateway to A Level

Solving equations forms a large part of A Level mathematics. Students will learn how to manipulate equations derived from a large range of functions and solve them using their understanding about the number of possible solutions in a given range. For future study, it is important that students understand the connection between graphical representations of functions and what the solutions mean, that is to say that for a function $y = f(x)$, in solving $f(x) = 0$ they are finding the intersection of the curve with the x -axis. They will also extend their work on quadratics to look at higher order polynomials, including cubics, and learn about other features of graphs.

The idea of an inverse function will also be covered in greater detail at A Level when students will learn about the domain and range of functions as well as the restrictions that must sometimes be placed on these for an inverse to exist.

Finally, the work on numerical methods will be covered in more detail at A Level and students will learn a wider range of iterative methods. Understanding why iterative methods are needed for functions that we can’t easily solve sets the groundwork for future work on this topic.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 2 Question 17
- Chapter 3 Question 4
- Chapter 6 Questions 1, 7
- Chapter 8 Questions 19, 20
- Chapter 10 Question 7

Higher

- Chapter 1 Questions 13, 14, 15
- Chapter 2 Questions 3, 12, 13
- Chapter 3 Questions 1, 9
- Chapter 7 Question 7
- Chapter 8 Question 16
- Chapter 10 Question 3, 4

Homework Book

Foundation

- Chapter 8

Higher

- Chapter 8

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 14 (F) / 20 (H) worksheets (+ solutions)
- 9 (F) / 13 (H) animated widgets
- 19 (F) / 28 (H) interactive walkthroughs
- 6 (F) / 10 (H) auto-marked quickfire quizzes
- 6 (F) / 10 (H) auto-marked question sets, each with four levels
- Auto-marked chapter quiz

9F / 9H Angles

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know the sum of the interior angles in a triangle;
- understand and be able to use notation conventions for labelling angles and lines, including naming a line, for example DE, and an angle, for example FED;
- know the properties of 2D shapes;
- have the ability to use basic arithmetic to find complements to 180° and 360° ;
- know how to form and solve linear equations with the unknown on one side of the equation (**Foundation only**);
- know how to form and solve linear equations where the unknown is on both sides of the equation (**Higher only**).

Additional useful prior knowledge

- How to measure an angle using a protractor.
- What a variable is and how it can be used to produce a formula.

Learning outcomes

Foundation

Section 1

- To recall knowledge of basic angle facts including: vertically opposite angles, angles on a straight line and angles around a point.
- To be able to apply basic angle facts to find the size of missing angles in various scenarios.

Section 2

- To recall knowledge of angle facts relating to parallel lines including: corresponding angles, alternate angles and co-interior angles.
- To be able to apply basic angle facts and those relating to parallel lines to find the size of missing angles in various scenarios.

Section 3

- To understand a proof for the sum of the interior angles of a triangle being 180° .
- To understand a proof for the exterior angle of a triangle being equal to the sum of the opposite interior angles.

Section 4

- To be able to calculate the sum of the interior angles of any polygon.
- To be able to calculate the size of a single interior angle of a regular polygon.
- To be able to calculate the size of a single exterior angle of a regular polygon.

Higher

Section 1

- To recall knowledge of basic angle facts including: vertically opposite angles, angles on a straight line and angles around a point.
- To be able to apply basic angle facts to find the size of missing angles in various scenarios.

Section 2

- To recall knowledge of angle facts relating to parallel lines including: corresponding angles, alternate angles and co-interior angles.
- To be able to apply basic angle facts and those relating to parallel lines to find the size of missing angles in various scenarios.

Section 3

- To understand a proof for the sum of the interior angles of a triangle being 180° .
- To understand a proof for the exterior angle of a triangle being equal to the sum of the opposite interior angles.

Section 4

- To be able to calculate the sum of the interior angles of any polygon.
- To be able to calculate the size of a single interior angle of a regular polygon.
- To be able to calculate the size of a single exterior angle of a regular polygon.

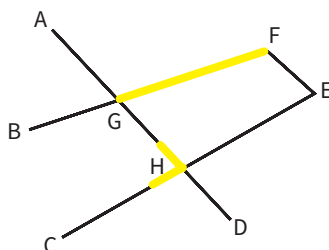
Vocabulary

vertically opposite angles, transversal, corresponding angles, alternate angles, co-interior angles, interior angles, exterior angles

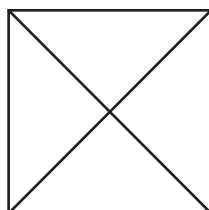
(Higher only) supplementary angles

Common misconceptions and other issues

- Students struggle with formal language and notations in geometry. For example, when a vertex is labelled B, students often refer to the angle as B forgetting that there are at least two possible angles this could be, one less than 180° and the other greater than 180° . Using starters based on notation that involve highlighting angles and lines given their labels, can help identify and correct these errors at the beginning of the chapter. For example:



- Students fail to use the correct reasoning, for example stating 'because they are "Z" angles' rather than 'because alternate angles are equal'. Students also think that it is enough to state 'opposite angles are equal' rather than using 'vertically opposite angles are equal'. Learning these reasons is not easy and is particularly challenging for students with weak literacy. This is a fundamental requirement for success with this topic and time should be spent learning the definitions by heart. It can be useful to have laminated A4 sheets on the walls around the room reminding students of these useful facts and they may also serve as a memory prompt if students have a visual memory based on where they are placed in the room. In addition, you could try testing the reasons in the same way as you would test spelling.
- Students often struggle to cut up the polygons when working on the investigation in Section 4. Students will happily say that there are four triangles in a square because when they join up the vertices with diagonals they get the following:



It helps to tell students that they must cut the polygon up into triangles using the minimum number of lines possible or tell them that no additional lines may cross each other.

- Students often misunderstand what the exterior angle of a polygon is, thinking that it is actually the entire outside angle at a vertex of a shape rather than the angle formed by extending the edge of the shape. Consequently they often think that the interior and exterior angles add to 360° rather than 180° . The correct definition of an exterior angle just needs to be reinforced regularly.
- Students get lost in their working and don't know what their values mean or cannot clearly define their answer. Encouraging students to always set up an equation, clearly stating what their unknown is on the left hand side, so building on the good practice from *Chapter 8 Equations*, helps improve this. It is important to model good practice when working out examples on the board even, when time is short.

Hooks

- A nice starter to this task, which recaps one of the first angle facts that students need to know, is **Round and Round and Round**. Having watched the first two stages you can pose a question based on where the dot would end up if it turned through $30,000^\circ$ (nrich.maths.org).
- A fun game that students should really enjoy is **Estimating Angles** (nrich.maths.org).

SECTION 1F / 1H: ANGLE FACTS

Section 1 considers the basic angle facts that students need to know, including the sum of angles on a straight line, sum of angles around a point and that vertically opposite angles are equal. Problems challenge students to set up and solve linear equations.

(Higher only) Extension problems require students to solve equations with unknowns on both sides of the equation.

Prompting questions

While working on the material in this section good prompts for promoting discussion might be:

- When skateboarding/scootering, what does it mean for you to have done a 360/180? How does this help us with knowing what angles around a point/on a straight line add to? *If you do a '180' you have turned around to look from one end of a straight line to the other end of the line. If you do a '360' you turn all the way round, so angles around a point add to 360°.*
- What is your unknown? What do you know about the angles in the diagram? Can you set up an equation to show things that are equal? *These are general prompting questions without a specific answer.*

Exercise 9A(F) / 9A(H)

While working through the exercise, good prompts for students might be:

- **Q2(F) / Q2 and 6(H)** How many unknowns do you have? Can you identify which angle will be easiest/which you have enough information to find? *These are general prompting questions without a specific answer. Depending on the angle facts used, students may suggest one angle is easier to find than another. These questions are designed to encourage students to break the problem down into smaller parts.*
- **Q3(F) / Q3(H)** What do we know about angles on a straight line? Why does this contradict the image you have? *Angles on a straight line add to 180° and these angles do not.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Spot the reasoning error. A starter or plenary could be statements in which students are asked to spot the error, for example 'opposite angles are equal' where the writer has forgotten to state '**vertically**'.

Assessment ideas

- Ask students to make 'how to' revision guides, including: rules, language to remember and examples. This makes a good homework assessment task that comes in useful in future revision lessons.

SECTION 2F / 2H: PARALLEL LINES AND ANGLES

Section 2 focuses on the three angle facts based on angles formed from a line transversal to parallel lines: corresponding angles, alternate angles and co-interior angles. Students must use these rules to find missing angles and give correct reasons to support their solutions.

(Higher only) The additional challenge for higher students in this chapter is understanding a proof that co-interior angles are supplementary. In addition to finding missing angles, students also need to use these new rules to decide whether diagrams have parallel lines or not.

Students tend to struggle to recall the facts correctly, particularly given there are three of them. You may find it useful to focus on only one parallel line rule, for example 'alternate angles are equal'. Armed with that and 'vertically opposite angles are equal' and 'angles on a straight line sum to 180°', students are able to correctly reason, with a few more steps, any angle they are trying to find.

Prompting questions

While working on the material in this section good prompts for promoting discussion may be:

- What letters are you looking for to identify angle facts? *Z, C, F shapes to find angles.*
- What are the geometric reasons for these letters? *Alternate angles are equal, corresponding angles are equal and co-interior angles are supplementary.*

- What other angle facts, aside from those, in this section might be useful here? *Angles on a straight line add to 180° and angles around a point add to 360° .*
- How many unknowns do you have? Can you identify which one will be easiest/which you have enough information to find? *Often a good rule of thumb here is to work through the unknowns alphabetically.*

Exercise 9B(F) / 9B(H)

While working through the exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** Can you label any parallel lines? Can you extend any straight lines to make the letters (Z, F, C) you are looking for easier to spot? *Encourage students to label and extend the vertical lines to help identify the angle fact to be used.*
- **Q6-7(F) / Q6-7(H)** Can you point to angle CEG? *Make sure that students know what this notation means and they don't think it just means the angle at E as there are many to choose from (this can be repeated for angle DCF in Q7 as well).*
- **Q9(H)** What must be true for two lines to be parallel? *The three parallel angle facts.*

Starters, plenaries, enrichment and assessment ideas

Assessment ideas

- Again, ask students to make 'how to' revision guides, including: rules, language to remember and examples. This makes a good homework assessment task that comes in useful in future revision lessons.

SECTION 3F / 3H: ANGLES IN TRIANGLES

In *Chapter 5 Properties of shapes and solids*, students learned that the sum of the interior angles in a triangle is 180° . This section offers two proofs for that fact: the sum of angles in a triangle and that the exterior angle of a triangle is equal to the sum of the two opposite interior angles. Students then use these two rules to solve additional angle problems.

Prompting questions

While working on the material in this section good prompts for promoting discussion may be:

- Are there any properties unique to your type of triangle that could be useful? Is it an isosceles, equilateral, right-angled or scalene triangle? *In some triangles we have additional information such as knowing two base angles are equal or all angles are equal, which can help in problem solving.*

Exercise 9C(F) / 9C(H)

While working through the exercise, good prompts for students might be:

- **Q2-5(F) / Q2-5,7-8(H)** How many unknowns do you have? Can you identify which one will be easiest/which you have enough information to find?
- **Q7(H)** What do we need to form a convincing proof? *Logical argument using formal geometric reasoning, clearly labelling our angles and lines.*
- **Q8(H)** Have you got more than one triangle involved in your diagram? How are they connected? *Some are similar so share the same angles, others are just next to each other so other angle facts, for example angles on a straight line add to 180° , can also be used to solve the problem.*

Starters, plenaries, enrichment and assessment ideas

Plenaries

- 'Angles noughts and crosses' works well as a whole class plenary in which the class is split in half to play as two teams against each other. This provides an opportunity for discussion and splitting the class into two teams adds a competitive element to increase motivation. Given simple diagrams, students have to identify the angle and give the correct geometric reason for it. Standard game rules apply, so suggest a tricky one for the centre of the grid. This could also be combined with rules from other sections in this chapter.

Assessment ideas

- Ask students to make ‘How to...’ revision guides, including: rules, language to remember and examples. This makes a good homework assessment task that comes in useful in future revision lessons.

SECTION 4F / 4H: ANGLES IN POLYGONS

Section 4 takes the knowledge about the sum of the interior angles in a triangle and develops it to form a general rule for the sum of the interior angles of any polygon through an investigation (Exercise 9D). This is then extended to derive a rule for the sum of the exterior angles of polygons. The ‘work it out’ section makes a good peer assessment task with prompts for promoting good discussion about common errors students make.

It is worth noting that students often find it very challenging to draw polygons with more than four sides so, if using the investigation in Exercise 9D, having a sheet of pre-printed polygons for students to annotate would be advisable.

As in Section 2, this material is something students tend to find very challenging to remember. To counteract this you may prefer to focus on one rule: that exterior angles sum to 360° . From this students can, with a few extra steps, solve any other polygon angles problem.

Prompting questions

Exercise 9D(F) / 9D(H)

While working through the exercise, good prompts for students might be:

- Q1(F) / Q1(H)** How many triangles are in your polygon? Have you used the minimum number of lines possible? *There should always be two fewer triangles than the number of sides; you may wish to ask students to reason why this must always be the case.*
- Q1(F) / Q1(H)** Have you crossed any of your lines to find the number of triangles? *They shouldn't have or they are adding more angles to the sum (they would need to take away 360° for each internal crossing).*
- Q3-5(F) / Q3-5(H)** What patterns have you spotted? *For example, number of triangles goes up by one each time.*
- Q6(F) / Q6(H)** Can you write your formula down in words? *Number of sides, subtract 2, multiplied by 180° for each triangle and divided by the number of angles in the shape.*
- Q6(F) / Q6(H)** What is the variable of the formula? *The number of sides.*
- Q6(F) / Q6(H)** What stays constant in each calculation you do? *Subtracting 2, multiplying by 180, and so on.*

Exercise 9E(F) / 9E(H)

While working through the exercise, good prompts for students might be:

- What do you know? Can you use any of the formulae you have learned for interior and exterior angles? We can use both for any question but often one will be quicker than the other. *For interior sum use $180(n - 2)$ for exterior angles use $\frac{360}{n}$ (where n is the number of sides).*
- Q3-5(F) / Q2-7(H)** Is the shape regular or irregular? How does that effect how you can solve the problem? *Irregular shapes limit the amount of options we have when solving.*
- Q5(F)** Can you define your unknown using algebra and set up a formula? *Suggest that students sum the angles and equate to the sum of the interior angles, and so on.*
- Q5-7(H)** Can you draw a sketch to help you see what is going on? *This may be required for students to see a way ‘into’ the problem. It may also help by making them consider whether the polygon is regular or not.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- Tessellations make an interesting investigation and students could now think about how angle facts can help them know whether a regular shape tessellates or not.
Here are some interesting additions to the problem:
Semi-regular Tessellations involves tessellating two or more regular polygons together. It also has a very good

interactive program for testing ideas (nrich.maths.org).

Another interesting investigation is to explain why we can tessellate any quadrilateral. See, for example **Simple Quadrilaterals Tessellate the Plane** (cut-the-knot.org).

- For students who enjoy tessellating you may wish to extend them by looking at Penrose tilings, so further consolidating their knowledge of angles.
- This is a good opportunity to explore what shapes you get if you take your interior angle of a regular polygon formula: $\frac{180(n-2)}{n}$ and use non-integer values. What does a 2.5-agon look like? You can use graphing software or traditional protractor and pencil methods to explore these shapes. If you have a particularly able class, you could ask: What does a π -agon look like?

Assessment ideas

- Again, asking students to make ‘how to’ revision guides, including: rules, language to remember and examples. This makes a good homework assessment task that comes in useful in future revision lessons.
- The first question of the chapter review would make a good starter to a revision lesson on this topic if projected on the board or printed out for students to work on.

Prompting questions

While working through the chapter review, good prompts for students might be:

- **Q3-6(F) / Q3-6(H)** Are you dealing with a specific shape? Is it regular or irregular? *This information is given in different ways for different questions and students need to get used to ‘lifting’ this information from a diagram or from the introduction to a question.*
- **Q9(H)** Have you got any parallel lines in your diagram? (Hint: you may have to use properties of shapes to help you here.) Can you extend any of the lines in your diagram to help you spot anything you hadn’t previously? *Encouraging students to extend parallel lines to make the angle facts easier to spot so that they can apply them to problems involving polygons, often makes a big difference to their ability to see what can be done.*
- **Q8(F) / Q8(H)** How many unknowns do you have? Can you identify which one will be easiest/which you have enough information to find? *y is probably the easiest angle to find first using their angle facts.*

Topic links

Previous learning

This chapter connects directly to *Chapter 5 Properties of shapes and solids*, in which students learned about properties of shapes and that the interior angles of a triangle add to 180° . It also uses the concepts learned in *Chapter 8 Equations* regarding the solving of equations, in particular linear ones.

Future learning

Angle facts will be revisited in *Chapter 30F/31H Similarity* and *Chapter 29F/30H Congruent triangles*, in which students will use alternate angles, co-interior angles and corresponding angles to geometrically reason about congruence and similarity. Angle facts will also support problem solving in *Chapter 32F/33H Trigonometry*. In *Chapter 34H Circle theorems*, students will be able to use basic facts, from section 1 and 3, to investigate and deduce new theorems based on angles formed by chords and tangents to circles.

Gateway to A Level

These basic facts are still used in problem solving at A Level. ‘Vertically opposite angles are equal’ and ‘the sum of angles on a straight line is 180° ’ are both used in work on vectors, and numerous basic facts and circle theorems are used in coordinate geometry problems on the (x, y) plane. In addition, students will extend their knowledge of complementary angles in trigonometry and complex numbers in order to solve equations and correctly use Argand diagrams.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 2 Questions 2, 18
- Chapter 3 Questions 2, 10
- Chapter 8 Question 11

Higher

- Chapter 2 Questions 14, 15, 23
- Chapter 3 Questions 2, 3, 10
- Chapter 5 Question 15

Homework Book

Foundation

- Chapter 9

Higher

- Chapter 9

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 10 worksheets (+ solutions)
- 7 animated widgets
- 10 interactive walkthroughs
- 4 auto-marked quickfire quizzes
- 4 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- Angle facts

10F / 10H Fractions

CHAPTER INTRODUCTION

What your students need to know before this chapter

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know the meanings of the words denominator, numerator, common denominator, multiple, factor, equivalent and reciprocal;
- be able to use formal and informal methods and algorithms, both mental and written, for the four operations;
- be able to find the lowest common multiple and highest common factor of a set of numbers;
- know and be able to apply the laws of operations (BIDMAS);
- know that 'of' refers to the operation 'multiply'.

Learning outcomes

Foundation

Section 1

- To apply knowledge of factors and multiples to simplify fractions and identify equivalent fractions.

Section 2

- To apply the four operations to fractions.
- To apply knowledge of the four operations to solving problems involving fractions.

Section 3

- To calculate fractions of amounts.
- To express one number as a fraction of another.

Higher

Section 1

- To apply knowledge of factors and multiples to simplify fractions and identify equivalent fractions.
- To apply and explain an algorithm to find the median fraction.

Section 2

- To apply the four operations to fractions.
- To apply knowledge of the four operations to solving problems involving fractions.

Section 3

- To calculate fractions of amounts.
- To express one number as a fraction of another.

Vocabulary

common denominator, reciprocal

Common misconceptions and other issues

- Students often make the mistake of adding the denominators when adding two fractions. Using the following example to show them why this is incorrect will help them to understand why this doesn't work: $\frac{1}{2} + \frac{1}{2} \neq \frac{2}{4}$. Students should be able to recognise that $\frac{2}{4}$ is equivalent to $\frac{1}{2}$ so they cannot end up with one half when they added two halves together! This misconception is also applied when adding two fractions with different numerators. Students may think that $\frac{2}{5} + \frac{1}{3} = \frac{3}{8}$
- When multiplying two mixed numbers, students might think that they can multiply the integers and multiply the fractions together then find the sum of the result, for example: $2\frac{1}{3} \times 3\frac{1}{2} \times (2 \times 3) + (\frac{1}{3} \times \frac{1}{2}) = 6 + \frac{1}{6} = 6\frac{1}{6}$. Instead, students could think about how they could apply their knowledge of expanding double brackets to assist or convert their fractions to improper fractions before they begin.

Hooks

- **Fractions Countdown.** This is a great way of assessing what students know and, if played throughout the topic, students can see the progress they are making as they become better at it (nrich.maths.org).

SECTION 1F / 1H: EQUIVALENT FRACTIONS

This section introduces equivalent fractions and then builds on this to introduce simplifying fractions.

For Foundation students the first part of the explanation introduces the idea of equivalent fractions by thinking about pieces of pizza. For students who may struggle with the concept of equivalent fractions, it would be useful to have pizzas (or cardboard equivalents) cut into different sized pieces to demonstrate this. Students can then physically work out how many sixths make a third, and so on.

Prompting questions

While working through the exercises, good prompts for promoting discussion might be:

- What types of numbers would lead to a fraction that is impossible to simplify?

Exercise 10A(F) / 10A(H)

While working through the exercise, good prompts for students might be:

- **Q6(F) / Q4(H)** All of the fractions in the question simplify. What properties of the numerators or denominators make this possible? *The numerator and denominator have a common factor.*
- **Q6 (H)** If students are struggling to explain why it works, or with the concept of mediant fractions, then direct them to investigate with fractions that have a numerator of 1. This will help them to generalise much more easily.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- For students who may struggle with visualising equivalent fractions, a fraction wall can be very useful. Students can create their own (using squared paper) or there is an **interactive version** (visnos.com).

Enrichment activities

- Odd one out. Give students a collection of fractions where all but one of them are equivalent. Students should justify their answer and explain it mathematically.
- Fractions and music. Students can use the link between music and fractions to contextualise equivalent fractions. They can use the facts below to create their own clapping rhythm. For example, students could be given four bars to fill with notes. They can create their clapping rhythm using this idea, ensuring that they have used all four bars.

Musical fractions

	Whole	
	Half	
	Quarter	
	Eighth	
	Sixteenth	

SECTION 2F / 2H: OPERATIONS WITH FRACTIONS

This section introduces adding, subtracting, multiplying and dividing fractions.

Prompting questions

While working through the exercises, good prompts for promoting discussion might be:

- What is wrong and what is right about the following calculation: $3\frac{1}{3} \times 9\frac{2}{5} = (3 \times 9) \times 9\frac{2}{5} = (3 \times 9) + (\frac{1}{3} \times \frac{2}{5}) = 27 + \frac{2}{15} = 27\frac{2}{15}$? *Half of the required calculations have been completed. However, the following two calculations are needed: $\frac{1}{3} \times 9$ and $3 \times \frac{2}{5}$. So the final answer is $27 + \frac{2}{15} + 3 + \frac{6}{5} = 21\frac{1}{3}$.*
- When dividing fractions, why is it possible to multiply the first fraction by the reciprocal of the second? *Try searching the internet for 'Why does keep change flip work?' for videos to explain this concept.*

Exercise 10B(F) / 10B(H)

While working through the exercise, good prompts for students might be:

- Q2(F) / Q1(H)** Why might it be better to convert mixed numbers to improper fractions before multiplying or dividing them? *There are fewer calculations to do, for example just multiply the numerators and then denominators.*
- Q7(F) / Q5(H)** In what order should you do the calculations? Why is it important to apply the laws of BIDMAS? *Students should apply the laws of BIDMAS so that they get the correct answer.*

Starters, plenaries, enrichment and assessment ideas

Starters

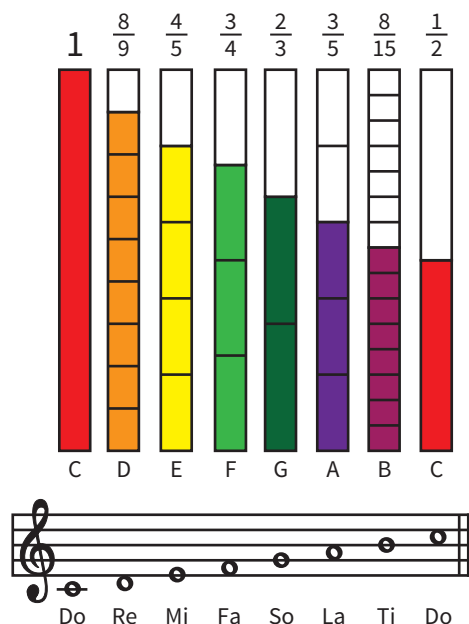
- (Foundation only)** This section starts with a useful activity that provides a visual way of demonstrating how to multiply fractions. This idea can be turned into a kinaesthetic activity with the use of some paper and colouring pens. The PDF '**Model the concept of multiplying fractions through paper folding**' explains the activity very well (schoolonwheels.org).

Starters or plenaries

- Number Loving Adding and Comparing Fractions Top Trumps.** A game for students to play. Most students know how to play top trumps. This particular activity instructs students to compare and add fractions (numberloving.com).

Enrichment activities

- Improving Learning in Mathematics **Mostly Shape and Space – SS3 Dissecting a square** is available from the National STEM Centre Archive (nationalstemcentre.org.uk).
- NRICH Fractions Jigsaw.** This activity is an alternative way to practise the four rules with fractions rather than use a traditional textbook exercise (nrich.maths.org).
- NRICH Peaches Today, Peaches Tomorrow.** An excellent activity to practise finding fractions of amounts (nrich.maths.org).
- Music and fractions. Students can create their own musical instruments and practise multiplying fractions at the same time. Students will need eight glass containers that are the same size. Students will need to calculate how much liquid to put into each container based on the fractions below. Students could then play simple music with their creations!



- **Number Loving Adding Fraction Mystery.** An activity where the answer is not as easy as just adding two fractions together. Students will need to sort through the clues and work out how much money a person has after paying for bills, and so on. This is also a nice link to financial education (numberloving.com).

SECTION 3F FRACTIONS OF QUANTITIES / 3H: FINDING FRACTIONS OF A QUANTITY

This section focuses on calculating fractions of quantities and writing a quantity as a fraction of another.

Exercise 10C(F) / 10C(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Which operation should you choose when you see the word 'of'? *Multiply.*
- **Q3(F) / Q3(H)** Why is it important to have both parts of a question expressed in the same unit? *Fractions compare proportions. These proportions must be the same in order to compare correctly.*

Exercise 10D(H)

While working through the exercise, good prompts for students might be:

- NRICH have a great interactive activity, **Egyptian Fractions**. There are also accompanying teacher notes and solutions that might serve as a prompt or as extensions to the textbook questions (nrich.maths.org).

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- The first explanation introduces four statements to students that involve fractions. To build on this idea students could be shown a range of images from supermarkets where the calculations are incorrect. Students can then discuss why they are incorrect. Here is a simple example:



Enrichment activities

- ‘Would you rather’ activities. Pose questions such as: ‘Would you rather two thirds of three quarters or five sixths of four fifths?’ and so on.
- Make a card matching or a Tarsia puzzle. This can be done using **free software** (mmlsoft.com). There are many ready-made examples available on the internet and the **TES website** has a good collection to choose from (tes.co.uk).
- **NRICH Ben’s Game**. This is an excellent fractions based activity that requires students to record their working effectively. It also covers factors and multiples as well as calculations with fractions (nrich.maths.org).

Topic links

This topic links with almost every topic at GCSE and beyond. In *Chapter 7 Further algebraic expressions*, students needed to apply the four rules of fractions to simplifying expressions involving unknowns and then, in *Chapter 8 Equations*, they went on to solve equations involving numerical and algebraic fractions. When solving problems involving vectors in *Chapter 27F/28H Plane vector geometry*, students will need to be able to simplify fractions and in *Chapter 27H Surds* they will learn how to rationalise denominators of fractions. It is usual to express probabilities as fractions and students will meet this in *Chapter 23 Basic probability and experiments* and *Chapter 24 Combined events and probability diagrams*. A good understanding of fractions is also required for work with decimals (*Chapter 11 Decimals*), percentages (*Chapter 13 Percentages*), ratio (*Chapter 22 Calculations with ratio*) and substitution into formulae (*Chapter 14 Algebraic formulae*).

Previous learning

To work successfully with fractions, students need to have good basic number skills and be able to add, subtract, multiply and divide integers competently. These skills were practised in *Chapter 1 Basic calculation skills*. They also need to understand what is meant by a factor, a multiple, a prime number and the lowest common multiple of two or more numbers. These were covered in *Chapter 2 Whole number theory*.

Future learning

Fractions will be used for solving a variety of problems and calculations in most topics, especially when use of a calculator is not allowed. For example, treating a division as a fraction can enable the numbers to be simplified so as to make the calculation easier.

Gateway to A Level

Fractions feature in most A Level topics, in particular in work on algebraic fractions, functions, vectors, indices and surds. Also in trigonometry, angles will be measured in radians as fractions of π . When a calculator is not allowed, it can be easier to treat division as a fraction calculation in order to simplify the numbers by cancelling. When exact answers to a calculation are required, it is better to give them as fractions.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 2 Question 19
- Chapter 3 Question 11
- Chapter 5 Questions 1, 2, 3
- Chapter 7 Questions 1, 18

Higher

- Chapter 2 Question 16
- Chapter 3 Question 4
- Chapter 5 Question 2
- Chapter 7 Question 8
- Chapter 9 Question 5

Homework Book

Foundation

- Chapter 10

Higher

- Chapter 10

GCSE Mathematics Online

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- 3 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

11F / 11H Decimals

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- have a concrete understanding of place value;
- know how to multiply and divide by powers of 10;
- know how to compare decimals and arrange them in order of size;
- recognise the fractional equivalents of common decimals;
- be able to use formal or informal methods to divide numbers that lead to decimals;
- understand how to round numbers to various significant figures or decimal places.

Additional useful prior knowledge

- How to compare and order integers.
- How to complete the four rules with integers.
- Place value and the concept of using tenths, hundredths, thousands, and so on, to describe parts of a whole.
- How to carry out calculations involving money.

Learning outcomes

Foundation

Section 1

- To apply knowledge of place value to convert between decimals and fractions and order fractions and decimals.

Section 2

- To apply knowledge of rounding to estimate answers to calculations that involve decimals.
- To be able to add, subtract, multiply and divide decimals.
- To use a calculator to complete more complicated calculations that involve decimals.

Higher

Section 1

- To apply knowledge of place value to convert between decimals and fractions and order fractions and decimals.

Section 2

- To be able to add, subtract multiply and divide decimals.
- To use a calculator to complete more complicated calculations that involve decimals.

Section 3

- To convert recurring decimals to fractions.

Common misconceptions and other issues

- Students may believe that when multiplying by 10 you simply add a zero (similarly taking away a zero for dividing). This may lead to some problems when multiplying and dividing decimals, for example 1.56×10 becomes 1.560.
- Students may find it challenging to divide a number by a decimal less than one. They may not understand the reason why it makes the number larger. It might be useful to ask the student to convert the decimal to a fraction and then complete the calculation so that they see why the number gets larger.
- When estimating calculations, students may round decimals that are less than one to 1. For example, if a student was estimating the calculation $345 \div 0.79$ to $300 \div 1$ instead of $300 \div 0.8$.

Hooks

Search for 'The decimal song' on the internet. This is a fun way of introducing what a decimal is while emphasising the importance of place value.

SECTION 1F / 1H: REVISION OF DECIMALS AND FRACTIONS

This section reviews students' understanding of decimals and fractions.

Prompting questions

Exercise 11A(F) / 11A(H)

While working through the exercise, good prompts for students might be:

- **Q3(F) / Q3(H)** How could you use your prior knowledge of $\frac{1}{3} = 0.\dot{3}$ to work out what $\frac{1}{9}$ would be as a decimal? $\frac{1}{3} \div 3$
- $= \frac{1}{9}$ so dividing $0.\dot{3}$ by 3 would give the decimal equivalent of $\frac{1}{9}$.
- **Q4(F) / Q4(H)** Why is it useful to make all decimals the same length? Which column should be compared first?

Starters, plenaries, enrichment and assessment ideas

Starters

- To help students to visualise the decimals when comparing them in Exercise 11A, try using a **Zoomable Number Line** (mathsisfun.com).

Enrichment activities

- Improving Learning in Mathematics **Mostly Number – N1 Ordering Fractions and Decimals** is available from the National STEM Centre (nationalstemcentre.org.uk).
- **Converting fractions to decimals game**. Students can play against other students who are also logged in (mathplayground.com).

SECTION 2F / 2H: CALCULATING WITH DECIMALS

Section 2 introduces adding, subtracting, multiplying and dividing decimals.

Prompting questions

Exercise 11B(F)

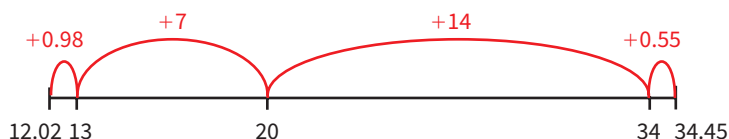
While working through the exercise, good prompts for students might be:

- **Q1-3(F)** Ask students how they decided on how many significant figures to round their numbers to and why.
- **Q1-3(F)** Students could compare their results for each question and decide whose estimate was better.

Exercise 11C(F) / 11B(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Students should round their values to 1 significant figure to estimate.
- **Q2(F) / Q2(H)** This method assumes that students can use the bus stop method. They may struggle to line their work up or lack the basic understanding of where the decimal point should be. Instead, they might like to think of the questions as a fraction. For example write $24 \div 0.6$ as $\frac{24}{0.6}$ and then find an equivalent fraction where the denominator is an integer by multiplying by $\frac{10}{10}$. This gives $\frac{24}{0.6} \times \frac{10}{10} = \frac{240}{6} = 240 \div 6 = 40$.
- **Q2(F) / Q2(H)** The chapter focuses on the long method for addition and subtraction. Some students may prefer to use the number line method for both of these. For subtraction, this is called the difference method and can be seen below:



- **Q9(F) / Q9(H)** Explain how to use the fraction button on your calculator to complete this calculation. If you aren't using a calculator why might it be best to convert $\frac{2}{15}$ into a decimal?

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- The book describes multiplying and dividing decimals by 'moving the decimal place' or placing it back in. If students are struggling to comprehend this they may find it useful to use a place value grid to exemplify why the numbers move. For example, if a number becomes 100 times bigger each digit moves two columns to the left.

Multiplication

Th	H	T	u	t	h	th	
thousands	hundreds	tens	units	tenths	hundredths	thousandths	
		2	8	3	4		$\times 10$
	2	8	3	4			
		3	1	0	6	9	$\times 100$
3	1	0	6	9			

The decimal point does NOT move. The numbers move to the left in multiplication.

Enrichment activities

- Gelosia method of multiplying decimals. Search the internet for 'Lattice (Chinese) multiplication with decimals tutorial'. This can be used as a procedure or students can begin to explain why it works.
- **NRICH Does This Sound About Right?** This activity introduces a series of statements for students to investigate by estimating the calculations (nrich.maths.org).
- **Stick on the Maths.** Understanding the effect of multiplying and dividing by numbers between 0 and 1 (Level 7, CALC 2) (kangaroomaths.com).

Plenaries

- Take-away menu maths. Give students a local takeaway menu (preferably one with set menus). Students can work out whether or not the set menus offer good value for money. They can then create their own and describe the savings.

SECTION 3H: CONVERTING RECURRING DECIMALS TO EXACT FRACTIONS

This section focuses on converting recurring decimals to fractions.

Prompting questions

Exercise 11C(H)

While working through the exercise, good prompts for students might be:

- **Q2(H)** What do you notice about the decimal $\frac{3}{6}$? What might you need to do to this fraction in order to make Nazeem's theory correct? $\frac{3}{6}$ is not simplified. All fractions must be written in their simplest form.

Exercise 11D(H)

While working through the exercise, good prompts for students might be:

- **Q2(H)** All of the examples use decimals where all of the digits after the decimal point recur. In that case, it is easy to see how the decimal relates to the fraction. What about if not all of the digits recur? For example, how does $0.5\dot{6}$ relate to the fraction now? *If all of the digits recur then the denominator will be 9 if one digit recurs, 99 if two digits recur and so on. If only the second digit recurs then the denominator will be 90 and so on.*
- **Q3(H)** What is the difference between the decimals $0.\dot{2}\dot{1}$ and $0.2\dot{1}$? How is the procedure different for converting these decimals to fractions? Why?

$0.\dot{2}\dot{1}$ is $0.2121212121\dots$ and $0.2\dot{1}$ is $0.211111111111\dots$ Students will have to find a different multiple of x for each question. For $0.\dot{2}\dot{1}$ it'll be $100x - x$ and for $0.2\dot{1}$ it'll be $100x - 10x$

Starters, plenaries, enrichment and assessment ideas

Starters

- Students might be asked to prove that a recurring decimal converts to an exact fraction. It is important to point out some algebraic conventions, for example that x is a variable and that it is important to state 'let $x =$ ' and so on.

Enrichment activities

- **NRICH Repetitiously.** This task will introduce converting recurring decimals to fractions (nrich.maths.org).
- **Stick on the Maths.** Understand the equivalence between recurring decimals and fractions (Level 8, NNS1) (kangaroomaths.com).

Topic links

Decimal fractions are part of almost all topics at GCSE and beyond. There is obviously a strong link between fractions, percentages and ratio.

Previous learning

Students should be competent at applying the four rules of calculation with integers, be able to order integers and have an understanding of place value. They should be familiar with decimals from money calculations and work on metric units.

Future learning

Students will need to be able to carry out calculations involving decimals in most topics. In particular, decimals multipliers are a useful way of calculating the value after a percentage change (*Chapter 13 Percentages*) and most calculations involving metric units (*Chapter 12 Units and measurement*), perimeter (*Chapter 15 Perimeter*), area (*Chapter 16 Area*) and volume (*Chapter 21 Volume and surface area*) will use decimal values. When using a calculator, students will need to be able to round decimal answers to a suitable degree of accuracy, for example two decimal places or three significant figures, and this is dealt with in *Chapter 17 Approximation and estimation*. The use of lower and upper bounds is also covered in this chapter.

Gateway to A Level

Decimals are used in any calculations set in context, for example when using the kinematics formulae in mechanics or when finding the magnitude of forces. They are used extensively in statistics when analysing data. Numerical methods are used to achieve approximate solutions to equations that cannot be solved by direct methods and these may be given as decimals to, for example, three significant figures.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 6 Question 2
- Chapter 8 Question 12
- Chapter 10 Question 14

Higher

- Chapter 2 Question 17
- Chapter 5 Question 16
- Chapter 8 Question 17
- Chapter 9 Question 15
- Chapter 10 Question 10

Homework Book

Foundation

- Chapter 11

Higher

- Chapter 11

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 5 (F) / 6 (H) worksheets (+ solutions)
- 9 (F) / 10 (H) animated widgets
- 9 (F) / 15 (H) interactive walkthroughs
- 2 (F) / 3 (H) auto-marked quickfire quizzes
- 2 (F) / 3 (H) auto-marked question sets, each with four levels
- Auto-marked chapter quiz

12F / 12H Units and measurement

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to multiply and divide by powers of 10;
- understand index notation, for example $10^2 = 10 \times 10$;
- be able to use inequalities appropriately in mathematical statements (**Higher only**);
- know how to find the volume of a cuboid;
- be able to measure accurately using a ruler and protractor.

Additional useful prior knowledge

- Non-calculator methods for multiplying decimals.
- Calculations involving related sums.
- Knowledge of rounding to 1 or 2 significant figures.
- Rounding to estimate solutions to problems.
- Work with equivalent fractions including cancelling to simplest form.

Learning outcomes

Foundation

Section 1

- To be able to convert metric units for capacity, mass and length.
- To be able to convert metric units of area and volume.
- To understand units of time are not metric.
- To be able to convert units of time and solve related problems.
- To be able to convert currencies using scale factors.

Section 2

- To be able to convert compound measurements.
- To be able to use formulae for compound units:
 $Speed = \frac{Distance}{Time}$, $Density = \frac{Mass}{Volume}$, $Pressure = \frac{Force}{Area}$ and
 to find any one of the variables given values for the other two.

Higher

Section 1

- To be able to convert metric units for capacity, mass and length.
- To be able to convert metric units of area and volume.
- To understand units of time are not metric.
- To be able to convert units of time and use to solve related problems.
- To be able to convert currencies using scale factors.

Section 2

- To be able to convert compound measurements.
- To be able to use formulae for compound units:
 $Speed = \frac{Distance}{Time}$, $Density = \frac{Mass}{Volume}$, $Pressure = \frac{Force}{Area}$
 and to find any one of the variables given values for the other two.

Section 3

- To be able to read and use scales on maps including both line/bar scales and ratio scales.
- To be able to form scales to construct scale drawings to fit a given dimension.
- To be able to read and use bearings in scale drawings.

Section 3

- To be able to read and use scales on maps including both line/bar scales and ratio scales.
- To be able to form scales to construct scale drawings to fit a given dimension.
- To be able to read and use bearings in scale drawings.
- To understand the connection between a bearing of B from A and A from B on a given line segment.

Vocabulary

conversion factor, exchange rate, scale factor

Common misconceptions and other issues

- Students have problems multiplying and dividing decimals by powers of 10. This is due to the misconception that $\times 10$ means you add a zero onto the end of the number (and other similar products with powers of ten). For students who have this issue it would be useful to revisit multiplying and dividing by powers of 10 with a place value grid.
- Students often perform calculations with values in different units for example centimetres and metres. To combat this, remind students that all units must be the same and try to stress the importance of estimation and checking to see whether their answer seems sensible.
- When using scale factors for working with 2D and 3D shapes, students do not always realise that they need to be working with scale factors squared and cubed for area and volume calculations. Ask them to revisit the original values in their calculations and convert the lengths involved to squared or cubed units, for example:

$$\begin{array}{ccc} 2\text{ m} & \boxed{12\text{ m}^2} & \longrightarrow & 200\text{ cm} & \boxed{120000\text{ cm}^2} \\ & 6\text{ m} & & & 600\text{ cm} \end{array}$$

to force the acknowledgement of an error in their solution. When introducing this topic you could use overlays of grids to turn 1 cm^2 into 100 mm^2 and consider the related sums as used in the section notes of the textbook, for example:

$$\begin{array}{ccccc} 2\text{ cm} & \times & 6\text{ cm} & = & 12\text{ cm}^2 \\ \downarrow \times 10 & & \downarrow \times 10 & & \downarrow \times 10^2 = 100 \\ 20\text{ mm} & \times & 60\text{ mm} & = & 1200\text{ mm}^2 \end{array}$$

You can also use the following table when making notes with students on this topic:

Length	Area	Volume
$\times 10 \left(\begin{array}{c} \text{mm} \\ \text{cm} \end{array} \right) \div 10$	$\times 10^2 \left(\begin{array}{c} \text{mm}^2 \\ \text{cm}^2 \end{array} \right) \div 10^2$	$\times 10^3 \left(\begin{array}{c} \text{mm}^3 \\ \text{cm}^3 \end{array} \right) \div 10^3$
$\times 100 \left(\begin{array}{c} \text{cm} \\ \text{m} \end{array} \right) \div 100$	$\times 100^2 \left(\begin{array}{c} \text{cm}^2 \\ \text{m}^2 \end{array} \right) \div 100^2$	$\times 100^3 \left(\begin{array}{c} \text{cm}^3 \\ \text{m}^3 \end{array} \right) \div 100^3$
$\times 1000 \left(\begin{array}{c} \text{m} \\ \text{km} \end{array} \right) \div 1000$	$\times 1000^2 \left(\begin{array}{c} \text{m}^2 \\ \text{km}^2 \end{array} \right) \div 1000^2$	$\times 1000^3 \left(\begin{array}{c} \text{m}^3 \\ \text{km}^3 \end{array} \right) \div 1000^3$

- Students multiply when they should divide or vice versa when converting units. To encourage the correct decision, focus on the size of the units and whether you would expect there to be more or fewer of them. For example centimetres are smaller than metres so you would expect there to be more of them, so $5\text{ m} \rightarrow \text{cm}$ needs $\times 100$ not $\div 100$. $\rightarrow \text{cm}$ needs $\div 100$ not $\times 100$.
- Students struggle to remember which power of 10 they need to use. This is because they confuse how many centimetres there are in a metre and how many millimetres there are in a centimetre and so on for the most common conversions. Try to give students a point of reference for measurements of length, for example, using their rulers to check cm to mm or knowing that a metre is roughly the length of an exam desk so must be more than 10 cm. You could also focus on the meaning of the prefixes (e.g. *kilo* is linked to the Greek word for 1000, *centi* is connected to the Greek word for 100) and connect to other words students are familiar with, such as century. Asking students to make a poster or laminated A5 revision card for them to keep could also be successful.

- Students treat time like they would the rest of the metric system and hence write 3.5 hours = 3 hours 50 mins or similarly 3 hours 30 mins as 3.3 hours. Returning to basics and asking students to convert 0.5 hours or 'half an hour' into minutes can help them identify this as well as considering simple fractions of an hour such as $\frac{1}{10}$, $\frac{1}{3}$, $\frac{1}{6}$ and so on.
- Students struggle to convert currencies where they are working 'backwards', in other words where they are told £1 = \$1.70 and are asked to convert \$500 into pounds. Combat this by using ratio conversion notation to remind them which way they are going, not always formally writing the division underneath but always reinforcing the question: What am I multiplying by? For example:

$$\begin{array}{c}
 \text{£1 : \$1.70} \\
 \times ? \quad \text{£300 : \$510} \quad \times 300 \\
 \text{£300} \div \text{£1} = 300 \\
 \text{so } \times 300 \\
 \\
 \text{£1 : \$1.70} \\
 \times 294.18 \quad \text{\$294.18 : \$500} \quad \times ? \\
 \$500 \div \$1.70 = 294.1176 \dots
 \end{array}$$

- Students struggle to work with rearranged forms of $S = \frac{D}{T}$ (and similar formulae for density and pressure). A good way to deal with this is to work with the formula triangle, in which students cover up the variable they are looking for (see Section 2 notes for more details).
- Students confuse the units when working with compound measures and don't know what their units of their value are. Persistently reinforcing that the units for the compound measurement are the units given in the values being used helps with this.
- Students confuse multiplication and division of the scale factor when working with scale drawings. For advice on how to deal with this see the point above on converting currencies.
- Students forget what a bearing is and how it compares to a given angle. Reminding students that bearings always start from North and must always contain three digits (perhaps as a fill-in-the-gaps definitions starter, for example, 'Angles that are always measured from North are called _____' and so on) is often enough to deal with this issue.

Hooks

For all the material covered in this chapter students are often most engaged when it is personal to them or topical. For example, when looking at scale drawings you could focus on maps of the area where they live or, when considering compound measures, you could look at the formula for speed, distance and time by focusing on recent sporting footage or reminisce on the London 2012 games.

SECTION 1F / 1H: STANDARD UNITS OF MEASUREMENT

Section 1 covers various units of measurement including length, capacity, mass and time. The first part considers the different metric units of measurement for length, capacity and mass and connections in the names used before considering the most common conversions required. Having looked at one-dimensional units the section then goes on to consider how these conversion rates are applied to area and volume problems through the use of related calculations, which could be supported with specific examples students could investigate to deduce their own rules for these conversions.

The second part starts by covering time and the various forms it can be written in (decimalised and using standard units). There is another list of conversion facts for students to learn and it is worth considering how they are all connected for example, $1 \text{ hour} = 1 \times 60 \times 60 = 3600 \text{ secs}$ as you go via minutes rather than straight to seconds. There are also opportunities to explore informal non-calculator methods for converting decimalised hours to hours and minutes using fractions (tenths). The penultimate part of this section considers the differences between the 12 hour clock and 24 hour clock. In the final section of the second part, unitary conversion methods for money are looked at in the context of exchange rates.

Prompting questions

Exercise 12A(F) / 12A(H)

While working through the exercise, good prompts for students might be:

- **Q3-4(F) / Q3-4(H)** What has to be the same for you to perform calculations with the measurements? *The units.*
- **Q4(F) / Q4(H)** Why might a sketch be helpful? *Because then I am holding fewer facts in my head.*
- **Q5(F) / Q5-6(H)** What do you have to remember to do to the conversion factor if you are dealing with area? *Square the conversion factor.*
- **Q5(F) / Q5-6(H)** What do you have to remember to do to the conversion factor if you are dealing with volume? *Cube the conversion factor.*
- **Q6(H)** What needs to be the same for me to compare the values? *The units.*
- **Q7(H)** Which part of the diagram would be 170 mm deep? *The sink only, not the draining board as well.*

Exercise 12B(F) / 12B(H)

While working on the material in this section good prompts for promoting discussion may be:

- Use the questions from the problem-solving framework in the Student Book:
 - What have you got to do?
 - What information do you need?
 - What information don't you need? (This is a hard question for weaker students as they often think everything they are told is relevant and may not be able to generalise to deal with the wider problem.)
 - What maths can you do? (Worth elaborating on this question to ask what information they can use first.)
 - Have you used all the information? (This could be a tough question for those incapable of identifying what information is irrelevant.)
 - Is it correct? (Worth changing to: Is it in the range of answers you were expecting?/Is it close to your estimate?)
- While working through the exercise, good prompts for students might be:
- **Q2(F) / Q2(H)** What time facts do you need? *Days in week etc.*
- **Q4(F) / Q4(H)** What do you need to ensure is the same when working with measurements? *The units.*
- **Q5(F) / Q5(H)** Which exchange rate do you need? *The one for the currency in the question.*
- **Q5(F) / Q5(H)** Are you converting using the rate given or working backwards to convert? How does this make a difference? *It changes whether you are multiplying or dividing.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- The skills in this section can be combined with measuring and accuracy with a series of tasks related to measuring objects in the classroom and converting to the units required in the question, for example the length of the desk or mass of pencil case.
- Card sorts can be used effectively with this topic to support weaker students and identify common misconceptions. Additional challenge can be added by removing some of the pairs and giving students blank cards to fill in.
- There is a series of 'We can work it out' tasks that are appropriate for the material on time available from **Association of Teachers of Mathematics** (atm.org.uk). The tasks could be replicated easily as they are a series of information and question cards for students to work through where some information is useful and some is irrelevant. Most are concerned with trips and include transport times and costs within the calculations.

Starters or plenaries

- An appropriate starter or plenary for this topic would be a series of short questions based on something personal to the students such as their bus timetable or school day timings.

SECTION 2F / 2H: COMPOUND UNITS OF MEASUREMENT

Section 2 introduces students to compound measures and considers a range of scenarios in which compound measures would be used, starting with money. After this, speed, density and pressure are considered in detail and there are step-by-step examples on converting these compound measures.

Prompting questions

Exercise 12C(F) / 12C(H)

While working on the material in this section good prompts for promoting discussion may be:

- What information is required to solve this problem? *What do I know? What do I need?*
- While working through the exercise, good prompts for students might be:
- **Q2(F) / Q2(H)** Is there any information that is irrelevant to this problem? *The number of hours Henry works in the week is unimportant.*
- What compound measure can I form from the information given in the question? How can I convert it to the one required for the answer?
- **Q4-6(F) / Q4-8(H)** Which variable are you looking for, speed, distance or time? How can a formula triangle assist in working out what calculation you need to do?
- **Q9(H)** What do you think the different parts of the time stamp are in units? *It's a marathon so hours, minutes and seconds.*
- **Q10(H)** What do the units have to be for you to work with the measurements? *The same units so if the compound measure relates to hours the times the car travelled for must be converted to parts of an hour.*

Exercise 12D(F) / 12D(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** What is the volume of each block? 1cm^3 – in opening line of question.
- **Q2(F) / Q2(H)** Do I have the volume of the cube? *No, just the side length measurement so I need to find the volume first.*
- **Q2-4(F) / Q2-4(H)** What do you need to ensure is the same when working with measurements? *The units.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- **Escalator.** A challenging problem that will require students to set up and solve equations based on proportional speeds (or if they can spot a trick, a simple idea based on proportion) (nrich.maths.org).
- **All in a jumble.** A card sort (electronic version included) that explores appropriate measurements for different scenarios. There are a few imperial units involved but these could either be removed or used as an extension (nrich.maths.org).
- **Olympic Measures.** This a very similar idea with an Olympics theme (nrich.maths.org).
- **Triathlon and Fitness.** This a simple exercise using compound measures that requires simple conversions of time and distance (nrich.maths.org).

SECTION 3F / 3H: MAPS, SCALE DRAWINGS AND BEARINGS

Section 3 covers reading and using scale drawings as well as bearings. Students are required to read and use scales given in both bar and ratio form as well as form appropriate scales for their own drawings. As in the previous sections students will be required to convert metric units. The final part of this section introduces students to bearings and uses them in scale drawings. They will be expected to apply the fact that, for any given line segment AB, the bearing of B from A is connected to the bearing of A from B by $\pm 180^\circ$ (**Higher only**).

Prompting questions

Exercise 12E(F) / 12E(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** What type of scale are you dealing with, bar or ratio? How does this change how you deal with the question? *Bar will require measuring.*
- **Q2(F) / Q2(H)** What units should your final answer be in? Do you need to compute any conversions to achieve this? *All answers should be given in kilometres so some conversion may be required.*
- **Q3(F) / Q3(H)** If you are unsure which will be the larger map what could you do to test it? *Test a length and convert to see.*
- **Q5(F) / Q5(H)** What formula is needed here to find the average speed? $S = D/T$.
- **Q7-8(F) / Q7-8(H)**: Lay out your conversions as a ratio with an unknown (see example above in misconceptions section). Do you need to multiply or divide by the scale factor?

Exercise 12F(F) / 12F(H)

While working through the exercise, good prompts for students might be:

- **Q2-3(F) / Q2-3(H)** How could we describe a 'clumsy' scale? *One with large numbers that are hard to multiply/divide by or numbers that are not whole.*
- **Q2-3(F) / Q2-3(H)** How do we 'correct' a clumsy scale? *Round up to 1 or 2 significant figures.*
- **Q2-3(F) / Q2-3(H)** What do you need to include in all scale drawings? *A scale.*
- **Q3(F) / Q3(H)** To produce a new scale drawing to fit the dimensions of your exercise book what do you need to know? *The actual dimensions of the classroom block – suggest students draw a sketch in their books first to note any dimensions needed.*

Exercise 12G(F) / 12G(H)

While working through the exercise, good prompts for students might be:

- **Q1-3(F) / Q1-5(H)** Where is a bearing always read from? *North.*
- **Q1-3(F) / Q1-5(H)** How many digits should a bearing always have? *Three.*
- **Q4(H)** Can you sketch the bearings given and see if you can identify how the second bearing you are looking for is connected to the first? *Suggest students extend the lines to see the parallel lines.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- An opportunity to revisit *Chapter 6 Construction and loci* through scale drawings is given in this NRICH task **Where am I?** in which students have to find out where in the UK the market square is, given the clues (nrich.maths.org). You could give students a variety of maps to work with to give them practice of different scales; you may even wish to give them a map with no scale and a single distance from which they can calculate the scale.
- (Main activity) Students are given a plan on an A4 sheet of paper and are required to create this plan on a field with the correct dimensions using cones as markers. Alternatively, students are required to make scale drawings of the classroom or rooms in their own homes. Students can then share and compare their plans to assess if the correct scales have been used. For example, if one student has a table that takes up twice as much space as another student's table ask, 'Why is that?' Are they using different scales or has something gone wrong?
- Another personalised way for students to work with scale drawings is to supply them with a series of maps of their local area with different scales. They can then compare the scales given for an area they know and gain an understanding of why scale drawings are needed.
- Students could investigate and form their own conclusions about bearings on line segments and revisit angle facts, from *Chapter 9 Angles*, to create proofs of the patterns they spot.

Starters or plenaries

- Students could be given a starting point on a blank sheet of paper and asked to complete a series of movements according to instructions that involve bearings. The students can then 'connect the dots' to form a picture that could be self-checked using a transparent overlay.

Topic links

Previous learning

There is scope to revisit *Chapter 6 Constructions and Loci* by working on a series of loci problems where the image is drawn to scale. In addition, angle facts based on parallel lines, from *Chapter 9 Angles*, can be explored again in this section with bearings.

Future learning

Conversion of metric units will be required in *Chapter 15 Perimeter*, *Chapter 16 Area* and *Chapter 21 Volume and surface area*.

Students will use what they have learnt about average speed again in *Chapter 39 Interpreting graphs*.

Scale factors will come up again in *Chapter 30F/31H Similarity*.

There are opportunities to check students' accuracy of measurement by revisiting some of the final problems of the chapter (and similar scale problems) by applying new conceptual knowledge relating to right-angled triangles.

Gateway to A Level

There are few direct links to this topic in the core part of the A Level course but an understanding of scale and accuracy will be needed for applied problems as well as an ability to convert units fluently. However, in mechanics students continue to work with compound measures and build upon their knowledge of speed, velocity and pressure.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 2 Question 8
- Chapter 5 Questions 4, 7
- Chapter 7 Questions 2, 10, 11
- Chapter 8 Question 21
- Chapter 10 Questions 2, 3, 15

Higher

- Chapter 5 Questions 3, 7
- Chapter 7 Question 1
- Chapter 8 Question 18

Homework Book

Foundation

- Chapter 12

Higher

- Chapter 12

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 10 worksheets (+ solutions)
- 4 animated widgets
- 13 interactive walkthroughs
- 5 auto-marked quickfire quizzes
- 5 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

13F / 13H Percentages

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to multiply and divide by powers of 10;
- know how to express a fraction in its simplest form;
- know how to express a percentage as a decimal.

Additional useful prior knowledge

- That 'percent' means 'out of 100'.
- Some basic fraction, decimal and percentage equivalents, for example $\frac{1}{2} = 0.5 = 50\%$.
- How to find a fraction of an amount.

Learning outcomes

Foundation

Section 1

- To be able to convert between fractions, decimals and percentages.

Section 2

- To use fractions, multipliers or calculators to work out percentages of amounts.
- To be able to express a quantity as a percentage of another.

Section 3

- To calculate percentage increase or decrease.
- To calculate the original amount given the value after an increase or decrease.

Higher

Section 1

- To be able to convert between fractions, decimals and percentages.

Section 2

- To use fractions, multipliers or calculators to calculate percentages of amounts.
- To be able to express a quantity as a percentage of another.

Section 3

- To calculate percentage increase or decrease.
- To calculate the original amount given the value after an increase or decrease.

Common misconceptions and other issues

- Students may believe that when multiplying by 10 you simply add a zero (similarly taking away a zero for dividing). This may lead to some problems when converting a percentage into a decimal and vice versa.
- Since dividing by 10 is the correct calculation for finding 10% students might think that this works for any percentage, for example, to find 20% divide by 20. This could be because students fail to see a percentage as a part of 100. They might not understand that they would have to divide the quantity by 5 because there are five 20s in 100.
- When writing a quantity as a percentage of another, students may fail to see that the quantities are in different units.
- When calculating percentage increase or writing a quantity as a percentage of another, students might fail to understand that a percentage can be higher than 100%. For example, when writing 80 as a percentage of 50 students may not understand the answer of 160%. Instead, they might calculate 50 as a percentage of 80 and get 62.5%.
- Students may misunderstand how to calculate a decrease followed by another decrease. For example, they might think that if you receive 50% off followed by another 50% you will not have to pay for the item!
- Students may think that because 100 reduced by 25% is 75 that 75 increased by 25% is 100. They may fail to see that 25% of 100 is not the same as 25% of 75.

- Students may know that if 100 is reduced by 20% that takes it down to 80. However, they may then believe that to get back from 80 to 100 you merely add on 20% rather than the 25% it should be.

Hooks

Introduce students to where they might see percentages in their day to day life. The ‘**Fractions Out Shopping**’ video from National Stem Centre shows many example (stem.org.uk). As well as this you could show photos that display discounts that have been calculated incorrectly. Students can then discuss and correct these. For example:



SECTION 1F / 1H: REVIEW OF PERCENTAGES

This section reviews the students’ understanding of the equivalence between fractions, decimals and percentages.

Prompting questions

Exercise 13A(F) / 13A(H)

While working through the exercise, good prompts for students might be:

- Q4(F) / Q4(H)** Why might it be useful to have all of the quantities written in the same form, for example all percentages? *In order to compare them.*
- Q5(F) / Q5(H)** This question mentions the fraction $\frac{2}{3}$. What is the difference between 2.6 and $2.\dot{6}$ (and even 2.66667)? *2.6 is a recurring number.*

Other activities, assessment ideas and rich tasks

Starters or plenaries

- To convert a percentage to a fraction or a decimal, students might find it helpful to have a place value grid.

Enrichment activities

- NRICH. **Matching Fractions, Decimals and Percentages**. A game similar to pairs that gets students to match equivalent fractions, decimals and percentages (nrich.maths.org).
- NRICH. **Doughnut Percents**. A silent group work activity that asks students to match equivalent fractions, decimals and percentages (nrich.maths.org).
- SMILE Mathematics. **Equivalent fractions, decimals and percentages** (final page of pack 1). Students are required to match a percentage with a pictorial representation and a decimal fraction (nationalstemcentre.org.uk).
- Just Maths. **FDP Who dunnit?** Students are required to work through the clues to work out who dunnit. The clues range from ordering fractions, decimals and percentages to working out whether a statement is true or not. A great activity that gives students a purpose for converting between fractions, decimals and percentages (nationalstemcentre.org.uk).
- Just Maths. **Extreme fractions, decimals and percentages!** A collection of questions with a real life context (justmaths.co.uk).

SECTION 2F / 2H: PERCENTAGE CALCULATIONS

This section introduces various ways to calculate a percentage of an amount and how to write one number as a percentage of another.

This chapter gives the students three different ways of calculating a percentage of an amount. The second method is called the ‘decimal method’. Students might like to know that this is also called the multiplier method and is very useful for completing questions with multiple calculations.

Prompting questions

Exercise 13B(F) / 13B(H)

While working through the exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** How are you going to make sure you are accurate when calculating these? *Don't round any numbers until the end.*
- For each question, students could discuss how to round their answer and decide why.

Exercise 13C(F) / 13C(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Why is it important to make the units the same? *To write one number as a percentage of another the units must be the same*
- **Q8(F) / Q8(H)** Students may find it interesting to investigate the label further. What percentage of the ingredients per serving are listed?

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- Just Maths. **Ever Wondered Why?** This is a code breaking activity that practises calculating percentages of amounts (justmaths.co.uk).
- Just Maths. **Percentages of amounts top trumps.** (Justmaths.co.uk)
- Just Maths. **Percentage of an amount – Connect 4.** (Justmaths.co.uk)
- NRICH. **One or Both.** A challenging activity where students have to work out how many people took an exam if they know how many people passed both and the percentage of students that passed each exam (nrich.maths.org).
- Mr Barton Maths. **Percentage Snake Board Game.** (Mrbartonmaths.com)

SECTION 3F / 3H: PERCENTAGE CHANGE

This section focuses on increasing and decreasing an amount by a given percentage and finding the original amount if you know the value after an increase or decrease.

The reverse percentages section shows students how to calculate a reverse percentage algebraically. If students are struggling, there are a couple of other ways in which you can demonstrate this. If the bike had 10% off then £108 represents 90%, students can now think about the helpful percentages they can find, such as 1% or 10%. For example, students could divide £108 by 9 to find 10% and then multiply this value by 10 to find 100%. Another way that students could calculate this is to divide by the multiplier. For example, if £108 represents 90% then the original value can be found by completing the calculation $£108 \div 0.9$.

Prompting questions

Exercise 13D(F) / 13D(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** If you did the calculation 48×0.14 , what have you calculated and what calculation would you have to do to complete the question? *You've calculated 14% of 48, which you then need to add on to 48.*
- **Q2a(F) / Q2a(H)** Explain why the calculation 68×0.86 represents the question £68 decreased by 14%. *0.86 is 100% decreased by 14%.*
- **Q7(F) / Q7(H)** Explain why the following calculation would work for the first part of the question: 2500×0.9575 . *100% reduced by 4.25% is 95.75%. 95.75% written as a decimal is 0.9575.*

- **Q7(F) / Q7(H)** What single calculation could you do to find the value of the shares at the end of month 2?
 2500×0.9718625 ($2500 \times 0.9575 \times 1.015$).
- **Q9(H)** Prompt students to look carefully at the amount of crimes reported and the context in which they were reported. *They should notice that the largest increase came from the area with the smallest population.*

Exercise 13E(F) / 13E(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** For all of these questions, what percentage would be useful to calculate first? How would you find this percentage? *It is useful to find 1%. To calculate this divide the value by the percentage you were given. For example, if 8% is 120 g then 1% is $120 \div 8 = 15$ g.*
- **Q3(F) / Q3(H)** What percentage of the original price did Misha pay for the DVDs? 80%.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Mr Barton Maths. **Increase and Decrease Bingo**. An activity which tests students' knowledge of writing an increase or decrease as a multiplier (mrbartonmaths.com).
- Stick on the Maths. **Percentage Increases and Decreases**. An activity that requires students to match a statement to an answer (kangaroomaths.com).

Assessment ideas

- **Percentages Project**. This is a project that tests all objectives and focuses on students' financial literacy (drive.google.com).

Enrichment activities

- Improving Learning in Mathematics **Mostly Number – N7 Using percentages to increase quantities**. This activity requires students to increase or decrease an amount of money by a given percentage. The students then need to match this to the same calculation written in words, as a fraction and as a decimal (nationalstemcentre.org.uk).
- National Stem Centre. **MP3 Player**. This task has many objectives (graphs, probability and functions) as well as percentage increase/decrease (nationalstemcentre.org.uk).

Topic links

Previous learning

Students will need to be competent at carrying out calculations involving the four rules with integers. For work with multipliers then should be able to multiply and divide with decimals. For working out percentages without a calculator they will need to know how to multiply and divide whole numbers and decimals by powers of 10. They should also be able to find simple fractions of amounts, such as one half, one quarter, one tenth or three quarters.

Future learning

A future chapters that builds on students' understanding of percentages is *Chapter 33F/35H Discrete growth and decay* which introduces the idea of compound interest or depreciation.

Probability may also be expressed as a percentage (*Chapter 23 Basic probability and experiments* and *Chapter 24 Combined events and probability diagrams*).

Gateway to A Level

Percentages are particularly useful in Statistics for calculations involving probability, but calculations involving percentage change or writing one value as a percentage of another can occur in many aspects of the A Level course.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 4 Question 4
- Chapter 6 Question 8
- Chapter 7 Question 19
- Chapter 8 Question 22

Higher

- Chapter 4 Question 11
- Chapter 6 Questions 1, 22
- Chapter 7 Questions 9, 10
- Chapter 8 Question 4

Homework Book

Foundation

- Chapter 13

Higher

- Chapter 13

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 11 worksheets (+ solutions)
- 6 animated widgets
- 12 interactive walkthroughs
- 6 auto-marked quickfire quizzes
- 6 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

14F / 14H Algebraic formulae

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know the order in which mathematical operations are processed (i.e. BIDMAS);
- be able to use common formulae used in mathematics and science, for example $A = \frac{1}{2} \text{ base} \times \text{perpendicular height}$;
- understand the meaning of an unknown, variable and constant;
- know the difference between an expression and an equation;
- be fluent in algebraic manipulation: simplifying, substitution, expanding, factorising and solving;
- know what a 'coefficient' is.

Additional useful prior knowledge

- Number work including operating with negative numbers, rounding and working with fractions.
- Calculator skills, namely use of extra brackets in calculations to force your calculator to do the correct calculations.
- Formulae relating to physical events, for example the equations of motion (**Higher only**).
- Substitution into trigonometric functions.

Learning outcomes

Foundation

Section 1

- To be able to write formulae to represent real life contexts.

Section 2

- To be able to substitute numerical values into formulae.

Section 3

- To be able to rearrange formulae to change the subject.

Section 4

- To be able to use formulae from the topic of kinematics.
- To be able to work with formulae in a variety of contexts.

Higher

Section 1

- To be able to write formulae to represent real life contexts.

Section 2

- To be able to substitute numerical values into formulae.

Section 3

- To be able to rearrange formulae to change the subject.

Section 4

- To be able to use formulae from the topic of kinematics.
- To be able to work with formulae in a variety of contexts.

Vocabulary

formula, subject, substitute, evaluate

Common misconceptions and other issues

- Algebraic manipulation is a major part of this chapter and all the misconceptions associated to *Chapter 3 Algebraic expressions* and *Chapter 8 Equations* are relevant here.

- **(Higher only)** In addition to this, students' are required to have strong skills in operating with fractions. A well-chosen starter with questions on addition, subtraction and simplifying algebraic fractions will help to draw out any issues so that key points, for example cancelling factors not terms, can be addressed before starting this chapter.
- A common misconception relevant to this chapter is to confuse the variables in a formula with any constants present, for example when using π . To address this, you could identify the constants and variables for each formula you use before working with the formula. This would fit into Section 2, which looks at substitution into formulae.
- When changing the subject of a formula, students sometimes simply swap the letter they have as the original subject with the one they wish to change it to. To address this misconception, you could ask them to try substituting some values into the original formula and then into the new formula, so that they see that this will not work. A good formula to test this with is the one for converting between degrees Celsius and degrees Fahrenheit.
- Many students do not know what a variable is and are unwilling to substitute different values into a formula. To look at this, you could go back to the idea of a 'function machine', where different inputs are selected and this gives different outputs. That is, if you vary the input, the output varies.
- 'It is an equation, so it can be solved.' This can cause students problems because sometimes after substituting numerical values into a formula they may need to solve the equation they have formed, so that they can find what value the variable must take, given the numerical values of the other variables. These problems could be resolved by giving students a simple rule to follow: solve equations that have one variable but rearrange a formula that has more than one variable.

Hooks

The use of formulae in the everyday world is prolific. From the useful formulae used to calculate speed or the energy of a particle right through to the less useful formulae about the best way to make **Perfect cheese on toast** (independent.co.uk) or to win the **World Cup** (aperiodical.com).

SECTION 1F / 1H: WRITING FORMULAE

Section 1 introduces the idea of creating a formula for a given situation. It may be useful to refer back to the 'hooks' section above. Throughout this section, students will be required to read and understand the sentence first, which might prove challenging for some. For example 'y is three less than x' might cause confusion from a literary point of view (see below for possible prompting questions).

Higher students will be required to construct some formulae that they are familiar with but will also need to read a large amount of information in a question in order to construct a formula that they will then be required to apply.

Prompting questions

Exercise 14A(F) / 14A(H)

While working through the exercise, good prompts for students might be:

- **Q1(F)** You could start by identifying what each of the variables is for. How could you present this on the page to help us construct a formula? *Maybe a word equation first could be useful before replacing the words by letters.*
- **Q2(F)** If 'y is three less than x', can you construct a numerical example with $y = \dots$ and $x = \dots$? *If $y = 10$ then $x = 13$. So $y = x - 3$.*
- If you have a context you are familiar with, for example the area of a triangle, you should write down what you know first. What information is given in the question? How can you organise this on the page? *This will be context based but noting down the information as it is read is a useful way to draw out and record the variables for the question.*
- **Q3(H)** Where have you needed to use this information before? What question did you use it to solve? *(This question has scope for recapping and revisiting questions that use each of the formulae.)*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Variations of the questions from Exercise 14A could be used to start or review content of a lesson.

Enrichment activities

- An exchange rates activity where students are required to construct formulae to change between different currencies. This can also form part of an ICT lesson where the students use a spreadsheet to accomplish this.
- This section could be linked with *Chapter 39H Interpreting graphs* using the NRICH activity **How Do You React?** The students are required to come up with their own formula for this situation (nrich.maths.org).
- Links can be made back to Section 2 with the activity, **Making Maths: Make a Pendulum** and an investigation in which you drop an object from varying heights and find the time taken to fall. This data can then be plotted and students can be asked come up with a formula to describe this situation (nrich.maths.org).

SECTION 2F / 2H: SUBSTITUTING VALUES INTO FORMULAE

Section 2 formally introduces the idea of substituting into a formula. Students will, however, be familiar with what a formula is and will have examples of formulae they have used (which are a nice way to start off a sequence of lessons on this topic). It is important in this section to emphasise the key vocabulary, particularly 'evaluate'. The 'Work it out' question is a nice way to assess what misconceptions will be present; specifically, the misconceptions that have been highlighted in *Chapter 3 Algebraic expressions* and *Chapter 8 Equations*. Exercise 14B allows for consolidation of this introduction.

Prompting questions

- While working through the 'Work it out' exercise, good prompts for students might be:
 - In what order do you do the operations? *BIDMAS*.
 - How can you work out from the formula what the units of the answer should be? *u is a velocity and a also gives a velocity, so our answer is a velocity.*
 - How could we set this calculation out on the page? *Write down the list of variables and equate them to any numerical values you know. Then substitute the values in and evaluate the formula to find the final velocity of the body.*

Exercise 14B(F) / 14B(H)

While working through the exercise, good prompts for students might be:

- **Q2(F)** This question could be used to reinforce some of the formulae the students are required to learn. When would you use the formula in part e? *The volume of a cylinder.*
- **Q4-6(F)** These questions are all in a context and thus you could ask: What units is our answer going to be in? How can you tell?
- What are the common errors made when substituting values into a formula? *Ignoring BIDMAS and the rules of negative numbers.*
- **Q10(H)** This question is based on a pendulum and a question you may ask is 'How can you tell what the units must be for the subject of this formula given the calculation you need to do?' *2π is dimensionless but length is in metres (m) and acceleration due to gravity is measured in metres per second squared (m/s^2). You therefore have the square root of $m/m/s^2$ which equals seconds (s). This is called dimensional analysis and is done by engineers and scientists.*

Starters, plenaries, enrichment and assessment ideas

Starters

- **(Higher only)** Before using Question 12 from Exercise 14B you could use this NRICH task, **Making Maths: Make a Pendulum**, to investigate what happens to the time for one complete swing when you change the length of the pendulum. This activity then provides a way to evaluate their experiment by using the formula given in Question 12 and the students could make suggestions as to why the times may be different (nrich.maths.org).

Enrichment activities

- This section provides scope for embedding some investigations to discover a mathematical formula for a specific situation. One such activity that can work well for both foundation and higher students is the NRICH task **Pick's**

Theorem. Dotted square paper is required and you can control the level of difficulty by restricting what your students look at, for example how many shapes can be described with four dots in their perimeter and none in their interior. What are the areas of these shapes? (Nrich.maths.org.)

- A simple substitution game based on the idea of a flow chart with different formulae at each stage but requiring the answer from the previous stage will give a fluency exercise but without having to repeat the style of Question 1.

SECTION 3F / 3H: CHANGING THE SUBJECT OF A FORMULA

Section 3 introduces the idea that you may not always want the formula in the form you are given. There are times when you wish to rearrange a formula to make one of the other variables the subject. This section may give rise to the misconception that you can 'solve a formula', so any statements that draw on the similarities with solving equations should be carefully stated. However, when looking at numerical values, if you choose to not change the subject of the formula but instead substitute numerical values in first, you will be solving an equation to find the value of the variable that is not given.

Prompting questions

While working through this section, good prompts for promoting discussion might be:

- How can you set out your method for changing the subject of your formula on your page? *You need to make sure you are clear about what algebraic manipulation you are doing to your formula at each stage. Setting it out as in the example in the book is helpful.*
- How can you check to see if you have correctly rearranged your formula? *You could substitute some values in.*
- How can you correctly determine the units for your answer? *Context of the question (F) or dimensional analysis (H).*
- **(Higher only)** What algebraic manipulation will you need to do if you have several occurrences of the variable you wish to change the subject of your formula to? *Need to expand, collect the variable you require and then factorise that variable out to leave only one occurrence of it.*

Exercise 14C(F) / 14C(H)

While working through the exercise, good prompts for students might be:

- **Q1-2(F) / Q1(H)** How can you set out these questions on the page to help when you perform our algebraic manipulation? *You can look at the mathematical processes that are required in each part and then find the opposite mathematical process required.*
- **Q9(H)** How can you approach this question? *You could substitute the values and then solve the equation for L or you could rearrange the formula to make L the subject first and then substitute the values in.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Exercise 14C Question 1(F) would work well as a starter or as a plenary.
- Have a sheet of incorrectly rearranged formulae. Ask students to spot the errors and correct them.

Enrichment ideas

- **(Higher only)** This section could be used to recap or revisit rearranging a quadratic equation to make x the subject. A useful resource can be found here: **Proof Sorter – Quadratic Equation** (nrich.maths.org).

Assessment ideas

- The use of mini-whiteboards for a short assessment task on this topic may prove useful, for example a slideshow where a formula is flashed up and students must change it to the required subject.
- The same idea can be extended to revise substitution after rearranging by constructing a relay where the previous answers are used to calculate the next as they work through 10 different types of formula. This activity can work well to reinforce their calculator skills and knowledge of rounding if you include non-exact answers at some stages.

SECTION 4F / 4H: WORKING WITH FORMULAE

Section 4 now applies all the skills gained in this chapter. The students can consolidate their knowledge of formulae using the common ones that they will need to remember for their examination. The questions cover a range of topics including Fahrenheit/Celsius conversion, perimeter, area, surface area, Pythagoras' theorem, volume and the trigonometric functions.

(Higher only) This section introduces the idea of further formulae that are used in mathematics; a specific example of a recurrence relation is given. Exercise 14D offers a range of formula related problems that require an additional process to occur.

Prompting questions

Exercise 14D(F) / 14D(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Can you answer this question without looking at the formulae above?
- How should you set out these questions on our page? *It is a good idea to record the variables you have, if you have a numerical value for them and the formula you can use. You may then need to rearrange your formula. It is then a good idea to substitute the values into your formula first before evaluating it.*
- **Q2(H)** Where have you seen questions like this before? *(This question has scope for recapping and revisiting questions that use each of the formulae you construct.)*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Turn Question 1 from Exercise 14D into a starter or plenary.

Enrichment activities

- Make connections to other areas of mathematics because the use of formulae is hugely important and a formula has no meaning unless each of the variables has a context. You could use a space in your classroom for a board dedicated to formulae and their use. Each pair of students gets a formula that they must remember. They are then required to provide an example using the formula, work out a good way to remember the formula and know at least one rearrangement of the formula. They then share their formula with the class.
- **(Higher only)** Exploring recurrence relations can be a good way to consolidate the use of a formula and link back to a topic that the students have seen before (*Chapter 4 Functions and sequences*). The students can investigate the recurrence relation given in the 'Did you know?' box at the start of section 4 for different initial values of the variable Z_0 and values of the constant C .

Assessment ideas

- Ask the students to write a revision guide containing the formulae they are required to remember and those that they are given and required to use, so as to aid their revision.

Topic links

Previous learning

Since this is a chapter on formulae it has connections to all areas of mathematics in which you are required to calculate an answer given several input values. During this chapter, your students will be revisiting the process of calculating areas, perimeters, volumes and solutions to quadratic equations as well as using trigonometric functions, Pythagoras' theorem and equations of straight line graphs.

Future learning

Work on substituting values into formulae links to interest calculations covered in *Chapter 33F/35H Discrete growth and decay* and to work on writing formulae related to exponential decay in the Higher *Chapter 35H*. The kinematics formulae introduced here (equations of motion) will be used in *Chapter 39H Interpreting graphs*. Also in *Chapter 41H Transformations of curves and their equations*, Higher tier students are introduced to the use of $f(x)$, and learn that the formula that produces a given curve is called a function and acts in the same way as a formula.

Gateway to A Level

Students looking further ahead will need to be fluent in their use of all the formulae mentioned in this chapter, particularly those relating to equations of motion and calculus topics. A Level students will be required to rearrange formulae with the variable appearing several times, for example a formula containing a quadratic term, say x^2 , to make x the subject. Further to this, recurrence relations will be tied together with different types of sequences where students will be required to recognise more than one way of rearranging the subject to give a recurrence relation to employ numerical methods.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 5 Question 8
- Chapter 6 Questions 9, 10
- Chapter 7 Question 12
- Chapter 8 Question 1
- Chapter 10 Question 8

Higher

- Chapter 6 Question 2
- Chapter 7 Question 2
- Chapter 10 Question 20

Homework Book

Foundation

- Chapter 14

Higher

- Chapter 14

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 3 worksheets (+ solutions)
- 1 animated widget
- 5 interactive walkthroughs
- 3 auto-marked quickfire quizzes
- 3 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- Required formulae – Foundation
- Required formulae – Higher

15F / 15H Perimeter

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know the properties of shapes;
- know the properties and labels associated with circles;
- understand how to convert between metric units;
- be able to form algebraic expressions and formulae from geometric problems;
- understand how to change the subject of a formula;
- be able to simplify by collecting like terms;
- know that angles around a point sum to 360 degrees;
- know how to round to a given or appropriate accuracy.

Additional useful prior knowledge

- How to simplify expressions.
- How to expand single brackets.
- How to substitute a value into a formula.

Learning outcomes

Foundation

Section 1

- To calculate the perimeter of a given simple shape, including the use of properties of triangles, quadrilaterals and regular polygons.
- To understand that the perimeter of a shape is its boundary and what a boundary is for a composite shape where a smaller shape has been removed from the centre of a larger shape.

Section 2

- To know and use a formula (either $C = \pi d$ or $C = 2\pi r$) for the circumference of a circle to find the value of one variable given any other, for example D given C .

Higher

Section 1

- To calculate the perimeter of composite shapes.
- To form expressions and equations for the perimeter of a given shape and solve these equations to find unknown lengths.

Section 2

- To know how find the arc length of a given sector and hence the perimeter of this shape.

Section 3

- To use known perimeter formulae from Section 1 and 2 to solve contextual problems.

Vocabulary

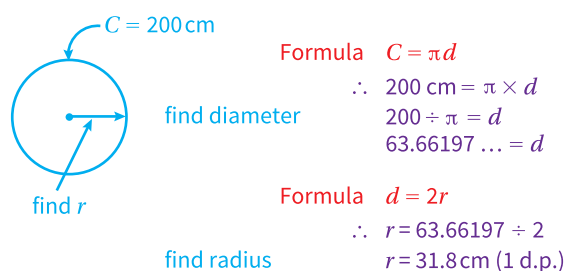
perimeter, sector, arc

(Higher only) subtended

Common misconceptions and other issues

- Students struggle to work with formulae involving π , particularly where rearranging is required. This is because students treat π as an unknown rather than an irrational number. Suggesting students replace π with 3.142 should not affect the accuracy of their solutions too much when working with length and can help some students whose solving/rearranging skills are weak.

- Students struggle to find π when working with different equipment and don't realise they can use an approximation on most calculators. If you do not have class sets of calculators for students to use you tend to be reliant on students bringing in their own equipment and often this is borrowed from a friend or shared in class which means students struggle with the different options to enter π on different calculators. Students should be encouraged to invest in and bring their own calculator. Ask them to write down the steps for getting π on their calculator in their books at the beginning of a series of lessons on circles.
- Students miss lengths when calculating the perimeter of a given shape; this is a more prominent problem when students are working with composite shapes. Encourage students to sketch the shape they are finding the perimeter of and make sure that every side is labelled with a length. In addition, encourage students to write out the sum they are calculating so that missed lengths can be more easily identified.
- Students don't include all boundaries in the perimeter. This tends to happen when working with composite shapes where a section has been cut out of the middle of a larger shape or when working with circle sectors. Reminding students that the perimeter is the boundary of the shape and using analogies such as: 'Imagine the shape is a fish tank, where would the glass have to go to keep the water in?' In addition, asking students to draw around the boundary of the composite shape they are finding the perimeter of can help with this.
- Students confuse area with perimeter. This is a common problem when students just learn a series of formulae for different shapes. Reinforcing the idea of perimeter as a boundary and in most cases a sum (even though the circumference of a circle is the sum of a length an irrational number of times, students don't often seem to see it this way) can help some students. It is also important that students spend some time working on a series of problems in which they are asked to find both area and perimeter so that they understand the difference.
- Students confuse diameter and radius or don't see the solution through when there are multiple steps to the problem. Sometimes this can be perpetuated by having two formulae for the circumference of a circle; at other times having two formulae can prevent it. Encourage them to always draw a sketch of the circle they are using and keep track of their working by laying it out in a column, annotating each step as they solve the problem, for example:



- Students confuse perimeter with arc length and either miss the two radii when finding the perimeter or include them when they are given the task of calculating the arc length only. You could try working on a series of problems that require both and having starters that ask students to identify the difference between the arc length of a sector and the perimeter either through a matching exercise or a filling in the gaps task. In addition, students could have a simple highlighting task given a series of sectors where they highlight either the arc length or perimeter as directed.

Hooks

- In Section 2 the notes introduce π as the ratio that connects the diameter and circumference of a circle. You may like to give students time to investigate this for themselves and have a collection of cylinders for students to measure. Paper measuring tapes can be very helpful as a resource when students conduct this investigation but a piece of string and a ruler can also be used.
- When working with sectors you may prefer to start with area by, for example, considering the area of a slice of pizza where the circle has been cut into sectors of equal size. You can use this to deduce the idea of fractions of the whole circle to find the area of a sector and from this connect to arc length.

SECTION 1F / 1H: PERIMETER OF SIMPLE AND COMPOSITE SHAPES

Section 1 focuses on a variety of perimeter problems involving polygons, including simple composite shapes. Students find the perimeter of these shapes using known properties and information given through notation (e.g. equal sides marked) to find unknown lengths. An emphasis on viewing the perimeter as the boundary of the shape is used to support students when working with composite shapes. Students also form expressions, given relationships between the sides of a shape, and use these expressions to form equations that they can then solve to find unknowns.

Prompting questions

Exercise 15A(F) / 15A(H)

While working through the exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** What do the units of the measurements have to be in order to calculate with them? *The same so some metric conversions may have to take place first.*
- **Q3(F) / Q3(H)** Sketch the shapes first and label all lengths you know.
- **Q3(F) / Q3(H)** What properties of the shape are useful to use in forming an expression for the area? *Look out for words like 'regular' and think about what lengths are the same.*
- **Q7(F) / Q7(H)** Think about what these shapes have in common. Do any of them have more than one name? *For example, a square is a rectangle, to deduce that all fit the properties of a parallelogram so all can use the parallelogram formula.*
- **Q7(F) / Q7(H)** Which side lengths are often unknown on a trapezium? *The sloped edges.* What mathematics do we need to find the length of these edges? *Pythagoras' theorem or trig if given an angle.* Can we always find the perimeter of a trapezium? *No, not if we don't have enough information. We need more than just the height; we need to have either an angle or additional information relating to how the length of the base is split into the base of the triangle(s). This problem is useful for highlighting to students that not all trapeziums are isosceles trapeziums.*
- **Q8(F) / Q8(H)** Is there any information in the question that is unnecessary? *The diagram gives a maximum distance for the fence posts but the question is setting them 2.8 m apart.*
- **Q9(H)** Sketch the paths run by each group of runners. Can you label every length?
- **Q9(H)** What formula can you use to calculate the average speed? $S = \frac{D}{T}$. What do the units have to be for total length if working with km/h? *Kilometres so they will need converting from meters.*

Exercise 15B(F) / 15B(H)

While working through the exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** Can you sketch the shapes given from the information to identify which lengths are needed to calculate the perimeter?
- **Q2(F) / Q2(H)** Try highlighting all boundary lines to calculate the total perimeter.
- **Q2(F) / Q2(H)** Do lines drawn inside the shape count as boundary lines? *No, they just indicate properties, which can help us find additional shapes.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- **Can they be equal?** This is a simple task that focuses on rectangles only but helps reinforce the concept that area and perimeter aren't connected (nrich.maths.org).
- **Changing Areas, Changing Perimeters** is another simple perimeter task that gets students to consider what increases the perimeter of a shape and how this connects to similar ideas of area. For stronger classes this is a nice starter task that could reintroduce the topic of perimeter (nrich.maths.org).

Starters

- **(Foundation)** Give students a square that is cut up into smaller rectangles of varying sizes and ask them to form a shape that has the largest perimeter. This task may seem simple, but some students will not rotate their rectangles so that the longer side is part of the perimeter of the final shape.
- Produce a series of flash cards on which shapes are drawn and ask students to name the shape and state any properties they know about their side lengths, for example a square has all sides the same length while an isosceles triangle has only two sides the same length. How do we denote this on each shape?

SECTION 2F / 2H: CIRCUMFERENCE OF A CIRCLE

Section 2 focuses on circles and sectors of circles. The first subsection looks at the two formulae for the circumference of a circle (using either the radius or the diameter) and how they are related to each other. In addition, students use their solving/rearranging skills with the formulae to find either the radius or diameter given the length of a circumference. The second subsection considers sectors of circles and derives the formulae for the arc length by considering it as a fraction of a whole circle, the size of which depends on the angle formed between the radii. This is extended to finding a formula for the perimeter of a sector by adding the lengths of the two radii to the arc length.

Prompting questions

Exercise 15C(F) / 15C(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** What variables have you been given the value of in the question? Which formula connects the variables you have and what you are looking for? *Either $C = \pi d$ or $C = 2\pi r$.*
- **Q1(F) / Q1(H)** What can you say about the length given in terms of x ? *It is a variable length so an expression for the perimeter will need to be given instead and the expression should be in simplified form.*
- Can you sketch the problem being described to help support your workings?
- **Q4(F) / Q4(H)** What boundaries are involved in the perimeter the disc has been cut out of? *The boundaries include both the edges of the square and the circumference of the circle.*
- **Q5-7(F) / Q5-7(H)** Start by filling the information you have into the circumference formula. What do you need to do to find your unknown length? *Solve.*
- What is π ? *A number, so we can treat it in the same way we would any other value.*
- **Q8(F) / Q8(H)** How far does the end of a minute hand on a clock travel in one hour? What maths can you do to find the distance it travels? *A full circle, so find the circumference.*
- Have you rounded your solution to an appropriate number of decimal places? *Students should round to the same accuracy as the values given or to 1 more degree of accuracy and state what they have rounded to.*

Exercise 15D(F) / 15D(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Have you been given the angle inside the sector or outside? How might you find the angle required? *Students need the angle inside the sector and can use complements to 360 degrees to find it.*
- **Q2(F) / Q2(H)** What is the difference between the perimeter and arc length of a sector? *The perimeter includes both the arc length and the radii.*
- **Q2(F) / Q2(H)** In part f, what is the angle indicated by a small square? *A right angle so 90 degrees.*
- **Q2(H)** What parts of shapes do you have in parts g–j of the question? Can you split the shapes up and label all lengths you know? *Encourage students to deal with each part of the shape separately to prevent confusion between the radius and diameter, which in some diagrams are represented by the same length for different parts of the composite shape.*
- **Q2(H)** Which lengths form the boundary of the shape? Which ones will be unnecessary? *If students use the advice given in the point above they may include more than the boundary line so suggest they use a highlighter to keep track of the lengths they need.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- **Arclets** is a nice way to explore the mathematics students have been working on in this chapter and revisit some of the angle facts they know, as well as further develop their generalisation skills (nrich.maths.org).
- **Rollin' Rollin' Rollin'** is nice for showing students that their initial ideas may not always be correct and links into some other work students have done on loci if they map the path the centre of the moving circle makes as it rolls around the static circle (nrich.maths.org).
- **Triangles and Petals** is another lovely task, in a similar vein to the one in the previous point, that considers the path a shape makes when rolled around one that is equal to it. It gives students opportunities to recap on previous work on regular polygons and angles and makes a good investigation assessment task (nrich.maths.org).
- A card sort can help to increase confidence in using the arc length and perimeter formulae to find any unknown variable, and offer students appropriate differentiation as they can use their preferred method to match the cards. By removing half of a pair you can force the use of solving an equation when finding the radius or angle.

SECTION 3F / 3H: PROBLEMS INVOLVING PERIMETER AND CIRCUMFERENCE

Section 3 places a greater emphasis on perimeter problems for composite shapes, drawing on the new material in section 2 and the revision material in Section 1. These problems are often longer with more parts and students will have to identify what information they need from a series of diagrams.

Prompting questions

While working on the material in this section good prompts for promoting discussion might be:

- What information do you need to solve the problem? Can you identify it in the problem? *Encourage students to sketch the problems and label information required. The use of highlighters might be advisable here.*

Exercise 15E(F) / 15E(H)

While working through the exercise good prompts for students might be:

- **Q5(F) / Q6(H)** Can you sketch the problem in 2D and label all known values?

Starters, plenaries, enrichment and assessment ideas

Starters

- The questions used in the review exercise would work well as starters (of one or two questions) to future lessons to keep the ideas of this chapter fresh.
- NRICH has a similar starter problem based on the **London Eye**, which could be a simpler intro to the series of questions at the end of the exercise (nrich.maths.org).

Topic links

Previous learning

Students are required to use their knowledge of properties of shapes and angle facts to solve many problems in this chapter; particularly as the problems get more complicated. This chapter doesn't directly follow on from any previous learning, but it uses a wealth of knowledge gained from *Chapter 5 Properties of shapes and solids* so as to produce formulae for the perimeters of various 2D shapes.

Future learning

Perimeter is not directly extended in another chapter in this book but students will continue to use and revisit the ideas covered here. They will gain new skills in *Chapter 31F/32H Pythagoras' theorem* and *Chapter 32F/33H Trigonometry* and apply these to problems involving perimeter. In addition, sectors will be revisited in *Chapter 16 Area*, where students will use the same principles that they used for calculating perimeter for finding the area of a sector and further investigation into properties of circles will be covered in *Chapter 34H Circle theorems*.

Gateway to A Level

Arc lengths of curves are an abstract extension to the GCSE topic of perimeter and appear in the further mathematics curriculum. Students will also revisit arc length of a circle by combining the mathematics of Section 2 with new knowledge of radians.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 1 Questions 1, 10
- Chapter 2 Questions 3, 4, 20
- Chapter 3 Question 5
- Chapter 4 Question 2
- Chapter 9 Question 5
- Chapter 10 Question 4

Higher

- Chapter 3 Questions 5, 17
- Chapter 5 Question 8
- Chapter 6 Question 23
- Chapter 10 Question 21

Homework Book

Foundation

- Chapter 15

Higher

- Chapter 15

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 3 worksheets (+ solutions)
- 2 animated widgets
- 7 interactive walkthroughs
- 2 auto-marked quickfire quizzes
- 2 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

16F / 16H Area

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know the properties and definitions of polygons, particularly triangles and quadrilaterals;
- know the properties of circles;
- be able to convert between metric units of length;
- be able to convert between metric units of area;
- be able to substitute into algebraic formulae and expressions;
- be able to solve linear and quadratic equations formed from area formulae to calculate the value of the unknown (and recognise that where lengths of shapes are involved the value of the unknown must be positive);
- **(Higher only)** be able to simplify algebraic expressions including expanding the products of binomials and cancelling common factors in fractions;
- **(Higher only)** be able to use Pythagoras' theorem to find the length of the hypotenuse.

Additional useful prior knowledge

- To know that finding the square root gives the initial value that had been squared.
- To rearrange a formula to change the subject (some students prefer this option to substitution followed by solving, particularly where there is a repetition in the calculation).
- To know that multiplication is commutative and hence products can be calculated in any order (area of a triangle) but that some calculations are distributive, for example the sum of the two parallel sides must be found first when calculating the area of a trapezium.
- To appreciate that dividing by a value is the same as multiplying by the reciprocal of the value, for example multiplying by a half is equal to dividing by 2.
- To be able to calculate fractions of amounts.
- To know how to calculate the perimeter of a shape.
- To know and use formula for the circumference of a circle.

Learning outcomes

Foundation

Section 1

- To know and use the formulae for calculating the area of rectangles, triangles, parallelograms and trapeziums.
- To identify how composite shapes have been formed using these four shapes and use the formulae to calculate the total area of the composite shape.

Higher

Section 1

- To know and use the formulae for calculating the area of rectangles, triangles, parallelograms and trapeziums.
- To identify how composite shapes have been formed using these four shapes and use the formulae to calculate the total area of the composite shape.
- To form algebraic expressions for the area of a shape given expressions for lengths of the shape.

Section 2

- To know and use the formula for calculating the area of a circle.
- To adapt this formula to find the area of a sector given the angle formed at the centre between the radii using fractions of the whole.

Section 3

- To split composite shapes into the sum of known shapes from Sections 1 and 2.
- To recognise that the area of some composite shapes can be found by subtracting known areas from a larger shape.

Section 2

- To know and use the formula for calculating the area of a circle.
- To adapt this formula to find the area of a sector given the angle formed at the centre between the radii using fractions of the whole.

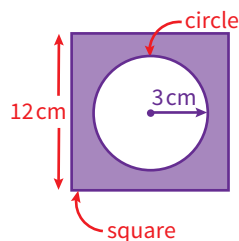
Section 3

- To split composite shapes into the sum of known shapes from Sections 1 and 2.
- To recognise that the area of some composite shapes can be found by subtracting known areas from a larger shape.

Common misconceptions and other issues

- Students do not state the units when finding the length of an unknown. In some cases this is due to a lapse in accuracy but in others this is because they do not understand that the units must be consistent for all measurements in a calculation, which would enable them to deduce the unit of the length they are finding if they are given the square units used for the shape's area. To combat this, often the most successful idea is to constantly reinforce good practice by always stating the units of every measurement used, even when it takes longer to go through an example, and penalise those who do not follow this by circling their work. You could even introduce shorthand-like common literacy marking abbreviations for spelling and grammar.
- Students don't use the perpendicular height when calculating the area of a triangle or parallelogram; they use the length given for the sloping side. This may be due to one of the following reasons: either the orientation of the shape confuses students and they use the sloping length as the value of the base, or the shape given has its perpendicular height labelled outside of the shape due to the side chosen as base (like the third example of a perpendicular height given in the section notes where we have an obtuse angled triangle and the chosen base is not the longest side). We encourage students who suffer from this to rotate the page so that the base (identified from the given right angle) is always horizontal and hence the height is the vertical distance given.
- Students don't recognise a trapezium with rightangles as being a trapezium, instead thinking that all trapeziums have two sloping sides. When giving notes on trapeziums you can give two diagrams, one the more traditional isosceles trapezium and the other a right-angled one. Encourage students to highlight the parallel lines to spot them. Appropriate starters based on properties of shapes can also be used to reinforce this, for example 'Who am I?' or 'Guess who?', with different shapes in different colours. Another method is to reveal part of a concealed shape to show different properties one at a time so that students are reminded through an indirect method that a right-angled quadrilateral with a pair of parallel lines is still a trapezium.
- Students struggle to remember the different formulae for finding area. This may be due to the different labels given lengths in each formula, for example length and width or base and height. There are several videos available online to assist in memorising the formulae, including 'The circle song' and 'The trapezium song'. Students could have one 'Area formulae' page in their books or make their own laminated revision cards showing supportive diagrams, highlighted using bright colours, which are useful for saving time in future lessons and for revision.
- Students confuse the order of operations when calculating the area of a circle, that is they calculate $(\pi r)^2$ rather than πr^2 . We have found that students who struggle with this prefer to use the formula $A = \pi \times r \times r$ or $A = \pi \times r^2$. The addition of the multiplication sign seems to prevent confusion with the order of operations here.
- Students struggle to identify the known shapes that make the composite shapes and miss parts of the shapes out of their calculations. Encourage students to sketch the composite shapes in their books and label the known shapes that make it up (as seen in the first example in the notes section of the chapter). Ask them to state what they intend to do symbolically to prevent the error of adding when subtraction is required and vice versa, for example:

Find the shaded area



shaded area = square – circle

$$\begin{aligned}
 &= 12 \times 12 - \pi \times 3^2 \\
 &= 144 - 9\pi \\
 &= 115.7 \text{ cm}^2 \text{ to 1 dp}
 \end{aligned}$$

Hooks

Area is a topic students have probably seen several times before they reach their GCSE course. However less able students, in particular, frequently have a poor grasp of what it means to measure area. Reintroducing this topic therefore has opportunities to be creative in terms of the units used to measure a 2D space. For example the space a student's hand measures on an A4 page could be measured in terms of dots, the workspace on the table could be measured in terms of pencil cases. These ideas could then be extended to set measurements, for example $10 \text{ cm} \times 10 \text{ cm}$ squares or a metre square, before focusing on the formal formulae for finding the area of triangles, quadrilaterals and circles in this chapter.

SECTION 1F / 1H: AREA OF POLYGONS

Section 1 focuses on the area of rectangles (squares), triangles, parallelograms and trapeziums. Each formula is considered and its links to the other formulae stated, for example the area of a parallelogram comes from the sum of the areas of two identical triangles and hence is given by $2 \times \frac{1}{2} \times b \times h = b \times h$. The first subsection focuses on rectangles and triangles only and extends students to consider composite shapes formed using these two types of shape. Students may struggle to identify the perpendicular height in some of the diagrams and should be encouraged to reorient the page by identifying the right angle so that the perpendicular height is always vertical. The second subsection extends to the use of the formula for the area of a parallelogram and from this the formula for the area of a trapezium. Interspersed throughout the exercises in this section are problems that focus on contextual area problems that require additional skills, including conversion of metric units.

Prompting questions

Exercise 16A(F) / 16A(H)

While working through the exercise, good prompts for students might be:

- **Q2(F) / Q1(H)** How do we identify which length is the height? *We look for the right angle.* Try rotating the page so that the base is horizontal. Which of the given lengths is the base of your triangle? Which is the height?
- **Q3(F) / Q2(H)** What formula are you using? *Area of a triangle so $A = \frac{1}{2}bh$*
- **Q4(F) / Q3(H)** What shape(s) is a kite made up of? *Two triangles or four right-angled triangles.* Can you sketch these triangles separately and label their dimensions?
- **Q4(F) / Q3(H)** What property do kites have that can help you? *It has a line of symmetry so two identical triangles.* Can you sketch the single triangle you need to find the area of? *Should be a triangle with base 10 cm and height 2.5 cm.* Can you derive a formula for a kite given what you have discovered in this question? *Half the product of the diagonals.*
- **Q5-6(F) / Q4-5(H)** What has to be the same when working with lengths and areas? *The units.*
- **Q7(F) / Q6(H)** How can I find two thirds of 2.7m? *Find one third by dividing by 3 and then multiply by 2 to find two thirds.*
- **Q7(H)** For part e, can you sketch the flag and annotate your diagram with the assumed proportions? To find the area of the golden arrow is challenging, think instead about what you can do to the area of the whole flag using the previous parts. *(Students can subtract the earlier parts of the question from the total area of the flag to leave the arrow.)*

Prompting questions

Exercise 16B(F) / 16B(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** How do we identify which measurement is the height? *We look for the right angle.* Try rotating the page so that the base is horizontal. Which of the given lengths is the base of your parallelogram? Which is the height?
- **Q3-5(F) / Q3-5(H)** What formula are you using? *Area of a parallelogram (Q3-4, Q5d) or rectangle (Q5b,e) so $A = bh$, area of a triangle (Q5e) so $A = \frac{1}{2}bh$, or area of a trapezium (Q5c) so $A = \frac{1}{2}(a+b)h$.* Write down the formula you are going to use and fill in the value of the variables you know, what can you do to find the length of the unknown? *Solve.*
- **Q6(F) / Q6(H)** What piece of information do I need to calculate the amount of fencing needed? *The perimeter.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Ask students to find as many triangles as possible with an area of 2. This task can be completed on square dotted paper (perhaps as a preparatory task for Pick's theorem, see below) and hence deduce that there are an infinite number of triangles that can be drawn on square dotted paper. This is a nice task for students who struggle to calculate the area of triangles when the slope requires the perpendicular height to be labelled outside of the shape.

Starters or plenaries

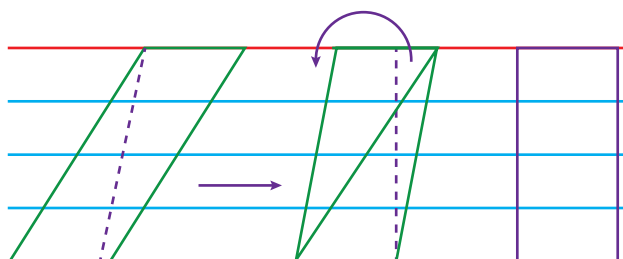
- **(Foundation only)** The first question in Exercise 16A is a nice discussion point for a class at either the beginning or the end of a lesson. Students could then be invited to offer their own suggestions of careers that may involve area calculations.

Plenaries

- Get students to match shapes to their area formula or answer quick-fire true or false problems when given a diagram of a simple polygon and its possible area.

Enrichment activities

- **(Higher only)** The flags in Exercise 16A could be printed on to A4 hand-out sheets and given a scale so that students can find the areas of the flags using their measurements and converting the lengths using the given scale (see *Chapter 12 Units of measurement*) rather than using the measurements given in Exercise A. This makes the task accessible for foundation students too.
- **Pick's theorem** is an excellent task for exploring the area of simple shapes and building students' confidence at dealing with areas involving part squares as they are given a grid to support their work. If you give students a series of areas to find and set them tasks based on finding as many shapes as possible with 12 perimeter dots, students will get purposeful practice in using the polygon formulae from this section and work on composite shapes. In addition, this is a good task for developing students' skills of pattern spotting and justification (nrich.maths.org).
- Some parallelograms cannot have a triangle cut off the end and joined at the other to form a rectangle. These parallelograms need to be cut along the diagonal and the two triangles joined along the original sloping sides and then this method repeated until a right-angled triangle can be removed and added to the other sloping side to form a rectangle.



- It is fairly simple to create an animation to show this using PowerPoint and students often can deduce their own formula if they see the animation first.

Assessment ideas

- Getting students to mark and correct another fictitious student's work which contains obvious common errors such as using the sloping edge rather than the perpendicular height when calculating the area or not giving the units with their answer. Students then have to make a list of the errors they find and how common they are. If students have recently had a homework task you could create this worksheet from their work or introduce peer assessment following the task.

SECTION 2F / 2H: AREA OF CIRCLES AND SECTORS

Section 2 focuses on area of circles and sectors. The formula for area of a circle is given alongside a recap of the connection between the radius and diameter. Sectors are treated as fractions of the whole circle and a formula is derived for the area of a sector in terms of the angle formed at the centre between the two radii.

Prompting questions

Exercise 16C(F) / 16C(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Have you been given the radius or the diameter? How does this affect your calculations? *If given the diameter students need to divide by 2 first.*
- **Q2-3(F) / Q2-3(H)** What is the area of the whole circle? What fraction of the circle do I have? How do I find that fraction? *By finding the angle at the centre out of the whole 360°.*
- **Q6(F) / Q6(H)** Which circle are you using to find the area of the landing zone? What do you know about it? *The one given by the sector with the radius 80 m.* Which circle are you using to find the area of the starting circle? What do you know about it? *That the diameter is 2.5 m.*
- **Q7(F) / Q7(H)** What other formula related to a circle can you use given the piece of information you have been given? *The formula for the circumference: $C = \pi D$.* How can this help you find the area? *It can help you find the radius, which is needed to find the area.*
- **Q8(H)** Can you sketch a plan of the tube to identify what information you know about the circle formed at the top of the tube? Which value is going to be easier to find first, the diameter or the circumference? *The circumference is the larger of the two arcs shown in the diagram. Once students have identified this they should realise the circumference is needed first. You may wish to suggest that students attempt to construct the tube out of paper to help them spot this themselves. These questions can be repeated for the base of the tube.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Simple starters based on circle sectors where the circle has been cut into equal sections (e.g. a pizza or clock) so that students are finding a quarter or sixth of a whole can be appropriate when wishing to recap the formula for the sector of an area because most students are not reliant on a formula when simple fractions are involved.

Plenaries

- 'Spot the error' plenaries work very well in circles work where examples involving common errors such as multiplying by π before squaring are given for students to spot and correct.

Enrichment activities

- **Curvy Areas** is particularly good for exploring compound shapes involving circles and parts of circles (nrich.maths.org). It also has opportunities to revise constructions at the same time. You may find that students are surprised when the areas are all equal. Students can then get experience in generalising these types of problems based on the number of sections the diameter is split into.
- **Bull's Eye** is a simple investigation for students who have recently learned to calculate the area of a circle (nrich.maths.org). The problem could be simplified for less able students by just asking students for the area of each ring first. A suitable follow up problem in a similar vein is **Blue and White** (nrich.maths.org). To simplify the problem for less able students you could give the square dimensions, for example 12 cm $\times$ 12 cm.

Assessment ideas

- Produce a matching activity based on giving students different pieces of information about three or four circles and cards with spaces to fill in with missing information. A group could contain: diagram, radius, diameter, circumference, area. Additional challenge could be added by repeating this task with the inclusion of sectors of circles, where cards could contain: diagram, radius, arc length, perimeter, area, angle formed at centre between both radii.

SECTION 3F / 3H: AREA OF COMPOSITE SHAPES

Section 3 focuses on different ways in which students can find the area of a composite shape by either summing the area of recognisable shapes that form it or by subtracting smaller, known shapes from a larger one. By encouraging students to sketch the composite shape, label the parts of the shape that they know and track their working with clear labels, students can have a lot of success with this section, even with the most challenging composite shapes. The exercise also places a deeper focus on contextual area problems.

Prompting questions

Exercise 16D(F) / 16D(H)

While working through the exercise, good prompts for students might be:

- Q1-2(F) / Q1-3(H)** What shapes can you see in the diagram? Draw on the boundary lines of each shape and consider the dimensions of each shape you are using. What are you doing in each case, summing the individual parts or subtracting one shape from another? *Encourage students to draw each shape involved in the calculation separately and write the dimensions of each shape on their diagrams so as to help them make the correct calculations.*
- Q1g(F) / Q1g(H)** Two semicircles have been placed together to create this composite shape, what can you tell me about the radius and diameter of each one? *The radius of the larger semi-circle is 4.3 cm, which is the diameter of the smaller semi-circle. Hence this value needs to be doubled and halved to find the other lengths.*
- Q2(F) / Q2(H)** If I have two equal semi-circles or four equal quarters of a circle what do I have overall? *A whole circle.*
- Q3(H)** What is the perimeter of a composite shape? *The total boundary.*
- Q3(H)** What do the marks on the triangle in the centre of the semi-circles mean? *That the triangle is an isosceles triangle and hence the angle formed by the centre line and the base of the triangle is a right angle. Why is this useful to us? Because we have right-angled triangles and we can use Pythagoras' theorem to find the length of the hypotenuse, which is the diameter of the other two semi-circles.*
- Q3-9(F) / Q4-11(H)** What information have you been given? What do you need to do? *Following the problem-solving structure in Section 3 of the textbook will support students who don't know where to start. Repeated exposure to these longer, more challenging problems will improve students' ability to tackle these problems on their own.*
- Q3-9(F) / Q4-11(H)** Can you use estimation skills to find an approximate value for the solution in order to check your working? What is the approximate value of π ? *π is roughly 3, so estimates can be formed based on this value.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Give students a selection of composite shapes and ask them to list the basic shapes that are used to form it, without doing any working or calculations. This can be quite a fun game at the beginning of the lesson and students can become very competitive.

Enrichment activities

- The NRICH activity **Floored** can be used to promote thinking about the shapes that are involved in a design and how they can be made using shapes we all know about. You could try this activity as it stands or change the floor pattern depending on the class you wish to use it with (nrich.maths.org). You might even consider using measurements in the first instance for those students who have less confidence when looking at proportion.

- The NRICH activity **Dividing the Field** looks at dividing a trapezium into two parts that have equal area, where each piece is also a trapezium. This is a nice activity to reinforce the formula for calculating the area of a trapezium and thinking about composite shapes (nrich.maths.org).

Assessment ideas

- Ask students, either individually or in small groups, to create model solutions to the longer contextual problems on large paper. These solutions can then be peer assessed by other groups/individuals in the class or shared for display and class feedback. Asking students to explain another's work can help them realise how important it is to show their working fully so as to explain their method clearly and not jumble up their calculations, when tackling multi-step problems.

Topic links

Previous learning

This can be a nice topic to revisit constructions (*Chapter 6 Constructions and loci*), particularly when working with circles and sectors. You could set tasks where students design their own piece of artwork based on circles and sectors and calculate the area of the different colours. In addition, many of these area problems can be based on scale drawings in which the material from *Chapter 12 Units and measurement* can be revisited, alongside the use of measurement conversions from this chapter.

There are also many opportunities throughout this chapter to work with the perimeter of shapes (*Chapter 15 Perimeter*) as well as the area and explore the connections (or lack of it) between them.

Future learning

The formulae learned and used in this chapter will be used again in *Chapter 21 Volume and surface area calculations* to find the area of the faces of a 3D shape.

Gateway to A Level

At A Level, students will encounter area scales when they are used in matrix transformations. They will also begin to study calculus and will learn how to use integration to calculate the area under a curve. It is here that many will first consider area to actually be the sum of infinitely thin strips rather than the number of square units that fill the space.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 2 Questions 9, 10
- Chapter 3 Question 12
- Chapter 4 Question 12
- Chapter 5 Question 9
- Chapter 6 Question 3
- Chapter 7 Questions 3, 4
- Chapter 8 Question 13
- Chapter 9 Question 13

Higher

- Chapter 2 Questions 18, 24
- Chapter 4 Question 1
- Chapter 6 Question 3
- Chapter 9 Questions 6, 11

Homework Book

Foundation

- Chapter 16

Higher

- Chapter 16

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 7 worksheets (+ solutions)
- 5 animated widgets
- 12 interactive walkthroughs
- 4 auto-marked quickfire quizzes
- 4 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

17F / 17H Approximation and estimation

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- have a strong understanding of place value;
- be confident using decimals in calculations, particularly those involving division;
- be able to make estimates for calculations to decide if a solution is reasonable;
- have an understanding of inequality notation (in order to help understanding of bounds).

Additional useful prior knowledge

- To understand how to round to the nearest whole number, nearest ten, nearest hundred etc.
- To be able to round answers to calculations where only integer values are possible.

Learning outcomes

Foundation

Section 1

- To be able to round to the nearest positive integer power of ten and apply this to some real-life examples.
- To round values to a specified number of decimal places.
- To round values to a specified number of significant figures.
- To truncate values and understand when this is useful to apply in context.

Section 2

- To apply the ability to round to one significant figure in order to estimate answers to more complex calculations without using a calculator.

Section 3

- To use inequalities and identify the lower and upper bounds for measurements and use these within calculations to find maximum and minimum solutions.

Higher

Section 1

- To be able to round to the nearest positive integer power of ten and apply this to some real-life examples.
- To round values to a specified number of decimal places.
- To round values to a specified number of significant figures.
- To truncate values and understand when this is useful to apply in context.

Section 2

- To apply the ability to round to one significant figure in order to estimate answers to more complex calculations without using a calculator.

Section 3

- To use inequalities and identify the lower and upper bounds for measurements and use these within calculations to find maximum and minimum solutions.
- Calculate the upper and lower bounds of a calculation (for discrete and continuous quantities).

Vocabulary

rounding, degree of accuracy, decimal place, significant figure, truncate, estimate, continuous variable, lower bound, upper bound, error interval

(Higher only) discrete values

Common misconceptions and other issues

- Many students think that estimation just means writing a ‘rough guess’ which doesn’t involve any calculation.
- Some students also think that rounding is an arbitrary process that simply follows the rule of ‘5 or more round up’, as opposed to understanding the reason for this. The decision about which way to round becomes clearer if you ask students to show the two possible answers from rounding up or down on a number line and identify their halfway point.
- Students commonly assume that multiplication always gives a larger answer. Include examples of multiplication with numbers less than 1 to challenge this perception.
- They also assume that division always gives a smaller answer. Again, provide examples of division with numbers less than 1 to counter this.
- It is easy to confuse rounding with truncation and think that rounding to decimal places means the same as truncation. Activities from Section 1 will clarify this.
- A common mistake is to write that 24 673 to two significant figures is 25 rather than 25 000, in other words just ‘losing’ the unnecessary digits and not maintaining place value. Encourage students to say the number out loud before rounding, to hear the place value expressed, for example ‘twenty four thousand, six hundred and seventy three rounded is twenty five to 2 significant figures’. This helps prompt them to realise their answer is not reasonable where place value has been lost.
- Students often think that when a solution is required to a specified degree of accuracy all answers obtained partway through the calculation should be rounded to this degree. Encourage use of fractions, surds and irrational numbers through calculations, or use of a calculator memory button or the ‘ANS’ key to minimise rounding until the final answer.
- When rounding to decimal places, some students erroneously think that each digit is rounded in turn with a compounding effect. For example rounding 3.4547 to 2 decimal places becomes 3.455 then 3.46 instead of the correct answer of 3.45 (2 dp). Again, representing numbers on a number line can help provide a visual stimulus, with the halfway value clearly identified, to avoid this type of error.

Hooks

The **Estimate180 website** has a different estimation question, using an image as a prompt, for 200 different days. It provides an excellent way of developing number sense in students and would provide a suitable introduction to this topic (estimation180.com).

SECTION 1F / 1H: ROUNDING

Section 1 focuses on different ways of rounding: rounding to the nearest positive integer power of ten; rounding to a given number of decimal places; the terminology of significant figures and how to apply rounding to these; truncating numbers at specific levels of accuracy, rather than using formal rounding techniques, and exploring when this might be applicable.

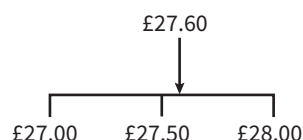
This section recaps much content that should have been covered in previous years. Establishing the areas your students are confident with is particularly important as it is easy to assume they should already understand, but in fact may have gaps or misconceptions. Equally, more able students could become quickly disengaged revisiting material they understood some years ago. Since the book provides a wide range of questions throughout Exercises A to D, using the Launchpad questions and then allowing students to select their starting points for some individual work would seem particularly relevant for this topic. Drawing special attention to the truncation method and its relevance is important, since this may not have been formally taught previously and can be incorrectly applied by students who think they are rounding to a given number of decimal places.

Prompting questions

Exercise 17A(F) / 17A(H)

While working through the exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** Can you suggest possible real-life examples for the figures you are rounding? *Obviously there is a multitude of possible answers here, but it could be good to use items in the classroom; for example they might be measurements of lengths of objects around the classroom, such as mini-whiteboards, pencils, and so on.*
- **Q3(F) / Q3(H)** Can you draw number lines to demonstrate why your rounding is correct? *For example:*



- How might the media use rounding to persuade the public? For example, which sounds more dramatic: The UK national deficit was £107.7 billion or £108 billion in March 2014? What is the increase in pounds if I round this to £108 billion? My original figure was presumably rounded anyway? What is the maximum possible increase that this rounding could have added on to the actual figure? *£108 billion sounds more dramatic, being an increase of 300 million. The lowest value that could have been rounded to 107.7 billion would be 107 650 000 000 so the maximum increase if rounded to £108 billion is 350 million! An additional point to make here is that in real-life numbers do tend to be rounded up, and not necessarily by mathematicians, so the rules may not be stringently followed.*

Exercise 17B(F) / 17B(H)

While working through the exercise, good prompts for students might be:

- What examples can you think of where amounts might be rounded to 1 or 2 dp? *Often seen in measurements such as mass and distance.*
- When might rounding to decimal places be inappropriate? *An inappropriate example might be rounding certain measurements which can only sensibly given as whole numbers because the decimal parts are hard to measure, for example angles or where very large numbers are involved and the decimal part is insignificant. Also, rounding to decimal places can be confusing when working with measurements not based on a base ten system, such as time (e.g. 0.25 hours might be better converted into 15 mins).*
- If I were drawing a scale map, where 1 cm represents 1 km, what effect would rounding to the nearest centimetre have? *The real-life measurements could be smaller or greater by up to 500 metres. Questions like this are useful to help students begin to think about bounds which come in a later section.*
- What is 1.899 rounded to 2 dp? What about 1.999? Why do examples like these make us think more carefully? *1.90 and 2.00; as a result of the nines we need to show the zeros to make it clear we have rounded to the required number of decimal places.*
- What is -2.75 rounded to 1 dp? *This should raise questions, since the answer depends on the method being used and it would be a good opportunity for students to research into different rounding methods such as half away from zero and half towards zero. Which method do you prefer and why? If we agree on a 'half round up' method, which is applied with the positive numbers, we would get -2.7 (1 dp) although rounding to the nearest integer we'd get -3. Rounding away from zero we get -2.8 to 1 dp. There are different systems and applications that might well advocate rounding towards zero.*

Exercise 17C(F) / 17C(H)

While working through the exercise, good prompts for students might be:

- When would using rounding to significant figures be useful? *This can be helpful with very large and very small numbers, when specifying a number of decimal places is irrelevant. In general, rounding to 3 or 4 significant figures should always give a reasonable answer, whereas rounding to 4 decimal places may not.*
- If 25 000 is the answer to a 'Round this number to ...' question, what might the question have been? *There is a range of possible answers, depending on levels of accuracy used. I'd ask students to state their rounding accuracy to each answer they give to help check their understanding. This thought-process leads nicely to the topics of bounds covered in Section 3, so does not need much detail at this stage, but exposing students to it now will help them later on.*

- Ask students to give examples of numbers that they find more difficult to round, for example 1.0005 to 2 sf or 0.000 873 to 1 sf. Discuss their queries to bring out any common misconceptions.
- Thinking challenge: Write down two different numbers that are the same when rounded to:
 - the nearest thousand and 1 significant figure (e.g. 2580 and 3100)
 - the nearest hundred and 2 significant figures (e.g. 190 and 203)
 - 2 decimal places and 2 significant figures (e.g. 0.345 and 0.353)
 - 5 decimal places and 3 significant figures (e.g. 0.001 234 and 0.001 229 99)
 - 1 decimal place and 8 significant figures (24.684 and 24.729).

Exercise 17D(F) / 17D(H)

While working through the exercise, good prompts for students might be:

- Why do we truncate numbers rather than rounding them? *If you learn programming, you'll need to learn about integer division, which is truncation for positive integers.*
- Give some examples to demonstrate when truncating is a relevant form of rounding. *Used in computing, some answers on calculators, and also tax returns as detailed in the point below.*
- Provide some functional examples where truncating has taken place, for the students to decide if they are or are not appropriate. For example, the number of tiles required for a kitchen wall (*depending on the number of calculations involved they are likely to be short of tiles*), number of glasses that can be filled from a bottle (*this is appropriate as truncation will give the number of whole glasses*), the sample size for a stratified sample (*again this is an appropriate use since you can have a decimal part of a person*), number of degrees needed for a category in a pie chart (*truncation throughout could leave you a few degrees short of 360 so rounding to the nearest integer is more useful*).
- When I fill in my tax return each year, the software uses truncation for all the values I enter, so for example an annual bank interest income of £25.83 is recorded as £25 and then tax is calculated. What do you think of this system? Who benefits – me or the tax office? *Me – perhaps surprisingly!*

Other activities, assessment ideas and rich tasks

Enrichment activities

- How many potential examples of rounding can students find in today's news headlines?
- **(Foundation only)** Improving Learning in Mathematics **Mostly Number – N3 Rounding Numbers**, available from the National STEM Centre Archive, will help to consolidate rounding to the nearest power of ten for students who find rounding to integers difficult (nationalstemcentre.org.uk).
- **NRICH Does This Sound About Right?** A good activity to ensure students are able to identify realistic estimations in context (nrich.maths.org).

Starters or plenaries

- **(Foundation only)** A quick starter: Can we afford our shopping? Give students a budget and then read out a range of items to buy and their prices. Ask students to keep an approximate running total in their head to decide roughly how much they should pay? Will any items need to be put back on the shelf?
- **(Foundation only)** For students who still struggle with the concept of place value and decimal intervals, the Interactive Teaching Programs from the old Primary Numeracy Strategy can be very helpful, particularly **Place Value** and **Decimal Number Line**. The full collection of these can be found on the internet (taw.org). Another good visual tool is the **number line 'zoom' tool** (mathisfun.com).
- Make a card matching or a Tarsia puzzle to match amounts rounded to specific levels of accuracy. This can be done using **free software** available from the internet (mmlsoft.com). (There are many ready-made examples available on the internet and the TES website has a good collection to choose from.)
- Create a 'follow me' loop where students read out a number and a rounding level of accuracy and the student with the answer on their card replies and gives the next question.
- The **Number Loving** website has a selection of card sorts and games, 'treasure hunt' and 'tick or trash' activities that could be used for group and class activities to consolidate and practise techniques beyond the textbook exercises (numberloving.co.uk).

SECTION 2F / 2H: APPROXIMATION AND ESTIMATION

This section covers estimation techniques using a calculator.

The worked examples include a question with three different possible estimations and their calculations. It would be useful for students to make connections with this type of problem and the number based ones that are predominant in the exercise, so asking students to write a possible scenario for the estimations, with sensible alternate calculations, could be a good way to develop higher order thinking around this topic.

Prompting questions

Exercise 17E(F) / 17E(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Are your answers to the estimations over or underestimates? *Since this is not always easy to predict, you could ask students to work out the precise answers to the calculations to double check.*
- **Q2(F) / Q2(H)** What are the percentage errors for each calculation? *Part a, percentage errors are: A +9.4%, B -6.5% and C +3.9%, all to one decimal place. Part b: A +3.3%, B -3.6% and C +10.2%, all to 1 decimal place.*
- **Q3(F) / Q3(H)** Why do you think rounding each number to 1 sf is suggested? *It's good for a general rounding level when estimating since it is not affected by the magnitude of numbers involved; it will always take into account place value, unlike when decimal places are specified. What are the limitations of this? It can make estimations very inaccurate, especially for large numbers, as shown previously. Can you give me an example when the actual answer is very different from the estimated answer? Can you give me an example where they are very similar? Ask students to calculate the actual answers and compare with their rounded ones to see this. When figures given are very close to the upper or lower bound for rounding this will create larger differences. This could link nicely to Section 3.*
- Given the students' observations from the above, what are the potential issues of rounding partway through a calculation? *They should see from checking the over- and under-estimates and percentage errors that premature rounding can greatly affect the answer in some cases.*

Other activities, assessment ideas and rich tasks

Starters or plenaries

- MARS Assessments **Division**. This is a quick task that will help to ensure students read worded problems carefully before applying estimation techniques (mathshell.org).
- Get students to mark each other's work. Provide some questions and written solutions where some answers are perfect, but others contain errors or inconsistencies such as a change to the rounding accuracy level. Ask students to mark the work and identify these errors.

SECTION 3F / 3H: LIMITS OF ACCURACY

This section introduces bounds of accuracy and error intervals; the higher textbook leads into associated calculations and considers both discrete and continuous quantities.

Before moving on to this section, ensure that students are confident using inequalities. Connect their understanding by considering the previous sections, where they obtained over-estimates and under-estimates through their calculations, since using bounds will enable them to find the maximum and minimum values of such calculations.

Prompting questions

Exercise 17F(F) / 17F(H)

While working through the exercise, good prompts for students might be:

- If a student's height is 1.7 m, what might their exact height be? *(Any height in the error interval $1.65 \leq h < 1.75$ i.e. lower and upper bounds 1.65 m and 1.75 m.)*
- Why is using the π button on the calculator better than assuming a value of 3. *Because it is not rounded to 2 decimal places therefore will ensure accuracy is maintained during calculations rather than being lost by premature rounding.*

Exercise 17G(H)

While working through the exercise, good prompts for students might be:

- **Q1(H)** and **Q4(H)** How would rounding figures up or down affect calculations involving subtraction? *To maximise the answer to $A - B$ do upper bound of A – lower bound of B , to minimise answer use lower bound of A – upper bound of B .*
- **Q2(H)** and **Q3(H)** How would rounding figures up or down affect calculations involving multiplication? *For numbers greater than 1, to maximise the answer, multiply all the upper bounds, for numbers less than 1 then using the lower bounds will give the maximum. To minimise use the reverse of this.*
- **Q5(H)** How would rounding figures up or down affect calculations involving addition? *To maximise the answer, use all upper bounds, to minimise answer use all lower bounds.*
- **Q7(H)** How would rounding figures up or down affect calculations involving division? *Again this depends on the magnitude of the numbers involved in the calculation. For numbers A and B where both are greater than 1, for the maximum of $A \div B$ do upper bound of $A \div$ lower bound of B ; to minimise the answer use the lower bound of $A \div$ upper bound of B .*
- **Q6(H)** Why do modern calculators give answers in terms of π or in surd form? *To show the precise answer. This can quickly be changed to a decimal answer but in doing so the calculator will have to perform some rounding or truncation. The precise form is very useful to go on to use in a subsequent part of the calculation, to avoid introduction of rounding errors partway through a calculation.*

Other activities, assessment ideas and rich tasks

Enrichment activities

- **NRICH Time to Evolve.** This encourages students to make sensible estimates through their calculations (nrich.maths.org).
- **(Higher only) NRICH More or less?** This asks students to consider a range of scenarios and decide whether a particular estimation would be over, under or exact (nrich.maths.org).
- **(Higher only) Nuffield Foundation Errors.** This uses examples in the context of shape and space to help students understand how big errors may be. Students are required to find the upper and lower bounds of measurements, given the level of accuracy used, and consider how possible errors accumulate (nationalstemcentre.org.uk).

Topic links

Previous learning

Place value and rounding through Key Stages 2 and 3

Future learning

Approximation and estimation are skills that are used regularly during GCSE. While they are particularly useful for a simple check that solutions are of a correct magnitude and for ensuring accuracy during calculations by avoiding premature rounding, they have direct applications within the following topics: using standard form notation (since this requires use of significant figures), see *Chapter 26 Standard form*; trial and improvement (since this requires finding approximate solutions to equations to a specified degree of accuracy, e.g. 1 decimal place, and it is necessary to use a greater degree of accuracy during the preceding calculations, e.g. 2 decimal places), see *Chapter 8 Equations*; algebraic inequalities (since error intervals are stated using these), see *Chapter 38F/40H Algebraic inequalities*; handling continuous data (where writing class intervals is required), see *Chapter 35F/37H Collecting and displaying data*.

Gateway to A Level

Estimation and approximation will be applied to a wide range of types of calculation throughout A Level, not only in mathematics but in any subjects where calculations or measurements are required. There are many situations where accuracy is important, for example working out concentrations in chemistry, or use of formulae in business studies and economics. Within the mathematics content, they have specific application in the Taylor and MacLaurin series and finding approximate solutions to equations by iteration.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 5 Question 5
- Chapter 6 Question 11
- Chapter 7 Question 5

Higher

- Chapter 2 Question 4
- Chapter 5 Question 9
- Chapter 6 Questions 13, 24

Homework Book

Foundation

- Chapter 17

Higher

- Chapter 17

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 6 worksheets (+ solutions)
- 3 animated widgets
- 12 interactive walkthroughs
- 5 auto-marked quickfire quizzes
- 5 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper

18F / 18H Straight-line graphs

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to generate terms in a sequence from a given rule;
- know how to identify coordinates of a given point;
- be able to manipulate and solve equations;
- be able to change the subject of a formula.

Additional useful prior knowledge

- To know how to substitute values into a formula.
- To understand how to plot points on a coordinate grid.

Learning outcomes

Foundation

Section 1

- To use a table of values to plot graphs of linear functions.

Section 2

- To identify the main features of straight line graphs and use them to sketch graphs.
- To sketch graphs from linear equations in the form of $y = mx + c$.
- To find the equation of a straight line using gradient and points on the line.

Section 3

- To identify lines that are parallel by considering their equations.
- To find the equation of a line parallel to a given line (perhaps passing through a known point).

Section 4

- To solve problems involving straight-line graphs.

Higher

Section 1

- To use a table of values to plot graphs of linear functions

Section 2

- To identify the main features of straight line graphs and use them to sketch graphs
- To sketch graphs from linear equations in the form of $y = mx + c$.
- To find the equation of a straight line using gradient and points on the line

Section 3

- To find the equation of a tangent that touches a circle centred on the origin

Section 4

- To solve problems involving straight-line graphs.

Vocabulary

function, coordinates, plot, gradient, y -intercept, x -intercept, coefficient, constant

(Higher only) reciprocal, tangent

Common misconceptions and other issues

- Students sometimes think the constant ($+c$) tells them the step increase (gradient) when plotting coordinates. It is vital to ingrain the understanding that the coefficient of x informs us of the gradient and that the ' $+c$ ' tells us the 'starting point' on the y -axis.

- Students occasionally think the ‘+ c’ tells them the starting point on the x -axis, rather than the y -axis. Substituting some values for x into the equation of the line will quickly demonstrate their mistake.
- When plotting lines, students often do not look at the big picture; if they have made a mistake in one of their y -value calculations, the line is no longer straight, but they will plot their points regardless. To combat this, ask students to stop and think about what they have done. If they know that an equation of the form $y = m + c$ gives a straight line, and their graph is not a straight line, there must be a mistake somewhere in their calculations!
- When introducing students to the use of the gradient and y -intercept to sketch graphs (i.e. sketching graphs from the equation) it is important to note that the gradient is the **coefficient** of x . Students often incorrectly state the gradient is $3x$, rather than 3 (if the equation were, for example, $y = 3x + 2$). This needs to be addressed to avoid misconceptions developing, and you can do so by drawing students’ attention back to the definition of the gradient (the steepness of the line) as a numerical value, not a function of x .

Hooks

Read out a rule (e.g. my y number is twice my x number) and ask students to think of pairs of numbers (x and y) that satisfy this rule. Plot these values on a graph, and ask students what they notice. Then ask students if they could predict any more pairs of numbers that would also satisfy the rule. Can they also write the rule in algebra? The following sets of equations would produce a good variety and provide the opportunity to look at the effect of gradient and y -intercept:

- *Gradient introduction:*
 - My y number is twice my x number.
 - My y number is three times my x number.
 - My y number is the negative of my x number.
- *y -intercept introduction :*
 - My y number is two more than my x number.
 - My y number is four more than my x number.
 - My y number is three less than my x number.
- *These are useful for introducing rearranging equations:*
 - x and y add up to three.
 - x and y add up to five.
 - x and two lots of y add up to seven.

SECTION 1F / 1H: PLOTTING GRAPHS

Section 1 focuses on plotting graphs. In the worked example, the first instruction is to ‘choose some values for x ’. Discuss with students what values of x would be sensible to choose. We can pick any as they are the input values for our function, but it makes sense to use consecutive values close to zero so that we can plot the points on a reasonably-sized pair of axes. It is important to continually stress that students should be careful reading scales on graphs: in particular, if axes do not start from zero or if the two axes use different scales.

Prompting questions

Exercise 18A(F) / 18A(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** What values of x are you using in your table? *Consecutive integer values close to zero (preferably including at least one negative x value).*
- Now you have your x and y values, what scale(s) do you need to draw on your axes? *Scales that accommodate the least and greatest values of x and y .*
- Do both axes need to have the same graduations on the scale? *No.*

- (If a student's graph is not a straight line) What do you notice about your line? Which value do you think may be incorrect? *The point that's not on a straight line with the others.*
- **Q2(F) / Q2(H)** Why could you not just use two points to plot your graph? *The function may not be linear; you may have calculated one of the y-values incorrectly.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Ask students to put a single point somewhere on their page. How many lines can be drawn through it? *An infinite number.* Now draw a second dot. How many lines can be drawn that go through both points? *Just one.* While two points may be sufficient to define a line, a third point adds some reassurance that a mistake has not been made.
- Be a 'human function machine'. Choose a function (a single operation, initially, perhaps doubling) but do not tell students. Get them to say numbers to you, which you apply the function to and tell them the output. Students should aim to guess the function with as few input values and guesses at the function as possible.

Plenaries

- Get students to explain how the y- and x-coordinates of the points that make up a line are related from looking at its equation. For example, $y = x + 4$ has the y-coordinate 4 more than the x-coordinate; $y = 2x$ means the y-coordinate is always double the x-coordinate.
- On a coordinate grid with four quadrants, pretend to be an inept student drawing the graph of a linear equation. Plot at least three points that are not in a straight line. Ask students how to join them up. Encourage students to identify the features that make an equation linear (e.g. no powers, no variables multiplied together) and that such an equation will result in a straight line. If points don't lie on a line, a mistake has been made (most likely because of poor substitution of numbers, especially negatives).

Enrichment activities

- Create a human graph. Using a narrow roll of paper (e.g. display board edging paper or toilet paper) make axes on the classroom floor. Write/print numbers on paper to make the scale on each axis. Call out/project an equation and ask students to create the graph for it. This could be done by asking one student to direct the task or asking a handful of students to find a pair of coordinates that fit the equation. If one student is directing the task, they should call on other students to help them create a table of values and then to stand on the correct coordinates on the grid. To reinforce the concept of a straight line, the students marking the coordinates could be given another roll of paper: if they do not create a straight line, it will be obvious who is in the wrong place because of where the paper bends.

SECTION 2F / 2H: USING THE FEATURES OF STRAIGHT-LINE GRAPHS

Section 2 focuses on using and identifying the gradients and intercepts of straight-line graphs (and includes finding the equation of a straight line through two given points). The chapter begins with introducing the key vocabulary (gradient, y-intercept, x-intercept). It is important that students are familiar with these terms and remember the correct definition of them so as to avoid confusing their meanings in their calculations.

Much teacher talk about gradient will rely on familiar concepts such as 'uphill'. It's vital that students realise that these descriptions make sense only if we consider a line being drawn from left to right.

Prompting questions

Exercise 18B(F) / 18B(H)

While working through the exercise, good prompts for students might be:

- **Q1a(F) / Q1a(H)** What does a gradient of $\frac{3}{2}$ mean? *For every 2 units to the right, go up 3. If I go 20 units to the right, how many up? 30.*
- **Q1b(F) / Q1b(H)** The line has a gradient of $-\frac{1}{3}$. What is the relationship between this line and one with a gradient of $\frac{1}{3}$? *The lines are equally steep, but one slopes 'uphill' and the other 'downhill'. If a mirror was put on the y-axis, they would be reflections of each other. NB: reinforcing that they are, in general, NOT perpendicular is important here.*

- **Q4a(F) / Q4a(H)** How does it change the result if we do ‘the first point take away the second’ versus ‘the second point take away the first’. *It doesn’t. The numerator and denominator will each be the same magnitude as before but the opposite sign (e.g. -3 versus 3). Because both numerator and denominator change sign, the gradient has the same sign as before.*

Exercise 18C(F) / 18C(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** How does the line $y = 3x + 2$ differ from the line in (a), $y = 3x - 2$? Playing devil’s advocate: ‘they both end in 2, so they cross the y-axis at 2’. *While the number at the end is the same, we need to pay attention to the sign in front of it.*
- **Q1(F) / Q1(H)** Which of these lines is steepest? *The line in (a), since 3 is the largest gradient.*
- **Q2a(F) / Q2a(H)** Rearrange $6x - 4y = 12$. What do you notice? *It is the same line as in Question 2a, since both sides of the equation have been doubled.*
- **Q3(F) / Q3(H)** Which of these lines is steepest? Why is this a more difficult question than for the lines in Question 1? *When lines are not in the form $y = mx + c$, it is not possible to immediately determine the gradient.*
- **Q3b(F) / Q3b(H)** Why might we write this equation $y = 3 - x$ rather than $y = -x + 3$? *They are equivalent, so we could! But many mathematicians will avoid starting an expression with a negative if it can be written in another way.*
- **Q6(F) / Q6(H)** How do you quickly know that there will be no constant term on the lines in (a) and (b)? *Because it goes through the origin, therefore the y-axis intercept is also at the origin.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Starters that review some of the prerequisite techniques would be sensible here:
 - Equivalent fractions
 - Subtraction from (and of) negative numbers
 - Division with (and of) negative numbers
 - Substitution into formulae
 - Rearranging linear equations
- Most of these tasks produce a single numerical answer (perhaps including fractions) and so would lend themselves well to be completed on mini-whiteboards for you to gauge understanding.

Starters or plenaries

- Call out a value for the gradient and y-intercept. Ask students to write the corresponding linear equation on a mini-whiteboard and hold it up for you to see. Repeat this with a variety of gradient and y-intercept pairs. This could be run as a competition (either seeing who could get the correct equations, or who can display the correct equations quickest).

Plenaries

- On a square grid, draw a line extending the length of the board (or paper). Draw on a number of different gradient triangles, both ‘above’ and ‘below’ the line. Show that all of these, once simplified, result in the same gradient.
- ‘Simon Says’ graphs. Call out/project linear equations and ask students to demonstrate them using their arms. Students should use their bodies as the y-axis. The differences in gradients should be seen by students changing the angle at which they are holding their arms. The change in y-intercept should be seen by students going on their tip-toes or bending down lower to the floor (for negative y-intercepts). You could give different equations to a few students at the same time and then ask the rest of the class if they have adequately demonstrated the differences/similarities, and if not how they could improve.
- Gradient of a vertical line: establish that as a line gets steeper, its gradient is a bigger number. What happens when the line gets really, really steep (i.e. close to vertical)? How about when it is actually vertical. Infinity might be a reasonable suggestion, but what does the gradient formula give us? The denominator would be zero, which would suggest that the gradient is undefined.

Enrichment activities

- The NRICH computer activity **Diamond collector** provides good practice at working out equations of lines in a game context that requires mathematical creativity (nrich.maths.org).

SECTION 3F: PARALLEL LINES / 3H: PARALLEL LINES, PERPENDICULAR LINES AND TANGENTS

In Foundation, Section 3 focuses on parallel lines. In Higher, perpendicular lines and tangents are also considered.

Prompting questions

Exercise 18D(F) / 18D(H)

While working through the exercise, good prompts for students might be:

- Q1a(F)** What would be the equation of a line that is parallel to the odd one out? *Any line of the form $y = x + c$.*
- Q2a(F) / Q1a(H)** What equation did you solve to find the value of a ? *The equation $2a - 3 = 3$.*
- Q4c(F) / Q3(H)** How did you prove that ABCD is a parallelogram? *Showed that opposite sides are parallel. Any other ways? For example, show diagonals bisect each other or that both pairs of opposite angles are congruent.*
- Q5(F) / Q4(H)** Why can't we write vertical lines in the form $y = mx + c$? *Because their gradient is undefined, therefore there cannot be a value for m . Instead they can be written in the form $x = a$ where a is the x -axis intercept.*

Exercise 18E(H)

While working through the exercise, good prompts for students might be:

- Q1(H)** Convince me that $y = 4x$ is perpendicular to $4y + x = -2$. *The gradient of the second line is $-\frac{1}{4}$. If you multiply $-\frac{1}{4}$ by 4 you get -1, hence they are perpendicular.*
- Q6(H)** Alternatively, how could you use Pythagoras' theorem to show the triangle is right angled? *Work out the lengths of the three sides. If the triangle satisfies Pythagoras' theorem, it is right-angled.*

Exercise 18F(H)

While working through the exercise, good prompts for students might be:

- Q1(H)** Tell me another point on the circle with a tangent of the same gradient. *On the opposite side of the circle at $(-3, 4)$.*

Starters, plenaries, enrichment and assessment ideas

Starters

- (Higher only)** A starter refreshing students' knowledge of reciprocals would be beneficial. For example, project six pairs of reciprocal numbers on the board, mixed up, and ask students to find the matching pairs, or call out a number and ask students to write the reciprocal on mini-whiteboards. Ensure that you include integers, which many students find hard to find a reciprocal for if their understanding only extends to 'a flipped fraction'.

Plenaries

- Write the equation of a line on the board, for example $y = 2x + 3$. Hold a 'closest unique bid auction' where students must write on their mini-whiteboards the equation of a line that is parallel/perpendicular to the given line and has an integer y -intercept. The winner is the student that gets closest to +3 without any other student in the room also having the same intercept.
- Pre-prepare pairs of equations that are parallel, perpendicular or neither. Students could hold up cards to indicate which of the three classifications they believe to be true.

Enrichment activities

- Get students to investigate the effect of plotting a line with a 'normal' reciprocal, rather than negative reciprocal, of a line. For example, plot both $y = \frac{2}{3}x + 7$ and $y = \frac{3}{2}x + 7$. How are they related geometrically? *These would be reflections in the line $y = x + 7$.*
- An unusual use of negative reciprocals is considered in the NRICH problem **Twisting and turning** (nrich.maths.org).

SECTION 4F / 4H: WORKING WITH STRAIGHT-LINE GRAPHS

Section 4 focuses on interpreting straight-line graphs. Students determine the equation of a straight line, calculate the gradient from given information and use graphs to model and solve problems, including simultaneous equations.

Prompting questions

Exercise 18E(F) / 18G(H)

While working through the exercise, good prompts for students might be:

- **Q4a(H)** Why would a value of $m = 1$, so $y = x - 2$ not intersect the first line? *Having the same gradient means it is parallel. Since it has a different y-intercept the lines are distinct, so will not cross.*
- **Q4a(H)** Would the line $y = 1.01x + 999$ intersect $y = x + 3$? They are nearly parallel, after all? *Yes, since any lines that are not parallel will intersect. For those lines, this happens at $(-99600, -99597)$.*
- **Q9(F) / Q11(H)** We are told that this shape is a rhombus. Is there a more specific name for this shape? *A square.* How could you prove this? *Since we know that the four sides are the same length (we are told it is a rhombus), we could just show that adjacent edges are perpendicular.*
- **Q9(F) / Q11(H)** Find the gradients of the diagonals of the rhombus. *0 and undefined.* Are the diagonals of a rhombus always perpendicular, or just this one because it's a square? *Always.*

Starters, plenaries, enrichment and assessment ideas

Starters

- If students are familiar with a method for solving linear simultaneous equations, get them to solve (for example) $x + 3y = 6$ and $2x - y = 5$. Students can then rearrange these equations into the form $y = mx + c$ and sketch them, noting that the point of intersection is the solution to the simultaneous equations. Is this a coincidence that just works for this pair of equations?

Starters or plenaries

- Ask students to choose an equation of the form $y = mx + c$, and plot it on a sheet of graph paper, writing the equation on the **back** of the sheet. These graphs can then be passed along to their neighbour and each student will have a different graph for which they can work out the equation (and check they are correct by looking on the reverse of the paper).

Plenaries

- Give students a graph with a hexagon (or other polygon) drawn on it. Ask students to work out the equations of the lines that make up the edges of the hexagon.

Enrichment activities

- The NRICH activity **Reflecting lines** gets students to think about the equation of lines that have been reflected in the y-axis. In particular, it should encourage students to think about what the gradient represents and how a reflection in the vertical axis changes the gradient (nrich.maths.org).

Topic links

Previous learning

This chapter provides practice at rearranging linear equations, for example to turn $2x + 3y = 25$ into the form $y = mx + c$. For questions where graphs are drawn for students, they should be encouraged to check that their rearranged equation is plausible, for example check that the gradient has the correct sign.

Pythagoras' theorem could be used in a number of these questions, especially those involving circles. Students could investigate Pythagorean triples that allow us to have circles that pass through points with integer coordinates, for example the point $(3, 4)$ on the $x^2 + y^2 = 25$ circle in the text.

Future learning

Chapter 19 Graphs of equations and functions extends the work on straight-line graphs, considering exponential and trigonometric graphs. Problems involving direct proportion are considered in *Chapter 34F/36H Direct and inverse proportion*. These problems can be represented or solved using straight-line graphs. Straight-line graphs are also

used in *Chapter 39H Interpreting graphs*. *Chapter 40H Algebraic inequalities* uses the skills of drawing straight lines to represent algebraic inequalities in the x - y plane.

Tangents to circles are considered in greater depth when investigating circle theorems in *Chapter 34H Circle theorems*.

Gateway to A Level

In linear programming (using graphs to find the optimal solution to a problem), we draw graphs with ‘constraints’ represented as straight lines. It is important to be able to do this quickly and accurately. These are frequently in the form $ax + by = c$, rather than $y = mx + c$. Finding and using the equations of tangents and normal is a major aspect of the use of calculus. We frequently know the gradient of the line as well as a single point it passes through, so finding the equation of the line relies on the ideas in this topic. At A Level we frequently use tangents to circles, which are straight lines. The concepts of gradients and coordinates are used heavily in vector methods, which are particularly helpful in solving problems involving straight lines in three dimensions.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 5 Question 10

Higher

- Chapter 5 Question 10
- Chapter 6 Question 4
- Chapter 8 Question 19

Homework Book

Foundation

- Chapter 18

Higher

- Chapter 18

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 20 **(F)** / 22 **(H)** worksheets (+ solutions)
- 10 animated widgets
- 25 **(F)** / 26 **(H)** interactive walkthroughs
- 9 auto-marked quickfire quizzes
- 9 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper
- 2 mm graph paper
- Axis grids from -10 to 10 in x and -10 to 10 in y

19F / 19H Graphs of equations and functions

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- be able to plot and interpret straight-line graphs including identifying gradients and y-intercepts;
- know how to solve linear equations to find the y- or x-coordinate given the x- or y-coordinate respectively;
- be able to identify or construct parallel lines given an equation of a straight line (in any form);
- **(Higher only)** be able to identify or construct perpendicular lines given an equation of a straight line (in any form);
- know how to generate a table of values from a given function;
- know how to find roots of a quadratic equation algebraically (including completing the square);

Additional useful prior knowledge

- To be fluent when calculating square and cube numbers.
- To be fluent when calculating with negative numbers.
- To be able to apply the laws of indices.
- To be able to set up and solve simultaneous equations.
- To know how to describe a line of symmetry as a straight-line equation.
- To know the properties and definitions of quadrilaterals.

Learning outcomes

Foundation

Section 1

- To be able to work fluently with equations of straight-line graphs.

Section 2

- To be able to identify and plot graphs of quadratic functions (i.e. parabolas).
- To find roots of quadratic equations from the x-intercept of the parabola of the quadratic equation that defines the graph.
- To know the features of graphs of quadratic equations.
- To be able to sketch parabolas.

Section 3

- To work fluently with cubic polynomials and their graphs.
- To be able to sketch cubic graphs.
- To work fluently to calculate reciprocals of numbers and plot functions involving reciprocals.
- To identify hyperbolas and match them to their equations.

Higher

Section 1

- To be able to work fluently with equations of straight-line graphs.

Section 2

- To identify and plot graphs of quadratic functions (i.e. parabolas).
- To find roots of quadratic equations from the x-intercept of the parabola of the quadratic equation that defines the graph.
- To know the features of graphs of quadratic equations.
- To be able to sketch parabolas.

Section 3

- To sketch cubic graphs.
- To work fluently to calculate reciprocals of numbers and plot functions involving reciprocals.
- To identify hyperbolas and match them to their equations.

Section 4

- To plot and sketch graphs from given functions.
- To recognise linear, quadratic and reciprocal graphs.

Section 4

- To plot and sketch graphs from given functions.
- To recognise linear, quadratic and reciprocal graphs.
- To identify and plot exponential graphs.

Section 5

- To represent a circle given its centre on the origin and radius r by a function.
- To identify equations of circles from their graphs.

Vocabulary

parabola, polynomial, reciprocal, hyperbola

(Higher only) exponential function, exponent

Common misconceptions and other issues

- Students find it difficult to make connections between the algebraic world and the geometry it can describe. Reinforce the connection between the two by referring to an equation of the form $y = f(x)$ and asking what graph we can get from it or, given a graph, asking how it could be described algebraically. A graph/equation here is taken to be one of straight-line/linear, parabola/quadratic or hyperbola/reciprocal, as given in the text.
- The main misconception when moving from equation to the graph has been students using their calculators to calculate a table of values for a quadratic function and incorrectly squaring negative numbers. This results in a graph that is not a parabola. Remind students of their number work and ask the question ‘what does happen if we square a negative number’
- When students move from graphing linear functions to quadratic functions they sometimes have a desire to connect the points with straight lines and consequently the plotted points of their quadratic are connected with line segments rather than a smooth curve. Some students struggle to connect the points with a smooth curve and they can be given additional practice by placing their plotted points on the grid into a plastic wallet and treating it like a whiteboard.
- Terminology is very important for this chapter and students often use equation when they mean expression. You can reinforce the correct terminology by using appropriate language at all times.
- Students may not have a good grasp of what the word ‘reciprocal’ means. A starter based on matching up reciprocal numbers could help to reinforce the meaning of the word. Further to this, basic number knowledge of cubed negative numbers is essential for this section in order to calculate a table of values. Again, a starter based on ‘tick or trash’ for the correct calculations for $(-2)^3$ or $- (2)^3$ or $-(2)^3$.
- **(Higher only)** Students’ understanding of what a function actually is can be weak. They can be familiar with ‘function machine’ from earlier material but this can sometimes confuse the situation here where function means something very specific. A function has an input given by a set of values, the domain, and has an output set of values, the range. Students need to use this language to become comfortable with it and must be corrected when using the incorrect vocabulary.
- **(Higher only)** $(x + y)^2$ is not equal to $x^2 + y^2$. This can be addressed by a series of counter examples.
- Something to be aware of: sometimes the language we use causes conflicts in students’ acquisition of knowledge. In this chapter it is down to the word reciprocal. When students learn about reciprocals in number they learn that it is the value you multiply a given number by to make 1, that is it is one divided by their starting number. However, despite the fact students will encounter more reciprocal functions in A Level trigonometry, in this chapter a reciprocal function refers specifically to those in the form $\frac{a}{x}$. It is not one divided by their given function.

Hooks

1. Graphs are a very useful way to display collected data. Being able to add a line of best fit to the data is very useful for predicting what might happen for a different value. Therefore, knowing what different types of graphs look like helps when matching different types of equations to the line of best fit. Have a program like GeoGebra open and allow the students to either make suggestions for the functions $f(x)$ to plot or give common functions (i.e. $ax + b$) and ask what they might look like before plotting.
2. René Descartes used algebra to describe geometry. This connection goes both ways and has allowed mathematicians and scientists to solve problems in algebra via a geometric solution and, vice versa, problems in geometry using algebra. On the most basic level, having a graph of a function $y = f(x)$ shows you straight away if you have any solutions to the equation $c = f(x)$, where c is a constant.

SECTION 1F / 1H: REVIEW OF LINEAR GRAPHS

Section 1 focuses on reviewing the material learnt in *Chapter 18F/18H Straight-line graphs*. The section goes through a recap of graphs of the form $y = mx + c$ and describing vertical and horizontal lines.

Prompting questions

While working on the material in this section, good prompts for promoting discussion may be:

- How many different ways can we represent a straight-line graph at this stage? *The classic equation $y = mx + c$, the more general $ax + by + c = 0$ and as an angle made with either the x-axis or y-axis at the point $(x, 0)$ or $(0, y)$ respectively.*
- How can we describe lines parallel to the x-axis / y-axis? *We can do this with equations of horizontal/vertical lines.*
- How does changing the gradient m in the equation $y = mx + c$ of the line change the graph? *The gradient of the graph controls how 'steep' the straight line is, taken from the positive x-axis. The larger the value of m the steeper the straight line is.*
- How does changing the y-intercept c in the equation $y = mx + c$ of the line change the graph? *The y-intercept controls where the straight line intercepts (cuts) the y-axis.*
- Why do we often prefer an equation of a straight-line graph in the form $y = mx + c$? *Having the straight-line graph represented by the equation $y = mx + c$ makes it easier to read off the gradient and y-intercept of the graph.*

Exercise 19A(F) / 19A(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Can we write down an example for each case? *For 1a $(3,3)$ must lie on the line.*
- **Q2(F) / Q2(H)** What pieces of information do we need to find an equation of a straight line? *The gradient and y-intercept are the most useful pieces of information at this stage.*
- **Q3(F) / Q3(H)** How do we find an equation of a line parallel to a given line? *We need to know the gradient, as the gradients of both lines must be equal.*
- **Q4(F) / Q4(H)** What is different about these equations compared with $y = mx + c$? *They are not in the same format, they are either $x = c$ or $y = c$, where c is an integer and hence given horizontal or vertical lines.*
- **Q6(F) / Q6(H)** What are the definitions of the different types of quadrilaterals?
- **Q7(F) / Q7(H)** How can we find an equation of a straight line given two points? *We need to calculate the gradient first.*
- **Q8(H)** How much information do we need to make a sketch of a straight-line graph? *We need to either know two points or a point and the gradient of the line.*

Starters, plenaries, enrichment and assessment ideas

Starters

- A matching activity for straight-line graphs. Equations and graphs are hidden behind card or moveable rectangles on an interactive whiteboard. Each student gets a chance to match up the graphs to an equation by revealing two cards each time. If they are correct the pair stay revealed, if not they are recovered and you start again.

Starters or plenaries

- A ‘tick or trash’ selection of slides where the students vote on whether the graph displayed is given with the correct equation.

PLENARIES

- A mathematical trail of straight-line graphs: starting with a specific equation and apply a series of ‘transformations’ to the equation to create new straight-line graphs with different properties. For example: start with $y = x$, we now want a line parallel to this but going through the point $(0, 3)$. Take $y = x + 3$ and now we want a straight line that has the same x -coordinate when the $y = 5$ but has a gradient of 3. The students can work as teams or individually and come out to the board to explain.

Enrichment activities

- The NRICH computer activity **Diamond collector** provides good practice at working out equations of lines in a game context that requires mathematical creativity (nrich.maths.org).
- The NRICH activity **Reflecting lines** gets students to think about the equation of lines that have been reflected in the y -axis. In particular, it should encourage students to think about what the gradient represents and how a reflection in the vertical axis changes the gradient (nrich.maths.org).

Assessment ideas

- Since this section is a review and recap of the material from *Chapter 18F/18H Straight-line graphs* students can form pairs and construct revision guides for the material they learnt in that chapter to help them consolidate their understanding before completing the exercises in this chapter. The guides could be given out (not necessarily to the same students who made them) and used to review previous learning.

SECTION 2F / 2H: GRAPHS OF QUADRATIC FUNCTIONS

Section 2 focuses on quadratic functions. The section starts by getting the students to plot quadratic equations from a table of values. This is then extended after Exercise 19B to identify the common features of the graphs (parabolas) the students have just plotted. The key features (turning point, axis of symmetry, y -intercept, x -intercept) are then calculated for a set of examples before inviting the students to do this in Exercise 19C. The section ends with worked examples of sketching quadratic graphs.

Prompting questions

While working on the material in this section, good prompts for promoting discussion might be:

- What is a parabola? The curve we get from graphs given by quadratic equations.
- What form can an equation for a parabola take? $y = ax^2 + bx + c$.
- How do we get ‘u’ shaped parabolas? How do we get ‘n’ shaped parabolas? *The coefficient of the x^2 term, a , in $ax^2 + bx + c$ controls this particular part of a parabola.*
- What happens when you square negative numbers? *They become positive.*
- How can you plot the graph of a function given a table of values? *The x and corresponding y values form a coordinate, all of which sit on the curve of the quadratic.*
- Where does the parabola cut the y -axis? *At the point where $x = 0$.*
- Does the parabola given by $y = ax^2 + bx + c$ cut the x -axis? What does this tell you about the quadratic $ax^2 + bx + c = 0$? *That it has either two intercepts (cuts the x -axis twice), one (touches the x -axis) or none (does not touch or cross the x -axis) solutions.*
- Where is the turning point on your graph? *The point where the gradient is zero known here as the vertex of the parabola (Higher only) or simply where the curve ‘turns’ (Foundation only).*
- What is our procedure for sketching a parabola given by $y = ax^2 + bx + c$? How can we start? *We complete the square on our quadratic expression $ax^2 + bx + c$, since this makes it easier to determine the coordinates of the maximum/minimum point. Or we could determine the shape of the parabola, given by $a > 0$ or $a < 0$. Or we could attempt to factorise the quadratic $ax^2 + bx + c$ to find the roots. Or we could find the y -intercept by substituting $x = 0$ into $ax^2 + bx + c$.*

Exercise 19B(F) / 19B(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** and **Q2(F) / Q2(H)** What happens when you square a negative number? *You get a positive number.*
- **Q3(H)** The questions in this section are excellent for promoting thought.
- **Q4(H)** If $y = f(x)$ what does it mean for $f(x) = 0$? *We want the intersection of the curve $y = f(x)$ with the line $y = 0$.*

Exercise 19C(F) / 19C(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** What does it mean to have an axis of symmetry? *The line about which the curve is symmetric.*
- **Q2(F)** How can we determine if the coefficient of the x^2 term is positive/negative?
- **Q2(H)** Where have you seen this type of form before for a quadratic expression? What happens when $x = h$? What happens when $x = 0$? *This is completing the square. When $x = h$ we have a maximum or minimum for the parabola. When $x = 0$ we get the y-intercept.*

Exercise 19D(F) / 19D(H)

While working through the exercise, good prompts for students might be:

- **Q1(F)** What is the basic shape of a parabola? What happens when x^2 has a coefficient that is negative? *The parabola is n-shaped rather than u-shaped.*
- **Q2(F)** Could we sketch a few examples?
- **Q2(H)** How could you use Question 1 to help answer this question? *Question 1 can give a guide to the type of features each parabola should have given its equation.*
- **Q3(F)** How do we do this? *We are looking at the intersection of the parabolas with $y = 0$.*
- **Q4(F) / Q2(H)** What is our process for sketching a parabola? *See bullet points before worked example.*
- **Q5(F) / Q1(H)** How could we group the graphs? What does this mean for each graph that can be paired together? *Some parabolas are n-shaped; some are u-shaped. This tells us that the coefficient of the x^2 term is either negative or positive respectively.*

Starters, plenaries, enrichment and assessment ideas

Starters

- ‘Guess the graph’ to get students in the mindset of recognising a parabola. Have a series of pictures that are either parabolas or not.

Starters or plenaries

- Having a few quadratic equations on the board, which need to be sketched within a certain timeframe. Points are awarded for accurate sketches and appropriate labelled information. This can be turned into a quick-fire game, with two students competing in front of the class.

Plenaries

- Have a series of graphs, with their equations, where some information is given from the list: turning point, y-intercept, x-intercept and axis of symmetry. The students need to decide if all information that can be given is given about the parabola. If it is not, they need to either find the missing information through calculations, or justify why it was not present in the first place.

Enrichment activities

- The NRICH activity **Parabolic Patterns** allows for the use of GeoGebra or a graphical calculator to give the students a chance to consolidate their recently gained knowledge of identifying an equation of a parabola (nrich.maths.org).
 - For Foundation students you may want fewer parabolas;
 - For Higher students you can increase the difficulty by attempting this activity from NRICH: **More Parabolic Patterns** (nrich.maths.org).

Assessment ideas

- Allowing the students to work in pairs to explore graphs of equations of the form $y = ax^2 + bx + c$ by varying the coefficients graphically first and then requiring them to discuss their paired work in a group of four. Once they have discussed what they have found as a group they can present or compile an A3 size poster on what they found. Either the presentations can be peer assessed or the poster can be displayed in the classroom and peer assessed. Key issues can then be summarised as a class and any misconceptions highlighted.

SECTION 3F / 3H: GRAPHS OF OTHER POLYNOMIALS AND RECIPROCALS

Section 3 focuses on sketching polynomials of degree higher than 2 and graphs of reciprocal functions. The word polynomial is brought in as key vocabulary in this section but should be introduced as early as possible when defining what a quadratic expression/equation is. During this section the common features of cubic graphs are discussed and sketching reciprocal functions from a table.

Prompting questions

While working on the material in this section, good prompts for promoting discussion might be:

- How do we write a general quadratic expression? $ax^2 + bx + c$, *a not equal to zero*.
- How can we write a general cubic expression? $ax^3 + bx^2 + cx + d$, *a not equal to zero*.
- What are the similarities and differences between cubic and quadratic graphs? *Both graphs always have a y-intercept. Cubic graphs always have at least one x-intercept. (Higher only: how could we prove this?)*
- What does the coefficient of the x^3 term in a cubic function control? *Either we get an increasing curve ($a > 0$) or a decreasing curve ($a < 0$).*
- What is the reciprocal of 3? $\frac{1}{x}$
- What is the reciprocal of x ? $\frac{1}{x}$
- What is the general form for a reciprocal function? $y = \frac{a}{x}$, *a not equal to zero*.
- What is the name of the graph we get? *A hyperbola.*
- What happens as the x values get closer and closer to 0? *If we approach from the positive x values, we get 1 divided by a very small positive number, which gives a very large positive number. If we approach from the negative x values, we get 1 divided by a very small negative number, which gives a very large negative number.*

Exercise 19E(F) / 19E(H)

While working through the exercise, good prompts for students might be:

- Q1(F) / Q1(H)** What is the result $-(-2)^3$? 8.
- Q2(F) / Q2(H)** Use Question 1 to help.
- Q3(F) / Q3(H)** What patterns have you noticed? *The negative cubic graphs are a reflection in the y-axis.*
- Q4(F) / Q4(H)** How do these two graphs look in comparison with $y = x^3$? *They are the same shape but moved up in the positive y-axis direction or down in then negative y-axis direction.*
- Q5(F) / Q5(H)** What patterns have you noticed? What can you say about $y = x^3 - 9$? *This curve will be like $y = x^3$ but moved down so that it crosses the y-axis at (0, -9).*
- Q6(F) / Q6(H)** What is happening each time to our basic cubic $y = x^3$? *It is being 'translated' up and down the y-axis.*
- Q7(H)** How could you keep track of all the substitutions required in these questions? *We could create an extended table to calculate each of the components.*

Exercise 19F(F) / 19F(H)

While working through the exercise, good prompts for students might be:

- Q1(F) / Q1(H)** What do we calculate first: $\frac{1}{x}$ or $x + 1$ for $y = \frac{1}{x} + 1$? *We do $\frac{1}{x}$ and then add one.*

- **Q2(F) / Q2(H)** Can we use Question 1 to help with this question? *We are taking the basic reciprocal graph $y = \frac{1}{x}$ and adding positive numbers to each of the y-coordinates.*
- **Q4(F) / Q3(H)** What does it mean 'line of symmetry'? How can we try an example?
- **Q5(F) / Q4(H)** What happens when we divide a negative number by a negative number? *We get a positive number.*
- How do these graphs compare with the graph of $y = \frac{1}{x}$ and $y = \frac{1}{x}$? *They are stretched or compressed versions of them.*
- **Q6(F) / Q5(H)** How can we use Question **5(F)** or **4(H)** to help us answer this question? *We have a set of graphs on the same grid we can use to compare the different functions.*
- **Q6(H)** Which values of x should we not use here? $x = -2$ for part b and $x = 2$ for part c.
- **Q7(H)** Can we predict what the graphs will look like before we plot them?
- **Q8(H)** Have we seen anything like this before? Are there any negative y-coordinates? *The quadratic graph $y = x^2$ has no negative y-coordinates.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- What type of graph? Have a series of graphs displayed based on the graphs from this chapter. The students must identify the correct type of graph.

Enrichment activities

- As an introduction to the topic of reciprocal graphs the NRICH activity **More Realistic Electric Kettle** could be used. This activity also has connection to a task that could happen in the students' science lessons and offers an opportunity to combine such a lesson. The students get a chance to plot a curve and find a way to figure out the equation of the line (nrich.maths.org).

Assessment ideas

- Students have learned about a new graph to add to their bank of common graphs in this section. Students could make a revision guide accompanied by a question for a reciprocal graph. They then give their guide and question to another student who answers the question and comments on the revision guide. The guides can be gathered together and stored for revision purposes.

SECTION 4F: PLOTTING, SKETCHING AND RECOGNISING GRAPHS

Section 4 focuses on consolidating the material from Sections 1 to 3. The section contains an overview of the material as bullet points and an exercise.

Prompting questions

While working on the material in this section, good prompts for promoting discussion might be:

- How do we tell which type of graph we have? Is it a line or curve? *The shape of the curve distinguishes certain graphs and if they are continuous functions (that is you can draw your graph by not taking your pencil off your graph paper).*
- How do we determine which type of algebraic function we have? *Polynomials differ from reciprocal functions. Polynomials can be distinguished by their highest power of x .*

Exercise 19G(F)

While working through the exercise, good prompts for students might be:

- **Q1(F)** How could we test some of the claims in the questions? *We can substitute values in or use examples given in exercises before this one.*
- **Q2(F)** What do the equations have in common? *They all give straight lines.*
- **Q3(F)** Which equations will give the same type of graphs? *(a), (b), (c) are quadratic functions, (d) is a cubic functions and (e) and (f) are reciprocal functions.*

- **Q4(F)** How do we find x-intercepts for quadratic functions? *We look where the line $y = 0$ crosses the parabolas.*
- **Q5(F)** What does it mean for a quadratic equation to have two roots? *This means the parabola given by the quadratic equation crosses the x-axis twice.*
- **Q6(F)** How can we categorise the graphs and equations? *Reciprocal, straight-line, quadratic and cubic graphs.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- The NRICH activity **Guessing the Graph** can be used for students to make their own data, plot the data and use their knowledge of the graphs of different functions to suggest possible curves to fit to the data (nrich.maths.org).
- This NRICH activity **What's That Graph?** can be used to draw connections between the physical world and the mathematics the students have covered in this chapter (nrich.maths.org).

SECTION 4H: EXPONENTIAL FUNCTIONS

This section defines what exponential growth is and what exponential functions are. Students will be exposed to the terminology of exponent and will need to be fluent at manipulating indices.

Prompting questions

While working on the material in this section, good prompts for promoting discussion might be:

Exercise 19G(H)

While working through the exercise, good prompts for students might be:

- What is the value of x^0 , where x is any number? 0.
- **Q1(H)** How should we connect the point calculated in our table of values? *With a smooth curve.*
- **Q2(H)** Which part of the graph $y = 4^x$ will be above/below the curve $y = 2^x$? Why? *The critical value here is $x = 0$. Any number raised to the power zero is 1. Thus, for $x > 0$ $y = 4^x$ 'dominates' $y = 2^x$. However, when $x < 0$ $y = 2^x$ 'dominates' $y = 4^x$.*
- **Q3(H)** How does this graph compare with the one from Question 2? $y = \frac{1}{4^x}$ is the image of $y = 4^x$ reflected in the y-axis.

Why? We can write $\frac{1}{4^x}$ as 4 to a negative power of x using the index laws.

- **Q4(H)** What happens if you raise 1 to any power? *It is still 1.*
- **Q5(H)** What value is the base of the exponent and why does this matter? *0.85, which is less than one so powers of 0.85 will produce smaller numbers than 0.85.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- An activity where students are asked to suggest an equation for a given graph. The class can debate the students' responses.

Assessment ideas

- The students can use mini-whiteboards to sketch the graphs of equations given to them.

SECTION 5H: CIRCLES AND THEIR EQUATIONS

Section 5 focuses on an equation of a circle with its centre on the origin and a radius of r . The students will be required to recognise circle equations and suggest circle equations for a given graph.

Prompting questions

While working on the material in this section, good prompts for promoting discussion may be:

- How are circles described? *One way is a locus of points a given distance away from a centre.*
- What will equations of circles look like for us? $x^2 + y^2 = r^2$.

Exercise 19H(H)

While working through the exercise, good prompts for students might be:

- **Q1(H)** and **Q3(H)** How can we tell when a point lies on any curve, not just a circle? *We can substitute the x- and y-coordinates into the equation.*
- **Q2(H)** and **Q4(H)** How can each of these questions help you answer the other? *Question 2 asks you to sketch the circles from an equation, while Question 4 asks you to determine an equation for each of the drawn circles.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Given a set of circles match them to their equations.

Enrichment activities

- Given a set of circles of different radii with centre on the origin and equations of the circles not centred on the origin, see if the students can:
 - firstly, identify which circle equations are based on which original circles (before moving them from the origin) using graphing software;
 - secondly, discuss and investigate how their equations are connected.

Topic links

Previous learning

This chapter links back right back to *Chapter 2 Whole number theory*, *Chapter 3 Algebraic expressions* and *Chapter 4 Functions and sequences* for the basic manipulation of numbers and algebra. There are also strong connections with *Chapter 18 Straight-line graphs* (which is recapped in this chapter).

(Higher only) There are also connections with *Chapter 6H Construction and loci*.

Future learning

The future chapters of *Chapter 33F/35H Discrete growth and decay* and *Chapter 34F/36H Direct and inverse proportion*.

(Higher only) This chapter has strong links with *Chapter 34H Circle theorems*, *Chapter 39H Interpreting graphs* and *Chapter 41H Transformations of curves and their equations*.

Gateway to A Level

A Level mathematics further develops the connections between algebra and geometry. Students who go on to A Level will be expected to be fluent in moving between descriptions of the function as a graph and as an equation, and will further their knowledge of circles that not centred on the origin. Further to this they will need to answer questions of the type: given two graphs $y = f(x)$ and $y = g(x)$ on the same pair of axes, what do the number of intersections mean for the solutions of the equation $f(x) = g(x)$?

From an application point of view, simple harmonic motion, projectiles and parabolic motion are studied and have foundations in this chapter.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 3 Question 13

Higher

- Chapter 3 Questions 11, 18
- Chapter 8 Question 5

Homework Book

Foundation

- Chapter 19

Higher

- Chapter 19

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 10 (F) / 21 (H) worksheets (+ solutions)
- 4 (F) / 8 (H) animated widgets
- 13 (F) / 21 (H) interactive walkthroughs
- 4 (F) / 7 (H) auto-marked quickfire quizzes
- 4 (F) / 7 (H) auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper
- 2 mm graph paper
- Axis grids from -10 to 10 in x and -10 to 10 in y

20F / 20H Three-dimensional shapes

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to identify common 3D objects;
- recall basic properties of polygons and common 3D objects;
- know how to accurately construct lines and angles using ruler and compasses.

Additional useful prior knowledge

- Understand the meaning of the terms face, vertex and edge.
- To know how to calculate interior angles in polygons.

Learning outcomes

Foundation

Section 1

- To apply what you already know about the properties of 3D objects.

Section 2

- To work with 2D representations of 3D objects.

Section 3

- To construct and interpret plans and elevations of 3D objects.

Higher

Section 1

- To apply what you already know about the properties of 3D objects.

Section 2

- To work with 2D representations of 3D objects.

Section 3

- To construct and interpret plans and elevations of 3D objects.

Vocabulary

isometric grid, plan view, elevation view

Common misconceptions and other issues

- Students sometimes muddle the meanings of face and vertex. Ensure the correct definitions are given at the start of the topic, and reinforced whenever talking about 3D objects.
- The difference between a prism and a pyramid can be difficult to grasp. To tackle this, ask students to think what would happen if you cut slices of the shape? Would they all be identical (like a stereotypical loaf of bread)? If so, the shape is a prism. A pyramid can be identified because it has an apex (point) where all but one of the sides (the base) meet.

Hooks

Many optical illusions 'work' by exploiting the limitations of two-dimensional representations of three-dimensional objects. Few maths teachers would pass up the opportunity to show students the work of MC Escher, for example the 'impossible staircases' in his lithograph **Relativity**. Many students are fascinated by such illusions and, even if they have seen a particular illusion before they can sit and feel smug that they 'know' the answer. You can search the internet for videos to provide useful hooks into this topic.

SECTION 1F / 1H: REVIEW OF 3D SOLIDS

Section 1 is a review of material students should already be familiar with. It provides a summary of the main properties of some polyhedra along with examples of how solids can be described.

Prompting questions

Exercise 20A(F) / 20A(H)

While working through Exercise 20A, good prompts for students might be:

- **Q3(F) / Q3(H)** Can you think of any other polyhedra that you might find being used in everyday situations?

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- There is much merit in students physically manipulating 3D objects. Resources such as **Polydron** are well regarded in helping students understand properties of the 3D objects (polydron.co.uk).
- NRICH activity **Redblue**. This activity looks at colouring the vertices of 3D shapes either red or blue, according to whether you travel an even (red) or odd (blue) number of edges from the original red vertex to reach them. It provides students with practice at using correct mathematical vocabulary and helps them to visualise the solids (nrich.maths.org).

SECTION 2F / 2H: DRAWING 3D OBJECTS

Section 2 focuses on drawings of 3D objects, both on plain and isometric paper. Having linking cubes (e.g. multilink) available to students can be useful when introducing this topic.

Prompting questions

Exercise 20B(F) / 20B(H)

While working through the exercise, good prompts for students might be:

- **Q3(F) / Q3(H)** Which face/cube are you going to draw first? *Students should try to draw the horizontal faces of the shape first. They might also find it easier to draw the front, highest, right-most cube to begin with, as this will have three faces showing, whereas others in the shape might only have one or two faces that need to be drawn.*
- **Q3(F) / Q3(H)** How many cubes might the first solid be made from? *Between seven (the ones shown in the diagram) and nine (there might be hidden cubes on the base, including one adjoining the bottom left cube).*
- **Q3(F) / Q3(H)** I knock over the third solid so that it falls backwards (i.e. four cubes are touching the table). Draw what it would look like.
- **Q4(F) / Q4(H)** The first solid is sitting on a mirror. Draw what it would look like (both the object and image).
- **Q5b(H)** Draw what the blue solid would look like if I remove all eight corner cubes. How many cubes would remain?
19.

Starters, plenaries, enrichment and assessment ideas

Starters

- When using isometric paper, it is vital that students have the paper in the correct orientation: portrait and landscape will result in different drawings. Students should typically orient the paper so there are vertical lines (but no horizontal). Many students really struggle using isometric paper: suggesting that they start by drawing a 'Y' shape for the front-top vertex of a cube helps them get started. It might be necessary to remind students that they should always draw along the grid lines.
- Give students linking cubes. How many different solids can be made from four cubes? What constitutes 'different'? *There are eight 'tetracubes'. Draw some of your tetracubes (differentiation possible here by selecting easier or harder solids). Draw the $4 \times 1 \times 1$ cuboid tetracube in as many different orientations as you can on isometric paper? (The eight tetracubes fit together to form a $4 \times 2 \times 2$ cuboid, but it is rather difficult to achieve!)*

Starters or plenaries

- The topic provides good opportunities for students to practise their visualisation skills. Using the solids made of cubes shown in the book, students could sketch the result of (a) objects being pushed over (b) cubes being removed (c) cubes being added (d) solids sitting on, or against, mirrors.

Plenaries

- Get students to think about the effects of representing 3D objects on 2D paper. What angle is formed on each face of a cube at the vertices? *90 degrees, since they are squares*. On isometric paper, if you draw a cube, what angles are formed? *120 (since three equal angles around a point) or 60 degrees (angles on a straight line)*. So, is it still a square?

Enrichment activities

- Most students enjoy using isometric paper to draw ‘impossible’ objects such as a Penrose Triangle or Rectangle.

SECTION 3F / 3H: PLAN AND ELEVATION VIEWS

Section 3 focuses on representing 3D objects with plan and elevation views. Students often struggle to accept that the plan view does not show if a side is sloping, and will draw trapezia rather than rectangles. It might be useful to have physical objects (e.g. a Toblerone® box or a trapezoidal prism) that students can look down on and/or take photos of to see for themselves. A teacher taking photos from a great height would emphasise the point more fully (e.g. from a mezzanine floor or a first floor balcony) since at this distance you are too far away to see the slope of the sides.

Prompting questions

Exercise 20C(F) / 20C(H)

While working through the exercise, good prompts for students might be:

- Q1(F) / Q1(H)** Can you sketch what these objects would look like from the front?

Starters, plenaries, enrichment and assessment ideas

Starters

- Satellite images (for example, from Google Maps or Google Earth) provide a ready source of plan views of surroundings that should be familiar to students. Projecting satellite imagery on the board and looking around the local neighbourhood can highlight how difficult it is to recognise even familiar landmarks from the air. For example, find the nearest playground and try to identify the swings. (Such imagery can also stimulate fruitful discussions about shadows.)

Starters or plenaries

- Project/display the floor plans of a property and ask students to describe the property. For example, is it a house, flat, bungalow? Does it have an extension on just one floor? What would the front of the house look like (in terms of where are the windows, front door, etc.)? Property-selling websites have a plethora of properties for sale, with floor plans and photos available to view (providing an opportunity to see if students were correct with their answers).

Enrichment activities

- Students could take three photographs (plan, front, side) of familiar objects. These should be shared with the rest of the class, who have to guess what the objects are as quickly as possible. Ideally students should be encouraged to show the least obvious one first to increase the difficulty.

Assessment ideas

- The Freudenthal Institute has many **interactive applets** that students can work through that assess their ability to interpret views and elevations. Since they are written in Java, students need to work on these individually or in pairs on desktop or laptop computers. They provide clear indicators for teachers to see how students are getting on. The ‘building houses’ and ‘colouring sides’ applets are particularly useful (fi.uu.nl).

Topic links

Previous learning

This topic provides an ideal opportunity to practise the concepts learned in *Chapter 5 Properties of shapes and solids*, in particular the vocabulary of 2D shapes and ideas of symmetry. This could be extended to include planes of symmetry of 3D objects.

Future learning

Volume and surface area of 3D shapes are covered in *Chapter 21 Volume and surface area* and students will need to be able to visualise in three dimensions to tackle problems that use Pythagoras' theorem in solids in *Chapter 31H Pythagoras' theorem*.

Gateway to A Level

The volumes of solids of revolution are considered at A2. These are solids formed by rotating a curve around some straight line, for example the sorts of objects that might be produced on a lathe.

Working with the three-dimensional solid called a parallelepiped is a common source of problems when using vector methods.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 8 Question 14

Higher

- Chapter 8 Question 6

Homework Book

Foundation

- Chapter 20

Higher

- Chapter 20

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 8 worksheets (+ solutions)
- 3 animated widgets
- 4 interactive walkthroughs
- 2 auto-marked quickfire quizzes
- 2 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper
- Square dotted paper
- Isometric grid paper
- Isometric dotted paper

21F / 21H Volume and surface area

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to identify 3D objects from a description;
- know how to calculate area and perimeter of 2D shapes;
- recall the definition and properties of 2D and 3D shapes;
- be able to apply Pythagoras' theorem to calculate unknown lengths in right-angled triangles.

Additional useful prior knowledge

- To understand what it means for two shapes to be congruent.
- To know the units of measurement and metric conversions for lengths, area and volumes.
- To know how to form and solve equations with an unknown from a given text.
- To be able to use mathematical formulae; either to substitute first and solve for an unknown or to rearrange to change the subject then substitute to calculate the value of the unknown.

Learning outcomes

Foundation

Section 1

- To calculate the volume of prisms (including cylinders).
- To calculate the surface area of prisms (including cylinders).

Section 2

- To calculate the volume and surface area of a cone.
- To calculate the volume and surface area of a sphere.
- To calculate the volume and surface area of composite 3D shapes.

Section 3

- To find the volume and surface area of a pyramid.

Higher

Section 1

- To calculate the volume of prisms (including cylinders).
- To calculate the surface area of prisms (including cylinders).

Section 2

- To calculate the volume and surface area of a cone.
- To calculate the volume and surface area of a sphere.
- To calculate the volume and surface area of composite 3D shapes.

Section 3

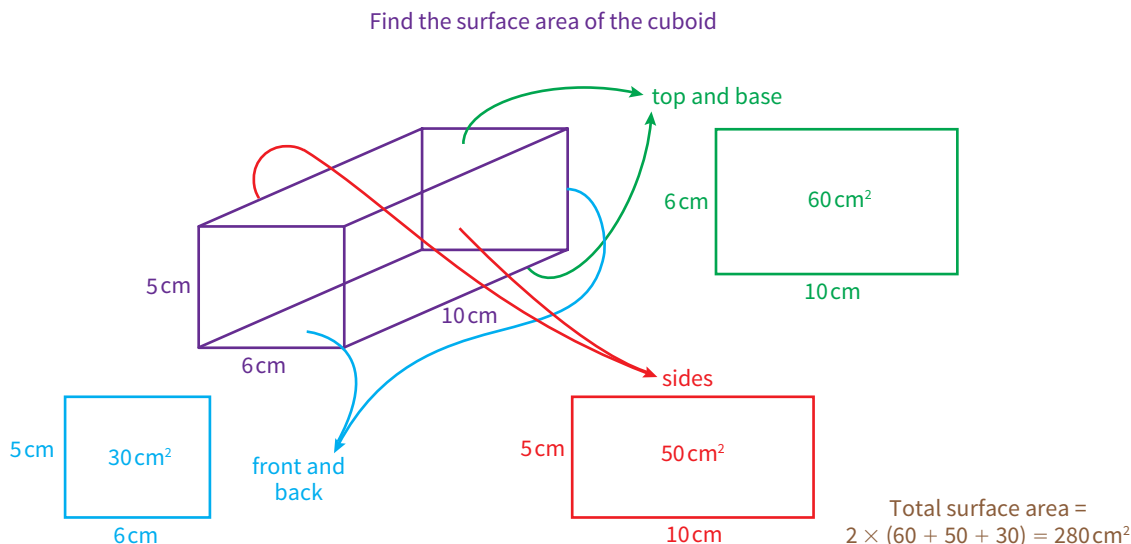
- To find the volume and surface area of a pyramid.

Vocabulary

right prism

Common misconceptions and other issues

- A common issue is confusing the units for area and volume when calculating surface area. When students have a 3D shape they often think that they need to use a unit of volume. Encouraging them to draw nets for each 3D shape they are required to find the surface area of will focus their attention on calculating area so they realise that they need the units of area. Some weaker students find sketching a net and identifying and labelling clearly the dimensions of each face on their net very challenging. For these students, rather than having the faces joined in the net of the shape, it might be more helpful to have arrows coming out from the faces of the shape with a sketch of each face and its dimensions, for example:



- For triangular prisms some students will not realise that they need to use the perpendicular height of the end triangle to calculate the area of the cross section when the question gives the slant height. A series of starters where the students are required to calculate the area of different 2D shapes with varying amounts of information on the diagrams will give them the opportunity to think about what pieces of information they need for each calculation they want to perform.
- Students often treat all three rectangular sides as equal, particularly for prisms with isosceles cross-section; a way to help them spot this is to suggest they make a net showing the dimensions before they calculate the surface area.
- When calculating with cones, students often get confused with slant height and perpendicular height, since the first is used in surface area calculations and the second when calculating volume. To address this, the students can create physical examples of cones that are labelled so that the students can unravel them. The slant height label is present on the unravelled cone while the perpendicular height can no longer be seen. This should help to make the connection that the slant height is needed for calculation of the surface area but not for volume.
- When working with spheres, students often confuse the formulae for surface area and volume. Having a matching activity that has the formulae for volume and surface area for each of the 3D solids they need and pictures of each solid with the title 'surface area' or 'volume' can be repeatedly used as a starter to increase familiarity with the correct formulas.
- Students can, in general, identify good places to split composite solids up. However, keeping track of all this information for each component solid requires good book keeping. Having a table to collect all the information for the volume and surface area calculations for each component solid is a useful way to address this issue.
- When calculating solid shapes that have been cut, for example a hemisphere, students sometimes forget to add in the cut surface in the surface area calculations in the same way that students often forget to include both radii in the perimeter of a circle sector. To tackle this problem you could have a series of solid shapes that you slice to create new solids and ask what 2D shape is formed when we make the cut and what contribution will this make to a surface area calculation for the new solid.

Hooks

- You might like to start a discussion on area with a packet of biscuits, using biscuits that are roughly one centimetre thick. Students first calculate the area of the cross section of their biscuit and then can discuss how many cubes would fit inside it and how they could describe the packet. Having roughly centimetre thick biscuits makes it a simple calculation to multiply the biscuit area by the number of biscuits in the packet to calculate the volume and, if students always consider volume to be broken up into these equally sized slices, the biscuit analogy can be recounted throughout volume problems.
- Acquiring 3D shapes with the same cross-sectional area and depth that can be filled (with either sand or water) is a nice way to get students thinking about connections between the volume of pyramids and prisms. There are many animations that can be found online to show this but it is much more fun to watch students guess how full the prism will be when pouring in the sand/water from the pyramid.

SECTION 1F / 1H: PRISMS AND CYLINDERS

Section 1 focuses on calculating volume and surface area of prisms and cylinders. The section opens with identifying what prisms the students will meet in the section and goes on to state how to calculate the volume for any prism. What follows is how to calculate the surface area of each prism and then the section finishes with a section on rearranging formulae and 'other prism'.

The second topic of the section, cylinders, is then explored starting with calculating the surface area of a cylinder from its net. The worked example following this opening gives the formula for calculating the volume of a cylinder.

(Foundation only) The problem-solving framework gives a detailed breakdown of how to answer an extended problem involving surface area and volume for a given question.

Prompting questions

While working through this section, good prompts to promote discussion might be:

- What objects in the room are prisms? What makes them a prism? *They have two parallel and congruent polygons end faces and a uniform cross section.*
- What does it mean for one side to be perpendicular to another? *The two sides are at right-angles to each other.*
- What units do we use for volume? *Our units are cubed.*
- What are the units for area? *Our units are squared.*
- How can we identify the cross section shape for calculating the volume of our prism? *If we have a prism, we are looking for two end faces that are congruent polygons.*
- Do all cylinders have the same sketch of a net? *Yes, the only lengths that change will be the radius (which will in turn change the circumference of the end faces) and the height of the cylinder.*

Exercise 21A(F) / 21A(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** What does the net of each of these 3D shapes look like?
- **Q2(F) / Q2(H)** How do we calculate volume of a rectangular prism? *We find the area of the any triangular face and multiply by the third length given.*
- **Q3(F) / Q3(H)** How do we calculate the volume of any square prism? *Given the area of the square cross-section (found by squaring the side length) we then multiply by the depth.*
- **Q4(F) / Q4(H)** Can we draw a sketch to help? *We have a rectangular prism.*
- **Q5(F) / Q5(H)** How do we calculate the volume of a triangular prism? What is the area of the cross sectional face? *We have the area of the triangle multiplied by the prism length. We require a perpendicular height for the triangle face.*
- **Q6(F) / Q6(H)** What do we wish to calculate here, is it a volume or an area? *We want the area of a rectangle.*
- **Q7(F) / Q12(H)** What do the markings on the diagram tell us about the shape? *That the sides with the same type of markings are equal in length.*
- **Q7(H)** Can we sketch a picture of the $\frac{3}{4}$ filled tin to help?
- **Q8(F) / Q13(H)** What are the shapes we need to calculate the volume of? *A rectangular prism and a cylinder.*
- **Q8(H)** What is the net of this water tank? *It is a cylinder so it has the same net.*
- **Q9(H) and Q10(H)** What 3D shape are we creating each time? *A cylinder.*
- **Q11(H)** What is the net of a cube? *It has six congruent squares.*
- **Q14(H) and Q15(H)** What would a labelled diagram look like for this question? *Each side would be labelled with one of the lengths given.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Guess the shape. Given a solid, students are required to find the 3D shapes that make up this composite 3D shape. Once they have identified all the basic 3D shapes in the composite shape, you can ask them ‘what would be the minimum amount of side length information you would need to calculate (i) volume; (ii) surface area?’

Starters or plenaries

- A calculation for surface area of a 3D shape is displayed on the board, but some of the calculation is missing or incorrect. The students must identify which parts of the solution are correct and what additional material is needed for each question.

Plenaries

- A matching activity where the students identify which volume formula to use with which 3D shape that is displayed.

Enrichment activities

- This NRICH activity **Cuboids** gives students a chance to experiment with how surface area changes when you change certain dimensions of the 3D shape. This activity is best used with a points system for students finding cuboids with surface area around the 100 square units goal (nrich.maths.org).
- Dan Meyer has an activity called **Popcorn Picker** based on creating two possible tubes from a piece of A4 paper, one where you roll the paper up from landscape, the other from portrait. His question is which tube gives you more popcorn? This is an excellent introduction to an investigation about maximizing volume given the minimum surface area which can follow on from maximising area given a set perimeter (threeacts.mrmeyer.com).
- The following is also a Dan Meyer creation called **Dandy Candies** and is an excellent task for weaker students as it focuses on cuboids only and the candies here are represented by cubes (24 for of them in fact so a good set of factors and a possible chance to revisit some number work at the same time). There are opportunities for thinking about the shapes that minimise surface area given a set volume by exploring other numbers of candies. If you wanted to simplify the task you could remove the ribbon from the calculations (101qs.com).

Assessment ideas

- Students design their own ‘exam style question’ for a calculating the surface area or volume of a prism or cylinder. They then swap with another student who tries their question. The students then come together to mark what they have done and suggest changes to either the question or to how the student answered it. These can be collected and used for revision purposes later.

SECTION 2F / 2H: CONES AND SPHERES

Section 2 focuses on cones and spheres. The formulae for volume and surface area of a cone are introduced and a link is made with earlier material on Pythagoras’ theorem. **(Higher only)** A problem-solving framework for a practical question on the volume of a cone is provided.

The middle part of the section is devoted to spheres. The formulae for the volume and surface area of a sphere are given along with a worked example.

The final part of the section looks at composite solids and how students can go about breaking the composite solid up to calculate volume and surface area.

Prompting questions

While working through this section, good prompts to promote discussion might be:

- What is the difference between the slant height and perpendicular height of a cone? *The slant height is the slope of the cone, while the perpendicular height is the distance between the centre of the base and the apex of the cone.*
- When would we use the slant height of a cone? *To calculate the surface area of a cone.*
- When would we use the perpendicular height of a cone? *To calculate the volume of a cone.*
- How do we define a sphere? *We use the radius r and it is the 3D loci of points that are r distance away from a given centre.*

- What are the differences between the surface area formula and the volume formula for a sphere? *The volume formula requires $\frac{4}{3}$ of the calculated value involving r^3 (a length multiplied by itself and then multiplied by itself again) while the surface area formula requires four lots of the calculated value involving r^2 (a length multiplied by itself).*
- How can you decide if you have included all the solids in your surface area or volume calculation for a composite solid? *You must first decide what 3D shapes you can split your composite solid into. Next, you can calculate the surface area and/or volume for each component. Finally, you need to decide which parts of the surface area calculation need to be included for each component. A table is a good way to display all this information, as given in the worked example.*

Exercise 21B(F) / 21B(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** How do we find the perpendicular height of a cone given the radius and slant height? How do we find the slant height of a cone given the radius and perpendicular height? *We can use our knowledge of Pythagoras' theorem on the displayed right-angled triangles.*
- **Q2(F) / Q2(H)** What are we assuming the shape of the moon is? *A sphere.*
- **Q3(F) / Q3(H)** What formulas will we need for each and what will the units be? *We will need the formula for the surface area of a sphere and require the square of the given units.*
- **Q4(F) / Q4(H)** What other piece of information do we require here and can we calculate it? *We require the perpendicular height of each cone and can use Pythagoras' theorem to calculate it for each cone.*
- **Q5(F) / Q5(H)** Can we draw a sketch to help display the information given? What pieces of information do we need to calculate for this question? *The radius and the perpendicular height.*

Exercise 21C(F) / 21C(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** How can we break up each of the composite solids? *Both solids have rectangular prisms, but part b also has half of a cylinder attached.*
- **Q2(F)** How can we break each of these shapes up into solids we know how to calculate the volume of? *The shapes are made up of prisms and cylinders.*
- **Q3(F)** What parts will make up the surface area of the hemisphere? *We have half the surface area of the sphere plus the area of the circle where the cut was made.*
- **Q2(H)** How is the capsule formed? *Two hemispheres and a cylinder.*
- **Q3(H)** How could we calculate the volume of this composite solid? *Either, you can form the separate prism and cylinder and calculate their volumes or use the prism like structure that the composite solid has. Calculate the area of the face (which is needed for the surface area calculation) and multiply by the length of the water tank.*
- **Q4(H)** Will the surface area of the cut shapes be greater or less than the original uncut metal blocks? *They will have a greater surface area, since we have now exposed some of the interior of the blocks.*
- **Q5(H)** What are the solids making up this composite solid? *There is a rectangular prism and a triangular prism. We could also calculate the cross sectional area of the trapezoidal prism that is given in the picture.*
- **Q6(H)** Can we form a picture to identify which lengths we require to calculate the volume?
- **Q7(H)** What type of prism do we have? *A trapezoidal prism, thus our cross section is a trapezium.*
- **Q8(H)** What would the headings of our table be? *We need the radius and perpendicular height for each solid.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Prism or not a prism? A game to round up or recap what the definition of a prism is. Various 3D shapes are displayed and the students must determine if it is a prism or not. An extension to this would be to include dimensions on the 3D shapes and also ask if they can calculate the surface area and volume.

Plenaries

- Give a variety of 3D solids on the board and ask the students to combine them to form a composite solid. Lengths are then added to the sides and the students must calculate the volume and surface area of the composite solid.

Enrichment activities

- **(Higher only)** Question 8 in Exercise C could form the basis of a class activity. Possible dimensions could be collected on the board as a class.

Assessment ideas

- The plenary described above can be turned into a peer assessment activity where pairs of students design a composite shape each and then exchange shapes to find the surface area and volume for their new shape.

SECTION 3F / 3H: PYRAMIDS

Section 3 focuses on the volume and surface area of a pyramid and gives a worked example for calculating these.

Prompting questions

While working through this section, good prompts to promote discussion might be:

- What makes a 3D solid a pyramid? *A pyramid has a base given by a polygon and each vertex of the polygon is connected by an edge to a single point above the base polygon.*

Exercise 21D(F) / 21D(H)

While working through the exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** What units will the volume of the pyramid be in? m^3 .
- **Q3(F) / Q3(H)** Can we draw a sketch for each pyramid?
- **Q4(F) / Q5(H)** How did you calculate the volume of the triangular-based pyramid in Question 3?
- **Q5(F) / Q7(H)** What are the components of this composite solid? *We have a cuboid of height 30 m – 1.5 m, with a square based pyramid on the top.*
- **Q4(H)** Can we create a labelled diagram with this information?
- **Q6(H)** What calculations have you performed to get your answer for part a?

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Prepare a worked solution for finding the surface area and volume of a pyramid, but make it so that this solution has errors and parts missing from it. The students mark this worked solution, highlighting errors and missed information. They then share what they have found with the class.

Enrichment activities

- Take **Q5(F) / Q7(H)** from Exercise D as a base and extend it to find the approximate volume of stone used to construct an Egyptian city. Your students could design their own city and calculate the volume and surface area of each of their buildings.

Topic links

Previous learning

Chapter 5 Properties of shapes and solids, Chapter 12 Units and measurement, Chapter 15 Perimeter, Chapter 16 Area and Chapter 20 Three-dimensional shapes all feed into this chapter and the ideas that were previously introduced are used extensively throughout it. Students might want to revisit *Chapter 16 Area* Exercise E, where nets are given to calculate areas but not constructed to give a solid. The natural question would be ‘what is the volume of each of the solids formed from the nets in Exercise E (if they were closed up)?’

You might also want to revisit the ideas from *Chapter 8 Equations* and *Chapter 14 Algebraic formulae* with unknowns given for some of the lengths in the solids.

Future learning

Chapter 22 Calculations with ratio and *Chapter 30F/31H Similarity* will use surface area and volume of solids to form a background for questions involving similar shapes that will be asked in those chapters. Having a strong understanding of surface area and volume of solids will remove any initial barriers students might have when introduced to the new ideas and allow them to concentrate on the new material while consolidating old topics.

Gateway to A Level

Students who go on to A Level will need to be fluent with all the formulae for volume and surface area of 3D solids. They will also see new ways to ‘discover’ some of the formulae they have met in this chapter when they study integration and volume of solids of revolution.

In addition to this, A Level students will be required to form equations with unknown lengths connecting a solid of given surface area but with a need to calculate the lengths that optimise the volume of the solid (differentiation).

For mechanics students, this topic appears when calculating the centre of mass of an object and moments of inertia.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 1 Questions 2, 3, 11
- Chapter 2 Questions 11, 12, 13
- Chapter 4 Questions 3, 5
- Chapter 5 Question 11
- Chapter 10 Question 9

Higher

- Chapter 2 Questions 5, 6, 25, 26
- Chapter 4 Questions 2, 12
- Chapter 5 Question 11
- Chapter 6 Question 14

Homework Book

Foundation

- Chapter 21

Higher

- Chapter 21

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 12 worksheets (+ solutions)
- 11 animated widgets
- 21 **(F)** / 22 **(H)** interactive walkthroughs
- 6 auto-marked quickfire quizzes
- 6 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- **(Foundation only)** Nets of common 3D solids.

22F / 22H Calculations with ratio

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to divide quantities using efficient methods;
- understand that multiplication is a more efficient form of repeated addition;
- know the multiplication tables and how to apply them;
- be able to write one amount as a fraction of another;
- know how to identify highest common factors;
- understand that multiplication is commutative.

Additional useful prior knowledge

- To be able to use proportional reasoning to solve problems set in context.

Learning outcomes

Foundation

Section 1

- To use ratio notation to write ratios for diagrams and word statements and to simplify ratios.

Section 2

- To divide a quantity into two or more parts given a specified ratio and to write the division of quantities into parts as a ratio.

Section 3

- To use a unitary method to solve ratio and proportion problems and relate ratios to fractions and linear functions in order to solve problems, including real-life ones such as conversions and scaling.

Higher

Section 1

- To use ratio notation to write ratios for diagrams and word statements and to simplify ratios.

Section 2

- To divide a quantity into two or more parts given a specified ratio and to write the division of quantities into parts as a ratio.

Section 3

- To use a unitary method to solve ratio and proportion problems and relate ratios to fractions and linear functions in order to solve problems, including real-life ones such as conversions and scaling.

Vocabulary

ratio, proportion, equivalent

Common misconceptions and other issues

Students might fail to see a difference between the ratios $m : n$ and $n : m$, not realising that the order in which the ratio is written affects how the quantities are shared. Ensure you stress that order is important. Use visual prompts to help address this, for example 4 : 5 red to green couldn't have 4 green and 5 red items.

- Students sometimes think that $m \div n$ is the same as $m : n$; this can often be seen in representations of probability where for example students might try to writing a probability of $\frac{1}{4}$ as 1 : 4. It is important to draw this out through discussion; reviewing the definitions of ratio and proportion provided in the chapter could be an effective way of establishing the students' current understanding and dealing with this misconception. To further develop this, ask students to write the fractions of shaded squares in the diagrams for Exercise 22A Questions 2 and 3 and compare these to the ratios they have identified.

- The multiplicative nature of ratio is not always understood and students may think that $n : m$ is equivalent to $(n + 1) : (m + 1)$ since 1 has been added to both sides. Practical sharing activities in given ratios using counters can quickly resolve this.

Hooks

- Maths Snacks ‘Bad Date’. An entertaining way of portraying ratio through the proportion of words spoken by each person on various ‘dates’. Search online for ‘maths snacks bad date animation’ to find the clip.
- (Foundation only)** Sharing chocolate bars. Place one bar on one table at the front, two bars on the next and three bars on the third. Invite students up in turn to choose where to stand for their share of chocolate. Be sure to ask each student in turn why they have chosen that position and how many segments of chocolate will they receive, how have things changed when another student has joined that table? See NRICH **Chocolate** for additional information and prompting questions (nrich.maths.org).

SECTION 1F / 1H: INTRODUCING RATIOS

Section 1 introduces writing related quantities in a ratio and simplifying and identifying equivalent ratios.

Prompting questions

- (Foundation only)** Which would you prefer: a glass of squash with a ratio concentrate : water of 2 : 7 or 4 : 14? *Both the same strength as equivalent ratios.*
- (Foundation only)** Does 4 : 14 as opposed to 2 : 7 necessarily imply that I would get more squash? *No as these are equivalent ratios; the ratio does not define the amount involved, simply the proportional share for each part. However, the quantity of squash may have increased due to a repeat sharing in the ratio of 2 : 7.*

Exercise 22A(F) / 22A(H)

While working through this exercise, good prompts for students might be:

- Q1d(F) / Q1d(H)** Suggest a different number of students on the trip that maximises the staff to student ratio and changes the girls to boys ratio to 2 : 3. *36 : 54 since 9 staff could have 90 students maximum.*
- Q2-3(F) / Q2-3(H)** (and see also the misconceptions section above) Write the fractions for the proportion of shaded squares in each diagram and compare these to the ratios. $2a \frac{6}{16} = \frac{3}{8}$, $b \frac{4}{12} = \frac{1}{3}$, $c \frac{5}{15} = \frac{1}{3}$; $3a \frac{4}{16} = \frac{1}{4}$, $b \frac{6}{12} = \frac{1}{2}$, $c \frac{6}{15} = \frac{2}{5}$.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- (Foundation only)** For students who need extra help visualising ratio, the Interactive Teaching Program **Fractions** from the Primary National Strategy can be helpful (taw.org.uk).
- (Foundation only)** NRICH **Rod Ratios**. Another interactive and visual tool to demonstrate ratio using Cuisenaire rods (nrich.maths.com).
- NRICH **Ratio Pairs**. An interactive card match of equivalent ratios (nrich.maths.org).

Enrichment activities

- The introduction to this section considers mixing paints and ensuring that the shade doesn’t vary through consistent application of ratio. This could be demonstrated in a practical way using paints, food colouring or squash in order to provide a visual and kinaesthetic demonstration to the textbook example. For example, an activity to create a ‘mocktail’ can be very engaging.

Assessment ideas

- Depending on the starting point of your students, it may be useful to supplement the initial exercise with further practice identifying equivalent ratios. Make a card matching pairs game or a Tarsia puzzle (this can be done using **free software** available from the internet) to link equivalent ratios (mmlsoft.com). There are many ready-made examples available on the internet and the TES website has a good collection to choose from.

SECTION 2F / 2H: SHARING IN A GIVEN RATIO

This section focuses on solving problems where quantities are shared in a ratio with two or more parts.

Prompting questions

Exercise 22B(F) / 22B(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Ask students to consider what fraction each share represents.
- **Q4(F) / Q4(H)** Why do you think the question used 350 g for the amount of pastry? *Ratio parts add to 14 which is a factor so convenient.* Suggest another total mass of pastry that would also results in whole number answers for the separate ingredients masses. *Any multiple of 14.*
- **Q8(F) / Q8(H)** I don't like dried fruit so have removed this from the recipe, leaving the other ratios the same. How much of each ingredient is needed now to make 600 g of tiffin? Why are the values no longer integers? *5 : 2 : 2 has 9 parts and 9 is not a factor of 600. Need $333\frac{1}{3}$ g of biscuit, $133\frac{1}{3}$ g of butter and also cocoa.*
- Ask students to group the questions in Exercise 22B into categories to identify similarities and differences. This should provide a discussion starting point to draw out the key details of what each question is asking. Students often find it difficult with ratio problems to spot this, and can easily fall into the trap of applying a procedural method to every question, where they add up the ratio parts and divide by that to find one share, even when this is not appropriate.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- **Language of ratio cards.** Available from Teachit Maths (as a free pdf if you register with the site). This asks students to categorise the types of question as simplifying, sharing in ratio or finding one quantity when others are known. It then provides additional questions for practice (teachitmaths.co.uk).

Enrichment activities

- **(Foundation only)** Interactive resource: **Thinking Blocks Ratio and Proportion.** Available as an app as well as through the website, this uses the 'box' method referred to in the textbook for solving ratio problems (mathplayground.com).
- **NRICH Mixing Paints and More Mixing Paints.** These would build very well on the simple textbook example given in Section 1 (nrich.maths.org).
- **(Higher only)** **NRICH Speeding Boats.** A good investigation that students could work on in pairs or groups to deepen their understanding of ratio and proportion (nrich.maths.org).
- **(Higher only)** **NRICH Escalator.** This has similarities to 'Speeding Boats' above and could be used as an additional activity. For both problems students may use a fraction approach rather than ratio so encouraging them to use ratio notation for an alternative solution would be useful (nrich.maths.org).
- **(Foundation only)** Using manipulatives such as counters is an effective way to represent sharing through ratio. Considering Worked Example 1, students should quickly see that repeat sharing of 1, 3 and 4 will take a very long time to achieve a grand total of 32 000. Ask them to consider the total number of counters after each additional round of 'dealing' out the shares. Can they suggest a method of working out how many each person will have when their total is 24, 32, ..., 320, 3200, 32 000? Link the use of counters to the box method described in the chapter so that students can see the worth of each counter is £4000.

Assessment ideas

- **Ratio and proportion 'Thoughts and Crosses'.** A useful way of presenting differentiated material. Use a 3×3 grid with easier questions in the first column and row, becoming increasingly difficult from left to right and top to bottom (you could use questions from a range of the exercises in the student book as an alternative way of presenting them). The students then have to play noughts and crosses to compete a row of questions to win.

SECTION 3F / 3H: COMPARING RATIOS

This section focuses on comparison of ratio using a unitary approach that links to conversion graphs and direct proportion.

Prompting questions

- How could students use the graph showing inches converted to centimetres, to work out the number of centimetres in 20 inches, 100 inches, 18 inches, etc.? *Since they are in direct proportion 20 inches is 2×10 inches so $2 \times 25 \text{ cm} = 50 \text{ cm}$. $100 \text{ inches} = 10 \times 10 \text{ inches} = 10 \times 25 \text{ cm} = 250 \text{ cm}$. $18 \text{ inches} = 10 + 8 \text{ inches} = 25 + 20 \text{ cm} = 45 \text{ cm}$. Ask students what methods they have used as they could have many alternative suggestions to these. Linking the graphs to what they already understand about the multiplicative nature of proportion is a concept students often seem to struggle with.*
- (Higher only)** The section on the golden ratio mentions that the ratios of consecutive numbers in the Fibonacci sequence tend towards the golden ratio as a limit. Ask students to calculate the first few ratios to investigate this.

Exercise 22C(F) / 22C(H)

While working through this exercise, good prompts for students might be:

- Q2-3(F) / Q2-3(H)** These questions are about conversion graphs. What other possible conversion graphs can the students suggest? *For example, pounds to dollars, miles to kilometres.* What would they need to know to plot these? What would the equations of the plotted graphs be? *Start encouraging students to make links between the ratio $1 : n$ and the linear function $y = nx$.*
- Q6(F) / Q6(H)** In this question we use the conversion 5 miles = 8 km. We could write this in a unitary way as $1 : n$. What would this be and can students explain why? *(The graph in the chapter review would provide one method to help establish this.)* Without doing this, what operations do we need to use to convert miles to km and vice versa? In France, the speed limit on motorways changes for wet weather, from 130 km/h to 110 km/h. Can you drive at 80 mph in dry weather or 70 mph in wet weather? *Yes then no: $80 \text{ mph} = 120 \text{ km/h}$, $70 \text{ mph} = 112 \text{ km/h}$.*

Starters, plenaries, enrichment and assessment ideas

Starters

- The Number Loving website's **Ratio and Proportion Mystery**. Students use clues to decide whether or not to buy a cake, using ratio and proportion along with a little bit of surface area. This is a good activity for consolidation of the topics in a real-life context (numberloving.co.uk).
(Search the resources section using: Key stage 4, Number, Ratio and proportion.)
- Currency conversions are a good example of $1 : n$ ratios in real life. Use an online currency converter to quickly establish current conversion rates and ask students to use this to convert between currencies. You could give students a list of items to purchase from different countries, along with the conversion rates (or ask them to research these themselves), and ask them what they would have to pay in local currency. A key question to ask here is if the conversion is given as $1 : n$, what do you do if the currency you have to convert is the ' n ' value? For example, if British pounds sterling to euros is $1 : 1.26$, how can you convert from euros to pounds (without just changing the currency converter!)?
- NRICH **Nutrition and Cycling**. A set of information cards and questions to solve requiring ratio and proportion calculations. This may be more appropriate for Higher tier students due to the quantity of information provided but the maths should be accessible for Foundation students too (nrich.maths.org).
- The section on the golden ratio mentions the link to art and architecture. You can obtain some useful images showing **artwork with rectangles illustrating the ratio** from the internet (goldennumber.net). These could be printed out for students to analyse.

Enrichment activities

- The ratio between our hands and wrist measurements are not the only ones that demonstrate the golden ratio. Research the Marquardt Beauty Mask for further ideas.
- Improving Learning in Mathematics **Mostly Number – N6 Developing Proportional Reasoning** available from the National STEM Centre Archive (nationalstemcentre.org.uk).
- NRICH **Golden Trail 1**. A collection of resources that lead students through a variety of examples where the golden ratio can be found (nrich.maths.org).
- **(Higher only)** Nuffield Foundation **Paper Sizes**. An investigation into the links between A and B paper sizes which could encourage students to create their own graphs representing the proportions they uncover (nationalstemcentre.org.uk).
- **(Higher only)** NRICH **Ratio and Dilutions**. This is a NRICH activity that provides a practical cross-curricular link to chemistry to help students connect their maths with other subjects. It has an interactive app that allows students to test out their concentration calculations and extends the principle to using two dilutions.
- Some students may find this quite difficult initially; it could be introduced using the much more simple **Mixing Lemonade** activity as a starter (nrich.maths.org).
- ‘Best Buy’ calculations. Find some current examples of supermarket offers, for smaller and larger packets of produce, and work out which offers the best value. Often this can demonstrate how some of the ‘Buy one get one free’ type offers are not as good value as they may initially seem.
- Give your class a recipe project, where recipes have to be scaled for the required number of recipients (i.e. the recipe is for 6 and they need 20). Use a shopping list with costs and quantities from a supermarket to ensure students order sufficient ingredients, and calculate the total cost of making their recipe. (This can link to the ‘Best Buy’ activity as students could work out the cost per gram etc. for ingredients and compare brands.) Students could independently research the information they require, or be given a prescribed list depending on your resources and time available. They might consider total costs on just the ingredients they use, or from the package sizes they have to buy, which then brings in calculations around wastage.
- Research and investigate the Penrose Tilings where proportions of the two tessellated rhombus tiles are in the golden ratio.

Topic links

Previous learning

Students should already know how to identify fractions from visual representations, write equivalent fractions and find fractions of quantities from the work covered in *Chapter 10 Fractions*. Division into equal parts was covered in *Chapter 1 Basic calculation skills*.

Future learning

The ideas in this chapter will be developed in *Chapter 34F/36H Direct and inverse proportion* and *Chapter 30F/31H Similarity*. There are also links with *Chapter 33H Trigonometry*, in particular using the sine rule. Irrational numbers and surds (from the golden ratio) are covered in more detail in *Chapter 27H Surds*.

Gateway to A Level

A Level students will need to know how to use trigonometry and the sine rule and be able to apply these to problems in Mechanics. Use of constants of proportionality are required for integration and solving differential equations and are involved in many contexts in Mechanics, for example, for an object moving along a rough surface, the frictional force is proportional to the normal reaction force (the constant of proportionality is μ , the coefficient of friction).

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 1 Questions 4, 12
- Chapter 2 Question 5
- Chapter 6 Questions 4, 12, 13, 14
- Chapter 7 Question 6
- Chapter 8 Question 15
- Chapter 10 Question 10

Higher

- Chapter 2 Question 7
- Chapter 7 Question 3
- Chapter 10 Question 5

Homework Book

Foundation

- Chapter 22

Higher

- Chapter 22

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 4 worksheets (+ solutions)
- 2 animated widgets
- 6 interactive walkthroughs
- 2 auto-marked quickfire quizzes
- 2 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper
- 2 mm graph paper
- Axis grids from -10 to 10 in x and -10 to 10 in y

23F / 23H Basic probability and experiments

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to calculate confidently with fractions, decimals and percentages;
- be able to convert between fractions, decimals and percentages.

Additional useful prior knowledge

- To be able to use the terms certain, likely, unlikely and impossible in context.

Learning outcomes

Foundation

Section 1

- To understand and use the vocabulary of probability.
- To express probabilities as a number between 0 (impossible) and 1 (certain), either as a decimal, fraction or percentage.

Section 2

- To understand that outcomes are equally likely if there is the same chance of each outcome occurring.
- To calculate the theoretical probability of a desired outcome.
- To calculate the probability of an event NOT happening.

Section 3

- To relate relative frequency to theoretical probability.
- To represent and analyse outcomes of probability experiments.
- To use tables and frequency trees to organise outcomes.

Section 4

- To calculate probabilities in different contexts.

Higher

Section 1

- To understand and use the vocabulary of probability.
- To express probabilities as a number between 0 (impossible) and 1 (certain), either as a decimal, fraction or percentage.
- To relate relative frequency to theoretical probability.
- To represent and analyse outcomes of probability experiments.

Section 2

- To calculate the probability of an event NOT happening.
- To understand that the probabilities of mutually exclusive events sum to 1.
- To use tables and frequency trees to organise outcomes, understanding that a frequency tree is not the same as a probability tree.

Section 3

- To calculate probabilities in different contexts.

Vocabulary

event, outcome, equally likely, random, mutually exclusive

Common misconceptions and other issues

- Students tend to be very familiar with vocabulary such as certain, probable, likely, 50/50 chance, unlikely, impossible, and so on, as it is used frequently in everyday conversations and they hear it in the media. However, often the words are used incorrectly, so it is important to identify misconceptions at the start of this topic.
- It is also worth considering how percentages are used in conversation and in relation to probability. Certain means a 100% probability; you can't have 110% probability but you can have 110% profit.

- Students often assume that possible outcomes are always equally likely; for example, students know that it is even for getting heads on a flip of a coin and will therefore assume that any event with two possible outcomes will have a 50/50 chance, such as the probability of passing or failing a test. Leading questioning and giving examples of an event with two possible outcomes that are not equally likely helps to dispel this misconception. For example, if there are two people running 100 m, do they both have an equal chance of winning? *They would need to be equally matched in terms of fitness and skill.* What if one of the people was Usain Bolt? *Usain Bolt is much, much more likely to win the race.*
- Packs of cards are often used in probability questions. It is worth checking that students actually know the cards that make a deck. It is always surprising how many students are not aware of how many cards are in a suit or how many suits are in a deck. Having packs of cards in the classroom will develop familiarity and allow students to trial different experiments to get a sense of the likelihood of different outcomes.
- Some students believe that some numbers are lucky while others are unlucky, or that rolling a six is harder than any other number because of its importance in some board games.
- There is a 'Gambler's fallacy' that after getting four heads in a row you are more likely to get tails on the fifth coin flip.

Hooks

- Use familiar games, like 'Guess Who?', 'Deal or no Deal' or 'Play your cards right' as prompts for discussion. Discuss strategies that students use and the probabilities at each stage.
- A game like **Dibingo** offers an introduction to both the vocabulary of probability and starting to work out the numerical probability of different outcomes (donsteward.blogspot.co.uk).

SECTION 1F: THE PROBABILITY SCALE

Foundation Section 1 reminds students of the specific language used with probability and then introduces the notation of $P(\text{event})$ before demonstrating how probability can be described using numbers between 0 and 1.

Prompting questions

Exercise 23A(F)

While working through this exercise, good prompts for students might be:

- Q2(F)** Susanne says the probability of her being late for school is 1.5; explain how you know that she is wrong. *The maximum probability is 1. 1 is certain.* Jamie says that the probability of him being late is less than zero. Explain why Jamie is incorrect. *The minimum probability is 0. 0 is impossible.* Robin says that he will either be late or on time, so he has a 50/50 chance of being late. Explain why Robin's statement could be inaccurate. *What could make him late? Surely it would depend on Robin's punctuality and organisation in general.*
- Q3e(F)** Where is it more likely to rain: in the rainforest or the desert? *Rainforest.* Why is this? *It rains every day in the rainforest, but very rarely rains in the desert.* Is there a 50% chance of rain tomorrow in the desert? *No.* Is there a 50% chance of rain tomorrow in the rainforest? *No.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Students write down something they think is certain, then discuss as a class if these events are actually certain or just highly likely. Repeat with something they think is impossible and something that they think has a 50/50 chance. This discussion identifies misconceptions and demonstrates the need for a scale.

Starters or plenaries

- Use quotes from celebrities or extracts from current newspaper articles that contain the language of probability as prompts for discussion. For example, a judge on a TV talent show says 'I am *certain* that you will have a number 1 single'; can they really be *certain*?

- Provide students with a set of cards with different events on them. Students should then order the cards by how likely the event is to happen. The cards can then be placed on a probability scale; this could be done as a small group or as a whole class on the board or by pegging the cards to a washing line. To enhance engagement in this task, the cards could be personalised to the class, for example 'Finlay will become a rock star' or 'Ethan will go to Spain' or 'Imogen will be given a merit in her next class'.

Enrichment activities

- If you can't have more than 100% probability why do the 'odds' on the Grand National add up to far more than 1? This can lead to an interesting discussion on how bookmakers make their money.

Assessment ideas

- Give each student in the class an outcome described in words, or probabilities given as a decimal, fraction or percentage. They need to put this on a probability scale shared by the whole class. This can provide useful practice in fraction, decimal and percentage conversion.

SECTION 1H: REVIEW OF PROBABILITY CONCEPTS

Higher Section 1 covers the concepts of the probability scale, relative frequency and theoretical probability. These concepts are the same as those covered in Sections 1, 2 and 3 of Foundation, but the complexity of the problems can increase significantly. In addition to fractions, decimals and percentages, probability offers the opportunity to revisit other topics, such as forming algebraic expressions and proof, area, or interpreting statistical graphs.

Prompting questions

While working on the material in this section, good prompts for promoting discussion might be:

- Why do we use experimental probability?
- Can you give an example of when calculating theoretical probability is not practical, but using relative frequency is?

Exercise 23A(H)

While working through this exercise, good prompts for students might be:

- **Q2(H)** How many times would you expect to roll a six, if you rolled a dice 300 times? *50 times*. If the frequency of a six was 54, do you think this is a fair dice? *Yes. 54 is approximately 50.*
- **Q3(H)** Why does the table start with a total of 2 instead of 1? *It is impossible to have a total of 1 when rolling two or more dice*. Why is there a higher frequency for getting a total of 7 than a total of 12? *There are lots of ways to get a total of 7, while there is only one way of getting a total of 12.*
- **Q4(H)** Is this a sensible sample size for the market research company to use? *While the relative frequency will be more accurate if you have a bigger sample size, this might not be cost effective or might be impractical.*

Exercise 23B(H)

While working through this exercise, good prompts for students might be:

- **Q3(H)** Extension – how would the probabilities change if instead of finding the product of two dice, it was three dice? Or instead of multiplying, Nick and Vijay found the difference or sum?
- **Q4d(H)** When do most families go on holiday? How is this reflected in the price of holidays at different times of the year? Consider term time vs school holidays. *It is more likely that families will go on holiday in the summer, particularly in August.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- The NRICH **Probability Stage 4 – Short Problems** is a collection of problems that are useful for starters or extensions (nrich.maths.org).

Enrichment activities

- The NRICH **Odds and evens** task can be approached using theoretical probability of combined events through systematic listing or sample space diagrams, or through relative frequency. It is worth considering how the rules of the game could change and what impact these rule changes would have on the probability (nrich.maths.org).
- The NRICH **Which Spinner?** task has an interactive activity that demonstrates that the more you repeat an experiment, the closer the experimental probability will be to the theoretical probability. It then challenges students to work out which spinners were used, given different relative frequency graphs (nrich.maths.org).

SECTION 2F: CALCULATING PROBABILITY

Foundation Section 2 begins with a concept that many students find difficult: that of equally likely outcomes.

Addressing the misconception that, for example, 'breaking your arm' can either happen or not, so the likelihood of each is a half will be important here.

While fractions, decimals or percentages are all equally valid ways of representing probabilities, for situations involving theoretical probabilities writing these as fractions should be encouraged as the natural choice. Correct notation is essential: '1 in 3' or 'fifty-fifty' would score zero in an exam, as would any use of ratio notation.

Prompting questions

While working through this exercise, good prompts for students might be:

- Most of an iceberg is hidden under water (if asked about the Titanic, most classes will have a student who will tell you this). If $\frac{3}{11}$ of the iceberg is above water, what fraction is below? $\frac{8}{11}$
What if 0.359 is below water? 0.641. What if 93% is above water? 7%.
- Is rolling a 5 mutually exclusive to rolling: an even number? *Yes*. An odd number? *No*. A prime number? *No*.
- Is it raining tomorrow mutually exclusive to it being sunny? *No*. *You can have both rain and sun on the same day and even at the same time*.
- Is winning a football match mutually exclusive to losing it? *Yes*.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- The probability of the complement of an event is straightforward to work out when probabilities are given as fractions. Doing the same activity with decimals is useful practice, especially for students who will go on to study statistics at a higher level and will need this skill when working with the normal distribution. (Hint: each place sums to 9 except the right-most digit.)
- Spinners with numbered sections that are also coloured can be helpful to reinforce the importance of considering whether events are mutually exclusive. For example, if a spinner is labelled 1–6, with even numbers red and odd numbers black, $P(\text{red or } 2)$ is $\frac{3}{6}$, not $\frac{4}{6}$.
- Alternatives to asking questions about vowels are 'dotted letters', for example there is a $\frac{3}{4}$ chance of picking a dotted letter from 'Fiji'. Reversing the problem and asking for a word with a $\frac{2}{3}$ probability of picking a dotted letter/vowel can add an additional challenge.

Plenaries

- Give students probabilities (e.g. $\frac{1}{6}$) and ask them to write outcomes that have that probability. Using denominators 2, 6, 13, 36, 52 should reinforce common situations involving dice, coins and cards.

Enrichment activities

- Students could research whether it actually is equally likely to have a baby boy or a baby girl. Does it matter where in the world you are? How does gender distribution change in a given country at different ages?

SECTION 2H: FURTHER PROBABILITY

Higher Section 2 begins by revising the idea of the complement to an event. As in Foundation Section 2, particular focus could be placed on quickly finding the complement of decimal probabilities, for example 0.109 and 0.891.

At Higher tier, more emphasis on the consequence of events being mutually exclusive is necessary. In particular, it is important to understand that we cannot just add two probabilities.

Most aspects of this section are discussed either in Foundation Section 2 above or in Foundation Section 3 below.

Prompting questions

While working on the material in this section, a good prompt for promoting discussion might be:

- I have cards that each show one of the letters of the word AUSTRALIA. Is picking a card with an A and a vowel mutually exclusive? *No.* What are some outcomes that are mutually exclusive? *For example, picking a card with an S and a vowel.*

Exercise 23C(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** During the next working week (Monday to Friday), Michelle manages to get a seat on the bus every day. How does this change the experimental probability of her having to stand? $\frac{58}{227}$ ($= 0.256$) compared to $\frac{58}{232}$ ($= 0.25$) which means the probability has slightly reduced (by 0.6%).
- **Q3(H)** In a game, this basketball player had 20 shots at goal. Approximately how many goals did the basketball player score? $(1 - 0.432) \times 20 = 11.36$, so 11 (or 12) goals.
- **Q5(H)** Is it realistic to assume that attendance of these different school clubs is an example of mutually exclusive events? *We can assume this if they are all run at the same time and students are therefore unable to attend more than one club.*

Exercise 23D(H)

While working through this exercise, good prompts for students might be:

- What happens when you add the frequencies on each branch of a frequency tree? How can you use this to check that you are filling in the frequency tree correctly? *The sum of the frequencies on the end of each branch should equal the total frequency.*
- **Q1(H)** Why would a hotel want this information? How can it use it to improve their facilities? *This should prompt student discussion.*
- **Q2(H)** What percentage of patients who thought they would need a prescription didn't actually need one? 30.95%. What percentage of patients guessed correctly about their need for a prescription? 66.67%.
- **Q4(H)** : Which do you think medical professionals would prefer when testing – a higher probability of a false positive or a higher probability of a false negative? Why? *This question should prompt student discussion.*

Starters, plenaries, enrichment and assessment ideas

- Please refer to Foundation Sections 2 and 3.

SECTION 3F: EXPERIMENTAL PROBABILITY

Foundation Section 3 introduces experimental methods of calculating probabilities. Although initially represented as fractions, the use of percentages or decimals is sensible to allow comparisons between experiments. Getting students to repeat experiments for a certain number of minutes (rather than a certain number of times) makes it likely that all groups will have carried out a different number of trials that can result in students suggesting this representation quite naturally. This should help students understand that the frequencies are relative to the total number of trials.

The section ends with two approaches to structuring results from experiments. The first, two-way tables, will be familiar to most students. The second, frequency trees, will be new to most students yet tests have shown that most students are able to make sense of them with limited intervention.

Prompting questions

While working on the material in this section, good prompts for promoting discussion might be:

- I flip a coin a number of times and get eight heads. Is it a fair coin? *It depends how many times you have flipped the coin.*
- I flip a coin eight times and get eight heads. Is it a fair coin? *Unlikely to be fair, however we won't be sure with this number of trials.*
- I flip a coin 16/15/1000 times and get eight heads. Is it a fair coin? *Maybe/maybe/no.*
- Which is a better game to play: one with a $\frac{3}{7}$ chance of winning or one with a $\frac{2}{5}$ chance? $\frac{3}{7} (\approx 0.429)$ is greater than $\frac{2}{5} (= 0.4)$ so the first game is better to play.
- I flip a fair coin 50 times and, amazingly, get 50 heads. If I flip it a further 50 times, how many heads can I expect out of 100? *As these are independent events you could argue that you can treat each 50 trials separately. Out of the further 50 times, you would expect to get approximately 25 heads, so 75 in total. Alternatively you could consider the 100 trials together, in which case you would assume that you would get 50 heads in total.* In 1913, the ball on a roulette wheel in Monte Carlo landed on black 32 times in a row. Do you think people bet for or against black?

Exercise 23C(F)

While working through this exercise, good prompts for students might be:

- Why do you get a head and a tail more often than two heads? *Because there are two different ways to get a head and tails, but only one way to get two heads.*
- Rory flips two coins a large number of times. If he managed to get two tails 63 times, can you estimate how many times Rory repeated the experiment? Can you estimate how many times Rory got two heads? Can you estimate how many times Rory got a head and a tail? *If Rory got two tails 63 times, he would get two heads approximately the same number of times (63) and he would get a head and a tail approximately double the number of times (126), so we can estimate that Rory repeated the experiment 252 times in total.*
- **Q2(F)** What is the experimental probability of getting tails with the same coin? $\frac{25}{60}$. Do you think this is a fair coin? *It looks like it is fair, but you need to repeat the experiment many more times to be sure.*
- **Q4(F)** What is the maximum value for probability? *You cannot have a probability greater than 1 as 1 is certain.* $\frac{6}{3} = 2$, so we know that Paul can't possibly be correct. Explaining why an answer is incorrect helps students to consolidate their knowledge and helps them avoid the same mistakes in their own work. What other mistakes could be made?
- **Q8(F)** and **Q9(F)** Why is the weather forecast for tomorrow going to be more reliable than for 5 days from now?

Exercise 23D(F)

While working through this exercise, good prompts for students might be:

- **Q1(F)** Why would a hotel want this information? How can it use it to improve their facilities? *This should prompt student discussion.*
- **Q2(F)** What fraction of patients who thought they would need a prescription didn't actually need one? $\frac{13}{42}$. What fraction of patients guessed correctly about their need for a prescription? $\frac{40}{60}$ (or $\frac{2}{3}$).
- **Q3(F)** Which do you think medical professionals would prefer when testing – a higher probability of a false positive or a higher probability of a false negative? Why? *These questions should prompt student discussion.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Biased dice and double-headed (or tailed) coins are readily available on ebay and can make useful props for whole-class experiments.

Starters or plenaries

- False imprisonment and unnecessary surgery are powerful contexts for demonstrating the usefulness of frequency trees. Experimental probability is always a fall-back when a problem is (too) hard to analyse theoretically. For example, the Monty Hall Problem has simulators such as **stayorswitch.com**.

Enrichment activities

- The NRICH **Which Spinners?** is an interesting task with helpful simulations (nrich.maths.org).
- Students often enjoy being devious, making spinners out of card and matchsticks with surreptitious Blu-Tack placed to bias it to one face. This can be most effective if the target is to make the bias subtle: for example, on a pentagonal spinner, the experimental probability of landing on the biased side should be no more than 25%.
- Getting across the idea that dice, cards and roulette wheels ‘do not have a memory’ will likely do your students more good than just in the mathematics classroom.

Assessment ideas

- Students identify common errors in solutions created by others. This will check their understanding of how frequency trees are constructed

SECTION 4F: MIXED PROBABILITY QUESTIONS / 3H: WORKING WITH PROBABILITY

This section brings together the concepts learned in earlier sections. Students are invited to consider statements that could be heard in everyday conversation and use theoretical probability or relative frequency to check the validity of the statement.

Probability questions can often be very wordy and some students struggle with knowing where to start. The ‘problem-solving’ framework in this section demonstrates how one of these questions can be broken down into steps. In this example, it asks for the final answer to be expressed as a percentage. Firstly, this provides an opportunity to highlight to students the importance of reading each question carefully and warn against losing marks in an exam for not expressing an answer in the desired format. Secondly, students might need practice in converting fractions and decimals into percentages.

In the exercise, students are given prompts in questions to draw a frequency tree or complete a two way table in order to sort information before calculating a probability. However, they might not always receive such prompts in exam questions. It is therefore worth providing opportunities for students to develop their processing skills by removing this support initially.

It is also useful for students to work out experimental probability from data presented in different formats, such as cumulative frequency curves or pie charts, as this gives an opportunity to increase their familiarity with these statistical graphs.

Prompting questions

Exercise 23E(H)

While working through this exercise, good prompts for students that should lead to discussion might be:

- Q3(H)** Which do you think medical professionals would prefer when testing: a higher probability of a false positive or a higher probability of a false negative? Why?
- Q7(H)** Part a draws out the misconception that students will assume that events are equally likely. Before answering part b, it is worth asking ‘Does it seem realistic that the probability of a baby being born with blood type AB is $\frac{1}{4}$?’

Exercise 23E(F)

While working through this exercise, good prompts for students that should lead to discussion might be:

- Q6(F)** You are given a ten day forecast with the chance of rain for each day. Which of these probabilities is the most reliable and why? Which of these probabilities is the least reliable and why?
- Q7(F)** Which do you think medical professionals would prefer when testing: a higher probability of a false positive or a higher probability of a false negative? Why?

Starters, plenaries, enrichment and assessment ideas

Starters

- The NRICH **Probability Stage 3 - Short Problems** is a collection of problems that are useful for starters or extensions (nrich.maths.org).

Starters or plenaries

- (Higher only)** Insurance companies calculate premiums based on the probability of an event happening. How could they work out what the premium should be for priceless pieces of art? The BBC reports a **news story** about a man tripping over his shoelaces in a museum and breaking several Ming vases (news.bbc.co.uk).
- Discuss other industries that will use probability.
- (Higher only)** The **NHS Blood Donation website** has a bar chart to show current stocks for each blood type (blood.co.uk). Ask students what probability questions could be asked based on this bar chart. For example, what is the probability of a blood donor being O positive? You have the blood type AB negative. How likely is it that there will be the correct type of blood available?

Plenaries

- Use one of the questions from Exercise 23E, the chapter review exercise or an exam-style question as a starting point. Ask students to think about other questions that could have been asked based on the same starting point.
- (Higher only)** Give students data for how late buses run and how late underground and overground trains run, presented in different formats. Ask students to make a decision on which form of transport is most reliable (i.e. has the smallest probability of being more than 10 minutes late) based on this data.

Enrichment activities

- The NRICH **Do you feel lucky?** task states some of the advice and tips people offer to increase the chance of winning a game and asks students to consider if it is good advice or not (nrich.maths.org).

Assessment ideas

- Give students a question with a lot of information to process before calculating a probability. Through peer assessment or group/whole class discussion, identify different methods used to sort the information (such as a frequency tree, two way table or list) and discuss the efficiency of each method.

Topic links

Previous learning

Students need to be able to work with fractions, decimals and percentages. These have been covered in *Chapter 10 Fractions*, *Chapter 11 Decimals* and *Chapter 13 Percentages*. It would also be useful to know how to calculate the area of a sector as this is linked to probability spinners. This is covered in *Chapter 16 Area*. Questions might also involve algebraic expressions and proof and require use of the skills from *Chapter 7 Further algebraic expressions*.

Future learning

These will apply the skills learnt in this chapter to the interpretation of pie charts, cumulative frequency curves, histograms and other statistical graphs in *Chapter 35F/37H Collecting and displaying data* and *Chapter 36F/38H Analysing data*.

Gateway to A Level

A Level statistics makes use of probability in hypothesis testing and this has wider applications in subjects such as psychology, economics and biology. Many situations can be modelled by the normal distribution and the symmetries associated with this are useful in making predictions of probabilities in a wide range of contexts.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 2 Question 14
- Chapter 5 Question 6
- Chapter 6 Questions 5, 6
- Chapter 8 Questions 2, 3, 4
- Chapter 9 Questions 7, 14, 15, 16

Higher

- Chapter 2 Questions 8, 9, 10
- Chapter 5 Question 4

Homework Book

Foundation

- Chapter 23

Higher

- Chapter 23

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 15 **(F)** / 14 **(H)** worksheets (+ solutions)
- 7 animated widgets
- 13 interactive walkthroughs
- 6 **(F)** / 4 **(H)** auto-marked quickfire quizzes
- 6 **(F)** / 4 **(H)** auto-marked question sets, each with four levels
- Auto-marked chapter quiz

24F / 24H Combined events and probability diagrams

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to carry out the basic operations on fractions and decimals;
- be able to apply methods of multiplication, including larger numbers;
- have an understanding of basic probability.

Additional useful prior knowledge

- To understand the terminology of probability, especially events, outcomes, and so on.

Learning outcomes

Foundation

Section 1

- To construct and use representations (tables, tree diagrams and Venn diagrams).
- To use the language and notation of basic set theory.

Section 2

- To use the addition rule, including an understanding of mutually exclusive events.
- To use the multiplication rule, including an understanding of independent events

Higher

Section 1

- To construct and use representations (tables, tree diagrams and Venn diagrams).
- To use the language and notation of basic set theory.

Section 2

- To calculate numbers of possible outcomes using the product rule for counting.
- To use the addition rule, including an understanding of mutually exclusive events.
- To use the multiplication rule, including an understanding of independent events

Section 3

- To use methods of conditional probability, including questions phrased in the form 'given that'.

Vocabulary

combined events, sample space, independent events, dependent events

Common misconceptions and other issues

- It is always surprising how many students are unfamiliar with how a deck of cards is made up. This lack of knowledge can prevent students accessing common probability questions.
- Similarly, it should be checked that students understand what the minimum score would be if two or more dice were thrown.
- Students can sometimes assume that all events are equally likely. Giving concrete examples should help dispel this misconception. For example, they might assume that it is equally likely that it will be raining or sunny tomorrow. The best way to address this is to then ask students if there is more chance of it being sunny or rainy in the desert or the

rainforest. Or show different spinners with the numbers 1, 2, and 3 on them, but in different sized sectors... will it still be equally likely to land on each number? *No, it's only equally likely if the size of each sector is the same!*

- Students are not always clear if an event is independent or conditional. For example, given the question 'If a person takes two yoghurts out of the fridge, what is the probability that they will both be the same flavour?' It is worth checking that they understand that the first yoghurt is not replaced.

Hooks

- Students could be challenged to beat the teacher at a dice game. The teacher needs a set of Efron's dice, ideally in four different colours (could be made with stickers on the faces). This special set of non-transitive dice mean it is always possible for the teacher to have the advantage over a student, even if the student chooses which dice they are going to use first.
- Similarly, a pair of Sicherman dice can be used against a pair of standard dice. The students can challenge the teacher and choose which pair of dice they wish to use. Sicherman dice have different numbers to a normal set of dice, but the odds of throwing every number is the same. Some more info on Sicherman dice can be found at **Wolfram MathWorld**. Given the choice between a set of Sicherman dice and a pair of normal dice, most students tend to go for the Sicherman dice as they assume they have a higher chance of winning. A sample space diagram will be particularly useful to demonstrate that there is the same chance of getting each number as a normal pair of dice.

SECTION 1F / 1H: REPRESENTING COMBINED EVENTS

Section 1 introduces a variety of diagrams that can be helpful in structuring solutions to probability problems involving combined events. Sample space (i.e. two-way) tables and tree diagrams will be familiar to many teachers and students. Venn diagrams will be new to many, although most students will have seen them used in simple cases for categorisation. Encouraging students to draw and shade diagrams will help them get to grips with some of the new notation.

Prompting questions

Exercise 24A(F) / 24A(H)

While working through this exercise, good prompts for students might be:

- Q1(F) / Q1(H)** If you double the number of red counters, while keeping the number of blue and yellow the same, has the sample space changed? *No, since the sample space tells us only the possible outcomes not their probabilities.*
- Q3(F)** Are you more likely to get two cards of the same colour or two cards of a different colour? *These are equally likely events.* Compare this sample space to a sample space diagram for the flip of two coins. Also compare with the example sample space for the roll of two dice; what are the similarities? *The sample space diagrams are all symmetrical.*
- Q3(H)** Would a tree diagram still be effective if I flipped four coins? five coins? *They generally become too cluttered with more than about 3 events, especially if each event has many possible outcomes.*

Exercise 24B(F) / 24B(H)

While working through this exercise, good prompts for students might be:

- Q1b(F) / Q1b(H)** : What is $n(A) + n(A')$? *The size of the universal set (i.e. $n(\xi)$).* What is $n(B) + n(B')$? *The size of the universal set (i.e. $n(\xi)$).*
- Q2(F) / Q2(H)** Nadia has 7 pairs of high-heels, of which 2 are red. How could this new subset be represented on the Venn diagram? *Will not overlap with the sports shoes, but will overlap with the red shoes.* What is the probability of choosing a pair of high-heels that are not red? $\frac{5}{20}$. Work out the probability that a pair of shoes chosen at random from her shoe collection will be neither red nor high-heeled. $\frac{9}{20}$. Work out the probability that a pair of shoes chosen at random from her shoe collection will be neither red nor high-heeled nor sports shoes. $\frac{6}{20}$

Exercise 24C(F)

While working through this exercise, good prompts for students might be:

- **Q3(F)** Would a tree diagram still be effective if I flipped four coins? five coins? *They generally become too cluttered with more than about 3 events, especially if each event has many possible outcomes.*
- **Q4(F)** At Christmas, Andrew also offers ‘Merry Christmas’ wrapping paper, red ribbon and a choice of gold, green or red gift tags. How many combinations are there now? 90. What is the probability that a customer will choose ‘Merry Christmas’ paper with silver ribbon and a red gift tag? $\frac{1}{90}$.

Starters, plenaries, enrichment and assessment ideas

Starters

- The ‘two dice horse game’ can be played by lining up 12 ‘horses’ (students) each with a number from 1 to 12. Two dice are rolled and the numbers showing are added together. That horse moves forward one step. Ideally the classroom tables are cleared to give a great demonstration that numbers ‘in the middle’ are more likely to come up. Horse number 1 will never move. Before the ‘race’ students could predict the winner. Repeating the ‘race’ a second time generally leads to significantly better predictions.

Plenaries

- After getting a student to draw their Venn diagram for one of the questions in Exercise 24B, others could be tasked with, for example, ‘creating a question with the answer $\frac{42}{80}$ ’. (Where ‘42’ will need to be chosen to be the size of a union/intersection/etc. of the student’s diagram and ‘80’ is the size of the universal set.)
- How can you remember that $\cup$ means union and $\cap$ means intersection?

Enrichment activities

- Students could determine the strategy for selecting one of Efron’s dice (described above) by constructing sample space diagrams for each pairing of dice from the set of four.
- Can you construct a Venn diagram for three sets? Four sets? (Not including the universal set.)
- If you could move (but not remove or add to) the spots on a dice, then rolled it against a normal dice, are you more or less likely to achieve the highest score? This is another activity that lends itself to sample space diagrams.
- NRICH problem **Fixing the Odds** (nrich.maths.org).

Assessment ideas

- Constructing Venn diagrams from a question is generally the lengthy part, so projecting ready-drawn ones can allow for quick assessment of students’ interpretation. For example, quick-fire questions which students complete on mini-whiteboards (‘what is the probability that a person eats only chicken’).

SECTION 2F / 2H: THEORETICAL PROBABILITY OF COMBINED EVENTS

Section 2 introduces the idea of combinations of events, in particular by adding and multiplying probabilities. However, in the Higher version, it first covers a method for calculating the number of possible outcomes using the ‘product rule for counting’.

Exercise 24C(H)

While working through this exercise, good prompts for students might be:

- **Q2(H)** I set up a competing lottery where you only need to match five numbers to win. However, you must choose from balls numbered from 1–59 rather than 1–49. Is my new lottery a better deal? *Yes, 1 in 5 million rather than ~14 million.* If you get three numbers in a lottery then you win £10. Why is the prize for that so low compared to the jackpot? *There are many more ways to get three numbers. There is approximately a 1 in 1000 chance of getting 3 numbers.*
- **Q6(H)** How does the answer change if you are only allowed to choose different fillings? *Number of possibilities decreases to 336.*

Exercise 24D(F)

While working through this exercise, good prompts for students might be:

- **Q1(F)** If you swap the yellow marble from the first box with the red marble from the second box. How will this affect the probability tree? What is the probability of choosing two red marbles now? *Impossible*. What is the probability of choosing a yellow and a blue marble now? $\frac{1}{9}$.
- **Q3(F)** Without adding to the tree diagram, can you work out the probability of getting at least one king if you draw four cards? Are you more or less likely to get at least one king if you draw a card four times rather than one time? *More likely; the more times you repeat the experiment, the more likely it will be to get at least one king.*

Exercise 24D(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** Nico's friend is on a different bus but amuses herself with the same game. She sees a road sign for LONDON. What probability questions could have an answer of $\frac{1}{2}$? $P(O \text{ and } N)$. $\frac{1}{4}$? $(P(L \text{ and } O) \text{ or } P(D \text{ and } O))$.
- **Q3(H)** I take just the hearts out of the pack of cards and shuffle them. I then turn them over one by one. What is the probability I turn over the ace, then the 2, then ..., then 10, J, Q, K? *1 in 13 factorial that is approximately 1 in 6 billion. Note that this is approximately the population of the planet, so if everyone tried this trick once, probably only one person would succeed.*

Exercise 24E(F)

While working through this exercise, good prompts for students might be:

- **Q2(F)** Nico's friend is on a different bus but amuses herself with the same game. She sees a road sign for LONDON. What probability questions could have an answer of $\frac{1}{2}$? $P(O \text{ and } N)$. $\frac{1}{4}$? $(P(L \text{ and } O) \text{ or } P(D \text{ and } O))$.
- **Q4(F)**: Which is more likely, getting the word 'bad' or the word 'dab'? *Equally likely*. What if another A and/or another D were added to the tiles, what is the probability of getting the words 'cad', 'bad' or 'dad' now? *The answer will depend on which combination of letters are added and which word is being considered.*
- **Q7(F)**: What is the probability of getting either green or orange? $\frac{8}{20}$ or $\frac{2}{5}$. What is the probability of getting both green and orange? $\frac{1}{20}$.

Starters, plenaries, enrichment and assessment ideas

Starters

- Project an image of a child's stacking toy (rings that fit over a post that subsequently reduce in size). 'My six-month-old child managed to stack them correctly. Are they a child genius?' Work through how many ways there are to select the first ring, then the second, and so on.

Plenaries

- Jake, what is the probability that it snows in your garden on Christmas day? $\frac{1}{5}$ or any sensible fraction. Donna, what is the probability that it snows in your garden on Christmas day? *The same-ish*. So what is the probability that it snows at Jake's house AND Donna's? *Expect* $\frac{1}{5} \times \frac{1}{5}$. Extend to more students, until we get that the class of 30 students all having a snowy Christmas day is $\frac{1}{5}$ to the power 30, or smaller than one in a quintillion. This is clearly wrong: what have we done wrong here? *We all live close to each other so it snowing in our gardens is not independent.*

Enrichment activities

- NRICH problem **Same Number!** (nrich.maths.org.)
- Consider why the PIN used for bank and credit cards is four digits long.
- Consider the infinite monkey theorem. Forget the entire works of Shakespeare, what is the probability that the first line of a Shakespeare play could be typed randomly? Or even the first word?

Assessment ideas

- Design a lottery game (how many numbers? how many must you match?) so that every one of the 50 million adults in the UK could have their own distinct ticket.
- **(Higher only)** Exercise 24C, Question 7 considers the number of different wraps that can be made. A simple internet search shows that there are millions of different possible sandwiches that can be made at a local deli sandwich shop. Investigate – is this correct?

SECTION 3H: CONDITIONAL PROBABILITY

Higher-only Section 3 focuses on conditional probability. Most students will find diagrams an enormous help here to structure their work.

Prompting questions

Exercise 24E(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** How would you use your diagram to calculate the probability of one girl and one boy? (*Need to calculate girl-then-boy and boy-then-girl and add them together.*)
- **Q3(H)** If the probability of rain on Friday is 0.21, what is the probability of sun? (*We cannot determine, e.g. since it can rain on a sunny day.*)
- **All questions** Why use a tree diagram for this question, rather than a Venn diagram (or vice versa)? What are the hints in the situation that suggest you use that diagram?

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- **Tree diagram match up exercise** can be downloaded from the TES website (tes.co.uk). Students match up pictures of boxes of shapes, tree diagrams and probability statements, that are partially complete. After matching the cards up, they need to fill in the missing information. This activity promotes discussion about independent or conditional probabilities.
- NRICH problem **One or Both** is a short problem that really lends itself to using Venn diagrams and extending to ask: ‘What is the probability of randomly selecting a student who got both questions correct?’ The question ‘Can you make a question like this of your own?’ at the end will promote interesting ideas from students (nrich.maths.org).

Enrichment activities

- NRICH problems **Prize Giving** and **Louis’ Ice Cream Business** (nrich.maths.org).

Topic links

Previous learning

This topic provides good practice of operations with percentages, fractions and decimals. In particular, practising non-calculator methods with fractions (e.g. cancelling before multiplication) could be encouraged. These skills were covered in *Chapter 10 Fractions*.

Students need to consider the similarities and differences between frequency trees and probability trees; frequency trees were introduced in *Chapter 23 Basic probability and experiments*. Venn diagrams are often used when calculating highest common factors or lowest common multiples after the prime factor decomposition of two or more numbers.

This method was covered in *Chapter 2 Whole number theory*.

Future learning

Some of the ideas from this chapter will be revisited in *Chapter 35F/37H Collecting and displaying data* and *Chapter 36F/38H Analysing data*.

Gateway to A Level

A Level statistics considers combinations and permutations in more detail and applies these to a range of contexts. Statistical techniques such as sampling and hypothesis testing have applications in many other subjects such as psychology and economics.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 1 Questions 13, 14, 21, 22
- Chapter 4 Question 6
- Chapter 6 Questions 15, 16
- Chapter 7 Question 13
- Chapter 9 Questions 17, 18
- Chapter 10 Questions 16, 17

Higher

- Chapter 1 Questions 5, 6, 16, 17, 18, 19, 20
- Chapter 4 Questions 5, 6
- Chapter 6 Question 5
- Chapter 7 Questions 4, 11
- Chapter 8 Question 20
- Chapter 9 Question 7
- Chapter 10 Questions 11, 12, 13

Homework Book

Foundation

- Chapter 24

Higher

- Chapter 24

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 6 (F) / 10 (H) worksheets (+ solutions)
- 2 animated widgets
- 9 (F) / 12 (H) interactive walkthroughs
- 2 (F) / 3 (H) auto-marked quickfire quizzes
- 2 (F) / 3 (H) auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper

25F / 25H Powers and roots

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to add and subtract integers mentally;
- know how to find the squares, square roots, cubes and cube roots of numbers;
- be able to find the reciprocal of a number;
- understand how to use the power, root and reciprocal buttons on a calculator;
- know the difference between the words 'evaluate' and 'simplify'.

Additional useful prior knowledge

- To know the first 12 square numbers and the first six cube numbers.
- To recognise square roots of the first 12 square numbers and cube roots of the first six cube numbers.
- To recognise powers of two.

Learning outcomes

Foundation Section 1

- To write a series of numbers multiplied together in index form.
- To write an exponent on a calculator.
- To understand zero and negative indices.

Section 2

- To apply the laws of indices for multiplying and dividing, and for powers of indices.

Section 3

- To calculate roots of a number fraction.
- To solve problems involving powers and roots.

Higher Section 1

- To write a series of numbers multiplied together in index form.
- To write an exponent on a calculator.
- To understand zero and negative indices.

Section 2

- To apply the laws of indices for multiplying and dividing, and for powers of indices.
- To work with fractional indices and understand the link to surds.

Section 3

- To estimate powers and roots of a number.
- To solve problems involving powers and roots.

Vocabulary

index, index notation

Common misconceptions and other issues

- When squaring a number students might double the number rather than multiplying it by itself, for example work out $7^2 = 14$. To address this you could pose two calculations to the students. One that calculates 7×7 and the other 7×2 . Students can discuss which they think is correct and why.
- If students are taught rules such as; $a^n \times a^m = a^{n+m}$ then they might incorrectly believe that $a^n \times b^m = a + b^{n+m}$ or ab^{n+m} . Asking students to evaluate the questions as well as simplify can overcome this. Students could be provided with all of the possible misconceptions and after they evaluate them using a calculator they will understand why they are not correct.

- When students are dealing with negative indices they might write the number 5^{-1} as -5 rather than $\frac{1}{5}$. If this is the case it would be helpful to recap their understanding of reciprocals.

Hooks

- Indices song. Search online for 'indices song'. This is a light-hearted way of introducing or reinforcing the topic. It also highlights some of the common misconceptions.
- Powers Countdown.** A twist on the popular numbers game (nrich.maths.org).

SECTION 1F / 1H: INDEX NOTATION

Section 1 focuses on how to write a number in index form and how to input a number in index form into a calculator. It also explains what negative indices are.

Prompting questions

Exercise 25A(F)

While working through this exercise, good prompts for students might be:

- Q1(F)** Can you think of any different ways to ask these questions? *The different question parts ask the same sort of question in different ways so students could suggest alternatives depending on which question part they are considering.*
- Q3(F)** Why is part n easier than part o? What can you do first to make part n even easier? *Simplify first.*
- Q3(F)** Explain how you should apply the order of operations to part q of this question. *Calculate the powers first, then multiply then add the two values together.*

Exercise 25A(H)

While working through this exercise, good prompts for students might be:

- Q1a(H)** What patterns can you identify in the powers of two? Can you use this to fill in the gaps? *Students should notice that, from zero, when the power increases it doubles the previous value. When the power decreases, the answer halves.*
- Q1b(H)** What is the mathematical name for what you have noticed? *The negative and positive values of the same number are reciprocals of each other.*
- Q4(H)** Why is part n easier than part o? What can you do first to make part n even easier? *Simplify first.*
- Q4(H)** Explain how you should apply the order of operations to part q of this question.
- Q6(H)** Explain why it would be incorrect to simplify part d to 3^{16} before evaluating the answer. *Because you can only simplify like this when multiplying terms with like bases.*

Exercise 25B(F)

While working through this exercise, good prompts for students might be:

- Calculate the powers first, then multiply then add the two values together.
- Q2(F)** Explain why it would be incorrect to simplify part d to 3^{16} before evaluating the answer. *Because you can only simplify like this when multiplying terms with like bases.*

Exercise 25C(F) / 25B(H)

While working through this exercise, good prompts for students might be:

- Q1d(F) / Q1d(H)** 3^{-2} is the reciprocal of what? *3^{-2} is the reciprocal of 3^2 .*

Starters, plenaries, enrichment and assessment ideas

Starters

- **Transum starter** – can you spot the mistake? A great quick starter that consolidates students' understanding of powers (transum.org). The mistake is actually in the text!
- A set of thought-provoking **indices resources by Susan Wall** are available from the National STEM Centre website (nationalstemcentre.org.uk). These include true or false, odd-one-out and matching activities that encourage student collaboration.

Enrichment activities

- The NRICH problem **Power Mad** gets students to investigate a range of patterns in powers (nrich.maths.org).
- The NRICH problem **A Biggy** will stretch students to think more deeply about the properties of numbers that have been raised to powers (nrich.maths.org).

SECTION 2F / 2H: THE LAWS OF INDICES

This section focuses on the following index laws:

- Multiplying numbers in index form;
- Dividing numbers in index form;
- Powers of indices;
- **(Higher only)** Fractional indices.

This section introduces all of the index laws students are required to know. Although the summaries are helpful to students it is important that they understand why the laws work so they do not use them incorrectly.

Prompting questions

Exercise 25D(F) / 25C(H)

While working through this exercise, good prompts for students might be:

- **Q1d(F) / Q1d(H)** What is the power of 5 when it is written on its own? $5 = 5^1$.
- **Q2(F) / Q2(H)** Why do you think other might students make a mistake with parts i, j and k of this question? What wrong answer do you think they might give? *Students might apply their knowledge of negative numbers incorrectly. For example, for Question 2j they might get an answer of 10^2 by simply subtracting the powers. Since subtracting a negative is the same as adding a positive then they should get 10^{10} .*

Exercise 25D(H)

While working through this exercise, good prompts for students might be:

- **Q2e(H)** Why is it useful to write $\sqrt[3]{3}$ as an index before simplifying? *Students can then simply multiply the indices when they are written in index form. For example $\left(\frac{4}{3}\right)^4 = 3^{\frac{4}{3}}$.*
- **Q3g(H)** What three steps should you complete to answer this question? *First find the reciprocal of 125. Then find the cube root, then find the 4th power of the result.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- **Number Loving's** 'Indices starter – True or False worksheet' is a quick starter to check students' understanding of use of powers and the basic index laws. Also from this site 'Simplifying Indices Catchphrase' makes a nice plenary where the whole class can contribute to solving questions and reveal a picture representing a known phrase (numberloving.co.uk). (The easiest way to find particular resources is to select KS4 and then click 'find' to get a full list in alphabetical order.)
- Teachit Maths **Power of y eliminator** is a good exercise to practise using the first three laws of indices to find a coded message (teachitmaths.co.uk). Registration to be able to download the pdf files is free.

- **Mr Barton's Maths** site has readymade Tarsia, matching cards and bingo activities for simplifying indices (mrbartonmaths.com).

Enrichment activities

- Improving Learning in Mathematics **Using Indices – N12** available from the National STEM Centre Archive contains a range of activities to develop the students understanding of the index laws (nationalstemcentre.org.uk).
- NRICH **Negative Power** is an interesting investigative problem involving negative indices and raising a power to a power (nrich.maths.org).
- **Powers and Roots Pack 2** from the National STEM Centre archive is a pack of SMILE activities exploring powers and indices and applying these in standard form. This could provide opportunities for additional practice and making connections to applications of indices (nationalstemcentre.org.uk).

SECTION 3F / 3H: WORKING WITH POWERS AND ROOTS

In Foundation, Section 3 focuses on finding the root of a number and recognising that finding the roots and working out powers are inverse operations. The section then goes on to look at problems in context.

In Higher, Section 3 focuses initially on estimating the value of powers and roots and then goes on to answering problems in context.

Prompting questions

Exercise 25E(F)

While working through this exercise, good prompts for students might be:

- Can you complete the table without using a calculator? What patterns do you notice? *The powers increase by one as they move to the right. For base two the values for each increased power are doubling.*
- What happens in each consecutive cell as you move to the right or left? *Base 2 is $\times 2$ moving to right, $\div 2$ moving left. For base 3, 4, and so on, moving right is multiplying by the base, moving left is dividing.*
- How could $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ be written involving powers of 2? *This should elicit $\left(\frac{1}{2}\right)^1, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3$, and so on, that would bring out discussion about the negative powers being reciprocals.*

Exercise 25E(H)

While working through this, good prompts for students might be:

- **Q1(H)** What square, cube, and so on, numbers do you know that are close the values in the question? *For example, for part a students know that the square root of 64 is 8 and the square root of 81 is 9. Therefore, they know the answer will lie between 8 and 9.*
- **Q4(H)** If m is 1, then what is a ? Now, find a as m increases. *If m is 1, a is 81. Students can use this as a useful starting point.*

Exercise 25F(F)

While working through this exercise, good prompts for students might be:

- Why doesn't this exercise ask you to find any sixth roots of numbers? How could you extend your table to be able to do this? *The table only goes up to raising to the power of 5. You can extend the table by an extra column and use the rules discussed above to fill in the values. Then it would be possible to answer questions such as what is $\sqrt[6]{64}$?*

Exercise 25G(F) / 25F(H)

While working through this exercise, good prompts for students might be:

- List as many examples of formulae you can think of in maths or science that use indices (e.g. πr^2). This could lead to an interesting discussion about which formula are more complicated to use and why, which leads in nicely to the types of problem covered in Exercise 25G.
- **Q1(F)** Which button on your calculator will work out the side lengths of a square if you know the area? *Square root button.*

- **Q8(F) / Q1(H)** What effect does changing the interest rate for the five years to 2.5% or 10% have on the final total from the investment of £2500? Does this surprise you? Now try changing the investment term from 10 years to 5 or 15. Use these investigations to write some financial advice on wise investments.

	5%	2.5%	10%
5 years	£3190.70	£2828.52	£4026.28
10 years	£4072.24	£3200.21	£6484.36
15 years	£5197.32	£3620.75	£10443.12

Starters, plenaries, enrichment and assessment ideas

Starters and plenaries

- **Aplus Click Maths** contains a wide variety of multiple choice questions on algebra, many of which involve indices and powers. As a sample of what you will find, two examples are: a) $27^8 = 81n$, find n ; and b) find the last digit of $3^{33} = ?$ (Aplusclick.com.)

Enrichment Activities

- **(Higher only)** For a challenge that extends toward A Level, try **RISP 35: Index Triples**, which investigates the effect of the order when writing numbers to the power of each other (risps.co.uk).
- Explore past functional maths paper questions (from exam board websites) for questions involving use of formulae where indices play a part, possibly in connection to area, volume, percentage calculations and units.

Topic Links

Previous learning

From Key Stage 3, students should be confident with square and cube numbers, and know the first 15 squares, 6 cubes and their corresponding roots. Geometrically, students should recognise the link between squaring a number and finding the area of a square from its length. Similarly they should make the link between the volume of a cube and its edge lengths.

Future learning

At GCSE the laws of indices will be required for successful calculations in *Chapter 26 Standard form*.

(Higher only) Indices link closely with surds (*Chapter 27H Surds*) since these are fractional indices and the laws can be used to justify statements such as $\sqrt{a} \times \sqrt{a} = a$.

Indices are also applied in various formulae, such as those for volumes of cubes and spheres (*Chapter 21 Volume and surface area*), direct and inverse proportion (*Chapter 34F/36H Direct and inverse proportion*) and calculations of compound interest (*Chapter 33F/35H Discrete growth and decay*).

Gateway to A Level

Students will learn to differentiate and integrate terms that involve indices and they will develop the laws of indices to work with logarithms, which are used extensively at A Level and beyond, linking, for example, to e, infinite series and solving differential equations. Indices are present in topics such as geometric series and binomial distribution and the ability to manipulate expressions involving indices is an important algebraic technique.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 4 Question 7
- Chapter 7 Question 14
- Chapter 8 Question 23
- Chapter 9 Question 21

Higher

- Chapter 4 Question 7
- Chapter 8 Question 21
- Chapter 9 Question 12

Homework Book

Foundation

- Chapter 25

Higher

- Chapter 25

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 8 **(F)** / 12 **(H)** worksheets (+ solutions)
- 2 **(F)** / 3 **(H)** animated widgets
- 10 **(F)** / 17 **(H)** interactive walkthroughs
- 3 **(F)** / 4 **(H)** auto-marked quickfire quizzes
- 3 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

26F / 26H Standard form

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to multiply and divide by powers of ten, applying their understanding of place value;
- be able to use the four arithmetic operations to calculate efficiently with decimals;
- know how to round to a given number of significant figures;
- understand that multiplication and division are commutative;
- know how to use the index laws when multiplying and dividing.

Additional useful prior knowledge

- To understand place value and rounding.
- To be able to round to a given number of significant figures.
- To be able to apply the index laws for multiplication and division.

Learning outcomes

Foundation

Section 1

- To apply understanding of multiplying and dividing by powers of ten to convert numbers to and from standard form.

Section 2

- To use a scientific calculator efficiently for standard form calculations.

Section 3

- To apply the laws of indices to multiply and divide numbers in standard form without the use of a calculator.
- To apply understanding of place value, and previously learned conversion between standard form and ordinary numbers, to add and subtract numbers in standard form.
- To solve problems, including contextualised ones, involving standard form.

Higher

Section 1

- To apply understanding of multiplying and dividing by powers of ten to convert numbers to and from standard form.

Section 2

- To use a scientific calculator efficiently for standard form calculations.

Section 3

- To apply the laws of indices to multiply and divide numbers in standard form without the use of a calculator.
- To apply understanding of place value, and previously learned conversion between standard form and ordinary numbers, to add and subtract numbers in standard form.
- To solve problems, including contextualised ones, involving standard form.

Vocabulary

power, index, ordinary number, significant figures

Common misconceptions and other issues

- $A \times 10^n + B \times 10^m = A + B \times 10^{n+m}$ (and ditto for subtraction). Practise converting numbers back to ordinary form and then performing the calculations, to ensure students see that this is not true.

- $A \times 10^{-n} = -A \times 10^n$. Use the number line with continuing patterns, suggested in the general prompting questions for Section 1, to reinforce that 10^{-n} is not a negative value. Enter examples on calculators to verify this and also to provide extra calculator practice.
- Students might still hold misconceptions related to the first index laws, thinking that $10n \times 10m = 10nm$ rather than $10(n + m)$ and similar for division. Again by encouraging students to see the patterns on the line in the general prompting sections, they can use these to test what the product of pairs of powers of ten are and deduce the correct rule for themselves.
- Some students might not fully appreciate the need for standard form, particularly in writing numbers such as 300 as 3×10^2 . There is a **controllable applet** available that will allow students to appreciate use of standard form in the context of our universe (htwins.net).

Hooks

1. 'Powers of Ten' video (search online for 'powers of ten sped up video'). This very short clip shows increasing distances in metres by an extra power of ten moving out from a picnic cloth into the atmosphere and space beyond (some versions also show smaller and smaller distances down to atomic level). There are other, longer, alternatives available but this will give a good starting point for discussions about representing very large numbers.
2. 'The Galaxy Song' by Eric Idle. This song contains a plethora of facts about the universe, incorporating very large numbers. Students could initially note down all the numbers they can from listening to the song. There are a range of compound measures used that could provide an interesting exercise to compare values by converting them to the same measure, for example miles per second and miles per day. They could also research to see if all the information is accurate. Note: The original Monty Python version contains some images that are not really suitable for this age group, and the final line of the song has one word you might want to avoid. There are versions online that simply show computer images of space to accompany the song. Alternatively you might wish to play the song without showing the accompanying video.

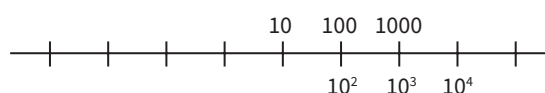
SECTION 1F / 1H: EXPRESSING NUMBERS IN STANDARD FORM

Section 1 introduces writing numbers in standard form and converting numbers in standard form back to ordinary numbers.

Prompting questions

To check students' grasp of working with indices and powers of ten before they work through the exercises, some good prompts might be:

- Use a scale like the one below to develop understanding of powers of ten in index form and also to demonstrate that negative powers involve division:



Ask students:

- What patterns can you see in the diagram? *Increasing power by 1 for each move to the right, multiplying by 10 each time. Going left, powers decrease by 1 and each value is the previous value divided by ten.*
- Can you add extra values? *Ensure $10^0 = 1$ and $10^{-1} = \frac{1}{10}$, and so on, are added.*
- What changes each time as you move along the diagram to the right? *Each new value is 10 times bigger than the previous one. The power increases by 1.*
- What changes each time as you move along the diagram to the left? *Each new value is 10 times smaller than the previous one. The power decreases by 1.*
- What are the rules for moving right or left? *Multiply/divide by 10.*

Exercise 26A(F) / 26A(H)

While working through this exercise, good prompts for students might be:

- **Q1k(F) / Q1k(H)** Why is the solution to this not 3.5×10^{-3} ? *It's key for students to recognise the importance of the place holding 0 between the 3 and 5 and see that as a result the answer is 3.05×10^{-3} .*
- **Q1m(F) / Q1m(H)** Why would an answer of 34×10^3 be incorrect, since this equals 34000? *It's not in the correct standard form notation.*

Exercise 26B(F) / 26A(H)

While working through this exercise, good prompts for students might be:

- **Q1c(F) / Q2c(H)** Can you give a real-life example where this number might be used? *Obviously there are many responses to this, but since the number is $\times 10^3$ this number could be an example of a metric conversion of kg to g, km to m, and so on.*
- **Q2f and g(F) / Q4f and g(H)** Using the two facts here, how far would the total journey from Jupiter to the sun and then to Earth be? *Since this type of calculation will appear later in the chapter this could be a good point to see if students convert to ordinary numbers first or try to use the standard form format. $7.7^8 \times 10^8 + 1.5 \times 10^{11} = 1.50778 \times 10^{11}$.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- The introduction to the section describes how to write numbers in standard form. Ask students the names of numbers such as 1 000 000, 1 000 000 000 and 1 000 000 000 000 first as a good way to check their knowledge of place value for large numbers and introduce the structure of naming these numbers i.e. million, billion, trillion, quadrillion etc. It can be interesting to discuss why the British system used to define one billion as one million $\times$ one million and why we changed this.
- **NRICH A Question of Scale.** An interactive sorting activity to link physical objects with their length in powers of ten (nrich.maths.org).
- A quick starter: write today's date (or a birthdate, etc.) in a creative way using index notation and standard form.
For example, my birthdate is $16^{\frac{1}{2}} / 2^3 / 1.97 \times 10^3$.

Enrichment activities

- Improving Learning in Mathematics **Mostly Number N4 – Estimating length using standard form** available from the National STEM Centre Archive (nationalstemcentre.org.uk).
- Converting standard form to ordinary numbers. Research data for very large and small numbers such as the diameter of an atom, size of the smallest virus, number of atoms in a human body, diameter of the Earth, and so on.
- **Instant Maths Ideas** (Standard Form) available at the National STEM Centre Archive. This excellent resource provides additional ideas or consolidation of the suggestions above and the exercises in the chapter (nationalstemcentre.org.uk).

SECTION 2F / 2H: CALCULATORS AND STANDARD FORM

Section 2 focuses on familiarising students with the buttons they need to use on their own scientific calculators.

Prompting questions

- How do you enter numbers in standard form? Does anyone near you have a different style button to do this? (If so, this is why being familiar with your own calculator is so important.) Does your calculator require the buttons to be pressed in a different or surprising order? *The order of entry is dependent on whether students have modern line display calculators or older/cheaper varieties. Stressing the importance that they find out how to use their calculator is vital.*
- Is the answer the same if you use the standard form button as if you type in $\times 10^n$? Why? *In theory it should be but again this can depend on the calculator make and ensuring value are entered in the required order for the operations.*

Exercise 26D(F) / 26C(H)

While working through this exercise, good prompts for students might be:

- **Q1a(F) / Q1a(H)** What is the difference between 4216^6 and 4216×10^6 ? *Interchanging between powers of ten and general use of indices might bring to light any lingering misconceptions and confusion between these so it's worth challenging at this point.*
- **Q1i(F) / Q1i(H)** Is $\sqrt{5.25} \times 10^8$ the same value as $\sqrt{5.25 \times 10^8}$? Explain your answer. *The second version is rooting the power of ten, which will reduce the magnitude to $\times 10^4$. This could link back to the index laws – it is an example of the third law where $(a^m)^n = a^{nm}$ since square rooting is raising to the power of $\frac{1}{2}$.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Students draw a step by step 'story board' for entering a specific calculation on their calculator. These could be displayed on a 'working board' for the duration of the topic.
- The **Number Loving** website has a selection of 'catchphrase', 'top trumps' 'tick or trash' and 'collect a joke' activities that could be used for group and class activities to consolidate and practise techniques beyond the textbook exercises (numberloving.co.uk). (Use search criteria of Key Stage 4 and the topic Integers, powers and roots to locate these easily.)

SECTION 3F / 3H: WORKING IN STANDARD FORM

Section 3 applies the use of standard form to a range of situations.

Prompting questions

Exercise 26E(F) / 26D(H)

While working through this exercise, good prompts for students might be:

- **Q1a(F) / Q1a(H)** Explain why you can rearrange the terms of the question $(2 \times 10^{13}) \times (4 \times 10^{17})$ to $(2 \times 4) \times (10^{13} \times 10^{17})$? Why is this method of calculation more efficient? *Multiplication is commutative, and this is useful as we can then see the application of index laws to work easily with these numbers.*
- **Q3a(F) / Q3a(H)** What makes the question $(3 \times 10^{12}) \times (4 \times 10^{18})$ more challenging than $(2 \times 10^{13}) \times (4 \times 10^{17})$? *Since 3×4 gives 12 students might write an answer of 12×10^{30} , so this will highlight any students not yet secure with the precise format required when writing a number in standard form.*

Exercise 26F(F) / 26E(H)

While working through this exercise, good prompts for students might be:

- Standard form is often used in conversions when using large and small units. How would you represent 1 megawatt in watts? *The prefix mega means one million so this is 1×10^6 watts.* How would you represent 1 nm in metres? *nm are nanometres and the prefix nano means one billionth, so this is 1×10^{-9} metres.*

Exercise 26F(H)

While working through this exercise, good prompts for students might be:

- **Q1a(H)** If you round 2^{30} to four significant figures, rather than 3 as requested, how does this affect the value? *1 073 741 824 to 3 sf is 1 070 000 000, and to 4 sf is 1 074 000 000, which makes a difference of 4 000 000. This is a good example to get students to consider the potential impact of premature rounding when working in standard form since 1.07×10^9 and 1.074×10^9 might appear deceptively similar. However, in the context of calculations with other very large numbers, the rounding might not have a significant effect on the solution.*
- **Q4(H)** Google gives the speed of light as 299 792 458 metres per second. What is the difference between this and the value given in this question? $3 \times 10^8 - 299\,792\,458 = 207\,542 \text{ m/s}$.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Before introducing the methods for multiplying and dividing numbers in standard form, it could be useful to give students some revision exercises on using the first two index laws (for multiplying and dividing by powers of ten) if they have not practised this for some time.
- A group task, done in silence to encourage awareness of others' requirements and cooperative learning, where each group member has to collect a set of four matching cards (one in ordinary form, one in standard form, one with the same value as the solution from addition or subtraction of two numbers in standard form, and one that is the solution from multiplication or division). Sixteen cards are dealt (four per person) and are laid in front of each person for all to see. Each group member may hand a card to another, being given one in return, until all members have a 'happy families' set of cards (i.e. matching ones of each category). An example matching set could be 50 000, 5×10^4 , $5.04 \times 10^4 - 4 \times 10^2$ and $(2 \times 10^7) \div (4 \times 10^2)$. The aim is to be the first group in the class to complete these successfully.

Enrichment activities

- NRICH **Big and Small Numbers in the Living World** combines estimation and large numbers with problem-solving skills to help contextualise the use of standard form (nrich.maths.org).
- **Space Math** from NASA, available from the National STEM Centre Archive, provides many activities and questions in a space context that involve using standard form. Each of these booklets contains a table at the start showing the content covered by each activity. Scientific notation features in many of them, particularly space math IV (nationalstemcentre.org.uk)
- **Little Green Men – The Long Way Round**, available from the National STEM Centre Archive requires students to make calculations based on the distances in the galaxy (nationalstemcentre.org.uk)

Assessment ideas

- MARS Assessments **A Million Dollars** is a useful written task to assess understanding. However, it is designed for an American audience therefore the units are imperial. (map.mathshell.org).
- MARS Assessments **Giantburgers** is a useful written task to assess understanding and problem-solving skills (map.mathshell.org).

Topic links

Previous learning

Students have encountered the use of powers, both positive and negative when working with the laws of indices in *Chapter 25 Powers and roots*. Also, a calculator will give an answer in standard form when it contains too many digits to fit on the display and students will have encountered this on many previous instances.

Future learning

Standard form is used whenever very large or very small numbers are involved in a calculation such as those for volumes of cubes and spheres (*Chapter 21 Volume and surface area*) or direct and inverse proportion (*Chapter 34F/36H Direct and inverse proportion*).

It is also used for the speed of light in calculations involving speed, distance and time (*Chapter 12 Units and measurement*) and students are likely to use standard form in Physics when looking at distances to distant stars and galaxies, the sizes of atoms or when working out the wavelengths of parts of the electromagnetic spectrum.

Gateway to A Level

Standard form is closely related to indices that are used extensively throughout A Level mathematics when working with functions. They are particularly useful in physics, chemistry and biology when working with very large values, such as the speed of light, or very small values, such as the size of a virus.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 7 Question 20
- Chapter 10 Question 18

Higher

- Chapter 2 Question 27
- Chapter 8 Question 22
- Chapter 10 Question 14

Homework Book

Foundation

- Chapter 26

Higher

- Chapter 26

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 3 worksheets (+ solutions)
- 2 animated widgets
- 7 interactive walkthroughs
- 2 auto-marked quickfire quizzes
- 2 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

27H Surds

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to round an answer to a required number of decimal places or significant figures;
- know how to use Pythagoras' theorem (from Key Stage 3 study);
- be able to write numbers as products of their prime factors (perform prime factor decomposition);
- know how to expand brackets and simplify expressions by collecting like terms (so as to have an understanding of basic algebra since manipulation of surds uses similar techniques);
- understand how to create equivalent fractions by multiplying or dividing both numerator and denominator by the same value.

Additional useful prior knowledge

- To be able to use the trigonometric ratios in order to solve more complex extension problems involving Pythagoras and trigonometry.
- To be able to find the circumference and area of a circle in order to tackle a few of the contextualised questions.

Learning outcomes

Section 1

- To use a calculator to approximate the values of numbers involving surds.
- To calculate exact solutions to problems using surds.

Section 2

- To simplify expressions containing surds.
- To manipulate surds when multiplying and dividing.
- To rationalise the denominator of a fraction.

Section 3

- To apply an understanding of surds to solve more complex problems.

Vocabulary

irrational number, surd, rational number, conjugate

Common misconceptions and other issues

- When students fail to spot a factor that is square they might assume a surd cannot be simplified, for example, for $\sqrt{75}$ they might write $\sqrt{5} \times \sqrt{15}$ and stop at this point. Encourage students to carry out full prime factor decomposition of the number each time so as to identify any primes raised to the power of 2 (or any even number) that would indicate square factors.
- Linked to the above, students might not fully simplify a surd because they have not spotted the largest square number that is a factor. A common example is a number like $\sqrt{48}$ where students reduce this to $\sqrt{4 \times 12} = 2\sqrt{12}$. Again, encouraging full prime factor decomposition might help.
- Students who do not fully understand how to simplify surds might follow a procedure and make errors that they do not realise are incorrect. For example, a student might say $\sqrt{28} = \sqrt{7 \times 4}$ and then present their final answer as $7\sqrt{4}$ or

$7\sqrt{2}$ as they have not grasped that the correct simplification is $\sqrt{7} \times \sqrt{4} = 2\sqrt{7}$. Double checking through use of approximation or a calculator during the early stages of students working on this topic might help.

- If students are simply presented with a set of rules concerning surds, they might incorrectly assume that, since $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ then $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ is also true.
- Manipulating surds might identify misconceptions remaining from simplifying expressions. A specific one to address is $(a+b)^2 = a^2 + b^2$, as if this still exists students will not understand the need for a conjugate expression when rationalising more challenging surds. For misconceptions like this, give students time to investigate which are true or false (the Improving Learning in Mathematics N11 resource suggested in Section 2's enrichment activities is ideal for this).
- When rationalising surds, students often forget to simplify the final fraction or are surprised when the answer is not in fractional form and assume they have gone wrong. Some quick practice or a starter on simplifying fractions before covering this could help.

Hooks

1. Ask students to try to draw a line that is exactly $\sqrt{2}$ cm long, (*the hypotenuse of an isosceles right-angled triangle with the shorter lengths 1 cm*) or $\sqrt{5}$ cm (*the hypotenuse of a right-angled triangle with the two shorter lengths of 1 cm and 2 cm*). As a challenge ask them whether they can draw a length of $\sqrt{3}$ cm. (*Here they could use a right-angled triangle with hypotenuse 2 cm and shortest length 1 cm*).
2. Build on the above by asking students to join two adjacent dots diagonally on isometric paper and use Pythagoras' theorem to work out the length ($\sqrt{2}$). What if they double the length of the line by extending it up to the next dot? *Its length must be $2\sqrt{2}$ but using Pythagoras again will give the solution of $\sqrt{8}$* . Ask students to continue to explore this in order to be able to suggest their own rules for simplifying surds such as $\sqrt{72}$.

SECTION 1: APPROXIMATE AND EXACT VALUES

Section 1 introduces the definition of a surd, how to work with them on calculators and use them in contextualised questions in order to obtain an exact solution.

Prompting questions

Exercise 27A(H)

The activities in Exercise 27A will help students to familiarise themselves with operating their calculators. With the wide range that are now available it is important to ensure students find all the relevant buttons, and compare with others, checking answers as they work through. This is particularly relevant if some students have older or cheaper calculators where the order of entry of calculations can differ and answers might not be given in surd form but rounded to fit the display.

While working through this exercise, good prompts for students might be:

- **Q1(H)** and **Q2(H)** What would the answers to each question be if they were rounded to 3 significant figures instead of 3 decimal places? This will give an opportunity for the revision of significant figures and could promote a discussion about a relevant degree of accuracy to give solutions to when this is not specified.
- **Q3(H)** Try changing Questions e and f to $2\sqrt{3} + 3\sqrt{3}$ and $-2\sqrt{5} + 3\sqrt{5}$. Ask students what they think the answer will be before checking on the calculator.

Exercise 27B(H)

While working through this exercise, good prompts for students might be:

- **Q3(H)** The solution to this might seem surprising initially. Ask students to suggest another similar question that would work in this way (e.g. $\left(\frac{25}{\sqrt{5}}\right)^2$).
- **Q5(H)** Work out the length of the plot without using surds. Try rounding your answers to a range of different degrees of accuracy (e.g. 1 dp, 2 dp, and so on). What effect does this have on the final answer for the diagonal length? In

what ways is using surds better? *Using surds, you can do all of this without a calculator and it removes the issue of rounding errors.*

- **Q6(H)** This question encourages students to think about the implications of rounding on calculating a selling price. Ask students to think of their own examples and create a question that illustrates how rounding could have an impact.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- In preparation for simplifying surds in the next section, a starter finding square roots of larger square numbers, using the method of prime factor decomposition, would be useful. For example, find $\sqrt{484}$. *Since $484 = 2 \times 242 = 2 \times 2 \times 121 = 2 \times 2 \times 11 \times 11 = 2^2 \times 11^2$ the square root must be $2 \times 11 = 22$.*
- Exercise 27B will help students to see the purpose of surds within calculations in order to consider the accuracy of their answers but requires students to be able to use Pythagoras' theorem. If they haven't covered this recently the Pythagoras chapter (from p60) in Colin Foster's book **Instant Maths Ideas: Shape and Space** will provide plenty of revision materials (nationalstemcentre.org.uk).

Enrichment activities

- The NRICH problem **Trice** can be solved using Pythagoras' theorem and leaving intermediate answers in surd form will allow students to obtain a precise solution (nrich.maths.org).
- NRICH **The Spider and the Fly** also uses Pythagoras' theorem and students could use surds but will need to convert them to approximate values to compare (nrich.maths.org).

SECTION 2: MANIPULATING SURDS

Section 2 covers all the key manipulations of surds, from simplifying them, to collecting like expressions, expanding brackets and rationalising fractions involving surds.

Prompting questions

- The introduction provides four rules that apply to surds, do students feel convinced? Is not having found a counter-example is sufficient to say the rules hold? Can they think of any ways to begin to generalise this? *For example, $(\sqrt{n})^2 \equiv \sqrt{n} \times \sqrt{n} \equiv n$, $\sqrt{n} = n^{\frac{1}{2}}$ and $n^{\frac{1}{2}} \times n^{\frac{1}{2}} = n^1 = n$ (for positive values of n).*

Exercise 27C(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** For each of the incorrect answers ask students what errors were made and therefore what possible misconception is being demonstrated.
- **Q3(H)** Ask students to suggest some other surds that must already be in their simplest form. *For example, the square root of any other prime number.*
- **Q5b(H)** To add to the prompt already given, if students are unsure ask them to compare this to $\sqrt{54}$.

Exercise 27D(H)

While working through this exercise, good prompts for students might be:

- **Q4(H)** Why can't $\sqrt{45} + \sqrt{8}$ be simplified? What do you notice about the surds involved in this question where further simplification is possible? *Look for suggestions of common factors; use prime decomposition to identify which surds could be simplified in this way.*

Exercise 27E(H)

While working through this exercise, good prompts for students might be:

- **Q4(H)** Which of these demonstrates use of a conjugate? *Part g.* Which mathematical technique already encountered in manipulating algebra is the basis for this technique? *Difference of two squares.* Can students write another question similar to these examples where they are certain the final answer will be an integer?

- **Q6(H)** For students feeling less confident ask them to write down the conjugate they will use for each question first, before attempting the rationalisation.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- **Teachit maths** (you can register for free and download pdf versions of their resources) has a pairs game for simplifying and matching simple equivalent surds (and the solutions are provided) (teachitmaths.co.uk).
- A light-hearted song summarising manipulations of surds to the tune of 'Grease is the word' is 'Root 3 is a surd' can be found by searching online for 'Root 3 is a surd, grease is the word'.
- **Surds Connect 4** covers basic simplification of surds and would be a good starter or plenary activity for paired work (justmaths.co.uk).
- Exercises 27C, 27D and 27E all contain a range of questions where students are asked to simplify surds. These could be used in a number of ways, such as to create matching cards, pairs, a follow me or a Tarsia puzzle in order to provide students with practice in a more interactive format where they can work more easily in pairs or groups.

Enrichment activities

- Improving Learning in Mathematics **Manipulating Surds – N11** available from the National STEM Centre Archive contains a range of activities such as true/false statements and dominoes to enable students to develop their understanding of how to manipulate surds (nationalstemcentre.org.uk).
- The **Number Loving** website has a differentiated 'collect a joke' activity for simplifying surds, when students can work on one of three different levels worksheets in order to find surd solutions that links to phrases that form a joke. The site also has a 'tick or trash' sheet for surds (numberloving.co.uk).
- For extra practice on the more complex manipulation of surds, the **MEI GCSE extension material 'Surds'**, available from the National STEM Centre, also makes the connection to the golden ratio (nationalstemcentre.org.uk).
- The NRICH problem **The Root of the Problem** will require students to practise rationalising denominators using conjugate surds in order to look for a solution.
- As an extension for students finding manipulation of surds quite straightforward, the NRICH task **Irrational Arithmagons** should prove quite a challenge (rich.maths.org).

SECTION 3: WORKING WITH SURDS

Section 3 embeds the skills used in the previous sections with a range of problem-solving questions involving the use of surds.

Prompting questions

Exercise 27F(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** Ask students to draw a different shape with the same perimeter as the ones in the question. They could also try this for area, which in some cases could be quite challenging!
- **Q7(H)** Why is it possible to solve this question? Could you write a question that looks similar but cannot be answered? Can you write one that is possible? *This factorises to $x(x + 1)^2$ so can then be simplified using basic surd manipulation. Therefore any cubic of the form $x(x \pm n)^2$ would also work.*

Starters, plenaries, enrichment and assessment ideas

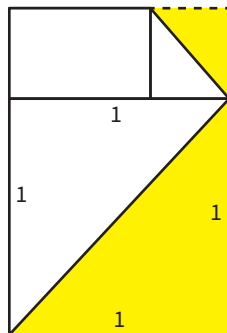
Starters

- Give students the section's example question to solve individually, before looking at the step by step solution in the book. Then ask their partner to review the solution and note any areas of method that seem unclear, that are implied rather than recorded explicitly or that are missing. Then ask students to look at the solution in the book and identify

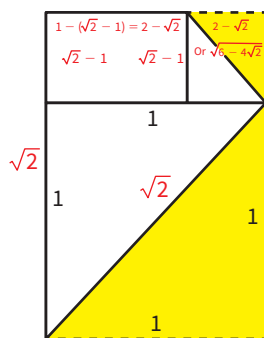
its strengths, and any points they might alter in their solution, recording how they could improve their method in the future.

Enrichment activities

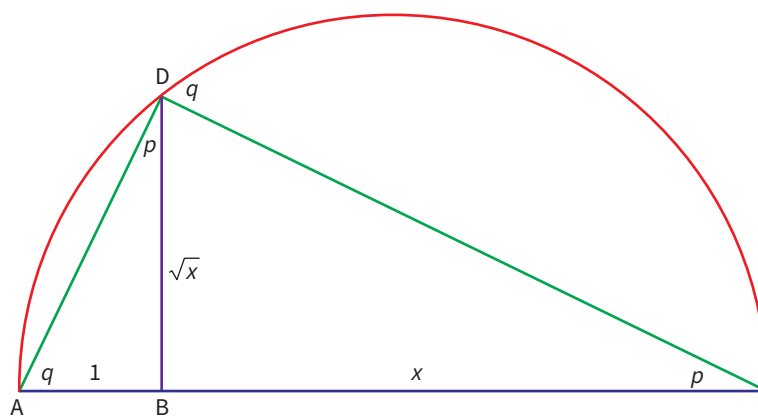
- This activity links well with Exercise 27F Question 4. Fold a piece of A4 or A5 paper to create a kite. Only two folds are needed: holding the paper in portrait, fold the bottom right vertex diagonally across to the opposite edge, bisecting the bottom left vertex. Then fold the top right vertex up to meet the horizontal edge created by the first fold. Tell students to assume the length of 1 for the shorter sides of the sheets of paper (see diagram below).



- How many other lengths can they find? What is the perimeter of the kite? Can they work out the area? (The long diagonal is $\sqrt{2}$ by Pythagoras. If they hold this against the long side of the rectangle of another sheet of paper (or fold the kite in half) they will see that this is the same length. A properties of A-type paper is that the ratio of side lengths is $\sqrt{2} : 1$. From this the dimensions below can be found:



- Perimeter = 4, Area = $2\sqrt{2} - 2$.
- Research the 'Geometric Square Root' construction technique described by Descartes (although not attributed to him). Create a presentation or leaflet explaining how the method works.
- A diagram showing the Geometric Square Root construction:



- Investigate the Babylonian tablet YBC7289 from c. 1700 BC, which provided a calculation for root 2 using their base 60 number system. $(1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \approx 1.41421297...)$

- For very able students a nice extension is to look at the formula for the Fibonacci sequence.

$$F_n = \frac{1}{\sqrt{5}} \times \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \times \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Topic links

Previous learning

This chapter relies on skills acquired from several previous chapters. In particular students should be proficient at squaring numbers, finding square roots and prime factor decomposition (*Chapter 2 Whole number theory*), using the laws of indices (*Chapter 25 Powers and roots*), using Pythagoras' theorem (*Chapter 32 Pythagoras' theorem*), cancelling fractions and equivalent fractions (*Chapter 10 Fractions*) and basic algebraic manipulation skills such as expanding brackets, simplifying expressions and recognising the difference of two squares (*Chapter 3 Algebraic expressions* and *Chapter 7 Further algebraic expressions*).

Future learning

Students will use surd form for giving exact answers to calculations involving Pythagoras' theorem and 3D Geometry (*Chapter 32 Pythagoras' theorem*) and for giving exact values of trigonometric functions such as $\sin 30^\circ$ (*Chapter 33 Trigonometry*).

Gateway to A Level

A Level students will continue to use surds to give exact answers to calculations such as the roots of quadratic equations or values of trigonometric functions. They will need to be able to give answers in their simplest form by rationalising the denominator. This skill will also be applied to complex numbers, in particular the method of using a conjugate to manipulate them.

Surds will be used when applying the binomial theorem in order to find approximate roots.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

- Chapter 4 Questions 13, 14
- Chapter 6 Questions 15, 25

Homework Book

- Chapter 27

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 5 worksheets (+ solutions)
- 6 animated widgets
- 16 interactive walkthroughs
- 5 auto-marked quickfire quizzes
- 5 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

27F / 28H Plane vector geometry

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know basic arithmetic skills including addition, subtraction, multiplication and division (for finding fractions of amounts) of both positive and negative numbers;
- know how to plot coordinates in all four quadrants, understanding that the x - and y -coordinates are distances in horizontal and vertical directions from the origin;
- understand basic ratio including connections to proportion (e.g. $2 : 3$ is $\frac{2}{5}$ and $\frac{3}{5}$) of the whole;
- know how to solve simple linear equations;
- know how to solve simultaneous linear equations.

Additional useful prior knowledge

- What the laws of associativity, commutativity and distributivity are and how they apply to basic arithmetic operations.
- Form and solve simultaneous equations.
- How to use Pythagoras' theorem to find the length of a line segment in 2D.
- How to use the tangent function to find angles in right-angled triangles.
- (Higher only) What a mathematical proof is.

Learning outcomes

Foundation

Section 1

- Represent vectors as a diagram or column.

Section 2

- Apply add and subtract vectors.
- Multiply vectors by a scalar.
- Recognise parallel vectors.

Higher

Section 1

- Represent vectors as a diagram or column.

Section 2

- Apply add and subtract vectors.
- Multiply vectors by a scalar.
- Recognise parallel vectors.

Section 3

- Use vectors to construct geometric arguments and proofs.

Vocabulary

scalar, vector, displacement, parallel vectors

Common misconceptions and other issues

Though in some ways this topic might appear straightforward, there are many possible misconceptions and students can easily pick up wrong ideas, so care is needed when first teaching this material.

- Confusing x - and y -values. When working with column vectors, some students confuse the x - and y -axis and which direction of movement the values refer to. Ask students to recall how coordinates work. Phrases such as 'x is a cross' and 'y to the sky' might prove helpful. You could also give students a coordinate grid to practise rewriting

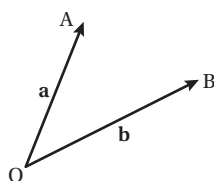
coordinates as a position vector from the origin, for example $A=(3,5)$ so $\overrightarrow{OA}$ or $\mathbf{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Be careful that this does not reinforce or introduce the misconception that all vectors must start from the origin (see next point).

- Assuming all vectors start at the origin. The term 'position vector' is not used in the Student Book, but teachers should be aware of the following definitions so they can spot if students begin to treat displacement vectors as position vectors:

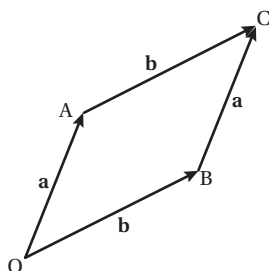
- position vectors describe the movement to a point on a grid and always start from the *origin* (in the student books coordinates rather than position vectors are used to locate positions on a grid).
- displacement vectors describe the movement between *any* two points on a grid, and therefore can start anywhere on the grid depending on the points in question.

A displacement vector can be equal to a position vector, but their starting points will be defined in a different way.

- Understanding that equal vectors are also parallel vectors. The definition of parallel vectors is when one vector can be written as a (scalar) multiple of the other, for example $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = t \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ where t is the scalar to be found. This is often described to students when they know how to multiply a vector by a scalar, as one vector is a multiple of the other. It is also important that students understand that equal vectors are also parallel; this is the case when the scalar is one.
- Understanding what $-\mathbf{a}$ means geometrically in relation to $\mathbf{a}$. Ask students to draw three vectors and then to draw their opposite vector on the same grid. For example, the opposite of the vector drawn '2-right, 3-up' would be '2-left, 3-down'. This should help students to identify that multiplying a vector by the scalar -1 results in moving in the opposite direction to the starting vector. This can also be explained by describing a series of vectors in a geometric problem as a network with a series of paths, for example:



- To find $\overrightarrow{AB}$ you can travel back along the arc $\mathbf{a}$ (i.e. $\overrightarrow{AO} = -\mathbf{a}$) and then along $\mathbf{b}$. This can be written as $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.
(Higher only) If using only displacement vector notation, you can write $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$.
- Be careful when using the idea of vector problems as networks because it can introduce new problems when working with midpoints and other values as proportional distances between two points, when students wish to stick to the grid only. For example, in addition to the information in the diagram above you are told that coordinate M is the midpoint of the line segment AB and are asked to find $\overrightarrow{OM}$. By considering $\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$ you can simplify to give $\overrightarrow{OM} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$. If students want to stick to the network and the drawn paths then this has little meaning to them, whereas with a strong knowledge of equal and parallel vectors students can understand that it is the start and end points and the movement between them that are important rather than the routes taken.
- Understanding that addition is commutative in vector arithmetic. This links into the previous point about wanting to use only the drawn paths in the diagram above. Given the same vectors above, additional vectors can be drawn on the diagram to give:



If students fail to understand the importance of parallel vectors in describing movement then they will fail to understand that $\overrightarrow{AC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OB} + \overrightarrow{BC}$, i.e. $\overrightarrow{AC} = \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

Hooks

1. A nice way to introduce the idea of a vector, and then ask students what they think it is, is through a clip of *Despicable Me* (movie), which you might be able to find online. It introduces the words 'direction', and 'magnitude' and produces a discussion point from which you can bring out the definition of a vector and how it differs from a scalar.
2. Another way to introduce vectors is to look at line segments on a grid joining two points (e.g. a map of an American city based on a grid/block network showing two points of interest). A discussion could be based on travelling 'as the crow flies' rather than by the grid network, and through this the discussion can move to the need for vectors to describe movement.

SECTION 1F / 1H: VECTOR NOTATION AND REPRESENTATION

This section introduces the basics of vectors including notation and representation. There are opportunities for students to go beyond what a vector is and how we can draw it to also consider what it means for vectors to be parallel, equal and opposite. These ideas are drawn out through the first exercise.

Prompting questions

While introducing the need for vectors, good prompts for promoting discussion might be:

- How can you describe any point on a grid/axis? *Using coordinates to define the distance from the origin in terms of the horizontal and vertical distance.*
- What are you describing its position in terms of? *Its distance from the origin.*
- What is the quickest way to travel from the origin to this point? *Diagonally or 'as the crow flies'.*
- What if you want to describe how to get from coordinate B to coordinate E? Why is this more challenging? *Need to give starting point and x and y movement, and hence introduce need for vectors.*

Exercise 27A(F) / 28A(H)

While working through the exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** What would the column vector be if the arrows were pointing in the opposite direction? *The same numbers with the opposite polarity.*
- **Q1(F) / Q1(H)** How else could you describe this 'opposite' vector? *It is the original vector multiplied by -1.*
- **Q2(F) / Q2(H)** Are there any other pairs of vectors that are of interest? What is interesting about them?
- **Q3(F) / Q3(H)** Could you draw on the right-angled triangle to show the movement represented by the vectors given when finding the new coordinate points? *For example, for point B – draw a triangle with base 2 units to the right from starting point and then 7 units up.*
- What is meant by opposite vectors? *Vectors that are linked by a scale factor of -1 (i.e. the same vector where each element has the opposite polarity so it describes a movement of the same distance in the opposite direction).*
- What is meant by equal vectors? *Vectors that describe movement in the same direction for the same distance.*
- What is meant by parallel vectors? *The movement is the same direction but they do not necessarily start and finish in the same place on the grid or move the same distance. One of the vectors will be a multiple of the other (i.e. they are connected by a scale factor).*
- Can you give 3 vectors and a coordinate point that together form a right-angled triangle?
- Can you do the same for an isosceles triangle?
- Can you do it for an equilateral triangle?

Starters, plenaries, enrichment and assessment ideas

Starters

- The ideas in the 'Hooks' section are nice ways to introduce this topic. Looking at a large coordinate grid (with or without the underlay of a city) would be a good way to replicate the example in Hook 2. Providing multiple examples of routes to take, for example, between two referenced points on the grid (defined in terms of their coordinates) that requires instructions such as starting at 'place 1' head east for three blocks then north for five. Whereas just saying head east for three blocks then north for five implies you start at (0,0).

Starters or plenaries

- In pairs, students play 'vector snakes and ladders', to help consolidate the use of column vectors. The board is designed so that the vectors placed on some squares on the board displace the counter in a helpful or not so helpful way. An alternative is to ask students to write the vectors that describe the movement the snakes and ladders give on a standard game board.
- A card game of 'follow me' or dominoes. A series of points on a grid are projected at the front of the class. One side of the follow me/domino card gives the label of one of the points and the other side gives a column vector, for example using the diagram for Question 2 in Exercise 27A(F) / 28A(H), you could have the following cards:

$$\mathbf{D} \quad \left(\begin{array}{c} 2 \\ -3 \end{array} \right)$$

$$\mathbf{H} \quad \left(\begin{array}{c} -8 \\ 4 \end{array} \right)$$

- Students read out their column vector and calculate the point it arrives at using the grid on the board. If another student's card has the letter of that point they then continue the game by reading out their column vector. In the example above, the next card would have the letter A. These cards can be made more challenging for use with Section 2 by giving a calculation to simplify first, for example:

$\mathbf{D}$	$\left(\begin{array}{c} 8 \\ -12 \end{array} \right) + 3 \left(\begin{array}{c} -2 \\ 3 \end{array} \right)$
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- NRICH Vector Journeys** task pulls out a lot of the above points that come through in Exercise 22A(F) / 23A(H) about opposite, equal and parallel vectors but there are many more options for exploration and introduction to addition of vectors (Section 2) in this simple, open task (nrich.maths.org).

SECTION 2F / 2H: VECTOR ARITHMETIC

This section introduces students to the basics of vector arithmetic including addition and subtraction of vectors and multiplication by a scalar. Students are not introduced to other forms of vector multiplication until A Level and it's probably worth explaining to students that this is one of many forms of multiplication with vectors.

Prompting questions

Exercise 27B(F) / 28B(H)

Whilst working through Exercise B, good prompts for students might be:

- Q1(F) / Q1(H)** How can you use a grid to help? How would these vectors look on the plane (end)? Starting at any grid point (on square paper) draw each vector successively to find the finishing grid point. Then join up the start and end points to reveal the single vector that describes the translation.
- Q2(F) / Q2(H)** What do you need for parallel vectors? *To find a scalar multiplier.*
- Q3(F) / Q3(H)** What does it mean when a combination of two vectors makes a third? *That the left-hand side combination for movements in the x-direction must be the same as the movement in the x-direction on the right-hand side of the equality sign. The same is true for the movements in the y-direction.*
- Q3(F) / Q3(H)** What do you need to set up in order to find the values x , y , z and t that create these vector equations? *Linear and simultaneous linear equations.*

- **Q4(F) / Q4(H)** If the ratio of AC : CB is 1 : 3, what must be the ratio of the movement in the x-direction of A to C and C to B? *The same, 1 : 3.*
- **Q4(F) / Q4(H)** If 20 is the movement between A and B in the x-direction, what proportion of 20 is movement from A to C and what proportion of 20 is movement from C to B? *We can look at the ratio for the x-direction movement as 1 : 3, with our total of 20 we have a ratio for the x-movement as 5 : 15 for AC and C to B.*
- **Q4(F) / Q4(H)** What fraction of the line segment AB is AC if the ratio of AC : CB is 1 : 3? *One quarter.*
- **(Higher only)** What different types of triangles do you know about and how do you tell them apart? *Equilateral, isosceles, scalene (consider the magnitudes of vectors) and right-angled (in 2D use, consider the gradients of the line segments).*

Starters, plenaries, enrichment and assessment ideas

Starters

- As mentioned in Section 1, the NRICH **Vector Journeys** task offers opportunities for students to discover addition of vectors through exploration. It also gives students concrete examples for drawing on right-angled triangles and forming a justification for why the x-values and y-values are summed when adding vectors (nrich.maths.org).

Starters or plenaries

- NRICH **Vectors – short problems** can be used as starters and plenaries (nrich.maths.org).

Enrichment activities

- The NRICH Vector Journeys task can be extended into a second problem **Vector Walk**, which explores all possible combinations of vector addition to arrive at a new coordinate point. By allowing multiple uses of the two vectors, students can also discover vector multiplication by a scalar as repeated addition (nrich.maths.org).
- Once students are able to add vectors they can explore those that sum to zero through a closed path (polygon). The NRICH task **Spotting the Loophole** offers opportunities to practise basic arithmetic on vectors while completing the task and further consolidates their knowledge through the use of diagrams (nrich.maths.org).

SECTION 3F: MIXED PRACTICE

This section gives foundation students the opportunity to practise the vector arithmetic of the previous sections in a variety of geometrical situations.

SECTION 3H: USING VECTORS IN GEOMETRIC PROOFS

This section explores the use of vectors to prove geometric results and applies vector arithmetic to solving problems including finding midpoints. This is the section where students display most misconceptions as they are able to apply procedure and manipulate vectors, but struggle to reconcile this with their geometric meaning to solve problems. In addition students' understanding of mathematical proof can limit their ability to successfully complete the problems in this section. Students often have more success when they are encouraged to sketch the vectors they are working with (note: this is a sketch and not an accurate representation of the vectors they are working on).

Prompting questions

Exercise 27C(F) / 28C(H)

While working through this exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** How do you reverse a vector? *We take the negative of it.*
- **Q2(F) / Q2(H)** If the vectors describing the movement between the vertices of the triangles are multiples of each other, what does this mean? *The triangles are similar.*
- **Q4(H)** How do you know when two vectors are parallel? *One vector is a multiple of the other.*
- **Q5(F)** How would you write down a vector equation using the movements between the points given?
Vector $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

- **Q6(H)** What do the vectors $\overrightarrow{DE}$, $\overrightarrow{EF}$ and $\overrightarrow{FD}$ tell you about triangle HIJ? *That it is a similar triangle to EFG.*
- **Q9(H)**
 - What diagram could you draw? *Assume parallel sides for the banks.*
 - How can you represent the flow of the river as a vector? *East to west flow is in the negative x-direction.*
 - How can you do the same thing for the swimmer? *We assume the swimmer will try and swim 1.5 m per second straight to the other side of the bank. So 1.5 in the positive y-direction.*
 - What will the resulting vector be for the movement? *The vector will look like $\begin{pmatrix} -3 \\ 1.5 \end{pmatrix}$.*

Starters or plenaries

- Getting the students to pick out the correct information from a diagram to construct vector equations is a useful starter or plenary once students have been done Section 1 and 2. A diagram could be displayed and the students must write down how to get from each vertex to each other vertex.

Enrichment ideas

- Students construct their own shapes from given vectors. The teacher then ask students to show, describe or **(Higher only)** prove something about their shape. To differentiate this, suggestions can be given that show certain properties of 2D shapes (i.e. the diagonals of a square are of equal length).

Assessment ideas

- Students could form a poster summarising the vectors material as a class. Each student contributes a sticky note to an area of the poster based on one of the topics or highlighted points in the chapter. The teacher then helps compile the poster, either on a white board or the beginnings of a wall display, and students comment on the points that have been drawn out or any that have been missed.

TOPIC LINKS

Previous learning

This topic provides a good opportunity to return to work on straight line graphs (*Chapter 18 Straight-line graphs*). There are opportunities to consider the connections between both a line segment, the vector that describes it, its gradient and its associated right-angled triangle. These are all extension tasks that help consolidate how all these concepts are connected and prepare students for future study at KS5.

Future learning

Column vectors are used in transformations (see *Chapter 41H Transformations of curves and their equations*) to denote a translation $\begin{pmatrix} x \\ y \end{pmatrix}$ in the 2D plane. Also, when students have covered the Pythagoras' theorem and trigonometry chapters (*Chapter 31F/32H Pythagoras' theorem, Chapter 32F/33H Trigonometry*) the content here can be revisited and students can use their new knowledge to look at magnitudes and directions of vectors.

Gateway to A Level

This is a relatively straightforward topic at GCSE that will be built upon at KS5. Having strong foundations in this concept will be necessary for students to extend their knowledge at A Level. In addition to column vectors and displacement vector notation, students will also learn to write and operate on vectors in their component form and extend in to 3D. They will also work mainly with position vectors and learn to write the equation of a straight line using a position vector, displacement vector and scalar, and also convert between Cartesian and vector forms of straight lines. Students might then also go on to learn additional vector operations, work with vector equations of planes and extend their linear algebra knowledge to matrices.

Students might also use vectors when they look at acceleration in a given direction and form vector equations of straight lines where the scalar represents time.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving book

Foundation

- Chapter 1 Question 15
- Chapter 3 Questions 3, 14

Higher

- Chapter 1 Questions 7, 21
- Chapter 3 Questions 12, 19
- Chapter 6 Question 16

Homework Book

Foundation

- Chapter 27

Higher

- Chapter 28

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 3 worksheets (+ solutions)
- 3 interactive walkthroughs
- 3 auto-marked quickfire quizzes
- 3 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper.
- 2 mm graph paper
- Axis grids from -10 to 10 in x and -10 to 10 in y
- Square dotted paper

28F / 29H Plane isometric transformations

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know what angles of 90, 180 and 270 degrees look like;
- know which way a clockwise and which way an anti-clockwise movement goes;
- be able to use the (x, y) coordinates on the 2D-plane fluently;
- know how to plot straight lines of the form $x = c$, $y = c$ and $y = x$ (where c is any number);
- understand what a column vector is (i.e. that it describes movement in the x and y directions).

Additional useful prior knowledge

- To have basic number skills for translations.
- To know the names of polygons, their definitions and properties.

Learning outcomes

Foundation

Section 1

- To carry out, identify and describe reflections.

Section 2

- To carry out, identify and describe translations using 2D vectors.

Section 3

- To carry out, identify and describe rotations.

Higher

Section 1

- To carry out, identify and describe reflections.

Section 2

- To carry out, identify and describe translations using 2D vectors.
- To find the centre of rotation by construction.

Section 3

- To carry out, identify and describe rotations.

Section 4

- To carry out, identify and describe combined transformations.

Vocabulary

object, image, congruent, similar, mirror line, perpendicular bisector, orientation

Common misconceptions and other issues

- Students often confuse the directions clockwise and anti-clockwise. Comparing these instructions to an analogue clock supports them in picturing this movement.
- It is often wrongly assumed that a reflection (mirror) line cannot cover part of a shape (see Foundation Exercise 28A, Question 3).
- Some students mix up the x - and y -axes. Phrases like 'x is a cross' and 'y to the sky' might be helpful in supporting students with this.
- Many students have problems when a reflection line is a diagonal (i.e. $y = x$ in the xy -plane). Directing students to turn the page so that the line of reflection is vertical makes this type of reflection far easier. In addition, students should be encouraged to ask for tracing paper in the exam as this can be very useful in supporting less able students.
- Students might confuse column vector notation with coordinates.

- It is easy to misremember which part of a column vector controls the x -direction movement and which controls the y -direction movement. Asking students which value comes first in a coordinate can help them realise which part of a column vector controls the horizontal and which the vertical movement.
- It can be difficult to remember which pieces of information are required for each transformation, for example a rotation needs a centre from which to rotate, an angle and a direction. Often the number of marks in an exam question can help with this and students should aim to give one piece of information for each mark.

Hooks

1. The mirroring of movement is a nice way to introduce reflections. Placing a metre ruler between pairs, get the one of the pair to 'mirror' the movement of their partner. You could start with the teacher mirroring a student.
2. The idea of a translation could be introduced by starting with a city in a grid layout (like New York) and overlaying a flat xy -grid. Ask the students 'How could we describe moving around a city like this?'
3. Illustrate rotation by photographing a ball on the end of a piece of string rotating around (or perhaps using a swingball). Display pictures of the ball in different positions. They are all pictures of the same ball and you can highlight that, although in different positions in each image, the ball is always equal distance away from the centre of rotation. There is a **Geogebra animation** that might also help to illustrate this (tube.geogebra.org).

SECTION 1F / 1H: REFLECTIONS

Section 1 focuses on reflections and introduces the idea of congruent shapes. The section uses the idea that reflections are mirror images of the object you start with.

The suggestion given above in the 'Hooks' section could be used to introduce this and the students could describe which things stay the same in a reflection and the differences between the object and the image.

While preparing the students for answering questions that involve performing reflections or finding equations of reflection lines, you could revisit some of the ideas in Chapter 18 *Straight-line graphs* on describing horizontal and vertical lines.

Prompting questions

Exercise 28A(F)

While working through this exercise, good prompts for students might be:

- What do we require to reflect an object to give an image? *We need to know where the reflection/mirror line is.*
- **Q3(F)** What happens if the reflection/mirror line goes through the shape? *Answer dependent on the shape but students sometimes struggle with this.*

Exercise 28B(F) / 29A(H)

While working through this exercise, good prompts for students might be:

- **Q2(F)** What do the lines $y = x$ and $y = -x$ look like? *They are diagonal lines with gradient one and negative one going through the origin.*
- **Q2(H)** What do the lines $y = x$ and $y = 1 - x$ look like? *One is diagonal line with gradient one going through the origin and the other has a gradient of negative one with a y -intercept of one.*
- **Q4(H)** What are definitions of a hexagon and octagon? *A hexagon has six sides and an octagon has eight sides.*

Exercise 28C(F) / 29B(H)

While working through this exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Does it matter which shape is the object and which is the image? *No. Why? Look back through your examples of reflections; does relabelling the object and image change the reflection/mirror line? No.*
- **Q2(F) / Q2(H)** How can we determine where the reflection/mirror line must be? *You can draw lines from the 'corners' or 'vertices' of the object to the 'corners' or 'vertices' of the image. The mirror line must be a perpendicular bisector for all the construction lines you have drawn.*

Starters, plenaries, enrichment and assessment ideas

Starters

- NRICH activity **Mirror, Mirror...** is useful as a starter activity before beginning other Section 2 transformations. The follow up task to this **...on the Wall** is useful and could form the main part of a lesson if you wanted to follow on from the Mirror, Mirror... starter and not go into transformations (nrich.maths.org).
- NRICH **Weekly Problem 34 – 2009** could also be used as starter for the material covered before Exercise 28B(F)/29A(H) (nrich.maths.org).

Plenaries

- Produce a ‘reflection game’ for use with an interactive whiteboard. Students can be challenged to shade in parts of different shapes to create a correct reflection using a given reflection/mirror line.

Enrichment activities

- Students can find symmetric objects around the school and discuss their properties. They could then be sent away with cameras to capture these images and share them with the class.
- **(Foundation only)** Take photographs of students’ faces and split the picture in half. They are then required to mirror their face. Extension problem: **One Reflection Implies Another** (nrich.maths.org).

Assessment ideas

- Produce a consolidation exercise in which students are required to reflect rectangles in specific reflection/mirror lines to form a word.

SECTION 2F / 2H: TRANSLATIONS

Section 2 introduces the idea of translation as a ‘slide’ of the shape from an object to an image. The beginning of the section stresses that a translated shape has the same orientation as the original and points out that translations are described using vectors. Fluency with *Chapter 27F/28H Plane vector geometry* is hugely beneficial here and it might be advisable to use a diagnostic starter (something along the lines of the ‘Work it out’ multiple choice question) on vectors.

Prompting questions

Exercise 28D(F) / 29C(H)

While working through this exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** Which part of the vector controls the movement in the x -direction? Which part of the vector controls the movement in the y -direction? *The top number in the vector controls movement in the x -direction and the bottom number controls movement in the y -direction.*

Exercise 28E(F) / 29D(H)

While working through this exercise, good prompts for students might be:

- **Q1-2(F) / Q1-2(H)** How do you move between each? *We have a movement in the x -direction and in the y -direction.*
- **Q1-2(F) / Q1-2(H)** If you are moving in the negative x -direction how do you represent that in the vector? *Negative number in the top part of the vector.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- An activity based on Question 2 in Exercise 28E(F)/29D(H), in which students come up with their own picture and instructions, is a great way to allow students to consolidate their knowledge of translations and provide scope for paired work. Once students think they have a vectors jigsaw, they give it to someone else to try.

Enrichment activities

- Following on from the NRICH activities mentioned in Section 1 (**Mirror, Mirror...** and **...on the Wall**), design a task that in which students discover that two consecutive reflections give a translation (nrich.maths.org).

Assessment ideas

- Produce a consolidation exercise in which students are asked to translate rectangles by different vectors to make letters that form a word. The word could be one from the key vocabulary list for this section, for example 'orientation'.

SECTION 3F / 3H: ROTATIONS

Section 3 focuses on rotations and introduces the idea that a rotation does not preserve orientation but does preserve congruence. The section opens with the three main pieces of information you need to define a rotation of an object to form an image: a given centre, an angle and a direction. The 'Work it out' questions can be used to determine if your students need a recap of 2D coordinates and the terms clockwise and anticlockwise, before starting this section. Exercise 28F(F)/29E(H) can then be used as a consolidation exercise.

Prompting questions

Exercise 28F(F) / 29E(H)

While working through this exercise, good prompts for students might be:

- Q1(F)** When is it acceptable to give only two pieces of information for rotating a shape? *When you rotate by 180 degrees, you do not require a direction to your rotation.*
- Q5(F) / Q3(H)** If you rotate a shape 270 degrees anti-clockwise, how far do you rotate in the clockwise direction? *90 degrees.*

Exercise 28G(F) / 29F(H)

While working through this exercise, good prompts for students might be:

- Q1(F) / Q1(H)** When describing rotations, what pieces of information do we need and why? *Centre of rotation to know where to rotate the object from, an angle of rotation to know how far to rotate it and a direction to the rotation so you know which way to rotate.*
- Q1(F) / Q1(H)** How can tracing paper help find the centre of rotation? *You can use the tracing paper because the image is congruent to the object. You can use trial and improvement by tracing over the object and trying different points as the centre of rotation.*
- Q3(H)** How can you find the centre of rotation? *Firstly, connect two pairs of corresponding points on the object and image. Secondly, use a pair of compasses to find the perpendicular bisector for each of the two lines given by connecting each pair of corresponding points. Finally, the place where the perpendicular bisectors meet is the centre of rotation.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Produce a 'spot the transformation' set of slides (can be used as a starter or plenary) based on the idea of the 'Work it out' multiple-choice question before Exercise 28G(F)/29F(H).

Plenaries

- A suitable plenary is playing noughts and crosses with key vocabulary. Students form two teams and pick a word on the noughts and crosses grid to define. If they get it correct they get to place a 'nought' or 'cross' in that grid. It can be a good idea to place a challenging word to define in the centre spot!

Enrichment ideas

- GeoGebra** provides a useful tool when teaching rotations (geogebra.org). A lesson in which students explore making patterns via rotations of a shape, such as NRICH **Attractive Rotations**, provides opportunities for students to consolidate their knowledge of rotations (nrich.maths.org). In addition, the patterns can be printed and swapped with other students so that they can determine how each shape was made, for example which rotations have been performed and where the centre of rotation was.

Assessment ideas

- Produce a consolidation exercise in which students are asked to rotate rectangles with different centres and by different angles to form a word. The word could be one from the key vocabulary list for this section, for example ‘angle’.

SECTION 4H: COMBINED TRANSFORMATIONS

Section 4 brings all the ideas of the chapter together and introduces higher tier students to combining a series of transformations and describing the single transformation that could be made to take the object to the final image.

Prompting questions

Exercise 29G(H)

While working through this exercise, good prompts for students might be:

- Q4(H)** This is a chance to get the students to ask their own questions about how to distinguish between transformations. Some questions that could help this process are:
 - What can you see straightaway has happened to your image? *It cannot be a rotation if the image has the same orientation as the object.*
 - How far away are the corresponding corners of your object and image? *If all corresponding corners are the same vector away then we can describe the transformation as a translation. For some objects, where the corners have not been pre-labelled, this method could yield a reflection.*
 - If you had some tracing paper, what could you do? *This could be useful to help with translating your object to the image or you can use it to help find a centre of rotation.*
 - What are the pieces of information that you need to write down in each case? *Rotation: centre, angle and direction. Translation: vector. Reflection: reflection/mirror line.*

Starters, plenaries, enrichment and assessment ideas

Plenaries

- Produce a consolidation exercise in which a word is formed by the transformation of rectangles using a variety of different transformations.
- NRICH’s **Transformation Game** is a good way to consolidate the chapter’s material, however it does also contain enlargements (that occur in *Chapter 30F/31H Similarity*) (nrich.maths.org).

Enrichment activities

- Produce a treasure map on a grid. Give a list of instructions to perform transformations on a cross (X) to find the treasure.
- The NRICH task **A Roll of Patterned Paper** combined with **One Reflection Implies Another** and **Rotations Are Not Single Round Here** provide good extension material. The NRICH page **Paint Rollers for Frieze Patterns** has an exposition on some of the ideas behind each of these problems (nrich.maths.org).

Assessment ideas

- A good, challenging assessment task is to ask students to create their own transformations jigsaws like the ones found online. After collecting in students’ work you can then pick the best ones to give to the rest of the class to complete. They are rarely flawless and getting students to correct their mistakes is just as valuable as a perfect jigsaw.

Topic links

Previous learning

Throughout this chapter students will be using some of the skills they developed in *Chapter 5 Properties of shapes and solids*, *Chapter 9 Angles*, *Chapter 18 Straight-line graphs* and *Chapter 27F/28H Plane vector geometry*.

Future learning

Congruency is dealt with again in *Chapter 29F/30H Congruent triangles* and a fourth type of transformation, enlargement, will be covered in *Chapter 30F/31H Similarity*.

Transformations of curves are covered at higher tier in *Chapter 41H Transformations of curves and their equations* where being able to describe a reflection/mirror line, a translation and a rotation will be useful.

Gateway to A Level

Matrices are the next level of transformations and appear at A Level. Matrices can be used to describe all the transformations mentioned in this chapter along with some further types of transformation that are studied in higher level mathematics.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 1 Question 23
- Chapter 5 Question 12
- Chapter 8 Questions 5, 24
- Chapter 10 Question 11

Higher

- Chapter 1 Question 27
- Chapter 5 Question 12
- Chapter 8 Questions 7, 23
- Chapter 10 Question 15

Homework Book

Foundation

- Chapter 28

Higher

- Chapter 29

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 4 worksheets (+ solutions)
- 2 animated widgets
- 3 interactive walkthroughs
- 1 auto-marked quickfire quiz
- 1 auto-marked question set, with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper
- 2 mm graph paper
- Axis grids from -10 to 10 in x and -10 to 10 in y

29F / 30H Congruent triangles

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to label equal angles and edges in shapes;
- know the basic angle facts, including those relating to polygons and parallel lines;
- understand how to justify geometric proofs using known geometric facts;
- recall the definitions and properties of triangles and quadrilaterals.

Additional useful prior knowledge

- To be able to use compasses to draw constructions and loci.

Learning outcomes

Foundation

Section 1

- To know what it means for two objects to be congruent.
- To know the conditions which imply that a pair of triangles are congruent:
 - SSS – three sides are the same in both triangles;
 - ASA – two angles and one side length are the same in both triangles;
 - SAS – two sides and the angle between them are the same in both triangles;
 - RHS – the hypotenuse and another side of a right angled triangle are the same in both triangles.

Section 2

- To be able to apply the conditions for congruency to a variety of situations.

Higher

Section 1

- To know what it means for two objects to be congruent.
- To know the conditions which imply that a pair of triangles are congruent:
 - SSS – three sides are the same in both triangles;
 - ASA – two angles and one side length are the same in both triangles;
 - SAS – two sides and the angle between them are the same in both triangles;
 - RHS – the hypotenuse and another side of a right angled triangle are the same in both triangles.

Section 2

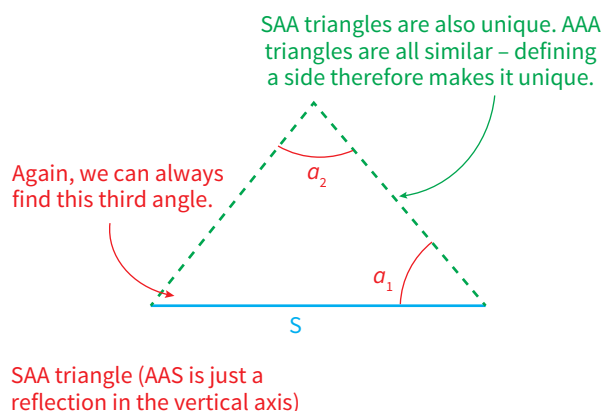
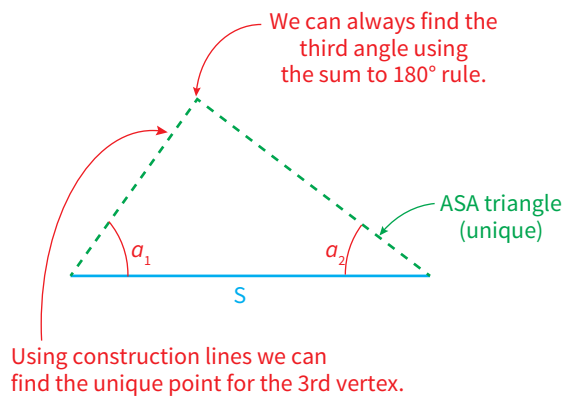
- To be able to apply the conditions for congruency to a variety of situations.

Vocabulary

congruent, included angle

Common misconceptions and other issues

- All issues from *Chapter 5 Properties of shapes and solids* and *Chapter 9 Angles*.
- Many students find notation difficult to grasp:
 - *Is labelling important?* Congruence of triangles only makes sense when the triangles have been labelled. When we say two triangles, say ABC and DEF, are congruent, what we are saying is that we can identify the sides $AB = DE$, $BC = EF$, $CA = FD$ and the angles formed between pairs of equivalent sides. Therefore, we need to label the triangles correctly and keep using the notation we have built up in *Chapters 5* and *9*.
 - *Is ASA the same as SAA or AAS?* There is a fundamental difference between ASA and SAA. This difference will not mean you cannot determine if two triangles are congruent, but there is a difference in the appearance of the triangles.



- For ASA the triangle has a baseline that represents the side and the two angles are formed at either end of the baseline. For SAA triangles the side given is opposite one of the given angles so students will have to find the third unknown angle to compare and form a correct argument for congruency (i.e. the triangles are not congruent unless the side of equal length is between the same two angles).
- Students can construct examples of these to convince themselves that a triangle defined by ASA might not be congruent to a triangle defined by SAA unless the side of equal length is between the same two angles.
- *Is SSA (or ASS) the same as SAS?* SSA triangle is the ambiguous case. Get students to draw a baseline of length S , then a circle of radius S at one end of the base and, finally, use a protractor to draw in a line at angle A at the opposite end of the baseline. This line segment will intersect twice with the circle giving a triangle that can be described by SSA that has either an acute or obtuse angle. This is therefore not the same as SAS which determines a unique triangle. However, SSA does determine a triangle uniquely when A is 90 degrees (see activity below).
- Confusing congruence and similarity has the potential to become an issue. Introducing the concept of congruence first and stressing that congruence is a way of determining if shapes are exactly the same will help reinforce these ideas with the students. Occasional starters (once similarity is learnt) where pairs of triangles are displayed and the students have to determine if they are congruent or not will also aid the distinction between these topics.
- The ability to write a mathematical proof will be crucial when answering the questions in Section 2 that apply the knowledge gained in the earlier section. Students can improve the presentation and style of mathematical proof by seeing model answers (either on paper or demonstrated by the class teacher) and critiquing incomplete or unclear solutions. For many students, it is appreciating the different forms that proofs can take and understanding what constitutes a full proof that causes problems. Encouraging students to rank proofs of the same problem from best to worst could help improve their awareness of what is required.

Hooks

1. Congruence is a powerful mathematical concept and is at the core to one of the main reasons mathematicians do research; to classify and understand. Knowing when two objects or (in the case here) two triangles are really the same allows us to narrow down the information we need to move on with a task or problem.

2. From *Chapter 6 Construction and loci*, we saw how to construct triangles from certain pieces of information. Knowing which pieces of information we need to uniquely define a triangle gives us a mathematical way to classify triangles. Once we know what information we require, we can compare triangles with the same information and state if they are congruent.

SECTION 1F / 1H: CONGRUENT TRIANGLES

Section 1 focuses on what congruence is for a triangle and what conditions need to be satisfied for two triangles to be proven to be congruent. This information is displayed in a table that clearly describes each of the conditions with an appropriate diagram.

Prompting questions

- What does orientation mean? *The position in space the object has.*
- How do we distinguish between pairs of sides with the same length? *We can use a series of markings on the sides. Sides that are the same length get the same number of markings.*
- How can we distinguish between angles that are the same? *We can use colours to identify the angles using the same colour for the same angle or we can use arcs to highlight equal angles. If more than set of equal angles exist, we can use multiple arcs to represent the sets.*

Exercise 29A(F) / 30A(H)

While working through this exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** How can we order our thoughts on the page? *We could form a table with the heading of each column one of SSS, SAS, ASA or RHS, then each pair of triangles can be discussed and categorised.*
- **Q3(F) / Q3(H)** Have we seen a picture like this before? *Tilting it so that the line segment AB is at the top gives a picture very similar to the one used for alternate angles between parallel lines. If the two line segments AB and ED are joined by a line segment AD at the same angle to each of the line segments AB and ED then AB and ED must be parallel. In addition to this, we have vertically opposite angles at C.*
- **Q4-9(F) / Q4-9(H)** What do we need for a mathematical proof? *We need to include reasons for each conclusion we come to.*
- **Q10(F) / Q10(H)** What are the properties of a rhombus? *All four sides of a rhombus are the same length, opposite angles are equal and the diagonals intersect at a right angle.*
- **Q11(H)** How many ways can we prove this? *We could use the properties of tangents to circles or we could use the properties of circles and parallel lines, for example.*

Starters, plenaries, enrichment and assessment ideas

Starters

- This NRICH activity **Triangular Tantaliser** can be used as an opening exercise to identify different triangles (nrich.maths.org). The students will have either:
 - discounted triangles that are congruent without justification; then you could ask ‘why did you decide not to include these other triangles?’ and ‘how do you know that they are *basically the same*?’
 - or included triangles that are congruent; then you could ask the students to look at them carefully.

Starters or plenaries

- This can be used as an opening task or as a consolidation exercise. A series of descriptions and/or pictures of triangles are displayed and the students are required to determine if there is enough information for each one to determine a unique triangle.

Enrichment activities

- This activity can be used to reinforce the material learnt in construction and loci. Give the students a series of pieces of information about a triangle, starting with every angle and every side and then gradually remove pieces of information. The students are then required to draw each of the triangles described, if they can! The activity has

natural plenary questions: ‘Which triangles could be drawn?’, ‘Which triangles are determined uniquely?’, ‘How many pieces of information do we need to determine a unique triangle and which groups of information?’

- Once the students have decided that ‘just three pieces of information’ are required to determine a triangle, you could ask ‘How many different arrangements of three pieces of information can we have and which are the ones that determine a triangle uniquely?’ (i.e. AAA, AAS, ASS, etc.) This brings in the ideas of combinations and you could even ask the questions: ‘Is SAS the same as ASS?’ or ‘Is SSA or ASS the same as SAS?’
- The condition RHS for two triangles to be congruent is not in the same form as the other three; why? *It arises in the case of ASS triangles, where the angle A is 90 degrees and thus there is no ambiguity normally associated with ASS triangles that can have either obtuse or acute angles.*

Assessment ideas

- Students can construct guides to congruence where they summarise the information from this section.
- Each student can construct his or her own simple ‘exam-style question’ on congruence for another student to answer. This activity can reinforce the ideas from the section from the point of view of both the writer of the question (you must be able to answer the question) and the student answering it.
- The first enrichment activity can be used in conjunction with this assessment idea. Once your students have determined the minimum information needed to determine a unique triangle (SSS, SAS, ASA), they can be given ‘proof’ that two triangles are the same. The students work through these ‘proofs’ and determine if they are correct. Through this, students connect their knowledge of determining triangles uniquely with deciding if two given triangles are congruent.

SECTION 2F / 2H: APPLYING CONGRUENCY

Section 2 focuses on applying the new information learnt on congruency from section 1 to a variety of situations. Throughout section 2, students are required to use material from *Chapter 5 Properties of shapes and solids* and *Chapter 9 Angles* as well as the new material. The section opens with a problem-solving framework that helps to break down a question into steps.

Prompting questions

- How many triangles can you see? *Identifying which triangles the students are looking at will help clarify what each question is asking.*
- How are the triangles labelled? *Labelling is very important in this chapter and students need to be able to describe triangles in a diagram using the correct labelling, in order to make comparisons.*

Exercise 29B(F) / 30B(H)

While working through this exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** How many triangles can you see? *Two.* Draw the two triangles separately and label everything you can. Are there any lengths they have in common? *They have one common length in the diagram (JL) and given the conventions of labelling we also know that $JM = JK$.*
- **Q2(F) / Q2(H)** How many triangles can you see? What is the definition of parallelogram? *We can see four smaller triangles and four larger triangles. A parallelogram has two sets of parallel lines.*
- **Q3(F) / Q3(H)** How many triangles can you see? *Two.*
- **Q4(F) / Q4(H)** What is required for an isosceles triangle? *The base angles must be the same.*
- **Q5(F) / Q5(H)** How many triangles do we have in this diagram that contain the identified angles? What other angles can we define in terms of what we already know? *We can describe the angles CEB and AEB using the angle fact that the sum of angles on a straight line is 180 degrees so hence both these angles are $180 - AED (= 180 - CED)$. Similar reasoning can be used to label angles EAB and ECB.*
- **Q6(F)** What do we know about the missing angles? *Angles ABC and ADC must be the same.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Display a diagram on the board for which the students are required to prove a certain property. Ask each student to write a solution and then exchange it with another class member who then reads through it and decides if it is correct. Finally, choose a student to describe the solution to the class.
- This NRICH activity **Hex** can be used as a quick starter. Once the students have found a solution, you can then ask which triangles in the diagram are congruent and why (nrich.maths.org).

Enrichment activities

- **(Higher only)** This NRICH activity **Triangles in a Square** can be used to help students apply their knowledge of congruence to prove that two line segments are the same. This task is one that might not ‘shout’ congruence and thus is very useful for students who find the exercises straightforward (nrich.maths.org).
- This NRICH activity **Making Sixty** can be used in the same way as the above task but can also be used with foundation students who can label the triangles they can find on a physical piece of paper and relate them to the conditions for congruence (nrich.maths.org).
- This NRICH activity **A Sameness Surely** can be used as another task for students who find the exercises straightforward (nrich.maths.org).

Assessment ideas

- Exam questions can be posted around the room. The students, in pairs, have several minutes with each exam question and they must continue to solve it from the point the proof has reached. This means that each successive pair continues from where the previous pair left off and hence their working must be clearly labelled. This activity is often not enjoyed by students the first time they do it but they do recognise quite quickly the importance of clear working and labelling.

Topic links

Previous learning

Chapter 5 Properties of shapes and solids and *Chapter 9 Angles* are extremely useful for this chapter and recapping their content, specifically for triangles, will allow students to draw connections.

Chapter 6 Constructions and loci provides a way into this chapter; see ‘Hooks’ above for more details.

Future learning

The following *Chapter 30F/31H Similarity*, takes congruence and moves to the more general topic of similarity. Students will need to be confident with congruence before moving onto similarity in order to not confuse the two.

Gateway to A Level

Students will be expected to have the idea of mathematical congruence in their minds when they tackle geometry questions. However, it is the mathematical proof skills that are consolidated and added to in this chapter that are essential for students going on to A Level mathematics. This work will be further extended to include more advanced algebraic proofs and reasoning, for example proof by induction.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 1 Question 16
- Chapter 3 Questions 6, 7
- Chapter 8 Question 25

Higher

- Chapter 1 Question 22
- Chapter 2 Question 28
- Chapter 3 Question 6

Homework Book

Foundation

- Chapter 29

Higher

- Chapter 30

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 7 worksheets (+ solutions)
- 2 animated widgets
- 6 interactive walkthroughs
- 2 auto-marked quickfire quizzes
- 2 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

30F / 31H Similarity

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to label angles correctly;
- understand what the symbols corresponding to parallel, perpendicular and equality look like on a diagram;
- be able to prove that two triangles are congruent;
- know how to solve simple linear equations;
- know how to recognise numbers in equivalent ratios;
- **(Higher only)** be able to calculate squares and cubes of rational numbers.

Additional useful prior knowledge

- To know how to work with ratio and proportion.
- To be able to use coordinates to define an object in the plane.
- To recall the definitions of the types of triangles and quadrilaterals.
- To be able to multiply and divide by rational numbers.
- **(Higher only)** To know how to calculate area and volume of 2D and 3D shapes.
- **(Higher only)** To know the dimensions that are needed for measurements of area and volume.

Learning outcomes

Foundation

Section 1

- To know what is meant by the phrase 'mathematically similar'.
- To be able to determine when two objects are mathematically similar.

Section 2

- To know what is meant by a 'mathematical enlargement'.
- To be able to enlarge a shape given a positive rational scale factor.
- To know what the centre of enlargement is.
- To be able to enlarge a shape given a scale factor and centre of enlargement.
- To determine a given centre of enlargement and scale factor from a diagram.

Higher

Section 1

- To know what is meant by the phrase 'mathematically similar'.
- To be able to determine when two objects are mathematically similar.

Section 2

- To know what is meant by a 'mathematical enlargement'.
- To be able to enlarge a shape given a positive rational scale factor.
- To know what the centre of enlargement is.
- To be able to enlarge a shape given a scale factor and centre of enlargement.
- To determine a given centre of enlargement and scale factor from a diagram.
- To be able to enlarge a shape given a negative rational scale factor.

Section 3

- To be able to determine similar polygons.

Section 3

- To be able to determine similar polygons.
- To be able to determine similar 3D shapes.
- To know the relationship between lengths, areas and volumes of similar shapes.

Vocabulary

scale factor

Common misconceptions and other issues

- All issues from *Chapter 5 Properties of shapes and solids*, *Chapter 9 Angles*, *Chapter 22 Calculations with ratio* and *Chapter 29F/30H Congruent triangles*.
- Labelling. Students need to be fluent in labelling triangles and general polygons in a consistent way. Poorly labelled polygons mean that students cannot refer to the sides or angles in a precise way. To address this issue, students should be exposed to good practice that they can model and they should also be given opportunity to critique poor labelling in a starter activity.
- Ratio and proportion. Students do not always use the correct terminology. To address this, students should pay particular attention to the 'Problem-solving framework' in each section. This should allow students to make the connection between a ratio of side lengths and the proportion the triangles are in.
- Scale factor and centre of enlargement. Students are generally quite happy with the concept and application of simply enlarging by positive numbers greater than one. Issues arise when students are required to construct their enlargements from a centre of enlargement. To address this with your students you could have a series of 'student-answered questions' on the topic of centre of enlargement by a positive scale factor and students could be asked to determine if (i) the correct centre of enlargement has been used or (ii) if the correct centre of enlargement has been found. The students then read and critique these answers to help identify common errors.
- Fractional scale factors. Some students find this hard simply because the word 'enlargement' is still used. Help these students by emphasising the word 'enlargement'; when using fractional scale factors.
- **(Higher only)** Negative scale factors. Students often can miss that the shape has been turned upside down. To address this, you could refer to work covered in *Chapter 27F/28H Plane vector geometry*. You can describe each vertex in your diagram by a vector given from your centre of enlargement. To find the enlarged image you can simply multiply each vector by the scale factor and the new vectors (using the centre of enlargement as the base point) will give the location of the transformed vertices under the enlargement.
- **(Higher only)** Volume scale factor. Students still struggle to see that multiplying the measurements of a 3D object has a cubing effect on the volume. Students could start with some basic cuboids and work out the effect that doubling and tripling the side lengths has on the area and volume of the cuboid.

Hooks

- The NRICH activity **Similar Rectangles** allows students a way into similarity. Here, the students are not given a definition of 'similar' but must deduce what this must mean for a rectangle. You could start with the dimensions of the object rectangle and give the area of the image rectangle and ask what the measurements of the image rectangle could be, given that the rectangles are similar. This also allows for an early conversation about area scale factors, which is useful for foundation students and essential for higher students (nrich.maths.org).
- Students are very happy to enlarge shapes. If one shape is an enlargement of the other, how much do they differ in their properties? This allows you to introduce congruence as a special case, where the scale factor is 1, and that all other cases create 'mathematically similar' shapes.

SECTION 1F / 1H: SIMILAR TRIANGLES

Section 1 begins by identifying the criteria for two triangles to be mathematically similar. The section outlines a problem-solving framework for determining unknown lengths for similar triangles using the ratio of their sides. It then uses the fact that the proportion of each triangle is the same. The section ends with an extensive exercise.

Prompting questions

- What does mathematically similar mean? *Two objects are mathematically similar if they are enlargements of each other. Two triangles are mathematically similar if they satisfy the criteria given at the start of Section 1.*
- How do we name angles and sides in triangles? *We use the convention that, if ABC is a triangle then the internal angle at A is described as angle CAB (formed between line segments AB and AC).*
- How do we find ratios between triangles? *We need to compare the length of one side in the original triangle with the corresponding length in the enlarged triangle.*
- What does it mean for two triangles to be in proportion? *It means that the ratio 1 : n of corresponding sides has the same n for each pair of sides.*

Exercise 30A(F) / 31A(H)

While working through this exercise, good prompts for students might be:

- **Q1(F) / Q1(H), Q4(F) / Q4(H), Q7(F) / Q7(H) and Q8(F) / Q8(H)** What do the arrows on the line segments mean? *They denote parallel sides.*
- **Q2(F) / Q2(H)** What conditions do we need for similar triangles? *See bullet points at start of Section 1.*
- **Q3(F) / Q3(H)** Can you prove that a statement is false by providing an example to show this is the case? *If you believe a statement not to be true then you can construct a pair of triangles that do not satisfy it.*
- **Q5(F) / Q5(H)** Have we seen these side lengths before? *They are Pythagorean triples.*
- **Q6(F) / Q6(H)** How do we calculate the proportion of each corresponding side? *We find the missing lengths using the same unity ratio 1 : n from the first triangle to the second.*
- **Q9(F)** Which sides cannot be which value? *The shorter lengths cannot be 10 or 12.*
- **Q10(F)** Which pairs of numbers (a, b) give the same fraction $\frac{b}{a}$? *(2, 6), (3, 9) and (4, 12) all give the same fraction $\frac{b}{a}$.*
- **Q9 and 10(H)** How many triangles are we looking at? *We are looking at triangles ADC and AEB for Question 9(H) and for Question 10(H) we have AB (top of boat) and AC (top of lighthouse).*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Questions Q9(F/H) and Q10(F/H) can be turned into an activity to start or finish a lesson by having a series of similar questions that can be answered by students on mini-whiteboards.

Enrichment activities

- This NRICH activity **Folding Squares** is a task that can be used to help students spot patterns and determine which triangles are similar using properties of a square and angles around a point. This activity can then be followed up with **Take a Square II** (nrich.maths.org).
- This NRICH activity **Napkin** is an open task involving similarity for students wanting to stretch their conjecturing skills (nrich.maths.org).
- This NRICH activity **Sitting Pretty** can be used to get students to form their own similar triangles that can be used (along with some other ideas connected with triangles) to prove the algebraic statement. This activity shows how some geometric properties can be used to prove something algebraic (nrich.maths.org).
- This NRICH activity **Two Ladders** is good for encouraging students to spot which triangles are similar and what scale factors they have (nrich.maths.org).

Assessment ideas

- Ask students to write a summary of how to determine if two triangles are similar. These can be kept as future revision resources.
- Students can create a problem-solving framework for finding unknown lengths in similar triangles. They can then test their framework by letting another student use it to answer a question. The students can then compare their frameworks and respond to comments.

SECTION 2F / 2H: ENLARGEMENTS

Section 2 focuses on enlargements. The section is broken down into subsections: the centre of enlargement; a fractional scale factor; (**Higher only**) negative scale factors; properties of enlargements; and describing enlargements.

Prompting questions

- What do we mean by ‘enlargement?’ *See first line in Section 2.*
- How can you find the centre of enlargement? *All vertices in the object have now been moved to a new place in the image by the same amount. We can connect the corresponding vertices in the object and image and extend them until they intersect. Where they intersect is the centre of enlargement.*
- What does the image of your object look like when it is enlarged by a positive scale factor larger than 1? *The image is larger than the object.*
- What does the image of your object look like when it is enlarged by a positive scale factor less than 1? *The image is smaller than the object.*
- (**Higher only**) What does the image of your object look like when it is enlarged by a negative scale factor? *The image is rotated 180 degrees in comparison with the object.*
- (**Higher only**) How do you find the centre of enlargement when you have a negative scale factor? *You do exactly the same as when the scale factor is positive, but when finding corresponding vertices you must remember that the image is rotated 180 degrees in comparison with the object.*
- What information do you always need to define an enlargement? *We need a scale factor and centre of enlargement.*

Exercise 30B(F) / 31B(H)

While working through this exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** What do you expect will happen to the object to form the image for each part? For parts a and c we expect the image to be larger than the object and for part b we expect the image to be smaller than the object.

Exercise 30C(F) / 31C(H)

While working through this exercise, good prompts for students might be:

- **Q1(F) / Q1(H), Q2(F) / Q2(H)** and **Q3(F) / Q3(H)** How do you enlarge from a point? *We need to draw line segments from the centre of enlargement O through the vertices of the object A, B, C, ... and extend these past the vertices of the object by the distance OA multiplied by the scale factor.*
- **Q4(F) / Q4(H)** What do you expect the relationship of image to the object will be? *The image will be smaller than the object and will be positioned between the object and the centre of enlargement.*
- **Q5(F) / Q5(H)** How do you know you have used the correct scale factor? *The side lengths of the image will be 1.5 times the side lengths of the object.*

Exercise 31D(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** What do you expect to happen to the object when it is enlarged by a scale factor of -1? *We expect the image to be upside down when compared with the object.*
- **Q2(H)** How far away will the vertices of the image be from the centre of enlargement? *They will be twice the distance that the object's vertices are from the centre of enlargement, but in the opposite direction.*
- **Q3(H)** What do negative scale factors between -1 and 0 do? *They give an image that is rotated 180 degrees and smaller than the object.*

- **Q4(H)** What do negative scale factors with modulus greater than 1 do? *They give an image that is rotated 180 degrees and larger than the object.*

Exercise 30D(F) / 31E(H)

While working through this exercise, good prompts for students might be:

- **Q2(F) / Q1(H)** How can you compare houses? *We can compare the lengths defined in each house.*
- **Q3(F) / Q2(H)** How can you determine the centre of enlargement? *We can connect the corresponding vertices in each object and image, and extend these line segments to find the point of intersection and hence the centre of enlargement.*

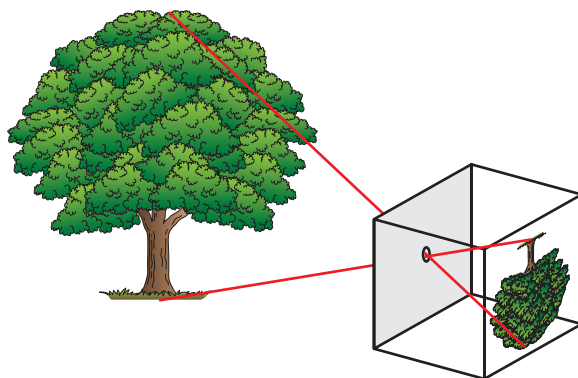
Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Pairs of images can be flashed up and the students need to determine if they are enlargements or not (this can be used with other forms of transformations).
- A sheet with randomly placed shapes from groups of similar shapes (scaled by fractional scale factors) can be given to the students who are required to group the shapes with the original shape that was used for each family and find the fractional scale factors that were used to create the other shapes in the family. (This activity could be presented with a selection of centres of enlargements as well so that the students need to identify which centre of enlargement belongs to which shape.)

Enrichment activities

- **(Higher only)** The NRICH activity **Matter of Scale** is a good task for students to consolidate their knowledge of enlargement. The task requires students to prove the general case and determine another proof of Pythagoras' theorem. It is strongly recommended that you try this activity yourself and read the 'Getting Started' notes before doing this in class (nrich.maths.org).
- This NRICH activity **Who Is the Fairest of Them All?** can be used to consolidate fractional scale factors, the need to describe where an object is being enlarged from and combined transformations (nrich.maths.org).
- **(Higher only)** The modern camera has been developed from the box camera or *camera obscura*. Light enters a dark box through a small hole. This creates an enlargement of the scene being photographed with a negative scale factor on the opposite side inside the box. Light sensitive film records the image.



This could provide a starting point for class discussion and students could be asked 'If you had a shoe box and used it as a *camera obscura* how far away from the Eiffel Tower would you need to be to photograph it? What scale factor image would it produce?'

Assessment ideas

- A series of enlargement questions can be printed on A4/A3 squared paper and the students work to complete each enlargement. Once done, the enlargements can combine to make a word that can then be checked. You can also instruct students to make their own versions of these puzzles which can be used in a later lesson and peer assessed.

SECTION 3F: SIMILAR SHAPES / 3H: SIMILAR SHAPES AND OBJECTS

Section 3 focuses on similarity for general polygons. The main point of the section is that for polygons with more than three sides, equal angles by themselves are not sufficient to prove similarity.

(Higher only) The section also looks at the effect of scale factors on area of 2D and volume of 3D shapes.

Prompting questions

- Why could quadrilaterals with the same angles not be similar? *See worked example in Section 3.*
- What do we need to check for similarity for general polygons? *We need all angles to be the same and their corresponding edges to be in proportion to each other.*
- **(Higher only)** Why has the area changed for each scale factor in this way? *For an area we have two 'lengths' multiplied together. Thus, the area of the object will be multiplied by the scale factor squared to form the image.*
- **(Higher only)** Why has the volume changed for each scale factor in this way? *For a volume we have three 'lengths' multiplied together. Thus, the volume of the object will be multiplied by the scale factor cubed to form the image.*

Exercise 30E(F) / 31F(H)

While working through this exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** How can we decide which pairs of shapes are similar? *We can compare the proportion of each corresponding edge.*
- **Q5(F) / Q5(H)** What do we know about a shape that is an enlargement of another shape? *Enlarged images are mathematically similar to the object you started with.*
- **Q6(H)** What do we know about the number 512? *The cube root of 512 is 8.*
- **Q7(H)** How many 48s are in 588 and how does that help? $588 \div 48 = \frac{49}{4}$ which is $\frac{7}{2}$ squared.
- **Q8(H)** What formula can we use to help find the height? *Volume of a cylinder is $\pi r^2 h$. We can find the height of the smaller tin and use this to find the height of the larger one.*
- **Q9(H)** If you increase each measurement by a factor of 3, by what factors will you increase the area and volume? *3^2 for area and 3^3 for volume.*
- **Q10(H)** What is the formula for the surface area of a sphere and the volume of a sphere? *Surface area = $4\pi r^2$ and volume = $\frac{4}{3}\pi r^3$. (These will be provided in an exam.)*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Produce a series of questions in which pairs of polygons are flashed up on the board and the students need to determine if they are similar or not.
- **(Higher only)** Display two 3D shapes along with some information about them (radius, height, side lengths, area, volume) and a statement 'similar' or 'not similar'. The students need to determine if the statement is correct and justify why they think that.

Enrichment activities

- This NRICH activity **Growing Rectangles** is a nice introduction to how scale factor affects area and volume. A more complex extension to this activity is the NRICH activity **Fit for Photocopying** (nrich.maths.org).
- This NRICH activity **Pent** is a more complex similarity task where students need to break down what they want to find into manageable chunks (nrich.maths.org).
- This NRICH activity **Look Before You Leap** is a nice open task that can be addressed using similar triangles but can also be tackled using a variety of other tools that are mentioned on the 'Solution' page (nrich.maths.org).
- This NRICH activity **A Shade Crossed** lets the students use similarity to assist their calculations involving area of a triangle (nrich.maths.org).

Assessment ideas

- Ask students to study exam-style questions that have been answered incorrectly and critique the solutions. This encourages students to think about their own responses to these types of questions.

Topic links

Previous learning

This chapter relies on *Chapter 29F/30H Congruent triangles*, *Chapter 5 Properties of shapes and solids*, *Chapter 9 Angles* and, for higher students, *Chapter 20 Three-dimensional shapes* and *Chapter 21 Volume and surface area*. This chapter provides a good opportunity to revisit some of this earlier material.

The language and calculations from *Chapter 22 Calculations with ratio* are also used in this chapter.

Future learning

Later chapters in this book such as *Chapter 31F/32H Pythagoras' theorem*, *Chapter 32F/33H Trigonometry* and *Chapter 34H Circle theorems* all use the material learnt in this chapter. For Pythagoras, students could explore the NRICH activity **Matter of Scale** to provide another proof of Pythagoras' theorem (nrich.maths.org). The notion of similar triangles is central to the ideas of Trigonometry and is used in some parts of the proofs of the circle theorems presented in *Chapter 34H Circle theorems*.

Gateway to A Level

The effect of increasing a length by a factor and how that changes the area and volume is studied in the topic of differentiation under rates of change. In addition to this, matrices can be used to describe enlargements of objects described in two and three dimensions.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 1 Question 17
- Chapter 2 Question 15
- Chapter 3 Question 8
- Chapter 8 Questions 6, 16

Higher

- Chapter 1 Questions 8, 23, 24
- Chapter 5 Question 17
- Chapter 8 Question 24

Homework Book

Foundation

- Chapter 30

Higher

- Chapter 31

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 8 (F) / 9 (H) worksheets (+ solutions)
- 3 (F) / 5 (H) animated widgets
- 6 (F) / 9 (H) interactive walkthroughs
- 3 (F) / 4 (H) auto-marked quickfire quizzes
- 3 (F) / 4 (H) auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper
- 2 mm graph paper
- Axis grids from -10 to 10 in x and -10 to 10 in y

31F / 32H Pythagoras' theorem

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- understand how to use labels in geometry correctly, particularly those that are unique to a triangle, for example vertices (A, B ...), sides (AB or, for triangles only, lowercase c as it is opposite vertex C) or the symbol for a right angle;
- know how to use a calculator to find squares and square roots;
- be able to round to a given accuracy (either significant figures or decimal places);
- **(Higher only)** be able to work with exact and approximate values of surds;
- know the properties and categories of triangles;
- be able to substitute into and rearrange formulae including those involving squares and square roots;
- **(Higher only)** recall the properties of polygons.

Additional useful prior knowledge

- To be able to convert between metric units.
- To be able to use area and perimeter formulae for polygons and circles.

Learning outcomes

Foundation

Section 1

- To know and use Pythagoras' theorem to find the length of the hypotenuse in any right-angled triangle.

Section 2

- To know and use Pythagoras' theorem to find any missing length in a right-angled triangle.

Section 3

- To use Pythagoras' theorem to show whether a triangle is right-angled or not.

Section 4

- To apply Pythagoras' theorem to 2D problems.
- To link Pythagoras' theorem to real-life skills for industry.

Higher

Section 1

- To know and use Pythagoras' theorem to find any missing length in a right-angled triangle.

Section 2

- To use Pythagoras' theorem to show whether a triangle is right-angled or not.
- To apply Pythagoras' theorem to 2D problems.

Section 3

- To apply Pythagoras' theorem to 3D problems.

Section 4

- To link Pythagoras' theorem to real-life skills for industry.

Vocabulary

hypotenuse, theorem, Pythagorean triples

Common misconceptions and other issues

- Students often identify the hypotenuse incorrectly by mistaking the length that is the longest (often due to the orientation of the triangle). Constantly reinforcing that the hypotenuse is the side opposite the right angle rather than just the longest side might help as the longest side can be hard to identify in some right-angled triangles when the orientation is misleading. An additional suggestion could be to rotate the page so that the right angle is placed at the bottom left or right corner; this can help students identify the hypotenuse as the sloping side.

- Students might misunderstand that the theorem connects the squares of the side lengths and forget to find the square root. This might be because Pythagoras' theorem is often introduced through the exploration of connections between square numbers. Reinforcing good practice such as always stating the theorem first, clearly setting out working in a column and lining up the equality signs might help. This consistency will leave students with their length squared equal to their answer, which can then be easily identified as an incomplete solution.
- Some students might struggle to rearrange the formula to find a side other than the hypotenuse. Given that there is only ever one unknown in these problems they can be solved through a series of 'What number is needed to...' questions rather than formally rearranging the formula. For example, $100 = ?^2 + 64$ so 'What number is needed to make 100 when I add 64?' This then gives $?^2 = 36$ so asking 'What number squared makes 36?' leads to the answer of 6.
- The theorem might be applied (incorrectly) to non-right-angled triangles. This tends to be a bigger problem when students are working with a combination of right-angled triangles and non-right-angled triangles, often after students have worked on trigonometry. Formally defining Pythagoras' theorem as 'given a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides' and getting students to learn this definition by heart, might help prevent this problem. In addition, a series of tasks that encourage students to use the converse of this statement which is that any triangle where the sum of the squares of two sides equals the square of the third is a right-angled triangle can help reinforce this condition for the use of Pythagoras' theorem.
- It is easy to confuse labels, for example, side a can be confused with the adjacent side when students have learnt about trigonometry, or when triangles are pre-labelled, for example p, q, r rather than a, b, c . Students relying on the formula in the form $a^2 + b^2 = c^2$ where c is the hypotenuse can encounter a problem when one of the other sides is labelled c or when the hypotenuse is labelled a or b . Again, formally defining Pythagoras' theorem as 'given a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides' prevents problems related to labelling alone. This way, Pythagoras' theorem can be written as a formula specific to the triangle given in the problem.

Hooks

Pythagoras' theorem can be introduced in many different ways, but it is nice to let students discover it for themselves to a certain extent. This can be done through an investigational approach using the NRICH 'Tilted squares' activity described below. It can also be done through an exploration of patterns in squares. You can set students the number problem: find square numbers that are the sum of two other square numbers. Students can then construct SSS triangles using these three linked numbers and see what conclusions they can draw.

- What is meant by a tilted square? A square where the side lengths do not lie on the grid but the vertices still do.
- How can you calculate the area of a new square that is tilted upwards by one? Fixing the problem to focusing on squares with a tilt of one, forms a pattern that students normally spot quite quickly. That is, when tilting a square of length n by 1 unit, the area of the tilted square is $n^2 + 1$ (i.e. 1 more than the area of the square when flat on the grid). From this, students might conjecture patterns for a tilt of 2, 3 and so on that can be explored.
- How can you calculate the area of a new square that is tilted upwards by two? We can calculate the area in the same way as before and also test our new theorem.
- Can you predict the area of a new square that is tilted upwards by three, four, and so on?
- How can you find the length of one of the square sides from this? The side length of the tilted square can be found by finding the square root of the area. This can be used to deduce Pythagoras' theorem based on the right-angled triangle formed by the tilt.

SECTION 1F: FINDING THE LENGTH OF THE HYPOTENUSE

For Foundation students, Section 1 introduces Pythagoras' theorem and focuses on how this can be used to find the hypotenuse given the other two sides of a right-angled triangle.

Prompting questions

Exercise 31A(F)

While working through this exercise, good prompts for students might be:

- **Q1(F)** How do you calculate the area of a square? *Base multiplied by height or base squared, and so on.*
- **Q3(F)** Which side is the hypotenuse? *The one opposite the right angle.*
- **Q4(F)** What do we know about the sides of a triangle from constructions? *The sum of the two shorter sides must exceed the sum of the longest side or they won't meet.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Get students to write a list of all the numbers from 1 to 25 squared to improve mental accuracy and recognition when dealing with squares.

Starters or plenaries

- Get students to spot Pythagorean triples given a list of the first 25 square numbers. This could be a continuation of the task above.

Enrichment activities

- Use the NRICH investigation **Tilted Squares** and the other interactive resources available to explore these squares and methods for calculating their area. Students sometimes struggle with the idea of a tilted square and the supporting animations make the idea very clear. Using dotted squared paper rather than standard squared paper might also make things easier. These tilted squares can be used to derive and prove Pythagoras' theorem in different ways should you wish to extend the most able students (nrich.maths.org).

Assessment ideas

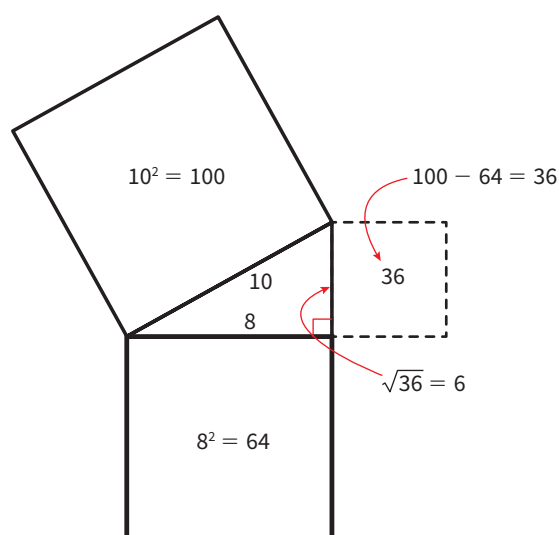
- Students could produce an information booklet aimed at describing what Pythagoras' theorem is and how they would use it.

SECTION 1H: UNDERSTANDING PYTHAGORAS' THEOREM / SECTION 2F: FINDING THE LENGTH OF ANY SIDE

The Higher chapter restates Pythagoras' theorem for right-angled triangles and applies it to finding missing lengths in right-angled triangles.

The Foundation chapter extends the Section 1 content to finding the length of any given side of a right-angled triangle.

For students less confident with algebraic manipulation, an alternative approach might be to draw diagrams similar to that shown below and fill in the gaps. This can be used to encourage students to check their answer is sensible (e.g. that the hypotenuse is the longest side).



Four tips for success when working with Pythagoras' theorem are offered in the Higher chapter:

- Write the formula.
- Show full working.
- Sketch the problem.
- Round your answer to the given degree of accuracy.

Students should be encouraged to follow this advice, particularly for the contextual problems in this chapter.

Prompting questions

- Once you've found the missing side, how can you check your answer is sensible? *The hypotenuse is always the longest length and the sum of the two shorter sides must exceed the largest.*
- If you square root both sides of $a^2 + b^2 = c^2$ why don't you get $a + b = c$? How could you convince someone in Year 8 about this? *Students might want to offer a counter-example or might instead wish to show that if $a + b = c$ then $(a + b)^2 = c^2$ and expand the brackets with an explanation that both sides of the equation must remain balanced.*
- Does it matter which side is labelled 'a', 'b' or 'c'? *No, you can use any letters you wish but the theorem must be adapted so that the sum always gives the hypotenuse squared, whatever letters you choose.*
- If Pythagoras' theorem for a particular triangle gives you ' $u^2 + v^2 = s^2$ ' what does the triangle look like? *The important thing is that the hypotenuse must be labelled s.*
- What about if the equation was ' $x^2 - y^2 = w^2$ '? *Here students will need to rearrange first and hence deduce that the hypotenuse is labelled x.*

Exercise 31B(F) / 32A(H)

While working through this exercise, good prompts for students might be:

- **Q1(F) / Q1-2(H)** Which side is the hypotenuse? *The one opposite the right angle.*
- **Q1(F) / Q1-2(H)** Does it matter which of the other two sides is a and which is b ? *No, they can have either label.*
- **Q1i-j(F)** What do you need to use Pythagoras' theorem? *A right-angled triangle and two side lengths so you will have to apply the theorem twice to find this information for the right-angled triangle with the missing side labelled.*
- **Q3(F) and Q4(F)** Sketch the right-angled triangle involved in the word problem. Which length is the unknown? *The most important thing here is identifying whether the unknown length is the hypotenuse or not.*

Starters, plenaries, enrichment and assessment ideas

Starters

- **(Foundation only)** Exercise 31B Question 5 is a matching exercise that could be produced as a card sort (either in electronic form or as a hard copy).

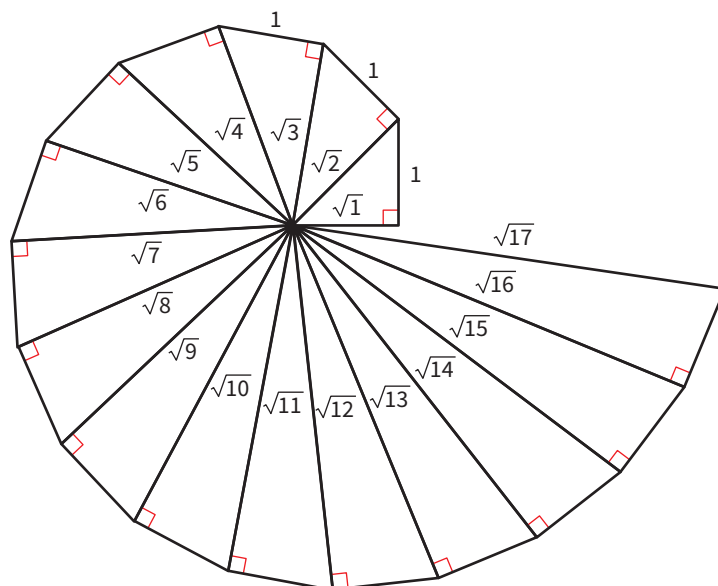
Plenaries

- Give students a series of Pythagoras' theorem problems and ask them to sort them into those where they need to find the hypotenuse and those where they need to find the length of a shorter side. This could also be used as an assessment task.

Enrichment activities

- Once students are confident using Pythagoras' theorem, an appropriate extension might be to consider different ways in which the theorem can be proved. There are several ways of doing this and you can search for these proofs online. NRICH has a few **Pythagoras proofs**, the first of which comes from the tilted squares problem (nrich.maths.org).
- NRICH's **Nicely Similar** is a problem in which students will have to think carefully about what information they have been given and label the lengths of the two right-angled triangles in their sketch carefully. They will need to compare the three triangles and form and solve appropriate equations to find the missing lengths. This problem can be used to revise similarity (nrich.maths.org).

- Students can explore what irrational values look like and their approximate size when working with surds by considering the right-angled triangle that has that length as a hypotenuse. For example, what does $\sqrt{2}$ look like on square dotted paper? How does it compare to 1 and 2? They can also use their work on similarity alongside their knowledge of right-angled triangles to simplify surds.
- Having done the above activity students might then wish to explore spirals such as the following. If you start with a 1 cm by 1 cm right-angled triangle and draw it carefully the root 16 line should be 4 cm long. It is not as easy as it sounds to get it right.



Assessment ideas

- Give students pieces of 'incorrect homework' and get them to identify the mistakes and correct the homework.

SECTION 2H: USING PYTHAGORAS' THEOREM / SECTION 3F: PROVING THAT A TRIANGLE IS RIGHT-ANGLED

This section considers the sets of numbers called Pythagorean triples. These numbers are used alongside the converse of Pythagoras' theorem that allows us to decide *if* a triangle is right-angled.

(Higher only) For higher students this section is extended to calculating missing lengths in order to calculate the perimeter or area of polygons.

Prompting questions

- A 3, 4, 5 triangle is right-angled. If you make each side longer by one, you have a 4, 5, 6 triangle. Is that triangle right-angled? *No, triangles are only similar if linked by a scale factor, so these triangles are not similar and so the 4, 5, 6 triangle cannot be right-angled.*
- Is it possible to draw a right-angled triangle with side lengths 4, 11 and 15? *No, the sum of the two smaller sides must exceed the largest, here we have a line because they are equal.*
- How many triples (a, b, c) of numbers less than 100 can you have such that $a^2 + b^2 = c^2$? How many can you find that are not multiples of each other? *Lists can be found online for Pythagorean triples.*
- How is the triple (3, 4, 5) connected to (12, 16, 20)? How are the two right-angled triangles with these side lengths connected? *They are similar and linked by a scale factor of 4.*

Exercise 31C(F) / 32B(H)

While working through this exercise, good prompts for students might be:

- Q2(F) / Q1(H)** How do we know which length is the hypotenuse? *The hypotenuse will always be the longest length so we can assume the largest value is the hypotenuse.*

Exercise 32C(H)

While working through this exercise, good prompts for students might be:

- How many right-angled isosceles are there? *An infinite number, but they are all similar to each other as the other two angles must be 45 degrees.*
- **Q6-7(H)** Sketch the right-angled triangle you are applying Pythagoras' theorem to separately and label all sides to prevent errors.
- **Q8-9(H)** Double check the question, are you finding the area or perimeter of the shape? Does this change how you are using Pythagoras' theorem? *No, because it is still a length that is missing, it is what you then do with that length that changes.*

Starters, plenaries, enrichment and assessment ideas

Plenaries

- Produce a 'tick or trash' PowerPoint. Show students pictures of triangles and either 'tick' if it's a right-angled triangle or 'trash' if it's not. This could be done with mini-whiteboards or with two different coloured cards.

Enrichment activities

- Get a piece of string and tie 12 equally-spaced knots in it. Can you make a right-angled triangle where a knot is a vertex? Why is it possible to do this with 12 knots but not 3? What other possible arrangements of knots can we have?
- In *The Wizard of Oz*, the scarecrow announces that 'the sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side'. Is the scarecrow always, sometimes or never correct?
- NRICH's **Liethagoras** activity offers an alternative theorem based on right-angled triangles for students to prove or disprove (nrich.maths.org).

Assessment ideas

- **(Higher only)** The questions in Exercise 32C can be enlarged to A3 sheets so that students can work on the problems together. They can then peer assess the model solutions using the check points for using Pythagoras' theorem given in Exercise 32A.

SECTION 3H: PYTHAGORAS IN THREE DIMENSIONS

This section extends the work on Pythagoras to problems in three dimensions, mainly looking at the length of diagonals of solids.

Prompting questions

Exercise 32D(H)

While working through this exercise, good prompts for students might be:

- Sketch the right-angled triangle you are working with. Which lengths do you know? *The answer will vary according to the question being worked on.*
- Does this sketch have to be in 3D? *No because a right-angled triangle is a 2D shape.*
- If you do not know at least two of the lengths of the right-angled triangle can you form another right-angled triangle to find one of those lengths? (In other words will you have to apply Pythagoras' theorem twice?) *The answer will vary according to the question being worked on.*

Starters, plenaries, enrichment and assessment ideas

Enrichment activities

- This NRICH activity **Rectangular Pyramids** gives the students a chance to work with Pythagoras' theorem in 2D and then extend it to 3D (nrich.maths.org).
- **(Higher only)** Once students have successfully mastered the application of Pythagoras' theorem, **The Spider and the Fly** makes a nice assessment task when the knowledge is combined with nets of shapes. A possible extension to this task that extends to 3D is: given that the fly is airborne and can take a more direct route, what is the shortest distance from the fly to the spider?

Assessment ideas

- Again, the questions in Exercise 32D can be enlarged to A3 sheets so that students can work on the problems together. They can then peer assess the model solutions using the check points for using Pythagoras' theorem in given in Exercise 32A.

SECTION 4F / 4H: USING PYTHAGORAS' THEOREM TO SOLVE PROBLEMS

Section 4 focuses on using Pythagoras' theorem to solve problems in real-life contexts.

Prompting questions

Exercise 31D(F) / 32E(H)

While working through this exercise, good prompts for students might be:

- What properties does an isosceles triangle have? *It has two sides of equal length.*
- How many right-angled isosceles triangles are there? *An infinite number, but they are all similar to each other as the other two angles must be 45 degrees.*
- Why are we doing step n in the six-step problem-solving approach? Why is it important? *It produces solutions that you have thought about and for which you have considered every piece of information in order to decide what you have to do. It also produces a solution that you can follow if distracted from your working and more importantly an examiner can follow in case of errors.*
- Sketch the right-angled triangle you are working with. Which lengths do you know? *The answer will vary according to the question being worked on.*
- If you do not know at least two of the lengths of the right-angled triangle can you form another right-angled triangle to find one of those lengths? (In other words will you have to apply Pythagoras' theorem twice?) *The answer will vary according to the question being worked on.*
- **Q3-4(F)** Sketch each right-angled triangle to which you are applying Pythagoras' theorem separately and label all the sides, to prevent errors.
- **Q5(F)** Double check the question. Are you finding the area or perimeter of the shape? Does this change how you are using Pythagoras' theorem? *No, because it is still a length that is missing, it is what you then do with that length that changes.*
- **Q12(H)** Design another piece of artwork based on right-angled triangles for a classmate to investigate.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Any of the problems in the exercise can be presented as a starter or plenary and more emphasis can be given to good practice in terms of quality of written communication.

Enrichment activities

- Get students to imagine that they are going to build a skate park with some ramps and need some pieces of wood to make the right-angled triangular ends for the ramps. Tell them that they can only order the pieces of wood online

with the length dimensions 3 m, 4 m, 5 m, 13 m and 15 m. They can add the pieces, for example $4\text{ m} + 4\text{ m} = 8\text{ m}$ but cannot saw any off! What are the lengths of the sides of the right-angled triangles can be made with these pieces?

Assessment ideas

- Enlarge questions to A3 size. Get the students to answer the questions in a group and then swap their answers with another group who then use a mark scheme to assess the work. Collect all the questions at the end and discuss common errors and the structure of some solutions.
- The questions from Exercise 31D/32E could be photocopied on to card or paper and used as a relay for pairs where students work together and only get to move on to the next question once they have correctly solved the problem they are on. This is a suitable activity if your aim is improving efficiency and fluency with Pythagoras problems.
- An assessed piece of homework could also be used to create a 'spot the errors' sheet if students need additional support. This could be compiled from a collection of errors identified when marking the students' work and can be shared with the class before handing their work back. Alternatively a 'bad' homework could be created for students to mark.

Topic links

Previous learning

The 'standard' diagram to demonstrate Pythagoras' theorem shows three squares on the sides of the right-angled triangle. If these squares were replaced with semi-circles, does it still work? Is the area of a semi-circle on the hypotenuse equal to the sums of the semi-circles on the other two sides? By exploring this problem you can link back to work on area of sectors of circles that was introduced in *Chapter 16 Area*. There are also many opportunities given in this chapter to revisit area and perimeter problems for regular and composite shapes now that students are better equipped with skills to find missing lengths associated with right-angled triangles.

Future learning

Chapter 32F/33H Trigonometry will increase students' knowledge of right-angled triangles and presents opportunities for them to solve problems that involve both trigonometry and/or Pythagoras' theorem. For many students, this will prove an additional challenge as they will have to select the most efficient method without direction. Pythagoras' theorem will also be applied indirectly in other problems involving right-angled triangles, for example finding the distance between two points given their coordinates in 2D and 3D.

Gateway to A Level

At A Level Pythagoras' theorem will be employed in many different ways. The most obvious of those is in coordinate geometry where students will use Pythagoras' theorem to find the length of line segments in 2D and 3D as well as finding the radius of a circle given the centre and a point on the curve and deducing the equation of the circle. Less direct, is its application to deriving trigonometric identities, notably: $\frac{\sin \theta}{\cos \theta} = \tan \theta$ and its use in deriving the compound angle formulae.

It will also be used to calculate the magnitude of vectors in 2D and 3D, and hence many other values associated with compound measures.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 3 Question 15
- Chapter 4 Questions 8, 9
- Chapter 5 Question 13
- Chapter 6 Questions 17, 21
- Chapter 7 Question 21
- Chapter 8 Questions 7, 17
- Chapter 9 Questions 19, 20

Higher

- Chapter 1 Question 28
- Chapter 2 Question 29
- Chapter 3 Questions 7, 13, 14, 20
- Chapter 4 Questions 8, 15
- Chapter 5 Question 13
- Chapter 6 Questions 17, 18
- Chapter 7 Question 12
- Chapter 8 Questions 8, 29, 30
- Chapter 9 Question 13
- Chapter 10 Question 16

Homework Book

Foundation

- Chapter 31

Higher

- Chapter 32

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 9 **(F)** / 8 **(H)** worksheets (+ solutions)
- 4 animated widgets
- 10 interactive walkthroughs
- 4 **(F)** / 3 **(H)** auto-marked quickfire quizzes
- 4 **(F)** / 3 **(H)** auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- Square dotted paper
- Foundation Exercise 31A Question 1 diagram

32F / 33H Trigonometry

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know the properties of triangles including notation conventions for angles and side;
- know how to round to a given accuracy and recognise the effect of rounding;
- understand what it means for triangles to be similar and how to use scale factors to calculate side lengths;
- be able to identify alternate angles in parallel lines and apply this knowledge;
- know how to apply Pythagoras' theorem to calculate unknown lengths;
- **(Higher only)** recall the properties of 3D solids;
- **(Higher only)** know the formula for calculating the area of a triangle.

Additional useful prior knowledge

- To be able to calculate with surds and manipulate expressions such as $\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{4}$.

Learning outcomes

Foundation

Section 1

- To use the trigonometric ratios given by the sine, cosine and tangent functions to find unknown lengths and angles in 2D right-angled triangles.

Section 2

- To know the exact ratios given by sine and cosine of 0° , 30° , 45° , 60° and 90° and the exact ratios given by the tangent function for 0° , 30° , 45° and 60° .

Section 3

- To know the difference between an angle of depression and an angle of elevation.
- To identify when the trigonometric ratios must be used instead of Pythagoras' theorem to solve 2D problems relating to right-angled triangles, including contextual problems.

Higher

Section 1

- To use the trigonometric ratios given by the sine, cosine and tangent functions to find unknown lengths and angles in 2D right-angled triangles.

Section 2

- To know the exact ratios given by sine and cosine of 0° , 30° , 45° , 60° and 90° and the exact ratios given by the tangent function for 0° , 30° , 45° and 60° .

Section 3

- To use the sine, cosine and sine area rules to solve problems relating to unknown sides, angles and areas in non-right-angled triangles.

Section 4

- To know the difference between an angle of depression and an angle of elevation.
- To identify when the trigonometric ratios must be used instead of Pythagoras' theorem to solve 2D and 3D problems relating to right-angled triangles, including contextual problems.

Section 5

- To identify and plot trigonometric graphs.

Vocabulary

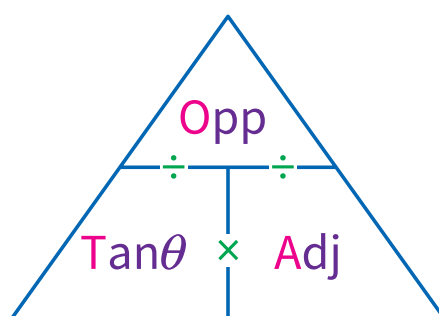
angle of elevation, angle of depression

Common misconceptions and other issues

- See notes on misconceptions and issues in *Chapter 9 Angles* relating to use of notation to identify angles and sides of a shape. In addition, students need to be familiar with the notation explicitly used for triangles where vertices are labelled with capital letters and the opposite sides are labelled with the same letter but lowercase, and they need to appreciate why this system might be more useful in certain cases.
- Some students fail to identify the hypotenuse incorrectly. (See *Chapter 31F/32H Pythagoras' theorem*.)
- Students struggle to identify and label the adjacent and opposite sides because they don't realise it depends on where the given angle is. Reinforcing the use of diagrams and asking students to label every scenario as a first step might help. Students should also be encouraged to use more than one diagram when more than one angle is used in a question.
- Students sometimes confuse the ratios and struggle to identify which function should be used. To combat this, it is often worth considering how the material is introduced to students in the first case, as it is hard to pinpoint whether their problem relates to identifying the correct function, labelling the sides correctly or something else. If students are exposed to only one ratio initially and use it to calculate missing lengths and angles it is then easy to go back to a right-angled triangle and discuss the other combinations of sides that could be used.

Other ideas could be using mnemonics such as SOHCAHTOA, which can be written as $\begin{matrix} O & A & O \\ S & H & C & H & T & A \end{matrix}$, to tie in to the use of formula triangles that, along with labelled triangles, can be used to identify the correct function.

- An additional, perhaps more radical way, is to ignore the three separate functions and instead encourage students to use the sine rule for non-right-angled triangles in all cases.
- Students work with the wrong mode for angle inputs on their calculator. When students are working with their own calculators it is worth doing an initial check by asking them to type in $\sin x$ with x as 40 say (or any other function for any value of x) to check everyone gets the same value. Note: non-exact values often work better here.
- Students struggle to rearrange the trig formulae. Using formula triangles to support memorisation of the formulae might help. Encourage students to write the formula triangle down and, by covering the variable they are solving for, write down the formula they will use.
- Students struggle to work with formulae triangles and do not understand how they are used. To assist students in gaining this understanding the form they are written in can be adapted, as shown in the diagram, to include more information.



- Students struggle to remember exact values for certain trig ratios. This material might form part of a non-calculator paper or might be tested in other ways which require it to be memorised. There are patterns in the ratios that might help with the memorisation. These can be seen in the table provided in the Student Book.
- **(Higher only)** Students need to recall the sine, cosine and area formulae for use in the exam as they will no longer appear on the formulae sheet. Some ways to help with this are:
 - Students could match the formulae with the title of the rule.
 - Students could play a game of true or false when presented with the formulae.
 - Students could be tested on the values like a spelling test and asked to write down the formulae.
 - Students could learn how to derive the formulae (though this is probably more time consuming for most).

Hooks

1. Similar triangles are used to explain where the trigonometric ratios come from at the beginning of Section 1. This idea can be developed in class by asking students to draw an angle of, for example 20° , create three right-angled triangles from this and calculate the perpendicular height divided by the base length for each triangle. This can lead to a discussion on similar triangles and why the ratio should always be the same. Students can then repeat this, finding the ratio for different sized angles, and use these values to solve standard trigonometry problems. This can really help students understand what the trigonometric functions are. You can then go on to explore the other ratios and explain how to find these values on a calculator.
2. You could introduce trigonometry by looking at practical applications, for example looking at aeroplanes coming in to land on a runway and angles of elevation and depression.

SECTION 1F / 1H: TRIGONOMETRY IN RIGHT-ANGLED TRIANGLES

Section 1 introduces the trigonometric ratios found in right-angled triangles. The first part focuses on where these ratios come from and how to use the trigonometric functions on a calculator (see Hook 1 for possible activity). Following this, the ratios are used to calculate missing lengths in right-angled triangles. The section then moves on to using the inverse functions to calculate missing angles. The use of formulae triangles is suggested to support students in identifying the function to be used.

Prompting questions

These prompts are useful throughout all exercises in this section.

- What is the hypotenuse always opposite? *The right angle.*
- Which side labels depend on the given angle? *The opposite and the adjacent. The hypotenuse never changes.*
- Sketch and label all triangles you are working on and indicate clearly which two sides are involved so that you can select which function you will need for each triangle. *We suggest you point students in the direction of the formulae triangles to help with this.*
- How can you identify the hypotenuse in the triangle? *It is the side opposite the right angle.*
- Make sure that you include the units with all solutions.
- How do you find the angle if you know the ratio of the side lengths? *You use the inverse functions to 'undo' the function and find the angle the ratio relates to.*
- How do you use the inverse function on your calculator? *Use the second function button or shift (this is the most common way).*
- How can you check that your answers are sensible? *Use knowledge such as the longest length in a right-angled triangle is the hypotenuse – is this still the case in yours? Draw a rough sketch of your triangle, do the dimensions seem right to you? (Note: this will rely on students having good estimation skills when working with angles.)*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- An initial activity to reinforce labelling could involve students labelling the adjacent, hypotenuse and opposite sides of right-angled triangles given an angle other than the right angle. You could then go on to discuss how labelling would work if all three angles are given (in this case it is often sensible to draw and label different versions of the same triangle or use different colours depending on which angle is being considered). This task could also be tied into a similar task based on revising standard notation for labelling sides and vertices (similar to the example given in the Foundation notes).
Given the later material on non-right-angled triangles, it might be appropriate to discuss with Higher students the notation explicitly used for triangles where vertices are labelled with capital letters and the opposite side are labelled with same letter but lowercase, and why this system might be more useful in certain cases.
- To help students identify the correct function to use when finding missing lengths and angles you could provide problems for students to sort into those requiring sine, cosine or tangent ratios. Alternatively, students could sort

them into problems in which the side length is required and those where the angle is required. (Note, these two tasks can be combined or you could simply ask students to categorise them and give them no guidance). Exercise 32C Q7-8(F) / Exercise 33B Q8-9(H) would be ideal for this task and the questions can be produced as cards that give space below for students to sketch the triangles before making a decision.

Plenaries

- Produce a true or false task in which problems have been matched to the function required to solve them. Students have to identify whether the correct function has been chosen or not.

Enrichment activities

- See first hook above.
- Consider different types of right-angled triangles, where different information is given (e.g. 1 angle, 2 sides, an angle and a side and so on). You can then ask students to discuss what can and can't be found, using trigonometry or Pythagoras' theorem. This can be used to help students see how both techniques can be used interchangeably based on the problem given and how more than one application of an idea can get you to the same solution in the end. This can prevent students worrying about making the correct initial decision. For Higher students these ideas can also be used to derive the trigonometric identities: $\sin^2 x + \cos^2 x \equiv 1$ and $\frac{\sin x}{\cos x} \equiv \tan x$ that will be used by students at A Level.
- **NRICH's Where Is the Dot?** This animation could be used to explore the trigonometric values for angles greater than 90° (nrich.maths.org).

Assessment ideas

- Students can create a 'how to' guide for using trigonometry in right-angled triangles. These can be assessed by you or their peers and kept for future revision or duplicated to provide notes for the class or a display.
- Ask students to mark an example of work (either a fictitious homework or assessment) and identify the errors that have been made before correcting the solutions. The work should include many of the errors identified in the misconceptions section above.

SECTION 2F / 2H: EXACT VALUES OF TRIGONOMETRIC RATIOS

Section 2 focuses on the values of the exact ratios given by sine and cosine of 0° , 30° , 45° , 60° and 90° and the ratios given by the tangent of 0° , 30° , 45° and 60° through the exploration of isosceles and equilateral triangles (and an implicit understanding of similarity).

Prompting questions

- What are the different types of triangle we can have? *Equilateral, isosceles, scalene, right-angled, right-angled scalene and right-angled isosceles.*
- Which types of triangle form a single family of similar triangles? *Equilateral and right-angled isosceles triangles are always similar to others because they obey the AAA rule.*
- Which angles are always present in these triangles? *Angles of 60° in equilateral triangles, and angles of 90° and 45° in a right-angled isosceles triangle.*
- What type of triangle has all angles of 60° ? *Equilateral. Every equilateral triangle is similar to the others.*
- How can we split an equilateral triangle to form a right-angled triangle? *By cutting the triangle into two halves.*
- How can we find the lengths of the sides of this right-angled triangle? *Knowledge of the equilateral triangle it came from and Pythagoras' theorem.*
- What other angles are present in this right-angled triangle? *A right angle and an angle of 30° .*
- What type of triangle has two angles of 45° ? *A right-angled isosceles triangle.*
- How can we find the lengths of the sides of this right-angled triangle? *Knowledge of Pythagoras' theorem.*

Exercise 33C(H)

While working through this exercise, good prompts for students might be:

- **Q4(H)** What three ratios can be found given a right-angled triangle? *The ratios given by sine, cosine and tangent, namely $\frac{\text{opposite}}{\text{hypotenuse}}$, $\frac{\text{adjacent}}{\text{hypotenuse}}$ and $\frac{\text{opposite}}{\text{adjacent}}$.* Given that you know the ratio in its exact form, try forming equations using the formula triangles that can then be solved to find the unknown value.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Students need to know these values and for some the best option might be to drill the values through regular tests. Starters or plenaries could be used in different ways:
 - Students could fill in a blank grid like the one given in the Student Book.
 - Students could match the trig function with input to its ratio.
 - Students could play a game of ‘true or false’ when presented with ratios and their trig function with input.
 - Students could be tested on the values like a spelling test and asked to write down the ratio given the trig function and input.
- If this reveals that students find it easier to remember the ratios in tabular form, they could be encouraged to replicate the table in any exam situation to read off when required.

Enrichment activities

- **(Higher only)** Why will some of the ratios be irrational? What is it about the right-angled triangles they are formed from that gives an irrational scale factor? Students can explore the different results of products involving irrational numbers and consider their results. This can be extended to consider proofs of these rules with a revision of rules for operating with fractions.
- Students can explore different equilateral and right-angled isosceles triangles to identify that the ratios will always be equal regardless of the side length values they choose. You might wish to discuss why this is the case and why the values chosen in the textbook are perhaps the best choice.

Assessment ideas

- See ideas for starters and plenaries above to test students’ knowledge.
- Again, ‘how to’ guides can be very useful in getting students to engage with the material and to support the class in remembering what is required by giving you material that can be duplicated for the class.

SECTION 3H: THE SINE, COSINE AND AREA RULES

Section 3 focuses on forming and using formulae to find unknown lengths, angles and areas in non-right-angled triangles. All formulae in this section will need to be memorised as they will not be given on a formulae sheet as in previous specifications.

Prompting questions

Exercise 33D(H)

While working through this exercise, good prompts for students might be:

- What would you use the sine rule for? *To find an unknown angle or length in a non-right-angled triangle when you have at least three pieces of information including an angle and the length of the side opposite it. Note: sometimes the sine rule will need to be used more than once to find a missing length or angle (or knowledge of angle properties will be required).*
- When is it better to use the formulae $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ than the rearranged form $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$? *When you are finding a length rather than an angle.*

- **Q1(H)** Sketch and label the sides of the triangle, do you have enough information to apply the sine rule? *Yes, opposite angles and side lengths are always given.*
- **Q2(H)** Sketch and label the sides of the triangle, do you have enough information to apply the sine rule? *No, opposite angles and side lengths are only given for one pair.* (Students will either have to apply the sine rule twice or use their knowledge of angles in a triangle to solve these problems.)
- **Q3-4(H)** Sketch and label the sides of the triangle, do you have enough information to apply the sine rule to find the first part? *Yes in both cases.*
- **Q3-4(H)** Do you need to use the sine rule to calculate the size of the third angle in the triangle? *No, you can just use the fact that angles in a triangle sum to 180°.*
- **Q5(H)** What angle fact can you apply to find the angle ABD? *That angles on a straight line add to 180°.*

Exercise 33E(H)

While working through this exercise, good prompts for students might be:

- What would you use the cosine rule for? *To find an unknown angle given three sides of the triangle or an unknown length given two sides and the included angle.*

Exercise 33F(H)

While working through this exercise, good prompts for students might be:

- When can you use the sine area formula for calculating the area of a triangle? *When we have the length of two sides and the included angle.*
- Label your triangle carefully to use the formula $\text{Area} = \frac{1}{2}ab\sin C$. Which vertex should be labelled C? *The one with the included angle.*
- **Q3(H)** How can you use the area formula to find the missing length XZ? *By filling in the information you know and solving to find the unknown length.*

Starters, plenaries, enrichment and assessment ideas

Starters

- An extended starter that can nicely introduce the sine rule for area is to give students a series of triangles to find the area of with different information, for example:
 - Q1 – Right-angled triangle with base and hypotenuse lengths given so students need to use Pythagoras' theorem.
 - Q2 – Right-angled triangle with height and hypotenuse lengths given.
 - Q3 – Isosceles triangle with base and slant-height given.
 - Q4 – Scalene triangle with base, angle (at either end of base) and slant height labelled as well as the perpendicular height shown dotted. They need to use the Sine ratio to work out perpendicular height.
 - Q5 – Scalene triangle with base, angle (at either end of base) and slant height labelled but not the perpendicular height shown dotted.
 - Further questions are in the same format as Question 5 but in different orientations.
- This helps students identify patterns in the calculations and hence helps the class arrive at a formula for the area of a non-right-angled triangle.

Enrichment activities

- For students who have used knowledge of triangles to deduce trig identities you could attempt to derive the cosine formulae with support by sorting a series of cards based on a derivation involving trigonometry and Pythagoras' theorem.

Assessment ideas

- Students will probably find it challenging to identify which formula to use to solve a problem. Give students a series of problems and ask them to categorise them by method of solving so as to help them improve this skill.

- ‘How to...’ guides can be very useful in getting students to engage with the material and supporting the class in remembering what is required by giving you material that can be duplicated for the class. These guides could include advice on how to identify which formula to use and an example of each formula being used to calculate the length of a side or angle (or area).

SECTION 3F: SOLVING PROBLEMS USING TRIGONOMETRY / 4H: USING TRIGONOMETRY TO SOLVE PROBLEMS

This section goes on to consider contextual problems requiring trigonometry to solve them. The idea of an angle of elevation and angle of depression is introduced and used. A similar problem-solving structure to other geometry chapters is offered for students to follow and the exercise brings in other concepts such as Pythagoras’ theorem.

Prompting questions

Exercise 32E(F) / 33G(H)

While working through this exercise, good prompts for students might be:

- **Q1(F)** What makes the angle an angle of elevation/depression? *When the angle we are looking at from our horizontal position is above/below the horizontal.*
- **Q2(F)** What other angles can you label in diagram A? *The angle of depression is also the angle of elevation from the point of view of the boat (alternate angles). The third angle in the triangle can also be found using basic angle facts.*
- **Q3(F)** Which side lengths do you have in diagram D? *The hypotenuse.*
- **Q5, 6, 10(F) / Q1, 2, 6(H)** Can you draw a right-angled triangle to model this situation? *Yes, it is possible to draw a right-angled triangle for each situation.*
- **Q7(F) / Q3(H)** How many right-angled triangles can you draw to model this situation? *Two.* Can you find the height directly? *No, we need to consider the height and distance from the base of the lighthouse to the first boat as unknowns and set up a pair of simultaneous equations.*
- **Q8(F) / Q4(H)** For part b, what new right-angled triangle can you now draw? *One where the base is 25 m and the height is the answer to part a.*
- **Q9(F) / Q5(H)** What information can you add to the diagram to give a right-angled triangle that you can use to find length DC? *We can add length DB and use the tangent function to find DC.*
- **Q11(F) / Q7(H)** How can Question 8(F) / Question 4(H) help here? *We are in the same situation, but this time we must clearly show how we do each calculation.*
- **Q8(H)** How do you write bearings? *Use three digits and measure clockwise from north.* What does the phrase ‘shortest distance’ mean? *It means the perpendicular line segment between the two parallel sides.*
- **Q9(H)** Split the pentagon into equally-sized isosceles triangles. What information is required to calculate the area of each of these triangles? *We can use the $\frac{1}{2}ab\sin C$ area rule so we need to know the angle at the centre ($\frac{360}{5}$) and the length of the line extending from the centre of the pentagon to each vertex. By cutting each isosceles triangle in half to form two congruent right-angled triangles and using right-angled trigonometry. (Note: there are other ways of approaching this problem and sketches should be encouraged.)*
- **Q10(H)** How can you model this situation? *We need to find two sides of the triangle SBD to be able to use trigonometry to find angle SBD.*
- **Q11(H)** Which right-angled triangles in the diagram will help with this? *Triangle DAB and triangle ENB.*
- **Q12(H)** How can you use right-angled triangles to help here? *The same method as in Question 11 can be used here* (Note: Having found the angle each face makes with the base, students might try to use the full power of trigonometry again to find the apex angle of a face of the pyramid. However, it is a useful exercise to see who did that and who simply subtracted their answer to part a from 90° .)

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Have a series of slides with ‘real-world problem’ on them. Students have to sketch a right-angled triangle that would help answer the question. Students could also be required to write down a relevant calculation.

Enrichment activities

- Take Question 9(H) as a starting point. Given a circle of radius 5 cm, what is the area of an equilateral triangle with vertices on the circle? You can then ask what the area of a square (or regular pentagon, regular hexagon, etc) with vertices on the circle will be. What happens to the area of the regular polygon as we increase the number of sides in comparison with the area of the circle?
- Students can make their own clinometer: try NRICH’s **Clinometer** (nrich.maths.org) or MEP: **Trigonometry Unit 4 Activity 4.3** (cimt.plymouth.ac.uk).
- A further enrichment task to connect ideas of trigonometry and construction could be to get the students to draw a triangle, draw on the medians to find a centre point and use this to draw the circumcircle. If they then apply the sine rule for any ratio the value will always be the diameter of the circumcircle: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = D$.

Assessment ideas

- Get students to construct their own questions that require the application of the trigonometric functions more than once and/or Pythagoras’ theorem to allow them to see how more complicated questions can be constructed from the ideas they have now learnt. Students can then exchange their questions and form model solutions.
- Critiqued answers to exam-style questions can be used to form ‘how to’ guides on the types of exam questions that the students might come across.

SECTION 5H: GRAPHS OF TRIGONOMETRIC FUNCTIONS

This section constructs graphs of sin, cos and tan functions from tables of values, introducing the notion of a period for a graph. The final part of the section moves on to recognising the trigonometric functions from their graph.

Prompting questions

While working on the material in this section, good prompts for promoting discussion might be:

- What happens at $\tan 90^\circ$? *From our right-angled triangle we can see that this would be undefined, since the sum of the interior angles of a triangle is 180° and we would have two angles that sum to 180° .*
- How could we remember some useful trigonometric function outputs for certain angles? *We can use a right-angled, isosceles triangle of side lengths 1, 1 and root two to describe the value of $\sin 45^\circ$, $\cos 45^\circ$ and $\tan 45^\circ$. We can use a similar technique for finding the output of trigonometric functions with input 30° and 60° using an equilateral triangle of side length 2 cut in half to create a right-angled triangle.*

Exercise 33H(H)

While working through the exercise, good prompts for students might be:

- **Q1(H)** What does it mean for a function to have a maximum/minimum value? *It means that the range of the function is always less than/greater than or equal to this value.*
- **Q2(H)** What do trigonometric graphs look like? *They are periodic.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- An activity where students are asked to suggest an equation for a given graph. The class can debate the students’ responses.

Enrichment activities

- The NRICH task **Tangled Trig Graphs** is meant for an extension as it stands but can be adapted for use with higher GCSE students by removing some graphs.

Assessment ideas

- The students can use mini-whiteboards to sketch the graphs of equations given to them.

Topic links

Previous learning

There are many opportunities to combine previous concepts covered at GCSE and connect geometric ideas. Opportunities should be taken to look at general problems based on right-angled triangles to encourage students to identify the different methods available to them in solving and what information is required to apply each one. Higher students have considered the graphical representation of each function and explored what it means to extend the angle beyond 90° in *Chapter 19 Graphs of equations and functions*.

Gateway to A Level

Trigonometry forms a large part of the A Level syllabus. Alongside the 2D and 3D problem-solving applications, students will explore the domain and range of each function in detail considering the features of each one. Students will learn to solve more complicated equations involving trigonometric functions and list all solutions in a possible domain. Knowledge of identities will be used to manipulate expressions and reciprocal functions will be used to extend these ideas.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 3 Questions 16, 17
- Chapter 10 Question 19

Higher

- Chapter 1 Question 29
- Chapter 2 Question 19
- Chapter 3 Questions 15, 21
- Chapter 6 Question 19
- Chapter 8 Question 31
- Chapter 10 Questions 6, 17

Homework Book

Foundation

- Chapter 32

Higher

- Chapter 33

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 9 (F) / 19 (H) worksheets (+ solutions)
- 6 (F) / 12 (H) animated widgets
- 14 (F) / 29 (H) interactive walkthroughs
- 5 (F) / 10 (H) auto-marked quickfire quizzes
- 5 (F) / 10 (H) auto-marked question sets, each with four levels
- Auto-marked chapter quiz

34H Circle theorems

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know the sum of angles on a straight line, interior angles of a polygon and around a point;
- know how to calculate an exterior angle (and what it is);
- know how to calculate the size of missing angles in geometry problems, including polygons and parallel lines;
- be able to identify congruent and similar triangles;
- know how to label the circumference and a diameter/radius of a circle;
- know how to find the circumference and area of circles and arc length and area of sectors;
- be able to add geometric reasoning to calculations that find missing angles;
- understand what a mathematical proof involves.

Additional useful prior knowledge

- To understand what the words perpendicular and bisector mean.
- To be able to use Pythagoras' theorem in a variety of settings.
- To know how to identify the different types of quadrilateral.
- To know how to form and solve equations with an unknown from a given text.
- To be able to use mathematical formulae; either by substituting first and then solving for an unknown or by first changing the subject and then substituting.

Learning outcomes

Section 1

- To review the names of parts of a circle.
- To be able to label angles correctly and refer to angles in a diagram involving a circle.

Section 2

- To learn how to prove the following circle theorems:
 - Angles subtended at the centre and at the circumference
 - Angles in a semicircle
 - Angles in the same segment
 - Angle between a radius and a chord
 - Angle between a radius and a tangent
 - Two tangent theorem
 - Alternate segment theorem
 - Angles in a cyclic quadrilateral

Section 3

- To be able to use the circle theorems.
- To be able to construct geometric 'proofs' using the circle theorems.

Vocabulary

subtended, cyclic quadrilateral

Common misconceptions and other issues

- Segment and sector. Some students can confuse the terms ‘segment’ and ‘sector’ when labelling a circle. To address this you could ensure that the correct terminology is used at all times and include references to these particular parts of the circle in any questions you ask.
- Correctly labelling angles on diagrams. This is a particular issue that can propagate into A Level mathematics. Making sure you use correct referencing for angles on diagrams to reinforce good habits and never refer to an angle by the vertex it appears at.
- Mathematical proof. Students’ ideas of mathematical proof can be varied. Here, when asked to prove a geometric property, a student needs a good model solution to know exactly what should be contained in a solution and exactly how to reference the theorems they can use without proof. Some students find the idea of proof very challenging and their lack of algebraic fluency prevents them being able to engage in the material properly. A way to help this might be to remove the algebraic focus initially by using coloured dots to identify equal angles (being able to identify and work with isosceles triangles can take students a long way). This use of colour can help students visualise the proof and the coloured dots can be replaced by letters in the last stage when the proof is complete. Using interactive whiteboard software to create these diagrams can also help students build proofs more easily.
- Angles subtended at centre and circumference. Some students use this theorem incorrectly for diagrams where the subtended angle is in the same minor arc as the two points we created the angle from. To find this angle, students should use a combination of the angle subtended at centre and circumference theorem and the angles in a cyclic quadrilateral theorem. To address this issue, you could provide two alternative solutions for the same problem, one using the correct method and one not. The students would then be required to pick apart both solutions thereby reinforcing the correct use of each theorem.
- Alternate segment theorem. This theorem often proves challenging for students. Continued exposure to this theorem with questions involving it will help to reinforce which angles are being referred to in the theorem.
- Drawing additional diagrams. Students don’t often feel they need to draw additional diagrams. However, when faced with inscribed triangles and cyclic quadrilaterals it is sometimes useful to sketch an additional diagram to display all the information they have been given or have derived from the circle theorems.

Hooks

Initially, you can give a peg-board (either a physical one or use the NRICH **Virtual Geoboard**) to investigate the shapes that can be formed inside the circle and what can be said about the angles. You can then gently direct students to consider certain shapes such as arrow-heads (nrich.maths.org).

SECTION 1: REVIEW OF PARTS OF A CIRCLE

Section 1 focuses on recapping the basic terminology students have already learnt in the context of circles.

Prompting questions

- Why do we need a convention for labelling angles? *Any vertex in a 2D shape will have at least two angles. We could say ‘the acute angle at this vertex’ for the case where there are only two angles, but if we have more line segments meeting at this vertex we need a way to distinguish which angle we are talking about.*

Exercise 34A(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** Can you draw a diagram for each of the statements? *Drawing diagrams might clarify the statement and help reinforce the names of the circle parts.*
- **Q2(H)** How are angles labelled? *If we want to label the angle at B enclosed by the lines BO and BC we use ‘angle OBC’ or ‘ $\angle OBC$ ’.*
- **Q3(H)** What shapes are we looking at here? *Similar triangles.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Display a circle on the board or as a cut-out and ask students to place the correct labels on the diagram to recap previously learnt material before moving on to Section 2.
- Display a series of circles with inscribed labelled triangles. Highlight angles in each diagram and ask the students to name them. Focus on using the correct method of labelling.

Plenaries

- Exercise 34A, Question 1 could be used as a card sort or as a matching activity on the board to assess if the students are ready to move on to new concepts.

Assessment ideas

- Students can construct a guide to the parts of the circle and swap with each another to comment on the content. This can then be glued (after incorporating peer feedback) into their books and referred to in the future.

SECTION 2: CIRCLE THEOREMS AND PROOFS

Section 2 focuses on the circle theorems, outlining them in the order given in the Learning outcomes (above). Each theorem is given with proof and on most occasions each proof makes use of the previous theorems.

Prompting questions

The following questions can be asked while teaching and working through the material associated with the proofs.

- Angles subtended at centre and circumference
 - How do you refer to the angles in the diagram? *If we want to label the angle at B enclosed by the lines BA and BC we use 'angle ABC'.*
 - If a triangle is inscribed in a circle with two radii, what type of triangle is it? *An isosceles triangle.*
- Angles in a semicircle
 - What is this theorem a special case of? *It is a special case of the angles subtended at the centre and at the circumference theorem.* (Note: you might wish to tell your students that mathematicians would refer to this 'theorem' as a corollary of the first.)
- Angles in the same segment
 - What can be added to the diagram to give the same situation as the angles subtended at centre and circumference theorem? *We can add in radii and then apply that theorem twice.*
- Angle between radius and chord
 - What sort of triangle(s) do you get by adding radii OC and OA? *An isosceles triangle.*
- Angle between the radius and tangent
 - What property does a tangent to a circle have? *It touches the circle only once.*
- Two-tangent theorem
 - What do you know about the two triangles ATO and TBO? *They are right-angled triangles and can be shown to be congruent.*
- Alternate segment theorem
 - Which angle is in the major segment? *The angle PRQ.*
 - Which previous theorems are being used in this proof? *Angles in a semicircle, angle between the radius and tangent, angles in the same segment.*
- Angles in a cyclic quadrilateral
 - How can you split up a cyclic quadrilateral to enable you to apply a previous theorem? *We can add in radii to the centre from two opposite corners of the quadrilateral and apply the theorem: Angles subtended at the centre and at the circumference.*

Exercise 34B(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** Which circle theorem is this? *Angles subtended at the centre and at the circumference.*
- **Q2(H)** Which circle theorem is this? *Angles subtended at the centre and at the circumference or angles in a semicircle.*
- **Q3(H)** What do you need to add to the diagram to be able to give a proof? *Two radii OD and OC.*
- **Q4(H)** Which theorem can you apply? *Angles subtended at the centre and at the circumference.*
- **Q5(H)** Which theorem do you need before you can look at the triangle ACB? *Angles in a semicircle.*
- **Q6(H)** Which theorem can you apply? *Angles subtended at the centre and at the circumference twice.*
- **Q7(H)** Can you set up an equation from this situation? *We know that we have a right-angled triangle (angles in a semicircle) and so we can let angle CBA = x. Thus, CAB = 2x and 3x + 90 = 180.*

Exercise 34C(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** What does the symbol $\perp$ mean? *Perpendicular.*
- **Q3(H)** Which angles are equal?

Exercise 34D(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** What angles can you label in this diagram and why? *We can use the alternate segment theorem to help label angles ABC and ACB.*
- **Q2(H)** What do you know about angles in a semicircle and angles at a tangent? *We have right angles for the angles RPQ, SPB and SPA.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Produce a series of starters based on the circle theorems in which students need to reconstruct the proofs of each theorem having been given an initial diagram. Adding prompts for each of the diagrams can differentiate this activity.

Plenaries

- Produce a series of flash cards with circle theorem questions on them and ask the students to identify which circle theorems they can apply to each question.

Enrichment activities

- There are several activities where the students can explore some of the theorems via examples such as those provided by GeoGebra and NRICH. Each of the files below can be used to form an investigative approach to each circle theorem before bringing the class together to discuss how they could prove each of the new ‘facts’ they have discovered (geogebraTube.org and nrich.maths.org).
 - **GeoGebra activity Angles subtended at centre and circumference** and NRICH activity **Subtended Angles**;
 - **GeoGebra activity Angles in a semicircle** and NRICH activity **Right Angles**;
 - **GeoGebra activity Angles in the same segment** or **Circle Theorem 5**;
 - **GeoGebra activity Angle between radius and chord**;
 - **GeoGebra activity Angle between the radius and tangent**;
 - **GeoGebra activity Two tangent theorem** or **Tangents from a point to a circle**
 - **GeoGebra activity Alternative segment theorem** or **Circle Theorem 3**
 - **GeoGebra activity Angles in a cyclic quadrilateral** and NRICH activity **Pegboard Quads**.
- There are also some useful resources from the **National Stem Centre** (nationalstemcentre.org.uk).

Assessment ideas

- Students find it difficult to know how much information to include in their solutions to circle theorem questions. Having a series of questions that are peer assessed in order for students to see what information their peers are providing helps students realise what level of detail is needed for each question.
- Ask students to construct a personal revision guide that can then be peer assessed by another student or assessed by you. This will help your students keep track of the circle theorems they will be required to know and provide a revision aid.

SECTION 3: APPLICATIONS OF CIRCLE THEOREMS

Section 3 focuses on applying the circle theorems to answer questions. A problem-solving framework outlines a method for tackling questions that require multiple applications of the circle theorems.

Prompting questions

- How can you organise your thoughts on the page when tackling circle theorem questions? *We can use the steps outlined in the problem-solving framework or/and we could produce a flow chart of steps to follow stating each step of our argument for another person to follow. Every time we find a new angle in the diagram we can add a new step to our flow chart with a justification using a theorem or a fact (each step on the flow chart must be notated with reasons).*

Exercise 34E(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** What is the setup in this diagram? *We have an isosceles triangle (given with two radii) and a chord bisected by a radius.*
- **Q2(H)** What is the set up in this diagram? *We have the same set up as needed for the alternate segment theorem and a chord bisected by a radius.*
- **Q3-4(H)** What is the set up in this diagram? *We have two tangents meeting at an external point.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- The 'work it out' activity is a great for students to assess their understanding of the circle theorems and to prompt discussion about what information is required for each solution.

Enrichment activities

- Pair up the students and ask them to answer circle theorem questions on large pieces of paper around the room. Each pair is only allowed one minute to look at the question before they are required to move on. This activity promotes good written communication skills since the students are required to pick up a solution where the previous pair left off and write answers to questions that they might have not necessarily started. When a solution is complete, students can then mark the next one they move to.

Topic links

Previous learning

The students will have previously studied material from *Chapter 5 Properties of shapes and solids*, *Chapter 9 Angles* and *Chapter 30F/31H Similarity* and will get a chance to apply this knowledge in the context of circle theorems.

Future learning

The students are now prepared to combine their knowledge of circle theorems and the work from *Chapter 7 Further algebraic expressions* and *Chapter 8 Equations* to answer more complex questions.

Gateway to A Level

Coordinate geometry of the circle forms a substantial part of A Level mathematics and students will be required to know the circle theorems to derive new information that they can use to solve questions involving new material.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES**Problem-solving Book**

- Chapter 2 Questions 20, 21
- Chapter 3 Questions 8, 22

Homework Book

- Chapter 34

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 11 worksheets (+ solutions)
- 5 animated widgets
- 8 interactive walkthroughs
- 3 auto-marked quickfire quizzes
- 3 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- Circle outlines

33F / 35H Discrete growth and decay

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to convert percentages to decimals;
- know how to find a percentage of a quantity using multiplication;
- be able to increase or decrease a quantity by a given percentage by multiplying by a suitable decimal;
- **(Higher only)** understand how to plot and interpret functions of the form $y = ab^x$, where b is a positive number and a , x are real numbers.

Additional useful prior knowledge

- To know how to extract information regarding percentage increase or decrease from a word problem.
- To be able to substitute values into a formula.

Learning outcomes

Foundation

Section 1

- To be able to calculate with simple growth, such as simple interest rates.
- To be able to calculate with compound growth, such as compound interest rates.
- To be able to solve word problems using simple and/or compound growth.

Section 2

- To be able to calculate with simple decay.
- To be able to calculate with compound decay, such as depreciation.
- To be able to solve word problems using simple and/or compound decay.

Higher

Section 1

- To be able to calculate with simple growth, such as simple interest rates.
- To be able to calculate with compound growth, such as compound interest rates.
- To be able to solve word problems using simple and/or compound growth.
- To be able to use the formula $y = a(1 + r)^n$ for compound growth.

Section 2

- To be able to calculate with simple decay.
- To be able to calculate with compound decay, such as depreciation.
- To be able to solve word problems using simple and/or compound decay.
- To be able to use the formula $y = a(1 - r)^n$ for compound decay.

Vocabulary

depreciation

Common misconceptions and other issues

- Some students find it hard to remember what simple and compound interest are. To combat this, linking the words to their literal meaning (simple because it is the simpler calculation; compound because the multiplier is being compounded each year) helps some students. For others, associating the words with a physical process can help.
 - For simple interest, do one calculation to find the interest and then add multiples of that to the original amount.

- For compound interest, set up a table with column headings 'amount at the start of the year' and 'amount at the end of the year'.
- Some students find it hard to know if they need a percentage increase or decrease. For students with weak literacy skills, there are lots of terms to use in this chapter. Having a vocabulary board with an explanation of interest, growth, depreciation, decay and exponential would allow students to access the material more easily.
- When calculating a percentage increase or decrease, some students prefer to calculate the amount of interest or decay first and then add or subtract. However, other students will be very happy with the concept of multiplying by 1.05 to increase an amount by 5%. Letting students find and use the way that makes sense to them does, on the whole, help with recalling this material in exam situations.
- Students often have difficulty calculating the initial amount given a percentage increase/decrease and a final amount. Working backwards often causes issues for students, especially in this topic where being able to rearrange formulae helps enormously. The most common misconception is that to find the original price after a 10% depreciation, you just add on 10% of the new price. To demonstrate that this does not work, have a series of examples that are 'incorrect'. Substituting the incorrect answer back into the question will reinforce that this does not work. In addition, always referring to amounts as a percentage of the original can help, for example when increasing by 10% you are finding $100\% + 10\% = 110\%$ of the original amount. When you have the price depreciated by 10% you have $100\% - 10\% = 90\%$ of the original amount. Connecting these ideas with unitary methods for scaling can also help students identify what to do in a question as the setup is always the same.

Hooks

1. Consider investments. Tell students that a high-profile billionaire wants to invest his or her money in a project but does not know which one will give the best rewards. Each project has varying amounts of interest, initial cost and maintenance costs. How can we help to advise what projects he or she should take?
2. Consider bacteria and epidemics. Given enough food, bacteria grow at an exponential rate. Show a piece of footage of bacteria growing and discuss with students how they think the bacteria are multiplying.

SECTION 1F / 1H: SIMPLE AND COMPOUND GROWTH

Section 1 focuses on simple and compound growth through and uses the example of interest rates. For Higher students the generic compound growth formula $q(1 + p)^n$ (initial investment q , growth rate or interest rate in a decimal form p and the number of years n) is used and exponential growth is discussed.

Prompting questions

Exercise 33A(F) / 35A(H)

While working through this exercise, good prompts for students might be:

- **Q1(F)** How do you calculate simple interest? *Simple interest is interest paid on the original amount and the same interest is paid for each time period.*
- **Q2(F)** How can you set out working for this question? *We can make a table, with the column titles 'amount at start of year' and 'amount at end of year'. For compound interest we use the amount at the end of the previous year as the starting amount for the next year.*
- **Q3(F)** How would you label the axes? *Time on the horizontal axis and total money in the account on vertical axis.*
- **Q4(F) / Q4(H)** What is the multiplier for a growth rate of 4%? *1.04.*
- **Q5(F) / Q5(H)** What does the word 'inhabitant' mean? *The people that live in that place.* What does the term 'model' mean? *A mathematical model that is a description of a situation using mathematical terms and concepts.*
- **Q6(F) / Q6(H)** Is inflation going to be treated as simple or compound growth? *Compound.*
- **Q11(F) / Q12(H)** What is the increase in price and what is that as a percentage of the original amount? *Increases by 40p and so a percentage increase of 160%.*
- **Q13(F) / Q14(H)** There is no starting value, does this matter? *No. Since we just want the number of hours for a colony of bacteria to double we could choose any number to start with. For example we could use an initial value of 1 and find out the how many times you have to multiply 1 by 1.05 to make 2.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Produce a ‘find the best deal’ activity, such as the one in Exercise 33A Q14 / 35A Q15. This will allow students to find the answers to each of the deals after which a whole-class discussion can take place to justify their choice.

Enrichment activities

- Growth rate is a huge part of mathematical modelling. Create a task based on epidemics, such as the ones from the **Motivate** website (motivate.maths.org). The summary on NRICH’s **Investigating Epidemics** webpage is a nice way to look at growth and, for Higher students, exponential growth (nrich.maths.org).
- Take a series of interest rate deals from various banks and get students to analyse which deal is best.

Assessment ideas

- Get students to mark and evaluate ‘student answers’ (formed from either a collection of assessed work or imitation student answers) to exam questions. In particular, the answers provided should highlight some of the misconceptions and issues given above.

SECTION 2F / 2H: SIMPLE AND COMPOUND DECAY

Section 2 focuses on simple and compound decay, through the main example of depreciation. For Higher students the generic compound decay formula $q(1 - p)^n$ (initial investment q , decay rate or depreciation rate in a decimal form p and the number of years n) is used and exponential decay is discussed.

Prompting questions

Exercise 33B(F) / 35B(H)

While working through this exercise, good prompts for students might be:

- Q1(F) / Q1(H)** How could you set this question out on the page? *We can make a table, with the column titles ‘amount at start of year’ and ‘amount at end of year’. For compound depreciation we use the amount at the end of the previous year as the starting amount for the next year.*
- Q4(F) / Q4(H)** Which one of the graphs can it definitely not be and why? *Graph C because it is increasing as time goes on.*
- Q5(F) / Q5(H)** How do you calculate this percentage decrease? *We need to know how much the original amount has decreased by and use this to calculate the percentage decrease using $\frac{\text{amount decreased}}{\text{original amount}} \times 100$.*
- Q8(F) / Q9(H)** Which value do you need to find two-thirds of? *The current value, £342.95.*
- Q8(H)** How can you work backwards to find the initial cost for row three, can you just add on 11% of the Year 1 value? *We cannot just add on 11% of the Year 1 amount. This is because $I \times (1 - 0.11)^1 = \30260 . Therefore, $I = \frac{\$30260}{0.89}$.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Display a series of different types of cars made in different years, with different depreciations, for example a classic car and a model made a couple of years ago. The students need to calculate the current price of each of the cars.

Enrichment activities

- The NRICH short activity for Higher students (or slightly longer activity for Foundation students), **Weekly Challenge 48: Quorum-sensing**, gives a novel question about decay (nrich.maths.org).
- Radioactivity is a natural occurrence of exponential decay and the students could look at data for a selection of radioactive materials then use it to answer a series of questions related to the age of an object.

Assessment ideas

- Get students to mark and evaluate ‘student answers’ to exam questions (formed from either a collection of assessed work or imitation student answers). In particular, the answers provided should highlight some of the misconceptions and issues given above.

Topic links

Previous learning

Foundation and higher students will be using their knowledge fractions, decimals and percentages (*Chapter 10 Fractions, Chapter 11 Decimals and Chapter 13 Percentages*). The graphs that are used in this chapter have been seen in their most basic form in graphs of equations and functions (*Chapter 19 Graphs of equations and functions*).

Future learning

For the remaining chapters in this GCSE book we have the general concept of direct and inverse proportion (*Chapter 34F / 36H Direct and inverse proportion*) for both foundation and higher students. For Higher students there is chapter on interpreting graphs (*Chapter 39H Interpreting graphs*), in which gradients of real world graphs are interpreted in context (making links back to the graphs showing growth and decay rates).

Gateway to A Level

Rate of change is the motivation behind differentiation and is used extensively at A Level. In addition to this, exponential growth is studied in more detail and leads to the exponential function (based on Euler's number e).

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- N/A

Higher

- Chapter 4 Question 3
- Chapter 5 Question 5
- Chapter 8 Question 8
- Chapter 9 Question 14

Homework Book

Foundation

- Chapter 33

Higher

- Chapter 35

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 4 (F) / 5 (H) worksheets (+ solutions)
- 1 animated widget
- 11 (F) / 18 (H) interactive walkthroughs
- 3 (F) / 4 (H) auto-marked quickfire quizzes
- 3 (F) / 4 (H) auto-marked question sets, each with four levels
- Auto-marked chapter quiz

34F / 36H Direct and inverse proportion

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to work with numbers including fractions;
- know how to write ratios and interpret them;
- know how many minutes there are in fractions of an hour;
- be able to perform a substitution into a formula;
- know what a reciprocal is;
- **(Higher only)** know how to square and square root numbers.

Additional useful prior knowledge

- To understand how money works and know some different currencies.
- To have a clear understanding of speed, distance and time.
- To know how to find the highest common factor and lowest common multiple of two or more numbers.

Learning outcomes

Foundation

Section 1

- To be able to use direct proportion to solve problems.
- To be able to use the unitary method to solve proportion problems.

Section 2

- To be able to solve direct proportion questions graphically.
- To be able to solve direct proportion questions using algebraic manipulation.

Section 3

- To be able to solve inverse proportion questions, based on $y = \frac{1}{x}$.

Higher

Section 1

- To be able to use direct proportion to solve problems.
- To be able to use the unitary method to solve proportion problems.

Section 2

- To be able to solve direct proportion questions graphically.
- To be able to solve direct proportion questions using algebraic manipulation.

Section 3

- To be able to solve direct proportion problems involving the square or square root of a variable.

Section 4

- To be able to solve inverse proportion questions, based on $y = \frac{1}{x}$.

Vocabulary

ratio, direct proportion, mathematical model, inverse proportion

Common misconceptions and other issues

- Some students will still not be comfortable with what a ratio is and will not understand that it relates to multiplication/division rather than addition/subtraction. To address this issue, allow students to make the error (e.g. 3 : 4 is the same as 2 : 3) and then show them pictorially that this does not make any sense.

- Some students do not understand that the constant of proportionality is the same for all parts of the question. Providing a longer exam-style question that requires them to keep using the constant they found in the first part of the question can expose this misconception and you can then address it through questioning. *'Is this the same situation as in part a? What does this mean for our equation relating our two variables?'*
- There is often confusion between the words ratio and proportion: this NRIC article **Ratio or Proportion?** discusses some of the misconceptions and issues surrounding these two words for KS3 and KS4 students(nrich.maths.org). Note: the article is written to increase awareness of the issues rather than to offer clarification to students.
- When students lack confidence in using fractions they often round too early when using the unitary method. Give an 'example' in which a student has rounded in the first calculation leading to large errors in the final answer to demonstrate the effect rounding has.
- Students can often get lost in the language used to set up a word problem involving concentration calculations. Looking at the 'proportion of each liquid used' and the 'ratio of the liquids in each container' can get confusing. Having concrete examples, where the students draw pictures to represent each situation will help to address this issue over time.

Hooks

1. Start with a times table, select a random part of it (e.g. the eighth term is 88 and the 12th term is 132) and ask students to find the first term and the fifth term. Follow this with the question 'what is the relationship between the term number and the value?' This is something students will have seen before, but now we can use the terms 'directly proportional to' and 'constant of proportionality'.
2. Take an example from *Chapter 30F/31H Similarity* involving similar rectangles and ask 'If all these rectangles are similar, how can we relate the length of each rectangle to the height?'

SECTION 1F / 1H: DIRECT PROPORTION

Section 1 focuses on direct proportion and introduces problems that arise as a result of two quantities varying in value but remaining in the same ratio.

Prompting questions

- What is a ratio? *A ratio is the relationship between two or more groups or amounts that explains how many times bigger one is compared with another.*
- Can you write down any ratios for the given situation? *General question for supporting students when making a start.*

Exercise 34A(F) / 36A(H)

While working through this exercise, good prompts for students might be:

- **Q4(F) / Q3(H)** How can you use the ratio of £1.20 : 100 g to help find the weight of jelly beans you can buy with £4.20?
We can create a new ratio, for £0.01 : $\frac{100}{1.2}$ g and use a multiple of $\frac{100}{1.2}$. An alternative is that both 1.20 and 4.20 have a common factor of 0.60. Divide 1.20 by 2 and then multiply by 7 to get 4.20, and do the same to 100 g to find the weight. This method leads students to consider the highest common factor aspect of the question.
- **Q5(F) / Q4(H)** How can you compare the two calls for best value? *We need to find ratios for each situation with the same number of minutes to be able to compare the costs. The easiest way is to find how much a one-minute call will cost each person.*
- **Q5(H)** How could you approach this question? *One strategy is to find out how much you can make based on 500 g of plain flour, since that is the easiest scaling to start with. Why? It is five times as much. Another strategy could be to work out which ingredient is the limiting factor.*
- **Q6(F)** You need to find the new amounts for different numbers of people, how can you do this efficiently? *Since we need to find a variety of different variations of the same recipe, we could find how much we need for one person and then multiply by the number of people we need. We do need to be careful and should keep the amounts for one person in the form of a fraction.*

- **Q7-8(F)** What are units of time in each case? Prompting question to make students think about the material from Chapter 12 Units and measurement: *that all units must be equal when calculating with values.*
- **Q9(F)** How can you calculate pounds from the number of dollars? *If we know how many dollars one pound is, we can divide the given number of dollars by this number to calculate the number of pounds.*
- **Q12(F) / Q10(H)** (Although this is not the same question, the idea is the same for each.) What unit of volume would be useful to help calculate the price per litre for each part? *If we look at the price of 250 ml, that is £18 price of 250r litre for each part? What unit of volume would behe prices for the other paint tins.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Create a 'collect the ratios' activity: a set of cards containing ratios (each one having a matching card with an equivalent ratio) are placed face down. The students reveal two cards at a time and must match pairs of equivalent ratios.
- Create a 'fill in the table' activity in which students are asked to complete a table, similar to Question 6 in Foundation Exercise 34A, with different amounts of information filled in. For students requiring more help, you could fill in all the rows for the first column (i.e. eight people). To make this more challenging, you can fill in different parts of the table (not the first column, but random cells in the table) and the students need to fill in the rest.
- **(Higher only)** Give students a series of quick fire 'do I have enough' questions (based on Question 5 in Higher Exercise 36A).

Enrichment activities

- Having a context to calculate direct proportion allows for fluency exercises to take place without pages of exercises. An activity based on NRICH **Triathlon and Fitness** provides a good challenge that works with the concepts of this section and includes mathematical modelling. You might even wish to get your class involved and produce your own data to use for this activity (nrich.maths.org).
- This NRICH activity **Dilution Series Calculator** can be used to help students deal with the complications that arise when looking at concentrations for liquids (nrich.maths.org).
- Pose a question with three variables, such as 'If it takes 8 men 3 days to build 10 metres of wall, how long will it take 5 men to build 12 metres of wall?'

Assessment ideas

- Students can design their own exam-style question, complete with mark scheme, and get another student to try it. The activity will make those designing the question think about what information is required in a question and make those who answer it consider how they should set out their working.

SECTION 2F / 2H: ALGEBRAIC AND GRAPHICAL REPRESENTATIONS

Section 2 focuses on direct proportion problems given graphically or generalised through the use of algebra (i.e. use of a formula).

Prompting questions

- When considering questions involving proportionality, what are the units for each of the objects involved? *Encourage students to identify in each case the units that are given in the question In particular, units can cause problems when you are moving between hours, minutes and seconds.*

Exercise 34B(F) / 36B(H)

While working through this exercise, good prompts for students might be:

- **Q2(F)** How can you use the graph to convert between pounds and lev? *We could find the £10 on the axis, then move up in a straight line to the line on the graph and then read across to find the number of lev at that point. Dividing by 10 will then tell us the number of lev in one pound.*
- **Q4(F) / Q2(H)** What does the word 'crowdedness' mean? *It means the number of people found in a given amount of space.*

- **Q5(F) / Q3(H)** How do you know which runner is going faster? *The graph for the faster runner is steeper than the other.*
- **Q7(F)** How can you write ‘the number of cars being produced is directly proportional to the number of days the factory stays open’ using algebra? $n = kd$.
- **Q9(F)** What does it mean for p and q to be directly proportional? $p = kq$ or $q = kp$. Does this help find a formula? *Yes, we can use the given numbers to find a value of k for our formula.*
- **Q10(F)** What initial ratio can you set up in this situation? $4 : 60$.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- The questions and activity in Work it out F34.1/H36.1 and F34.2/H36.2 would make an excellent starter.
- Give students a ‘word problem’ involving two variables that are directly proportional to each other. The students need to extract the correct information and write a formula for doing further calculations.

Enrichment activities

- Ask students to make a hypothesis linking two measurable quantities that they think are directly proportional. They then collect the data in order to provide evidence for/or against their hypothesis. This activity gets the students thinking about the data cycle, as well as the properties needed for two quantities to be directly proportional to each other.

Assessment ideas

- Students can design their own exam-style question, complete with mark scheme, and get another student to try it. The activity will make those designing the question think about what information is required in a question and make those who answer it consider how they set out their working.

SECTION 3H: DIRECTLY PROPORTIONAL TO THE SQUARE, SQUARE ROOT AND OTHER EXPRESSIONS

The Higher version of Section 3 focuses on directly proportional relationships that are non-linear (i.e. square and square root).

Prompting questions

Exercise 36C(H)

While working through this exercise, good prompts for students might be:

- Why is an initial set of values given in the opening for each question? *When we say ‘ y is directly proportional to the something of x ’ we mean that for pairs of values (x, y) , y is equal to a constant multiplied by the something of x . To find this constant we need some ‘initial values’ or ‘initial conditions’.*
- **Q1(H)** and **Q5(H)** What index do we use when we say something is cubed? 3.
- **Q2(H)** and **Q4(H)** What index do we use when we say something is square rooted? $\frac{1}{2}$.

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Display a series of tables with data in them and ask (possibly using mini-whiteboards) what the relationship is between the variables. You could also ask your students to give a formula for each situation.
- Display a series of word problems that describe directly proportional situations. Students need to write down a formula representing each situation.

Enrichment activities

- As indicated in the introduction to this section in the student book, the distance travelled by an object dropped from rest is proportional to the square of the time taken. This can be used as a base for a practical activity in which students collect data by dropping an object from different heights. This can then be graphed and analysed to either discover this relationship or find the ‘constant of proportionality’ for this formula.

Assessment ideas

- Students can design their own exam-style question, complete with mark scheme, and get another student to try it. The activity will make those designing the question think about the information that is required in a question and make those who answer it consider how they set out their working.

SECTION 3F / 4H: INVERSE PROPORTION

Section 3F/4H focuses on inverse proportion. More explanation is given in the Foundation book in Work it out 34.3.

Prompting questions

Exercise 34C(F) / 36D(H)

While working through this exercise, good prompts for students might be:

- Q1(F) / Q1(H)** What does this question have to do with inverse proportion? *Look back at Work it out 34.3 in which we fixed the area of the rectangle and varied one of the lengths. In this question, we have fixed the total spend but are varying the price of the souvenirs. The more expensive the souvenir the fewer we can buy.*
- Q2(F)** What does 180 in the formula mean? *The distance to Cambridge in miles.*
- Q2(H)** Given that we are told speed is inversely proportional to time, what is simplest equation that we can write down connecting these two variables? $s = \frac{k}{t}$. What does the 'k' part of our formula represent? *Since k is a constant, the only thing we are keeping constant is the distance.*
- Q5(H)** What does 'a is inversely proportional to the square of b' actually mean? *This means that a and $\frac{1}{b^2}$ differ by a single constant for any given connected values of a and b.*
- Q6(H)** What does 'r is inversely proportional to the square root of m' actually mean? *This means that r and $\frac{1}{\sqrt{m}}$ differ by a single constant for any given connected values of r and m.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Now that the students have seen situations where variables are directly linked or inversely linked, you can display a series of tables of data and ask (possibly using mini-whiteboards) if the variables are directly proportional or inversely proportional. To add an extra level of complication, you could ask students to give a formula.

Enrichment activities

- An enrichment activity for Section 2 and a good review before moving on to Section 3 could be **Zin Obelisk** from NRICH. This activity gets students to think about the types of information that have been given and then use their knowledge of proportion to answer the question (nrich.maths.org).
- Allow students to collect their own data for variables that are inversely proportional to each other and then get them to try and discover the relationships or formulae. Speed and time or tasks where the number of people reduces the time to complete a task might be good starting points.

Assessment ideas

- Give students a series of graphs with a mixture of direct and inverse proportional variables. Students need to sort them into the correct type and then provide a physical situation where that graph might occur. This can be displayed as a poster and other students can comment on the 'realistic' nature of the situations described and on the labelling as directly or inversely proportional.

Topic links

Previous learning

Students will have looked at ratio and proportional at KS3, but might have not used mathematical language to describe the situation. At GCSE, students will have used this language to describe ratio (in *Chapter 22 Calculations with ratio*), similarity (in *Chapter 30F/31H Similarity*) and will also have used it for trigonometry (in *Chapter 32F/33H Trigonometry*).

Through the GCSE content, students will have encountered various types of graphs and equations and this chapter provides a good opportunity to revisit straight-line and curved graphs.

In addition to the above, this chapter can be used to refresh ideas of data collection and representation, and also revise the language of sequences to help analyse the proportionality relationship between variables.

Future learning

Higher tier students will use the notion of a gradient in *Chapter 39H Interpreting graphs*; links can be made back to the terminology and representations used in this chapter.

Gateway to A Level

Students who go on to A Level mathematics will encounter proportionality in the description of rates of change and thus in differential equations.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 6 Question 22
- Chapter 7 Question 7

Higher

- Chapter 2 Question 22
- Chapter 7 Questions 5, 13

Homework Book

Foundation

- Chapter 34

Higher

- Chapter 36

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 7 (F) / 8 (H) worksheets (+ solutions)
- 5 animated widgets
- 6 (F) / 7 (H) interactive walkthroughs
- 3 (F) / 4 (H) auto-marked quickfire quizzes
- 3 (F) / 4 (H) auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper
- 2 mm graph paper
- Axis grids from -10 to 10 in x and -10 to 10 in y

35F / 37H Collecting and displaying data

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to divide (360) by a given number and then multiply, possibly rounding the result;
- be able to draw a circle using a pair of compasses;
- be able to measure angles accurately;
- know how to draw axes and choose appropriate scales;
- **(Higher only)** understand how to use inequality notation for class intervals.

Additional useful prior knowledge

- To be able to read information from a graph.
- To be able to group data into class intervals.
- To know the difference between discrete and continuous data.

Learning outcomes

Foundation

Section 1

- To be able to infer properties of populations or distributions from a sample, while knowing the limitations of sampling.

Section 2

- To be able to interpret and construct tables, charts and diagrams, including frequency tables and bar charts.

Section 3

- To be able to draw and interpret pie charts and pictograms for categorical data and vertical line charts for ungrouped, discrete numerical data.

Section 4

- To use tables and line graphs for time series data.

Higher

Section 1

- To be able to infer properties of populations or distributions from a sample, while knowing the limitations of sampling.

Section 2

- To be able to interpret and construct tables, charts and diagrams, including frequency tables and bar charts.

Section 3

- To be able to draw and interpret pie charts and pictograms for categorical data and vertical line charts for ungrouped, discrete numerical data.

Section 4

- To be able to draw and interpret histograms and cumulative frequency diagrams for continuous data and know their appropriate use.

Section 5

- To use tables and line graphs for time series data.

Vocabulary

population, sample, representative sample, discrete data, categorical data, continuous data

(Higher only) grouped data, class intervals, cumulative frequency, histogram

Common misconceptions and other issues

- Some students have difficulty drawing obtuse angles using a protractor. Encourage students to ‘sense-check’ their angle (for example an angle of 170 degrees should be bigger than a right angle), to ensure that they have used the correct scale on their protractor.
- Some students draw their axes incorrectly and, in particular, use inconsistent scales. Although providing students with pre-prepared axes can enable a greater variety of questions to be undertaken in a lesson, there is a clear need for students to practise drawing axes for themselves as many find it a challenge. Projecting some badly drawn graphs and asking what has gone wrong can be a powerful plenary.
- A common error in exams is misreading scales, so practice with different scaled axes and a variety of data is useful.
- Students can get confused by the use of everyday words that have precise statistical meanings, for example, ‘population’. Some will have difficulty understanding the idea that, for example, if we are interested in the feelings of Year 7 students about the girls’ toilets our population is just Year 7 girls. Using non-human examples can help here. If we are interested in the strength of climbing ropes we make in our factory, the population is all the climbing ropes we manufacture. This context also demonstrates the purpose of sampling: if we test every climbing rope until it breaks, we wouldn’t have anything to sell!
- **(Higher only)** Some students make the mistake of plotting cumulative frequency at the midpoint of each class interval. If the class intervals represent money (e.g. a class of £10–£15, with cumulative frequency 12) it can be helpful to think ‘the only thing we know for sure is that if I have £15 I could buy 12 of those things’. If I have £14, I cannot be sure what I can buy. Hence we plot at the end point of each class.
- **(Higher only)** Some students make the mistake of drawing histograms with frequency as height (rather than area). Students should be reminded that histograms with bars of unequal width would be unrepresentative if their height represented frequency.

Hooks

The classic photograph of a happy man holding a paper whose headline proclaims ‘DEWEY DEFEATS TRUMAN’ can be an excellent introduction to bias in sampling. A variety of versions of this photograph are available on the Internet by searching for the headline. Students naturally assume that the happy man must be Dewey (not the re-elected President Truman). They quickly come up with a variety of valid, historical reasons (typically ‘telephone surveys in the 40s would only include rich people who didn’t vote democrat’) for the biased sample.

SECTION 1F / 1H: POPULATIONS AND SAMPLES

Section 1 introduces the ideas of population and sampling and discusses some sampling methods. Students are likely to be familiar with the term ‘sample’ if they’ve ever been offered a free sample at a supermarket. They’re not allowed to eat the whole thing, just have a taste to see what it’s like.

Prompting questions

These questions should prompt open-ended discussion between students.

- I want to interview 20 people in a town, so I put a map up on the wall and throw a dart at it 20 times. I interview whoever lives at the place the dart lands. Is this fair?
- I want to interview 10 people who live in Birmingham, so I dial 0121 and seven random digits. Is this fair?

Exercise 35A(F) / 37A(H)

While working through this exercise, good prompts for students might be:

- **Q2b(F) / Q2b(H)** How would your answer change if one fewer/more person said they were interested? *It would change to 680 or 720.*

Starters, plenaries, enrichment and assessment ideas

Starters

- It is useful to review calculating with fractions. For example, ‘quick fire’ questions like ‘What is $\frac{2}{5}$ of 30?’ could be completed on mini-whiteboards.

Plenaries

- Discussing real uses of sampling is a useful way to end a lesson. For example, how could customs/police officers decide who to stop? (A discussion to be handled with great care, but an important one in our present security climate.)
- Most students will be familiar with claims in advertising along the lines of ‘9 out of 10 owners prefer...’. In the small print, the sample size will be shown. Do they think that, say, 45 people is enough to justify this claim?

Enrichment activities

- The **100 people website** presents some thought provoking statistics. Students could collect some data from their class and use proportional methods to increase them to 100 people (100people.org).
- A similar mistake to ‘DEWEY DEFEATS TRUMAN’ with Landon and Roosevelt was made by many pollsters, but not by George Gallup whose name will be familiar to many. Students could investigate Gallup’s method.

SECTION 2F / 2H: TABLES AND GRAPHS

Section 2 focuses on different ways that data can be displayed to help with understanding and interpretation. Students can sometimes be complacent about this topic as they have seen bar charts repeatedly since primary school and in other subjects. Using the ‘assess your starting point...’ section will help establish a need to revisit the topic. Also asking students to criticise misleading or badly-drawn bar charts is a useful way to check for misconceptions and emphasise key learning points.

Prompting questions

These questions should promote discussion among students.

- Why is it important that the bars are the same width in a bar chart?
- Why is it important that the pictures in a pictogram are the same size?
- Why is it important to have a uniform scale?
- Why are graphs formatted so that they could be misleading?

Exercise 35C(F) / 37C(H)

While working through this exercise, good prompts for students might be:

- Q3(F) / Q3(H)** What other diagram could be used to show the proportion of sales for the four companies? *A series of pie charts.* What benefits do these bar charts have? *Comparing sizes of the four categories between companies is more straightforward.*
- Q5(H)** Dana claims that more boys than girls got A*, so boys did better. Hannah claims that more boys than girls got D, so the boys did worse. Who is correct? *This is not straightforward to answer, even with a stacked bar chart. The boys got a higher proportion of the very highest grades, but the girls got more Bs.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Produce ‘spot the mistake’ activities or similar, where students assess, criticise and identify errors. This promotes useful discussion of common mistakes and encourages students to reflect on their own presentation of bar charts and pictograms. An example of this could be a hand-drawn bar chart that has been drawn without a ruler, where the axes have not been drawn on the grid lines of the squared/graph paper, where the scale on the y-axis is not uniform, spaces between the bars are not equal, and so on. Another example could be a pictogram where the key is not interpreted correctly for a partial picture, where there is no key or where the pictures are of different sizes.

Plenaries

- From the same set of data, get students to create three or four different bar charts with all but one of the bar charts drawn to be misleading. Graphs can be made misleading by changing the scale on the y-axis, not starting the scale at zero or having unequal bar widths (introduction to histograms). If different parties want to use the same data to support their opposing views, consider which of these bar charts they would prefer to use.

Assessment ideas

- The **S5 section of the standards unit** has a card sort activity to match up bar charts and pie charts (and box plots). This could be adapted to include pictograms, frequency tables or lists of data. It is useful to have some cards that do not match or some incomplete cards (greatmathsteachingideas.com)
- Ask students to draw flow charts to show what types of diagram might be reasonably constructed for a particular data set. They should consider both the type of data (discrete, etc.) as well as the purpose of the diagram (comparison, computation of statistics, etc.).

SECTION 3F / 3H:PIE CHARTS

Section 3 focuses on interpreting and drawing pie charts. A pie chart shows proportions, and so is a great way to reinforce multiplicative methods such as the unitary method.

Prompting questions

These questions should promote discussion among students:

- Is it important to always start the first sector at 12 o'clock? Does the order of segments matter?
- Draw two nearly identical pie charts to represent two countries, A and B, showing that around 20% of adults smoke (and 80% do not). Which country (A or B) has more smokers? Replacing 'A' with, for example, China and 'B' with a country with a small population (such as Belgium) should reinforce the point that pie charts show proportion not numbers.

Exercise 35D(F) / 37D(H)

While working through this exercise, good prompts for students might be:

- Q1(F) / Q1(H)** Did the angles you calculated add up to 360 degrees? *No. Why not? Because of rounding.*
- Q2(F) / Q2(H)** Which country has the biggest population? *You can't tell from the pie chart because pie charts show proportions not actual numbers.*
- Q3(F) / Q3(H)** I interview another 30 students and every one of them says their favourite leisure activity is reading. How would this change the pie chart? *The reading sector would get bigger and the others smaller.*

Starters, plenaries, enrichment and assessment ideas

Plenaries

- Give students a pie chart (without numerical labels) and the size of population. Ask them to work backwards to find the frequency of each sector. This will raise issues of bounds and accuracy.

Starters or plenaries

- The charts in the link **three pie charts** illustrate how difficult it is to compare two sectors of a pie chart and how, generally, a bar chart is a better representation of data. Hiding the bar charts and getting students to order the sectors is an effective way to use this graph (later revealing the bar charts) (upload.wikimedia.org).

Enrichment activities

- As an extension activity, students could devise a method to draw comparative pie charts (for two data sets) where the area is proportional to the size of the population. Doubling the population does not double the radius.

SECTION 4H: CUMULATIVE FREQUENCY CURVES AND HISTOGRAMS

This Higher only section starts with a discussion of cumulative frequency diagrams. One possible way of introducing cumulative frequency is described below as a starter activity.

Histograms are a form of diagram that most students struggle to interpret and to draw. Working with unequal classes can sometimes confuse students. These are covered in Q5 in the Chapter review and also Homework 37E and Chapter 37 Review in the Higher Homework Book.

Prompting questions

Histograms

- Is it still a histogram if the bars are the same width? *Yes! Most histograms used in reality (but not in GCSE Maths exams) have classes with equal width.*
- How would we plot a histogram where the final class doesn't have a specified end point (e.g. ages, where the final age is stated as 80+)?

Exercise 37E(H)

While working through this exercise, good prompts for students might be:

- Q1(H)** How many classes did you have? What were your boundaries? How did you decide on the boundaries? *The text suggests 5–10 classes, ranging roughly from 150–350, therefore class widths of about 20 is appropriate for 10 classes (or wider if fewer classes).*
- Q2(H)** Why do we plot cumulative frequency points at the (right-hand) end of the class interval? *To include all possible temperatures in that class interval.*
- Q2(H)** What difference does it make if the points of a cumulative frequency diagram are joined up with a straight line compared to a curve (an 'ogive')? *Any predictions we make might be different. Joining with straight lines 'linearly interpolates' within each group.*
- Q3(H)** How can cumulative frequency graphs be used to estimate the median value and the quartiles? *There are 60 results so the median will be the 30th one. Draw a horizontal line from the cumulative frequency of 30 across to the graph. Then go vertically down to find the median time. For quartiles, do the same for the 15th and 45th results.*

Starters, plenaries, enrichment and assessment ideas

Starters

- As an introduction to cumulative frequency, online shopping websites can provide a nice starter. For example, B&Q's website (**diy.com**) has a filter to narrow searches by price: it shows how many items are in each price band (in the example below it is kitchen worktops).

By price
£0–£50 (31)
£50–£100 (16)
£100–£500 (17)
£500–£1,000 (42)

- Project the filter for one particular category, and ask students questions like 'How many worktops can I buy if I have £50?' (31) 'How about if I have £49?' (*We don't know, because those 31 might all cost £50.*) There are also issues with overlapping boundaries, for example if a worktop does cost £50, does it appear in the first or second class?

Plenaries

- Deciles are like quartiles but there are ten and not four. How could we use a cumulative frequency graph to estimate the seventh decile? What might a percentile be? What does the tallest bar of a histogram represent? Strangely, it **is**

the modal class. When could this differ from the modal class worked out from a frequency table? See **Modal Class of a Histogram with Unequal Class Widths** (mathforum.org).

Enrichment activities

- Provide students with a partially completed frequency table and histogram (for example, with only one row and one bar given) and ask them to complete both.
- Provide students with an unlabelled histogram (no scale on the y-axis) and its total frequency, and ask them to work out the frequency of one bar.

Assessment ideas

- Ask students to construct model answers for examination questions on histograms, showing their working clearly.

SECTION 4F / 5H: LINE GRAPHS FOR TIME-SERIES DATA

This section is introduced by an example of a line graph showing daily temperature. Line graphs are used frequently in science and geography, so this topic creates an opportunity for cross-curricular links. Students sometimes have difficulty recognising which chart or graph is required in the context of other subjects.

Prompting questions

Exercise 35E(F) / 37F(H)

While working through this exercise, good prompts for students might be:

- **Q7(F) / Q7(H)** Look at the graph. What would it look like if the y-axis started at zero? *The detail would be very small since it would be squashed in the very top of the graph.* **Q7(F)** Using 'index numbers' like this question does is one situation where it is permissible to have the y-axis not starting at zero.
- **Q7(F)** Use this same graph to predict what will happen to the number of buses and coaches over the next 10 years. *The general trend is one of decrease, but there have been points over the last 10 years when the number of buses and coaches was higher than in 2002. We cannot be sure, but a steady decrease would be the most likely scenario.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Ask colleagues for data from a science experiment that students in your school have recently completed. This data could be represented using the techniques being developed in maths, which will generally differ from those used in science subjects. Graphs that will also be used in a recent or subsequent topic in another subject can also be displayed and discussed (e.g. for geography, graphs of things such as population, weather, etc.).

Plenaries

- Consider several predictions made from a time-series graph, discuss the relative merit of each statement and state which predictions will be more likely to be correct.

Starters or plenaries

- Time-series graphs are used regularly in the media. Students could be asked to bring in an example of a time-series graph and make up their own questions based on it. Another student could then attempt to answer these questions and then make up some more of their own. Examples of time-series graphs found in the media regularly include petrol prices, stock prices, or exam results.

Enrichment activities

- The NRICH activity **What's the Weather Like?** provides data for three different UK towns over the last 50 years and suggests suitable research questions (nrich.maths.org).
- This data was taken from the **Met Office website**, so this investigation could easily be adapted to your local area (metoffice.gov.uk).
- **(Higher only)** Question 7 gives traffic volume as an index number, which is the percentage change in traffic volume from 2002. An extension would be to examine index numbers, including Retail Price Index (RPI).

- **(Higher only)** Another extension, and a thought-provoking exercise, is to examine a time-series graph of world population and link it to exponential growth.

Topic links

Previous learning

Ideas from *Chapter 23 Basic Probability and experiments* are important to take into consideration when considering methods of sampling and potential bias.

Future learning

Methods of analysing data, for example averages, are covered in *Chapter 36F/38H Analysing data*.

Gateway to A Level

Histograms at A Level introduce added complexities. Firstly, end points might be less obvious and students might need to consider bounds. Secondly, the area of bars become proportional to frequency rather than equal to it.

The idea of area below a curve representing (or being proportional to) frequency becomes increasingly important as students begin to consider distributions such as the normal distribution.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 1 Questions 5, 18
- Chapter 4 Question 13
- Chapter 7 Questions 8, 15, 16

Higher

- Chapter 1 Question 9
- Chapter 7 Question 14

Homework Book

Foundation

- Chapter 35

Higher

- Chapter 37

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 17 **(F)** / 19 **(H)** worksheets (+ solutions)
- 7 **(F)** / 9 **(H)** animated widgets
- 7 **(F)** / 14 **(H)** interactive walkthroughs
- 5 **(F)** / 8 **(H)** auto-marked quickfire quizzes
- 5 **(F)** / 8 **(H)** auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper
- 2 mm graph paper
- Axis grids from -10 to 10 in x and -10 to 10 in y
- Circle outlines (for plotting pie charts)

36F / 38H Analysing data

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to use their calculator correctly and, in particular, be able to divide using brackets for the dividend (numerator) or use the fraction key;
- know how to find the mean, median, mode and range of a set of data;
- be able to plot coordinates on a set of axes;
- know how to decide whether a gradient is positive or negative;
- **(Higher only)** understand how to construct cumulative frequency diagrams.

Additional useful prior knowledge

- To understand inequality notation for class intervals.

Learning outcomes

Foundation

Section 1

- To calculate summary statistics from raw and grouped data.
- To compare two or more sets of data.

Section 2

- To identify why a graph might be misleading.

Section 3

- To construct scatter diagrams.
- To describe correlation.
- To draw a line of best fit.
- To identify outliers.

Higher

Section 1

- To calculate summary statistics from raw and grouped data.
- To compare two or more sets of data.
- To estimate quartiles from a cumulative frequency diagram.

Section 2

- To identify why a graph might be misleading.

Section 3

- To construct scatter diagrams.
- To describe correlation.
- To draw a line of best fit.
- To identify outliers.

Vocabulary

bivariate data, correlation, dependent variable, outlier

Common misconceptions and other issues

- Many students do not realise that modal is the adjectival form of mode.
- When using a calculator to work out the mean, some students type, for example, $14 + 15 + 15 \div 3$ rather than $(14 + 15 + 15) \div 3$. Use these numbers as the money that three friends each have. The incorrect mean of £34 is clearly nonsense.
- When calculating the mean from a frequency table there are several common errors:
 - Dividing by the number of classes rather than the total frequency. Draw a frequency table showing the number of passengers (from 1 to 5) in 100 taxis. If dividing by 5 rather than 100, the average number of people in a taxi will be nonsense.

- Simply adding up the frequencies and dividing by the number of classes. For the frequency table showing the number of passengers in 100 taxis, if students add up the values in the frequency column they will arrive at the total of 100. Dividing 100 by 5 (the number of classes) gives an answer of 20, which is clearly nonsense if the maximum number of passengers is 5.
- Adding up the values in the class column and dividing by the number of classes. In our taxi example, $(1 + 2 + 3 + 4 + 5) \div 5$. The answer won't be so obviously wrong.
- For a grouped frequency table, the student adds up the midpoints instead.
- Students might forget to order values before finding the median.
- Some might forget what to do if there are two numbers when working out averages: if it is the mode, we give both; if it is the median, we find their midpoint. Justify that we can't find the midpoint of vanilla and strawberry ice cream.
- Some students think that the range is a type of average. The range is a measure of spread and puts the average into context, especially when comparing data.
- **(Higher only)** When working out quartiles, if there are 60 items some students might give '15' as the lower quartile ($60 \div 4$) rather than the value of the 15th item.

Hooks

1. Hans Rosling's '**200 Countries, 200 Years, 4 Minutes**' video shows wonderfully animated scatter diagrams that describe the wealth and health of nations through the past two centuries (gapminder.org). This chapter describes how graphical representations can be used to make sense of real-world sets of data and this video really shows the power of such visualisations. In our notes on Section 3 we describe how students can create such graphs for themselves.
2. Students always engage more readily if the data they are analysing means something to them. Generate some data for the class to use such the percentage scores from a recent homework or test (with names removed). Analyse if they are improving their scores or find out if there is any correlation between their score on a number-based topic compared to an algebra-based topic.
3. Challenge students to answer 60 times table or other basic numeracy questions faster than teachers. Present the teachers' data in a frequency table so as to compare it with the generated raw data for the students.

SECTION 1F / 1H: SUMMARY STATISTICS

Section 1 reviews the three averages students must learn for GCSE. Many students struggle to remember which is which; most classes will have a variety of good mnemonics that students will be keen to share. Students studying for the Higher tier will also need to learn about quartiles. (Teachers should be aware that methods to work out quartiles are not well defined, with at least seven different methods.)

Prompting questions

Exercise 36A(F) / 38A(H)

While working through this exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Why is it an estimate of the mean/median/range rather than the actual value? *Because the data is grouped, we don't know the actual values.*
- **Q1(F) / Q1(H)** The midpoint $\times$ frequency for the first row is 37.5. What does this number actually represent? *An estimate that those 15 people together had a total of 37.5 days absent from work.*
- **Q1(F) / Q1(H)** If the company had a few members of staff signed off sick for longer periods of time, they might record this in a class with interval $d > 25$. How would you deal with this class when calculating an estimate of the mean? *We would have to choose a sensible midpoint. To do this accurately you would need more information about the length of absences.*
- **Q2(F) / Q2(H)** Why did you choose the class intervals you did? How will the estimated mean be affected if you made the class intervals bigger/smaller/unequal? *The answer to this will depend on the student, but should lead to some discussion.*

- **Q2(F) / Q2(H)** How does your estimate of the range differ from the true range? *It is an overestimate.* Will the estimate always be an overestimate? Yes, *unless there is a piece of data at the lowest and highest endpoints of class intervals.*
- **Q3(F) / Q3(H)** If I gave you another table with the same data, but written in cm rather than m, what would happen to the mean/mode/range? *It would be multiplied by 100.*
- **Q3(F) / Q3(H)** Oh no! The health club's height measurer is found to always give readings 2 cm too short. We add 2 cm to every height to compensate. What would happen to the mean? *Increase by 2 cm.* What would happen to the range? *Stay the same.*

Exercise 38B(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** If there are 30 members of a bingo club, which person(s) is the median? *The 15th and 16th.* Why is it all right to just read off the 15th value from the graph? *Because the data is grouped, we are making an estimate. With a large number of people it would make little difference.*
- **Q3(H)** How many trains took longer than the upper quartile? 24. How many took less time than the median? 16.
- What advantage does the IQR have over the range? *It is unaffected by any outliers.*

Exercise 36B(F) / 38C(H)

While working through this exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** If Ahmed and Bill's data wasn't about runs in cricket but was the number of strokes in golf, how would your conclusions change? *In golf a lower number is better so the position would be reversed.*
- **Q3(F) / Q3(H)** Which would you rather take: a bus that takes on average 10 minutes with a range of 20 or a bus that takes on average 12 minutes with a range of 1? *This question should prompt discussion between students.*
- **Q7(H)** The mean of these salaries is £93 300 as compared to the median of £81 500. Why is the mean so much higher? *The median does not take direct account of the two people who earn very (very) high salaries: £256 000 and £345 000. The mean, however, does. Is the mode a useful measure for this data? No, not really because most values only occur once. (Grouping the data and finding the modal class might be useful.)*

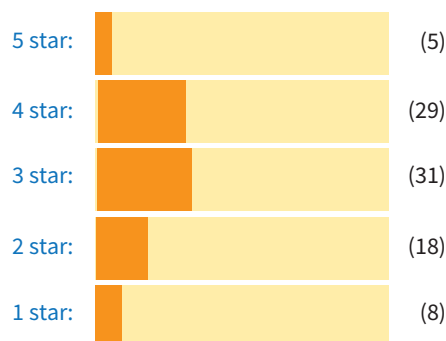
Starters, plenaries, enrichment and assessment ideas

Starters

- The Student Book's earnings example could be extended with some other occupations. Each of these could be written on a card and given to ten students. For the median, the students could arrange themselves in salary order. Students could also work out the mean. It should be clear to students that the 'workers' will want to use the mean in any argument to justify a pay rise, but that the median might be a more representative measure. Indeed, it can be helpful to think of the median as the 'wage of the average worker', rather than the 'average wage'.

Plenaries

- Amazon, and other shopping websites, allow customers to rate products on a discrete scale, typically 1–5. From this they calculate an average rating, usually rounded to the nearest half star. If you project the bar graph for a product students are interested in (music? film? game?), but hide the average rating, students could calculate the average. This also allows students to think carefully about rounding to an unfamiliar degree of accuracy: halves. When do they round up/round down?



- Show the class four different versions of a solution for the estimated mean from a grouped frequency table. Three of the solutions should be incorrect and each should highlight one of the common errors discussed earlier. Ask

students to spot which solution is correct and explain what mistakes have been made in the other examples. This could be done as a class discussion or students could work together to annotate a paper copy that could then be displayed in the classroom for future reference/as a reminder.

- A TV show ‘*2point4 Children*’ was named after a statistic that suggested that the average number of children per family was 2.4. Are any of your students’ families ‘average’? Is it possible to be ‘average’? You could calculate the average number of children in your class’s families and see how close it is to the current quoted average: 1.6 or so.

Enrichment activities

- Students could look at recent news stories that report on data to see whether the type of average is specified (it rarely is!). They could then go to the source of the data, to find for themselves how the analysis was actually done.
- Pick a future sporting event (it could be for a school team, football championship or an international event such as the Olympics). There is only one spot left on the team and the coach is unsure which player to choose. Give students the statistics for several players and tell them that, in a group, they need to use this data to present a case for one of the players. Each group has a different player to represent.

Assessment ideas

- The processes introduced in this section give lots of scope for calculation errors. Presenting students with ‘answers’ to questions that contain subtle mistakes is a good way of assessing their understanding of the processes and ideas in use.

SECTION 2F / 2H: MISLEADING GRAPHS

Section 2 explains different ways that graphs might be constructed which do not give a true representation of the underlying data. In some cases this might be through naiveté. In others, it might be an intentional distortion to enhance the claims of a product or service, or to justify a policy.

Prompting questions

Exercise 36C(F) / 38D(H)

While working through this exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** If you don’t have crime-rate data for 2001–2004, how could you legitimately draw the graph? Is it valid to join up the line? *It is probably not justified to join up the points as they imply a trend that may (or may not) be present. A graph with three data points over ten years isn’t very convincing!*

Starters, plenaries, enrichment and assessment ideas

Starters

The internet is a rich source of poor graphs, with many websites set up to dissect their failings. One with some good examples is **Statistics How To** (statisticshowto.com). Students are generally quick to spot how the graphs might mislead, which can then be covered more formally using the introduction and questions in the textbook.

Plenaries

- Question 1 shows what most people expect crime rates to do: go up. Showing the true data (e.g. search for ‘UK crime rates’ on the internet) can be surprising for many students (and teachers!).

Enrichment activities

- Students could seek out misleading graphs on the internet and present their favourite. The **Junk Charts** website is a good starting point (junkcharts.typepad.com).
- Give the class a data set, for example traffic accidents over the past few years. They should aim to create a graph that overemphasises the trend using the techniques they have learnt.

SECTION 3F / 3H: SCATTER DIAGRAMS

Section 3 introduces one of the most powerful statistical diagrams: the scatter diagram. At GCSE, lines of best fit are still done ‘by eye’. Finally, an informal idea of an outlier is discussed.

Note that more detailed discussion of interpolation and extrapolation is available in GCSE Mathematics Online.

Prompting questions

Exercise 36D(F) / 38E(H)

While working through this exercise, good prompts for students might be:

- **Q2(F) / Q2(H)** I am looking to buy a car with a 1.5 litre engine. How many miles per gallon would you expect? 58.5... from graph.
- **Q2(F) / Q2(H)** I am looking to buy a Hummer H1 with a 6.2 litre engine. Can you use the graph to predict its mpg? *Not accurately since there are no data bigger than 3 litres; we would be extrapolating.*
- **Q3(F) / Q3(H)** Which is the independent variable? *The temperature.* Which axis should this go on? *The horizontal (x) axis.*

Exercise 36E(F) / 38F(H)

While working through this exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** Which of the test scores did you decide is an outlier? *The 85.* What explanations are there for this result? *It might be real! It could be recorded wrongly (e.g. it was actually 58 but the digits were transposed). Lots more possibilities!*
- **Q3(F) / Q3(H)** National Rail considers trains to be 'late' if they arrive more than ten minutes after the published arrival time. If the normal time for the journey is 115 minutes, how many of these trains would be considered as late? 5. What is this as a percentage of the trains? 25%. Nationally, 91% of trains are 'on time'. Should the company running the train in the question be concerned? *Yes!*

Starters, plenaries, enrichment and assessment ideas

Starters and enrichment activities

- Students can use the **Gapminder world** website to produce similar graphs to Hans Rosling's '**200 Countries, 200 Years, 4 Minutes**' video described earlier. They are able to do this for a whole host of other variables. Higher attaining students can also experiment with logarithmic scales (gapminder.org).

Plenaries

- **Anscombe's quartet** consists of four sets of data that look very different when plotted on a scatter diagram, yet have identical statistical properties (such as line of best fit, correlation coefficient, mean, etc.) Groups of students could calculate these for just one of the four data sets. This quartet demonstrates the usefulness of graphical techniques even with statistical packages that can calculate summary statistics from raw data (en.wikipedia.org).
- The **spurious correlations** website is a great way to demonstrate the adage *correlation does not imply causation*. This shows a variety of sets of data that show high correlation but with no obvious causal link (e.g. 'marriage rate in Washington DC' against 'number of video games sold in the UK' is negatively correlated with $r = -0.995$). However, the data sets do not have scatter diagrams so students might draw these and create spurious predictions (tylervigen.com). Marcus du Sautoy explains another few examples of correlation vs causation at the end of this '**Painting with Numbers**' video, including the story of a US governor who believed that having books in the house would make a child smarter (tes.co.uk).
- Get students to draw a line of best fit for a well-plotted scatter diagram (e.g. printed for students). They should then all make a prediction from their graph. What is the range of values within the class?
- Show some very wrong looking lines of best fit, including one drawn through the origin when there is a negative correlation. Ask students to explain what is going wrong.

Assessment ideas

- If students can make sense of the **XKCD correlation** cartoon, they probably understand the difference between correlation and causation (xkcd.com)!

Enrichment activities

- Make links with the science department. Is there an experiment that students have done recently that would utilise a scatter diagram with a linear line of best fit? If students have drawn scatter diagrams in science, they could be scanned and then peer/self-assessed in their maths lessons.

Topic links

Previous learning

After the introduction of grouped frequency tables, this is a good opportunity to construct some cumulative frequency diagrams. Many students fall into the habit of plotting the cumulative frequency diagram at the midpoint of each class rather than the end point (because they use midpoints when estimating the mean). These graphs were introduced in *Chapter 35F/37H Collecting and displaying data*.

Future learning

Although it is rarely assessed, students could find the equation of the line of best fit and use this equation to predict data values, especially for extrapolated data.

Gateway to A Level

Students will learn how to calculate standard deviation, which is a more sophisticated measure of spread that accompanies the mean (like range and IQR accompany the median).

They will learn how to use linear regression, which is the process of creating an equation to represent a line of best fit that can be quickly used to make predictions.

Students will also learn how to calculate the product-moment correlation coefficient that gives a numerical value for the strength of a correlation.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 1 Question 6
- Chapter 2 Question 6
- Chapter 6 Question 18
- Chapter 7 Question 17
- Chapter 8 Question 8
- Chapter 10 Question 12

Higher

- Chapter 1 Questions 10, 25
- Chapter 2 Question 11
- Chapter 5 Question 14
- Chapter 6 Questions 6, 7, 20
- Chapter 7 Questions 6, 18
- Chapter 9 Question 8
- Chapter 10 Questions 7, 8, 18

Homework Book

Foundation

- Chapter 36

Higher

- Chapter 38

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 6 (F) / 10 (H) worksheets (+ solutions)
- 2 (F) / 4 (H) animated widgets
- 6 (F) / 11 (H) interactive walkthroughs
- 3 (F) / 5 (H) auto-marked quickfire quizzes
- 3 (F) / 5 (H) auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper
- 2 mm graph paper
- Axis grids from -10 to 10 in x and -10 to 10 in y

37F / 39H Interpreting graphs

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to distinguish between direct and inverse proportion;
- be able to calculate the gradient of a straight line;
- understand how to calculate the area of composite shapes.

Additional useful prior knowledge

- $\text{Speed} = \frac{\text{distance}}{\text{time}}$.
- $\text{Acceleration} = \frac{\text{speed}}{\text{time}}$.

Learning outcomes

Foundation

Section 1

- To construct and interpret graphs in real-world contexts.

Section 2

- To interpret the gradient of a straight-line graph as a rate of change.

Higher

Section 1

- To construct and interpret graphs in real-world contexts.

Section 2

- To interpret the gradient of a straight-line graph as a rate of change.
- To find and interpret the gradient at a point on a curve as the instantaneous rate of change.

Section 3

- To calculate and interpret the area under a graph.

Common misconceptions and other issues

- Some students transpose the numerator and denominator when calculating gradient. Most students will be happy with the idea that a horizontal line has zero gradient: this is only possible when the (zero) change of y is the numerator (otherwise it would lead to division by zero).
- Some students are unable to read from graphs correctly, especially from those with unequal scales. Ensuring that students are exposed to such graphs is important: they should make it their first step, when presented with a graph, to ensure they properly consider the detail of the axes.
- **(Higher only)** Some might have difficulty plotting tangents to curves. Imagining a surf board on a wave can often help.
- Students might have problems dealing with scales graduated in time (hours and minutes) and working with such quantities. Starters that get students to work with time in the form hh:mm can help: 'What is 0:55 + 0:12?', 'What is 11:23 – 2:51?'

Hooks

- The videos on **Graphing Stories** are a nice visual prompt to get students to consider how values (heights, speeds, distances, counts, etc.) change over time (graphingstories.com).

SECTION 1F / 1H: GRAPHS OF REAL-WORLD CONTEXTS

Section 1 focuses on interpreting and constructing graphs with a real-world context, notably distance–time and conversion graphs.

Prompting questions

Exercise 37A(F) / 39A(H)

While working through this exercise, good prompts for students might be:

- **Q1(F) / Q1(H)** When was the car travelling fastest? *Between 100 and 120 minutes. How do you know? The graph is steepest in that section.*
- **Q1(F) / Q1(H)** What does the negative gradient between 100 and 120 minutes signify? *The taxi is travelling towards the starting point.*
- **Q2(F) / Q2(H)** How do you think Monica was travelling? Walking, running, cycling or by car? *People typically walk at about 4 km per hour (i.e. 1 km takes 15 minutes). She travelled 1.2 km in 10 minutes, so she is going faster than walking. But probably not fast enough to be in a car, so she's running or cycling or walking briskly.*
- **Q2(F) / Q2(H)** The first section of the graph is a straight line. What does this signify? *Monica is travelling at a constant speed. Is this realistic? Probably not!*
- **Q2(F) / Q2(H)** Is there an alternative explanation for the horizontal line, rather than Monica remaining in the same place? *She is getting no further away from home, so she could be walking in a circle with her home as the centre.*
- **Q4(F) / Q4(H)** You get a message from central government that further help cannot reach the disaster centre for 14 days. How many people could you support? *Approximately 343 (days = $4800 \div \text{people}$).*

Starters, plenaries, enrichment and assessment ideas

Starters

- Draw a distance–time graph on the board with two different coloured lines (e.g. red and blue) to represent two different car journeys. Make the lines different (starting at different points on the y-axis, always increasing but with varying gradients). Get students to write and then read out descriptions of the journeys of two cars.

Starters or plenaries

- Students could be challenged to draw graphs representing the videos on **Graphing Stories** (graphingstories.com).
- NRICH activity **Mathsjam Jars** This activity asks students to work out which of ten differently shaped jars will fill up at the same time (given a constant rate of jam pouring) (nrich.maths.org).

Plenaries

- Given freezing and boiling point in Celsius (0 and 100) and Fahrenheit (32 and 212), students could draw a conversion graph that they can use to convert weather forecasts. If allowed to extend to negative temperatures, students could be asked 'at what temperature is the number in Celsius and Fahrenheit the same?' (*Negative 40 degrees.*)
- It is useful for students to have a 'feel' for the speeds of various forms of transport. This could be phrased in the form 'which of these is a sensible answer for a question that asked you to calculate the speed of a jumbo jet: 1000 mph, 500 mph, 200 mph, 100 mph, 10 mph?' Working in various units (e.g. mph, km/h, m/s) should be encouraged.

Enrichment activities

- Students could make their own video in the style of those on **Graphing Stories**. Videos could involve multiple students for example one crawling, one running and one hopping (perhaps in different directions) (graphingstories.com).
- If any members of staff have recently run a marathon, their split times (usually downloadable after the event) could be used as the source of a distance–time graph. When did they 'hit the wall'?

Assessment ideas

- The Standard's Unit **Interpreting Distance–Time Graphs** activity (A6) can be used to assess students' ability to move between multiple representations of situations that might be represented by a distance–time graph (nationalstemcentre.org.uk).

SECTION 2F/2H: GRADIENTS

Section 2 focuses on interpreting the gradient within the context of the problem: what do different magnitudes mean? What does a negative gradient suggest? Higher students are then introduced to the idea of using a tangent to the curve to approximate the gradient at a point.

Prompting questions

Exercise 37B(F) / 39B(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** Which tanks show the graphs being filled? *a and c.*
- **Q1(H)** Which tanks show the graphs being emptied? *d, e and f.*
- **Q1(H)** Which graph shows the tank being emptied, initially quite quickly then steadily slowing to a trickle? *e.*
- **Q1(H)** If you are dividing a change in height by a change in time, what unit could be used for the gradient of these graphs? *m per second, cm per minute, furlongs per fortnight, or some other length/time unit.*
- **Q3(H)** When does the parachutist reach ‘terminal velocity’? *Point 3.*
- **Q3(H)** When does the parachutist open the parachute? *Point 5.*
- **Q3(H)** Imagine at point 6 their parachute catastrophically failed. It takes them 3 seconds to open their reserve ‘chute. What would the new graph look like? *Phase 1 onwards would happen again.*

Exercise 39C(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** What are the units of gradient on this graph? *Metres per second, confirming that the gradient represents speed.*
- **Q1(H)** Was it easier to accurately draw a tangent at 0.5 seconds or 3.5 seconds? *Probably 3.5 seconds since it is not at the edge of the graph.*
- **Q2(H)** At what time would you expect the next low tide? *About 5 am since they occur roughly every 12.5 hours.*
- **Q3(H)** What are the units of gradient? *km/h². What does the gradient represent? Acceleration.*
- **Q3(H)** What does a negative gradient represent? *The walker slowing down, in other words ‘deceleration’ or negative acceleration.*
- **Q4(H)** If you continue the graph to the right (that is, *x* increases) how do you expect the gradient will change? *It will increase, since the graph gets steeper.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Ask students to plot a distance–time or speed–time graph of their day so far. They should have periods of time when they are not moving (i.e. during lessons), times when they are walking, and possibly times when they are moving faster (e.g. on a bus, cycling to school, running at a sports club). A couple of students’ graphs could then be presented to the class who could hypothesise about what the students were doing at different times.

Plenaries

- With mini-whiteboards, get students to draw blank distance–time axes. Ask questions such as ‘Draw a graph that shows: (a) speeding up, (b) moving at a constant speed, (c) slowing down’ and so on.

Enrichment activities

- **(Higher only)** Students could calculate the gradient at various points on the graph of x^2 and notice that it is always double the *x*-coordinate. This could lead into a discussion about differentiation.

SECTION 3H: AREAS UNDER GRAPHS

(Higher only) Section 3 focuses on calculating and interpreting the area under graphs.

Prompting questions

Exercise 39D(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** What shape would you use to work out the area under graphs a and b? *A (right-angled) triangle.*
- **Q1(H)** On graph c, how would you work out the area under the graph between 1.5 and 3 hours? *Either a triangle with a rectangle, or a trapezium. (Links to the trapezium rule.)*
- **Q1(H)** If we drew a single triangle from 0 to 4 hours to approximate the curve in part d with a straight line, would it be an overestimate or an underestimate of the true area? *An overestimate.* How could we improve our estimate? *Use additional triangles (trapezia).*
- **Q1(H)** We are finding the area of shapes whose heights are measured in km/h and widths in hours. If we multiply these two units together, what do we get? *km, confirming that the area represents distance.*
- **Q2(H)** For the period between 30 and 45 minutes, is there an efficient way to work out the area? *Since it is a straight line, treat it as a single trapezium.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Get students to calculate the areas of triangles, rectangles and trapezia. Extend this to compound shapes built from these shapes.

Plenaries

- This section uses speed–time graphs, since the area under a distance–time graph ('absement') is rather more difficult to interpret. Students could attempt to draw distance–time graphs from the given speed–time graphs.

Enrichment activities

- NRICH activity **Speedo** This problem has a defined distance the car has travelled and students are given information on the speed of the car in different scenarios. This problem could also be solved by using the constant acceleration (suvat) equations (taught at A Level) so provides an opportunity to introduce/discuss them (nrich.maths.org).
- Derive the constant acceleration equations from a speed–time graph. Students now know that the area under a speed–time graph is the distance travelled. By considering a straight-line segment of the graph, the formula $s = \frac{1}{2}(u + v)t$ can be found (where s is the distance travelled and u and v are the initial and final velocities respectively). Secondly, the gradient of a speed–time graph gives the acceleration, resulting in the formula $v = u + at$. From these two equations, the remaining three equations can be derived by eliminating a variable in three different ways (e.g. eliminating t from the two equations gives $v^2 = u^2 + 2as$).

Topic links

Previous learning

Features of straight lines, such as the concept of gradient, were discussed in *Chapter 18 Straight-line graphs*.

Proportional methods from *Chapter 34F/36H Direct and inverse proportion* are used, for example 'How many Indian rupees would you get for US\$600?' in the worked example in Section 1.

Methods from *Chapter 12 Units and measurement* are important, including working with time, for example calculating speed in miles per hour when a distance of 15 miles is covered in 20 minutes.

Breaking down compound shapes into rectangles, triangles and trapeziums to find their areas was covered in *Chapter 16 Area*.

Gateway to A Level

A Level students will learn how to use differentiation to find the gradient at a point on a curve.

They will also use integration to find the area under curves.

Students of Mechanics will make extensive use of the constant acceleration (suvat) equations.

The method of least squares or regression is used by students of Statistics to fit a model (often linear) to a set of data.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 1 Questions 19, 20, 24
- Chapter 3 Question 9
- Chapter 4 Questions 10, 14
- Chapter 6 Questions 19, 20
- Chapter 8 Question 18

Higher

- Chapter 1 Questions 11, 26
- Chapter 3 Question 16
- Chapter 4 Question 9
- Chapter 6 Questions 8, 9
- Chapter 7 Questions 15, 16, 17
- Chapter 8 Question 10

Homework Book

Foundation

- Chapter 37

Higher

- Chapter 39

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 5 (F) / 6 (H) worksheets (+ solutions)
- 3 animated widgets
- 9 (F) / 10 (H) interactive walkthroughs
- 2 (F) / 3 (H) auto-marked quickfire quizzes
- 2 (F) / 3 (H) auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper
- 2 mm graph paper
- Axis grids from -10 to 10 in x and -10 to 10 in y

38F / 40H Algebraic inequalities

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to solve linear equation;
- **(Foundation only)** be able to rank numbers in ascending or descending order;
- **(Foundation only)** understand to use the rules for operations with negative numbers;
- **(Higher only)** be able to solve quadratic equation;
- **(Higher only)** know how to use linear graphs.

Additional useful prior knowledge

- **(Higher only)** To be able to factorise quadratic equations allowing quick sketches to be drawn.

Learning outcomes

Foundation

Section 1

- To understand and interpret inequalities and using the correct symbols to express inequalities.

Section 2

- To use a number line to represent an inequality.

Section 3

- To solve linear inequalities in one variable and represent the solution set on a number line.

Section 4

- To solve problems involving inequalities.

Higher

Section 1

- To use the correct symbols to express inequalities.

Section 2

- To use a number line and set notation to represent an inequality.

Section 3

- To solve linear inequalities in one variable and represent the solution set on a number line and in set notation.

Section 4

- To solve quadratic inequalities.

Section 5

- To solve (several) linear inequalities in two variables and represent the solution set on a graph.

Vocabulary

inequality, number line

(Higher only) set, equalities

Common misconceptions and other issues

- When being asked to list integers that satisfy an inequality, students frequently leave out zero. They should be told that integers are the whole numbers and their negatives, including zero. A large number line on the wall might help to remind students of the existence of zero.
- Students might find it difficult to appreciate that $-3 < x$ is the same as $x > -3$, but reminding them that the x is on the 'big' side of the inequality sign should help.

- Many students struggle to remember whether they should ‘fill in the dot’ at the limits. It might help them to remember that the more ink there is in the inequality sign, the more ink there is in the dot (i.e. $\leq$ has more ink than $<$).
- (Higher only)** Students often find it difficult to work out which signs to use for the solution of a quadratic inequality. Remind students to keep the original inequality in their calculations until the point at which they have a factorised inequality of the form $(x + a)(x + b) > 0$, for example. At this point, it might be helpful for students to test a value of x between a and b to determine which inequality sign is appropriate for each of the factors. Considering the graph might also be helpful, as it visually demonstrates where the function changes from positive to negative (i.e. when crossing the x -axis).
- (Higher only)** As with ‘dots’ on number lines, when graphing linear inequalities many students easily forget whether a line should be broken or solid. Again, the idea of ‘more ink in the sign’ matching up to the line helps here: $\geq$ has more ink, so needs to be a solid line (versus $>$ and a broken line).

Hooks

Get students to draw a first quadrant grid, with both axes going from 0 to 5. Give students the following instructions, allowing time for them to mark off possible locations on their grid.

- My treasure is buried at a place with whole number coordinates.
- The x - and y -coordinates add up to a prime number. How many possible places might be treasure be buried? *17 points on the lines $x + y = 2$, $x + y = 3$, $x + y = 5$, $x + y = 7$.*
- The x -coordinate is between 2 and 4. *Now 9 possible points.*
- The y -coordinate is greater than 3. In how many places might my treasure be buried? *Two – at (2, 5) and (3, 4).*
- Give me a rule that would limit the locations to just one. *For example, the y -coordinate is 3 more than the x -coordinate (or $y = 2x + 1$, $y > 5$, $x < 3$, etc.).*

SECTION 1F / 1H: EXPRESSING INEQUALITIES

Section 1 focuses on expressing inequalities using mathematical symbols.

Prompting questions

Exercise 38A(F) / 40A(H)

While introducing the concept of inequalities, good prompts for promoting discussion might be:

- How are the inequalities $x > 6$ and $x \geq 6$ different? *$x \geq 6$ includes the number 6, whereas $x > 6$ does not.*
- What numbers satisfy the inequality $x > -3$? *-2, -1, 0, 1, 2, 3, ... Do these also satisfy $-x < 3$? Yes! It is the same inequality.*
- What happens to an inequality when you multiply or divide by a negative number? *The inequality sign changes. This can be shown by adding/subtracting the terms to swap them to the other side of the inequality (i.e. moving the terms, rather than changing the symbol). For example, for $-x < 3$, add x to both sides to get $0 < x + 3$, then subtract 3 from both sides to get $-3 < x$, which is the same as $x > -3$.*

While working through this exercise, good prompts for students might be:

- Q1(F) / Q1(H)** What has happened to the expression on the left side? *For example, a) add 4. Has the same happened to the right side? For example, a) yes. Then what happens to the inequality sign? For example, a) it stays the same.*
- Q3(F) / Q3(H)** What happens when you multiply or divide by a negative number? *The inequality sign changes.*
- Q4(F) / Q4(H)** What does the inequality symbol $\geq$ mean? *Greater than or equal to.*
- Q6(F) / Q6(H)** If you can only use integers or halves, how many values can x take? *9.*
- Q7(F) / Q7(H)** If $6 > x$, what is the biggest whole number x could be? *5. If $x > 2$, what is the smallest whole number x could be? 3.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Since solving linear inequalities relies heavily on the skills used in solving linear equations, students could start this topic with some practice. To match up to the questions in the section, it would be useful for students to identify the steps to 'build' an equation. For example, they could be given a 'starting equation' like $x = 5$, and need to identify the steps necessary to turn the equation into $1 = 11 - 2x$.

Starters or plenaries

- Distribute numbers to each student (or ask them to write a number between, say, -20 and 20). Call out/display inequalities and ask students to stand up if they have a number that satisfies the inequality given.

Plenaries

- Write an inequality on the board and ask students to come up with equivalent inequalities. For example, with $x > 4$ on the board, students could write $x + 1 > 5$, $x - 6 > -2$, $-2x < -8$.
- Give students a 'starting inequality' (e.g. $x > 5$) and a 'finishing inequality' (e.g. $9 < 2x - 1$). Students should work out what has happened to get from one to the other.

Enrichment activities

- The NRICH activity **Inequalities** uses inequalities in the formulation of a problem involving marbles that can be solved by logical thinking (nrich.maths.org).

SECTION 2F: NUMBER LINES / SECTION 2H: NUMBER LINES AND SET NOTATION

Section 2 focuses on using number lines to illustrate inequalities. It is worth explaining to students that we only draw the relevant section of the number line and the arrows at the end of the line show the line continues, so we aren't forgetting the rest of the numbers.

If an inequality is closed, that is to say there is an upper and lower limit to the range, a dot should be drawn on each end. Students might find it helpful to think about it as having arrows pointing towards each other (originating from the dots) but the second dot acts as a barrier to the arrow progressing.

The Higher version of this section also introduces set notation. This should not be a challenging idea for students, but might be unfamiliar and so will need a careful and precise introduction.

Prompting questions

- What could be a good scale for showing the inequality $x > \frac{1}{2}$? 0, 1, 2, 3, ... with $\frac{1}{2}$ marked on, or 0, $\frac{1}{2}$ marked on, or 0, $\frac{1}{2}$, $1\frac{1}{2}$, 2, A scale of 0, 5, 10, 15, ... would be inappropriate because it is difficult to accurately mark $\frac{1}{2}$ on it.

Exercise 38B(F) / 40B(H)

While working through this exercise, good prompts for students might be:

- Q1(F)** Where are you going to draw the dot? Is it open or solid? Which way does the arrow point? *Answers will depend on which question part is being answered.*
- Q3(F)** What is the lower limit of the inequality? What is the upper limit of the inequality? Are the dots open or solid? *Answers will depend on which question part is being answered.*
- Q3a(H)** Are there different ways we could write this inequality? *Yes, for example $\{x: 7 < x \leq 11\}$ or $\{x: 11 \geq x > 7\}$. Would $\{x: 14 < 2x \leq 22\}$ also be valid? Yes, but it would usually be best to write an inequality in its simplest form.*
- Q3d(H)** Which integers satisfy the inequality? $\{-1, 0, 1, 2\}$

Starters, plenaries, enrichment and assessment ideas

Starters

The **Inequalities and paper hats** activity on ATM gets each member of the class to make a paper hat and then write a distinct integer on it. The class line up in numerical order and then stand up or sit down based on a particular

inequality. This is a nice visual way to illustrate an inequality and feeds nicely into using number lines for this purpose, especially for ‘double’ inequalities such as $7 < x \leq 11$ (atm.org).

Plenaries

- The use of a number line for inequalities is a good opportunity to reinforce that real numbers all have a position on the number line. Display a number line, say, for $-4 < x < 3$. Then ask students whether these numbers satisfy the inequality $x = \sqrt{9}$, $x = \sqrt{8}$, $x = (-2)^2$, $x = -\pi$, $x = -\frac{8}{2}$, and so on.

Assessment ideas

- Call out inequalities and ask students to draw a number line (or, for Higher, use set notation) to represent them.

SECTION 3F: SOLVING INEQUALITIES / 3H: SOLVING LINEAR INEQUALITIES

Section 3 focuses on solving linear inequalities. Students are encouraged to use the same methods that are used for solving linear equations, but it is vital to reiterate the rules from Section 1 involving multiplying/dividing by negative numbers and swapping the sides of an inequality.

In Worked example 1, students must apply the same operation to all three sections of the inequality. Some students might find it difficult to solve the problem in this way, so they could alternatively consider the question as two separate inequalities and solve each of them separately before considering how they can re-form to make the final combined inequality solution (i.e. solving $5 \leq 3x + 4$ and $3x + 4 \leq 13$ should lead to $-3 \leq x$ and $x \leq 3$, which combine to make $-3 \leq x \leq 3$).

The Higher version of the section also includes solving word problems involving linear inequalities.

Prompting questions

Exercise 38C(F) / 40C(H)

While working through this exercise, good prompts for students might be:

- Q1m(F)** How did you remove the fractions on both sides? *Multiply through by 2.* What would you do if one fraction had 2 as a denominator and the other had 3? *Multiply through by 6.*
- Q2b(F)** Show that two ways of solving this inequality give the same result. *Get students to demonstrate dividing by negative 10 (and reversing the sign) versus adding 10x and subtracting 130.*
- Q3a(F)** Expand the brackets but write +2 rather than -2 when expanding the second bracket and ask ‘What have I done wrong here?’
- Q8a(H)** A student scored 20 marks on the second test. Is it possible for them to have passed? *No, since they would need 110 marks on test 1, which is impossible.*

Starters, plenaries, enrichment and assessment ideas

Starters

- A starter refreshing students’ memories of solving linear equations might be a useful way of leading into this section. This could be something as straightforward as a set of 5 linear equations, including questions with negative coefficients of x , brackets, x terms on both sides of the equals sign and fractions. Negative coefficients (e.g. $5 - 2x = 7$) can provide a way into a productive discussion about methods of ‘dealing’ with the negative coefficient. (Some students will move the $-2x$ to the RHS, others will divide through by negative 2.)

Plenaries

- Ask students to write five inequalities that all simplify to give $x > 3$. At least one should have brackets and one should have a negative coefficient of x .
- Ask ‘Does the inequality $3 < x < -3$ make sense?’ *No, there is no number that satisfies it: this is probably a poor combination of $x < -3$ and $x > 3$.*

Enrichment activities

- Ask students to consider what values of x would satisfy a double inequality such as $6 < 5x + 3 < 15$. Students can approach this by using the same operation on each of the three sections simultaneously, for example subtracting 3 leads to $3 < 5x < 12$, then dividing each section by 5 gives $3/5 < x < 12/5$. Alternatively students can break the problem down into its two constituent inequalities: $6 < 5x + 3$ and $5x + 3 < 15$, which can be solved as previously outlined.

SECTION 4F: WORKING WITH INEQUALITIES

At Foundation level, Section 4 focuses on working with inequalities and word problems. Students might struggle to identify the inequality necessary to represent 'at least'. By rephrasing this as 'the same or more than', students should be able to recognise it is asking for values higher than the given number.

SECTION 4H: SOLVING QUADRATIC INEQUALITIES

The Higher version of Section 4 focuses on quadratic inequalities. Students might find it helpful to sketch the graph for each quadratic, noting where it cuts the x -axis and when the line is above/below the x -axis (i.e. where the answer to the quadratic is positive/negative as required for the question).

Prompting questions

Exercise 38D(F)

While working through this exercise, good prompts for students might be:

- Q2(F)** What numbers are prime? 2, 3, 5, 7 Which of these numbers are even? 2. (*This should go in the overlap of the 'prime' and 'even' circles. Which of the even numbers are square numbers? 4 (so this should go in the overlap of the 'even' and 'square' circles).*)
- Q3(F)** *These questions require students to think carefully about how to write the inequalities before solving them. They might need help to interpret the questions. For example d) What happens first to a ? And then what happens? 3 is added, then it is multiplied by 2, so we write $a + 3$, then 2 lots of this is $2(a + 3)$.*
- Q4(F)** If you save 85% of what you earn each hour, what percentage of what you earn can you use to buy the video game? 15%. What is 15% of £16? £2.40. If this is how much you can save per hour and you call the number of hours h , how can you write the total amount saved? $2.4h$.

Exercise 40D(H)

While working through this exercise, good prompts for students might be:

- Q2b(H)** Can we join the two inequalities $x \leq 2$ and $x \geq 6$ into a 'double inequality', for example $2 \geq x \geq 6$? *No, because there is no number for x that satisfies both inequalities.*
- Q3d(H)** What are the two x -axis intercepts for $x(x + 3) = y$? *It can be written as $(x + 0)(x + 3) = y$. The x -axis intercepts are at $x = 0$ and at $x = -3$.*
- Q4d(H)** How can you make the coefficient of x^2 positive? *Multiply through by negative 1, but ensure the inequality sign is reversed giving $x^2 + 3x - 40 < 0$.*
- Q5(H)** What are the boundaries for each part of the inequality (i.e. where are the dots?) How can you write an inequality for each section of the line? *For example a) $x > 3$, $x < -2$.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Go through different ways of writing inequalities in words. For example, 'at most', 'no more than', 'at least', 'more than', 'fewer/less than'.
- Write some numbers on the board (lower than 100). Ask students to write down the numbers that are (a) square numbers, (b) cube numbers, (c) prime, (d) composite, and so on.

Plenaries

- Have some worded examples for which students must decide whether solutions should be limited to integer values. For example, ‘there must be more than 5 people for the training course to run’, ‘I hope to raise at least £5000 for charity’ or ‘I want to lose at least 1 kg on my diet this week’.

SECTION 5H: GRAPHING LINEAR INEQUALITIES

(Higher only) Section 5 focuses on graphing linear inequalities in two variables. This is the basis for linear programming, in which a number of constraints can be graphed and the optimal solution discovered.

Prompting questions

Exercise 40E(H)

While working through this exercise, good prompts for students might be:

- Q1(H)** Is the line in part c ‘steeper’ or ‘shallower’ than that in part a? *Shallower since $\frac{1}{2}$ is a smaller gradient than 1.*
- Q3(H)** Which point did you test? *Students might be encouraged to choose points that are straightforward to substitute, for example the origin (0, 0) or (1, 1). However, they should avoid any points on the boundaries of shaded regions.*

Exercise 40F(H)

While working through this exercise, good prompts for students might be:

- Q3(H)** Is (8, 0) in the shaded region? *No, since the blue line is broken it doesn’t include the point.*
- Q5(H)** Which points with integer coordinates are in the shaded region? (3, -3), (4, -2), (5, -1), (6, 0), (5, -2), (4, -4), (3, -4), (4, -3).

Starters, plenaries, enrichment and assessment ideas

Starters

- A review of how to efficiently sketch linear equations would be a good way into this topic. In particular, ensuring that students don’t confuse $x = 3$ (vertical line) and $y = 3$ (horizontal line) is essential.

Plenaries

- Given three inequalities, students could compare the two approaches of (a) shading ‘in’ the region satisfied by each inequality and (b) shading ‘out’ the region that does not satisfy the inequality. They can compare the relative merits of each method.

Enrichment activities

- Students could investigate simple linear programming problems that use graphical representations of inequalities.

Topic links

Previous learning

This topic relies heavily on skills learnt in previous topics. Most notably, students should be secure in their ability to solve and draw linear equations. These methods were covered in *Chapter 8 Equations* and *Chapter 18 Straight-line graphs*.

Gateway to A Level

Linear programming extends the work on graphing linear inequalities to find optimal solutions given a number of constraints.

Inequalities feature in many A Level questions and quadratic inequalities have much greater prominence.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

Foundation

- Chapter 2 Question 16

Higher

- Chapter 6 Question 21
- Chapter 10 Question 19

Homework Book

Foundation

- Chapter 38

Higher

- Chapter 40

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 4 (F) / 8 (H) worksheets (+ solutions)
- 3 (F) / 5 (H) animated widgets
- 6 (F) / 11 (H) interactive walkthroughs
- 2 (F) / 4 (H) auto-marked quickfire quizzes
- 2 (F) / 4 (H) auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper
- 2 mm graph paper
- Axis grids from -10 to 10 in x and -10 to 10 in y

41H Transformations of curves and their equations

CHAPTER INTRODUCTION

What your students need to know

Students should be confident with the items in the chapter's 'Before you start...' section. Specifically they should:

- know how to recognise the graphs of linear, quadratic and reciprocal functions;
- know how to sketch the trigonometric functions: $y = \sin x$, $y = \cos x$ and $y = \tan x$ and recognise the features of each including the period and amplitude;
- know how to manipulate quadratic expressions to factorised form and completed square form.

Additional useful prior knowledge

- To know the language used for features of quadratic functions including roots and turning point/vertex.
- To have knowledge of graphing software such as Geogebra or Desmos (freely available) to plot functions.
- To know the index laws including $a^0 = 1$.

Learning outcomes

Section 1

- To know the features of a quadratic function: axis of symmetry, roots and vertex, and identify these features from the sketch of a quadratic.
- To sketch vertical translations of quadratic functions.
- To sketch horizontal translations of quadratic functions.
- To sketch quadratic functions that have been translated in both the horizontal and vertical directions.
- To know the effect that translations have on the axis of symmetry and vertex of a quadratic.
- To use graph sketching to identify the effect of multiplying $f(x)$ by -1.
- To use algebraic manipulation skills to identify the features above and sketch any quadratic of the form $y = ax^2 + bx + c$.

Section 2

- To identify reflections and translations in the graphical representations of trigonometric functions.
- To sketch a transformed trigonometric curve for a given domain.

Section 3

- To sketch translations and reflections of cubic, reciprocal and exponential functions.

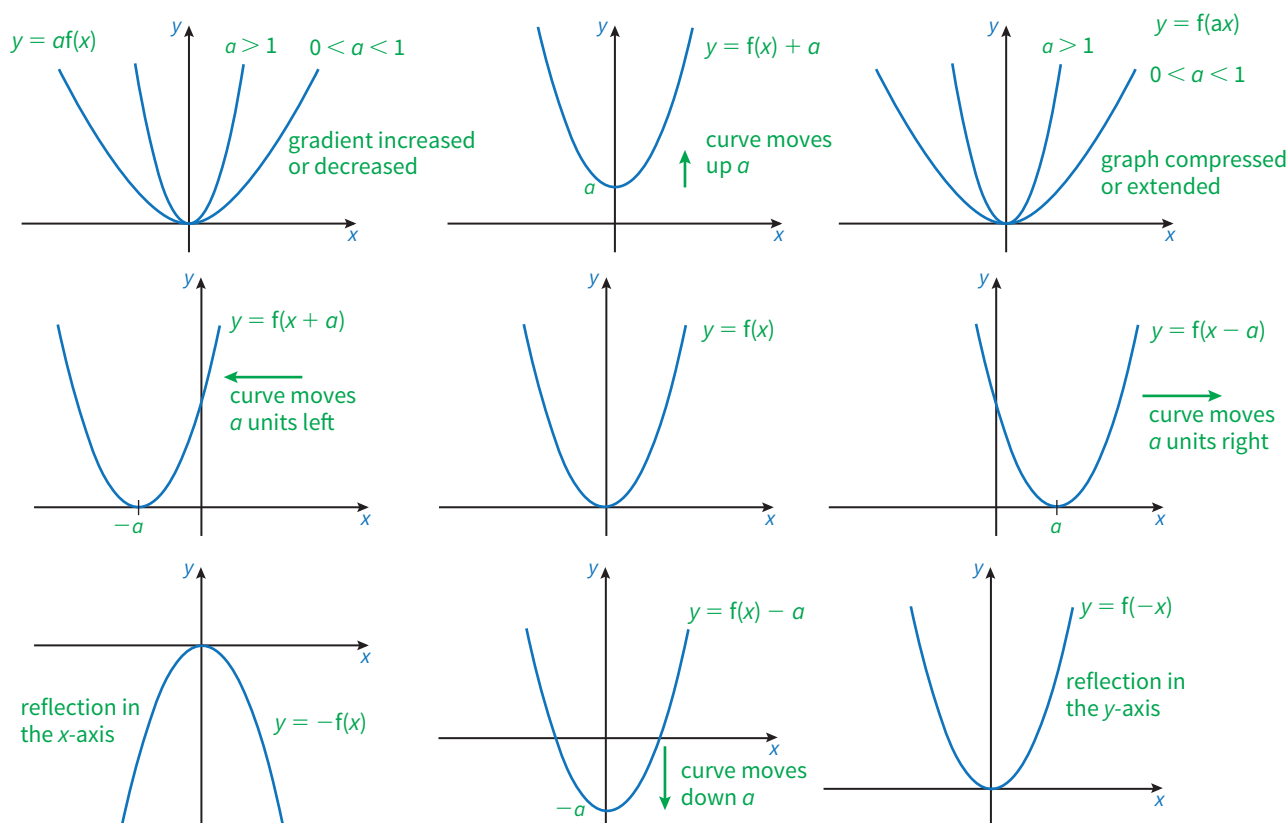
Section 4

- To apply the transformations learnt in this chapter to a variety of problems including identifying the effect of a transformation on a feature of a graph and finding the equation of a function once a transformation has been applied.

Common misconceptions and other issues

- Students most often try to remember a series of rules for transformations of graphs, for example $f(x) + a$ (translation a units up), $f(x) - a$ (translation a units down), $f(x - a)$ (translation a units right), and so on. When students try to remember a series of rules they often confuse movement left and right with each other. It is helpful to encourage

less able students to consider the transformations in two categories, one affecting x (inside the bracket) and one affecting y (outside the bracket), y acts as we would expect whereas the transformations affecting x do the opposite, for example $f(x + a)$ is a movement in the negative, not positive, x -direction. Reducing the number of facts for students to remember by combining rules to remove the distinction between positive and negative numbers, for example $f(x) + a$ where a is a translation in the y -direction so if a is positive it moves up and if a is negative it moves down can also help. In addition, encouraging students to use substitution to see what is happening to points on the graph might also help them identify the transformation applied. You can support your students further by supplying them with a crib sheet such as the following to include in their notes and by having further discussion of which transformations can be combined to give one rule.



- Students do not understand the difference between a sketch and a drawing/plot of a graph and when asked for a sketch try to plot the graph exactly by drawing in a grid and plotting coordinate points. Plotting coordinates might be useful initially for students to get an idea of what a graph looks like, but it should not be encouraged long term (particularly given the time it takes) and students should instead focus remembering the general features of these families of graphs, for example quadratics are 'u' or 'n' shape. Reinforcing what is required in a sketch and having a checklist for students to work through might also help. For example, a sketch consists of: a pair of axes; a pencil drawing showing the rough shape of the graph; and the points where the curve intersects the axes. For some sketches additional information will be required such as the coordinates of a stationary point, axis of symmetry or indication of amplitude; this will be clearly stated in the question though.

Hooks

The NRICH activity **Surprising Transformations** can be used to consolidate knowledge of straight-line graphs and the terminology needed for this chapter (nrich.maths.org). The question can then be asked: 'Can we tell what transformation has taken place to an equation of the form $y = f(x)$ just from the algebra?' or 'Can we tell what the new equation should look like from how the graph has been translated?'

SECTION 1: QUADRATIC FUNCTIONS AND PARABOLAS

Section 1 focuses on quadratics in the form $ax^2 + bx + c$ where $a = \pm 1$ and the ways in which each quadratic function can be thought of as a combination of translations and reflections of the function $y = x^2$. Through this, vertical and horizontal translations as well as reflections are introduced and their effect on the features of a quadratic defined.

Prompting questions

Exercise 41A(H)

- While working through this exercise, good prompts for students might be:
- **Q1(H)** Given any value of x , what is the smallest that x^2 can ever be? *0 because the square of a number is always positive. How does this help us identify the vertex of these quadratics? The vertex for these quadratics is always going to be the minimum because the coefficient of x^2 is 1, so we can find the minimum of these graphs by considering what happens when $x = 0$.*
- **Q2(H)** What do you do to each value of x in $y = -x^2$ to find y ? *We are squaring x and then multiplying it by -1 so all values will be negative.*
- **Q4(H)** What am I replacing x^2 with? $-x^2$. How can you use the graphs in Question 2 to help? *We can use the graphs from Question 2 and combine them with the transformations given in Question 1 to sketch the graphs.*
- What is the general shape of a quadratic where the coefficient of x^2 is -1? *An 'n' shape, rather than a 'u' shape. The graph $y = x^2$ has been reflected in the x -axis before it is translated.*

Exercise 41B(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** What form does a quadratic have to be in to easily identify the axis of symmetry? *Completed square form. How do you identify the axis of symmetry from a quadratic in this form? We look at the minimum of the graph by considering what makes $(x + a)^2 = 0$.*
- **Q1(H)** What is the value of x when the graph intersects the y -axis? $x = 0$.
- **Q3(H)** What are you expecting the shape of these graphs to be if you are multiplying $(x \pm a)^2$ by -1? *This gives a negative coefficient of x^2 so the graphs will all be 'n' shape.*
- **Q3(H)** Has the vertex of these quadratic graphs changed? *No. What can you say about the line of symmetry for each of these quadratics? This will also remain unchanged.*

Exercise 41C(H)

While working through this exercise, good prompts for students might be:

- **Q1-2(H)** What does the completed square form tell you about the features of the graph? *The vertex and axis of symmetry.*
- **Q1-2(H)** What do you need to do to find the y -intercept? *The graph intersects the y -axis when $x = 0$ so substitute this into the completed square form.*
- **Q1-2(H)** What do you need to do to find the x -intercepts? *The graph intersects the x -axis when $y = 0$ so solve the completed square form $= 0$.*
- **Q2(H)** What do you do first when completing the square? *Factorise out the coefficient of x^2 (so factorise out -1).*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Give students a series of transformations to perform on the parabola given by $y = x^2$ for example, first translate the graph three units in the positive x -direction, then reflect it in the x -axis, and so on. They have to perform all these transformations and give a sketch of the final parabola along with an equation for it.
- Spot the transformation: give students a series of parabolas that are transformations of $y = x^2$ and they need to write down which transformations were used.

Enrichment activities

- The NRICH activity **Parabolic Patterns** gives students graphs and asks them to find their equations using graphing software. An extension to this activity would be **More Parabolic Patterns**, which includes quadratic equations for y rather than x (nrich.maths.org).
- The NRICH activity **Which Quadratic?** is part of a larger selection of tasks based on team work. The aim is to encourage students to ask a series of questions where the answer is 'Yes' or 'No' to discover which quadratic equation the student has (nrich.maths.org).

Assessment ideas

- Ask students to create a revision booklet in which they outline the key features of the parabola given by a quadratic equation and the way they change as you apply different types of translations. For a more able group, students could each be given a different curve to consider and further discussion could take place regarding generalisations specific to the transformation and those that are only specific to the group of curves explored.

SECTION 2: TRIGONOMETRIC FUNCTIONS

Section 2 focuses on the application of vertical and horizontal translations and reflections of the functions $y = \sin x$, $y = \cos x$ and $y = \tan x$. The use of graphing software to support students' investigations of these transformations is recommended.

Exercise 41D(H)

While working through this exercise, good prompts for students might be:

- Q1a(H)** What are you expecting to happen to the graph of $y = \sin x$ when you add 2 to it based on your findings in Section 1? *That the graph will be translated 'up' in the positive y-direction.* Note: this question can be repeated for Question 1b and Question 2a/2b with the function and translation changed appropriately.
- Q4-5(H)** What happened to quadratic curves when we multiplied the equation by -1? *We got a reflection in the x-axis.* Use this to predict what the transformed graphs in Questions 4 and 5 will look like and check your sketch with ICT.
- What do your findings in this exercise show? *That all the transformations identified in Section 1 hold for any function.*

Starters, plenaries, enrichment and assessment ideas

Starters

- Ask students to sketch the key features of sine and cosine graphs when given different domains. This will get them thinking about the key features of sine and cosine graphs and at what values these features occur.

Plenaries

- Ask the students to sketch a sine or cosine graph for a given domain on mini-whiteboards. Then ask them to sketch a given transformation on the same axes.

Enrichment activities

- Give students a series of cards containing sketches of sine graphs that have been transformed. They are then required to match the graph with the correct equation. To increase difficulty (and awareness) you can select several of the graphs and state that the original graph was a cosine graph – and ask how would they then write the transformation (given that they already know what the sine graph transformation would be).

Assessment ideas

- Ask students to create a revision booklet in which they outline the key features of the trigonometric functions sine and cosine and how they change as you apply different types of translation.

SECTION 3: OTHER FUNCTIONS

Section 3 applies the transformations learnt in the previous two sections to graphs in the form: $y = \frac{1}{x}$, $y = a^x$ where $x > 1$ and $y = x^3$. Again in this section the use of graphing software is advised.

Prompting questions

Exercise 41E(H)

While working through this exercise, good prompts for students might be:

- Q1-3(H)** What transformation is being applied in each part: reflection or translation? What are you looking for to spot these? *Translations are indicated by the addition (or subtraction) of a value to the function, reflections are indicated by a change of sign (that indicates the function must have been multiplied by -1).*

- **Q2(H)** What is 1 divided by 0? *Typing this into a calculator produces a 'math error' why do we think this is? It is because we can't share 1 into 0 parts and hence the answer is undefined. What happens to 1 when we divide by a very small number approaching 0? Our answer gets infinitely large. How are these two features visible on the graph of the function $y = \frac{1}{x}$? The curve extends upwards to infinity close to 0 and there is no value of y for $x = 0$ as the curve does not touch the y-axis.*
- **Q3(H)** How would you describe the features of $y = 2^x$? How do you think they are related to the function it represents? *The curve crosses the y-axis at 1 this is because any number raised to a power of 0 is 1. The curve also tends to infinity in the positive x and y quadrant; this is because powers of 2 get very large very quickly. In addition the curve tends to 0 in the negative x positive y quadrant as we are dividing 1 by an exponentially increasing divisor as the power approaches negative infinity. The graph is also never negative; this is because there is no power that can be applied to make a positive number negative.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Give students a picture with a series of curves on the same pair of axes and they need to decide which curves are transformations of each other. This could be provided as an A3 printout with graphs of exponential functions with bases that are different.
- Start by giving a function. A series of transformations are flashed up on the board and the students need to sketch them on their mini-whiteboards. After a series of say five transformations, they need to write down the equation of the final curve they have.

Enrichment activities

- As is suggested in the section in the Student Book, using ICT to explore the graphs for each of the functions given is a useful exercise.

Assessment ideas

- Ask students to create a revision booklet in which they outline the key features of the functions and how they change as you apply different types of translation.

SECTION 4: TRANSLATION AND REFLECTION PROBLEMS

Section 4 offers a series of mixed problems to consolidate the material learnt in this chapter.

Prompting questions

Exercise 41F(H)

While working through this exercise, good prompts for students might be:

- **Q1(H)** How has the initial quadratic $y = x^2$ been transformed to give the graph $y = (x + 3)^2 - 8$? *It has been translated 3 units to the left and 8 units down. How far has the original function been transformed once it is also translated following the instructions in parts a and b? In part a it has been moved back to its original position, in part b it ends up 5 units to the left and 11 units down. What does this mean the equation of the curve is now? Part a: $y = x^2$ and part b: $y = (x + 5)^2 - 11$.*
- **Q3(H)** What does it mean for two parabolas to be congruent? *It means that they are the same size and shape in a different position on the x-y plane.*
- **Q3(H)** What form should a quadratic be in to be able to identify the features required for a sketch? *It should be in completed square form.*
- **Q4(H)** Is there more than one way you can answer this question? What is it about the trig functions sine and cosine that allows you to do this? *They are repeating functions with a period of 2π so there are many translations we can use.*
- **Q5(H)** What is being reflected in the y-axis? *The x values so we can replace x with -x.*
- **Q7(H)** If a graph has been shifted what type of transformation has taken place? *A translation, so we are adding (or subtracting) values to the original function.*

Starters, plenaries, enrichment and assessment ideas

Starters or plenaries

- Combine all the functions that have been covered in this chapter into a single ‘picture’. Students first need to identify which curves belong to the same ‘type’ of function. Then they need to decide what transformations have taken place to create the curves in each ‘type’ of function. It needs to be made clear which function you want to be the starting function or you could suggest that once they know the type, for example quadratic, they know that $y = x^2$ is the simplest quadratic function and so can work out how the others were created from this curve.

Enrichment activities

- Now that students have a selection of functions they know about, they can create their own transformation question and swap it with another student to try.
- If you wish to extend students and focus on multiple transformations, students can consider which transformations are commutative; students can apply one after the other in both orders and check the results. They can also consider which transformations are self-inversing (i.e. when applied twice give the curve they started with).

Assessment ideas

- Give two exam-style questions to pairs of students. Each student answers one and then marks the other. The conversation at the end of this activity can be focused on exam technique and layout of answers (as well as peer assessment).

Topic links

Previous learning

Students learnt about the main features of quadratic functions in *Chapter 19 Graphs of equations and functions*. The algebraic manipulation techniques required to rewrite a quadratic in different forms including completed square form were covered in *Chapter 7 Further algebraic expressions*. In addition, transformations and reflections were introduced through transformations of objects in the 2D plane in *Chapter 29 Plane isometric transformations*.

Gateway to A Level

Transformations of graphs feature at A Level with the addition of horizontal and vertical stretches. This work is further extended to composite transformations.

Knowledge of the way transformations can be identified, using both graphical representations of functions and the algebraic equations defining them, is fundamental to understanding the generalisation of several mathematical concepts learnt at A Level, including the use of calculus to calculate the area under a curve.

LINKS TO OTHER CAMBRIDGE GCSE MATHEMATICS RESOURCES

Problem-solving Book

- Chapter 6 Question 10
- Chapter 8 Questions 11, 25, 26, 27

Homework Book

- Chapter 41

GCSE Mathematics Online

- Student Book chapter PDF
- Lesson notes
- 6 worksheets (+ solutions)
- 5 animated widgets
- 6 interactive walkthroughs
- 3 auto-marked quickfire quizzes
- 3 auto-marked question sets, each with four levels
- Auto-marked chapter quiz

Time-saving sheets

- 1 cm squared paper
- 2 mm graph paper
- Axis grids from -10 to 10 in x and -10 to 10 in y

Literacy in mathematics – helping your students to do their best

This chapter provides advice and ideas on how you can help your students to become competent in mathematics, both in their GCSE course and beyond. It includes ideas to support learning around the assessment objectives, suggestions for enrichment and advice you can offer parents should they ask how they can support their child's maths learning.

EXTRACTING INFORMATION

Students are now able to extract information from a much wider variety of sources than in the past. You can teach them how to extract information from:

- tables and charts from both hard copy text and from the computer;
- spreadsheets from both hard copy and from the computer;
- PDF and word processed text in both hard copy and from the computer.

You might find the following approach useful.

Create your own compare and contrast grid, either generic or specific to a particular task. Choose data that represents mathematics found in situations students are familiar with or interested in, or that is appropriate to their studies in other mathematically-based subjects like science or economics; this develops their knowledge of topics they are studying, or about to study, and makes the information contextual and personal. From this data, students can extract and compare information, for example, which is the best bus/train/tram/coach to use from timetables.

Potential data sources include:

- Scientific studies found online. Students can extract and compare information about density, speed, atomic weights, interplanetary or interstellar distances, hardness of materials and so on. These lessons could be tied in with topics such as equation forming and solving, units and measures and standard form.
- Health information about Body Mass Index (BMI). However, this topic must be approached with sensitivity and can be taught in conjunction with health education, for example.
- Growth and decay patterns from either printed material or websites. This also helps in the study of indices, standard form, graph plotting and sketching, the idea of exponential growth and an appreciation of 'infinity'.
- Recipe books. Students can, for example, compare cooking times from different books, or quantities of ingredients for the same recipe in different books. These lessons could be tied in with topics such as ratio and proportion, scaling up or down and mass.
- Shopping and comparison websites, from which students can extract and compare information. These lessons could be tied in with topics such as value for money, general number work, fractions, decimals and percentages.
- Mobile phone tariffs, which can be accessed from newspaper adverts, airtime providers' own literature and websites. These lessons could be tied in with work on fixed or variable rates and value for money (e.g. advantages and disadvantages of Pay As You Go vs Monthly tariffs).

Extension activity

Using data extracted from the above suggested sources, students can identify coordinate pairs and plot them on a graph, either individually or as a group. These will generate both linear and curved graphs and enable students to have an appreciation of how functions look.

PROBLEM SOLVING

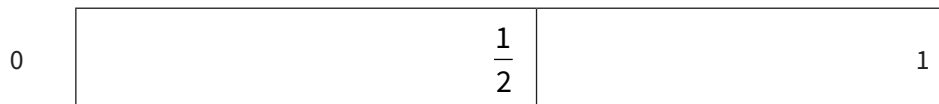
Problem solving can be used frequently throughout a topic. It can set the context for the topic, assess students' previous knowledge or develop their skills through the topic.

You should teach students to analyse the information you have presented them with, and select and extract the information required to solve the problem. This might involve understanding that the skill they need is embedded in another branch or topic. Students should then work through the problem to reach a solution. This may be 'multi-stage' and students should realise they must work through it step by step. The important thing for students to remember is that the problem is structured – that each step leads to the next and depends on the previous one. Information is gained gradually.

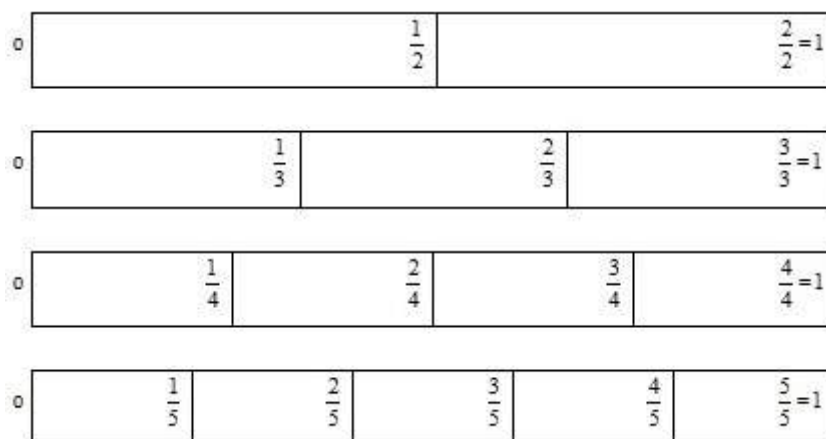
(Foundation only) Example activity: Fractions

Many Foundation students find fractions difficult to understand and master, and teachers often find the following problem-solving activity helpful.

- 1 Distribute a number of paper strips to the students.
- 2 Ask students how they could use their strip to represent the fraction $\frac{1}{2}$. Then show them how you would do it, for example by folding it in half.
- 3 Ask students to label their strip as shown:



- 4 Help students to solve the problem of how they could use other strips to fold them into 3, 4, 5, ..., and so on, equal parts.
(Some might need considerable help with this, although more able students might be imaginative in solving the problem of how to fold a strip into more difficult sections like 5, 7 or 11.)
- 5 When students have solved the problem of how to fold strips of paper into different fractional parts you can get them to put their strips on to a wall display vertically underneath one another to form a 'fraction wall' like the one below:



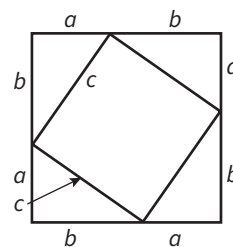
- 6 The students can then explore such things as equivalent fractions; comparing sizes of fractions and ordering fractions. It can also help them understand the very widely held misconception that 'the bigger the denominator, the bigger the fraction!'

(Higher only) Example activity: Investigating Pythagoras' theorem and Pythagorean triples

This problem requires students to prove Pythagoras' theorem, and there are a number of ways of going about this.

Proof 1

- 1 Students draw a square with another inside it as shown:
- 2 Students work through following steps:
 - i Area of 1 corner triangle; $[= \frac{1}{2}ab]$
 - ii Total area of 4 corner triangles; $[= 4 \times \frac{1}{2}ab = 2ab]$
 - iii Area of 'outside' square; $[= (a + b)^2]$
 - iv Area of 'inside' square; $[= c^2]$
 - v Area of 'outside' square = Area of 'inside' square +

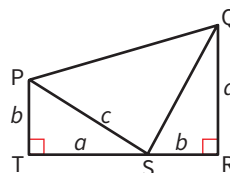


total area of 4 corner triangles; some practice at algebraic manipulation leading to:

$$a^2 + b^2 = c^2 \quad [(a + b)^2 = c^2 + 2ab \rightarrow a^2 + ab + ab + b^2 = c^2 + 2ab \rightarrow a^2 + 2ab + b^2 = c^2 + 2ab \rightarrow a^2 + b^2 = c^2].$$

Proof 2: two congruent right angled triangles PST and QRS

- 1 Students are given this diagram and you describe it to them.
- 2 Students work through following steps:
 - i Justify that the angle PSQ is a right angle;



- ii Area of full trapezium; $[= \frac{1}{2} \times (a + b) \times (a + b)]$
- iii Area of three small triangles; $[= \frac{1}{2}(ab) + \frac{1}{2}(ab) + \frac{1}{2}c^2]$
- iv Students equate their answers to i and ii; some practice at algebraic

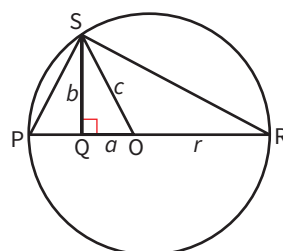
$$\text{manipulation leading to } \frac{1}{2} \times (a + b) \times (a + b) = \frac{1}{2}(ab) + \frac{1}{2}(ab) + \frac{1}{2}c^2$$

$$\rightarrow \frac{1}{2} \times (a^2 + 2ab + b^2) = ab + \frac{1}{2}c^2 \rightarrow (a^2 + 2ab + b^2) = 2ab + c^2 \rightarrow a^2 + b^2 = c^2.$$

[Some students realise that this diagram could be the bottom section of the diagram in Proof 1. Hence the investigation can be turned on its head and Proof 2, after Proof 1, can be used to 'prove' the formula for the area of a trapezium.]

Proof 3: circle theorems

- 1 Students are given this diagram and you describe it to them.
- 2 Students work through following steps:
 - i Prove that the triangles RQS and PQS are similar.
 - ii Students use their answer to i to prove that $a^2 + b^2 = c^2$.



[There are two common proofs of this investigation. The first one involves calling the angle SRO = x (say); then it follows that angle SPO is (90 – x) (angle in a semicircle) and, since triangle PQS is right angled, angle PSQ is x. Hence we have two triangles such that the angles are 90, x and (90 – x), making them similar. The second proof involves using 'angle at centre = 2 × angle at circumference' and so, if angle SRO = x, then angle SOQ = 2x. Also since triangle SOR is isosceles (c = r), then, by considering the three angles making up 90° at point S, the two triangles in question can be proved to be

similar. After either of these methods, we can end up with the relationship $\frac{SQ}{PQ} = \frac{QR}{QS}$ or $\frac{b}{r-a} = \frac{r+a}{b}$. Hence $(r-a)(r+a) = b^2 \rightarrow r^2 - a^2 = b^2$ or $a^2 + b^2 = r^2$ and since $c = r$, then $a^2 + b^2 = c^2$.]

Pythagorean triples

After an investigation into the proof of Pythagoras' theorem, students often like to investigate Pythagorean triples. Introduce the students to Pythagorean triples and encourage them to **systematically** find and record Pythagorean triples in 'families' (i.e. 3-4-5; 6-8-10; 9-12-15, etc. and then 5-12-13; 10-24-26; 15-36-39, etc. and then 8-15-17; 16-30-34; 24-45-51, etc.

Ask the students to investigate triples using websites and ask them to find a 'formula'. Hopefully they will come up with: 'For any two positive integers a and b, a Pythagorean Triple exists of the form $a^2 - b^2$; $a^2 + b^2$ and $2ab$.'

Students then investigate this formula. This could give them good practice at using spreadsheets that can also be used to record their results. Students could then investigate further such questions as:

- Why must every triple have at least one even number in it?
- Can you find a formula to generate triples where a must be odd?

Very able students could be asked to confirm the triple formula by evaluating $(2ab)^2 + (a^2 - b^2)^2 = (a^2 + b^2)^2$.

REASONING

To empower your students with good reasoning skills, you can teach them to:

- state the reasons for their results, for example, to realise outcomes and make decisions, to give opinions and to conjecture;
- use logical step-by-step efficient techniques to solving a problem and draw a conclusion;
- break down complicated problems into small steps and to work systematically;
- find counter-examples that do not fit the line of reasoning and explain why this is the case.

Sue Waring, in her book, *Can you prove it*, identifies four possible stages for students as they work towards a formal proof:

- Stage 1 Convince yourself (mental justification);
- Stage 2 Convince a friend (oral justification);
- Stage 3 Convince a pen-friend (informal written justification);
- Stage 4 Convince your mathematics teacher (more formal written justification).

(Foundation only) Example activity

Stage 1: Convince yourself

Ask students to draw a pair of intersecting straight lines.



Ask them to write down reasons why the vertically opposite angles are equal.

For example, students might write that the two angles 'look the same' or that any two adjacent angles are on a straight line so if you added the two together, whichever pair you chose you would get the same result.

Stage 2: Convince a friend

Ask students, in pairs, to talk through their arguments with each other. For example, they might say that the star and the black circle add up to 180 and so do the white circle and the black circle, so the star must be equal to the white circle.



Stage 3: Convince a pen-friend

Ask students to write down a 'proper' justification.

This stage is just an easy justification, like, for instance $\star + \bullet = 180$, $\bullet + \circ = 180$, so $\star = \circ$.

Stage 4: Convince your teacher

This would have to be a formal justification, probably using algebra.



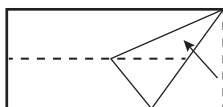
For instance: $w + x = 180$ (angles on a straight line); $x + y = 180$ (angles on a straight line);

$$w + x = x + y; \text{ so } w = y.$$

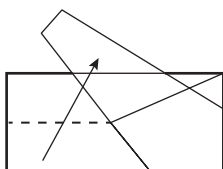
Similarly they would go on to prove that $x = z$.

(Higher only) Example activity: How to fold an equilateral triangle

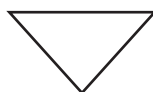
- Give students a piece of A4 paper and ask them to fold it down the middle parallel to the long side.
- Open it out and make a fold from the bottom corner to meet the central crease by folding from the other top corner.



- Fold from the bottom left hand corner until it meets the shorter of the two diagonal lines formed in the previous folding.



- Fold the top edge over to form a triangle.



- Ask students to describe what has happened and what shape they have made.

f. All will say triangle, of course, but a few might venture that the triangle is equilateral; which you confirm!

Ask them to write down a clear line of reasoning as to why the triangle is equilateral.

The reasoning could look like this:

- i The two folds meet at a point on the lower edge of the rectangle.
- ii These two folds, therefore, divide up the edge of the sheet into three equal angles.
- iii The angle at the folds along the edge is 180° .
- iv Therefore, the three equal angles must be $180 \div 3 = 60^\circ$, so one angle in the triangle is 60° .
- v The first fold to the central crease makes the triangle to the left of the right angled triangle on the right hand side into an Isosceles one.
- vi Hence, since we have one base angle of 60° , the angle to the left of that one is also 60° , leaving a vertical angle of 30° at the apex.
- vii Hence the right angle at the top right hand corner of the rectangle is divided into three angles of 30° each.
- viii When the second fold is made it envelopes two of those 30° angles into an angle of 60° .
- ix Hence the triangle is equilateral.

USING QUESTIONS TO ASSESS

Asking effective questions is important when using questions to assess, as is listening carefully and interpreting students' answers.

Good assessment has a real and instantaneous impact on learning, since it identifies the students who are both falling short and those who are exceeding the expected standard.

Key features of assessment for learning include:

- sharing learning objectives with students;
- helping students to understand the standard they are attaining. Interpreting their answers is the key to understanding their knowledge and misconceptions;
- providing guidance to students so that they can see their next steps and how to attain them;
- giving students the confidence to improve;
- providing the opportunity for teachers and students to review and reflect.

Targeting misconceptions in the classroom

Lessons should focus on known, specific difficulties. Focus on one problem or context and facilitate a variety of responses so that students can evaluate and learn from them. Properly structured lessons and questions give an opportunity for meaningful responses from students. You could include time for class discussion, allowing new ideas and concepts to surface, and remember to provide opportunities for students to secure what they have learned.

Be careful and sensitive in how you approach whole class discussions; some students are tentative and must be included in the process in a calm and encouraging environment.

Examples of effective questioning techniques

- How can we be sure that...? ... can be more effective than... What is...?
- What is the same and what is different about...? ... can be more effective than... Compare...
- Is it ever/always true/false that...? ... can be more effective than... Is this true or false?
- Why do... all give the same answer? ... can be more effective than... Are they all the same?
- How do you...? ... can be more effective than... Can you ...?
- How do you explain...? ... can be more effective than... Explain...

- What does that tell us about...? ... can be more effective than... tell me about...
- Why is... true? ... can be more effective than... Is ... true?

You might also find the table below helpful:

TOPIC	QUESTIONS	MORE EFFECTIVE QUESTIONS
CALCULATIONS	$4 - 6 = ?$; $5 + ? = 9$	What is the difference between addition and subtraction?
	$17 + 9 = ?$; $9 + 17 = ?$	Does the order you add numbers make any difference?
	5×8 ; 8×5	Does the order you multiply numbers make any difference?
	$6 \div 3$; $3 \div 6$	Does the order you divide numbers make any difference?
	$3 + 6 =$; $4 + 6 =$; $5 + 6 =$;	How do you add 6?
	$45 + 57 = ?$	What is wrong with $45 + 57 = 92$?
	$5 + 5 + 5 + 5 = ?$	Is it true that $5 + 5 + 5 + 5$ is the same as 4×5 ?
GEOMETRY AND MEASURES	What is the sum of the angles in a quadrilateral?	Explain why the sum of the angles in a quadrilateral is 360°
	Draw the lines of symmetry on the figure	Why does this figure have symmetry?
	Are these triangles congruent?	Why are these triangles congruent?
NUMBER	What is $\frac{1}{4}$ as a decimal?	What are the differences between fractions and decimals?
	What is three quarters of 30?	How do you find three quarters of something?
	Work out 45×0.15	Does multiplying always make things bigger?
	What is a half of 34?	Do you always get a whole number if you divide by 2?
	Is 15 a multiple of 4?	What can you tell me about multiples of 4?
	Work out: $5 - -6$; $-7 - 4$; $-6 + -8$.	What are the rules for adding and subtracting positive and negative numbers?
	Work out: 5×-6 ; -7×-4 ; -6×8 .	What are the rules for multiplying and dividing positive and negative numbers?
	Find the square root of 16	What can you tell me the square roots of positive and negative numbers?
	What is an odd number + an even number?	Why is an odd number + an even number always an odd number?
	What will the graph of $x + y = 5$ look like?	What can you tell me about the graphs of linear functions?
PROBABILITY	What is the chance of throwing a six on a dice?	Does any number on a dice have a better chance when the dice is thrown?
ALGEBRA	Expand $2(z + 4)$.	What happens when you expand $x(y + z)$?
	Factorise $x^2 + 5x - 6$.	How do you factorise quadratics?
	Simplify $2a + 3b + 4ab - a + 5b$.	What are the rules for simplifying expressions with like and unlike terms in them?
	Simplify $\frac{3}{\sqrt{5} + \sqrt{2}}$.	How do you rationalise a denominator?

RESOURCES TO ENRICH AND EXTEND LEARNING

There are a huge number of online resources that can be used to create rich lessons; some are listed below and some are referenced within the chapter notes, but undoubtedly there will be many others worth investigating. These websites are being updated constantly with new ideas so it is worth visiting them frequently for inspiration.

There are some students who will require extension work, even for the newer, more challenging GCSE, but this does not necessarily mean teaching them A Level topics in advance. Instead, additional challenging and stretching problems can be presented to help students build connections between concepts. The websites below that are appropriate for this, as well as for making students aware of wider developments in the area of mathematics, are indicated in the comments.

- **NRICH** has been referenced many times in the chapter notes to offer enriching tasks and activities for various topics in the Student Books. Their website offers a search by keyword, topic and key stage as well as offering a regularly updated curriculum mapping document for teachers to use. There are hints and solutions for most problems (solutions are sent in by students) and many problems also have additional notes and resources for teachers to use when planning a lesson. However, for students working independently there are also student pages arranged by key stage and places to seek advice in forums referred to as 'AskNRICH' (nrich.maths.org).
- **Dan Meyer** has created a series of resources for use in the classroom that are particularly useful in developing students modelling skills called '**Three-Act Math**'. These are all available freely online; however, the curriculum links refer to the American system. In addition to this, he also has an excellent **blog** that generates interesting discussion points and lesson material (blog.mrmeyer.com).
- The **Cambridge Maths Education Project (CMEP)** is an excellent place to find interesting resources based on the more challenging material in the GCSE. It is also an ideal place to send students interested in studying mathematics at A Level and beyond as the mathematics they will study is sorted into strands and the progression of ideas is clear.
- **Numberloving** is another interesting blog that presents great ideas for explaining the concepts of mathematics as well some great resources for use in the classroom (numberloving.com).
- **Plus Maths** is an excellent website aiming to bring current developments and Maths news to a wider audience. The articles are accessible and ideal for kindling students' interest in mathematics beyond GCSE (plus.maths.org).
- As well as offering tours and talks, **Maths in the City** has a number of resources that could be adapted for lessons about the mathematics visible around students every day and they could be encouraged to photograph what they see and upload it to the website (mathsinthecity.com).

ADVICE TO OFFER PARENTS

Many parents ask what they can do to support their child in studying for their GCSE. Aside from the usual advice relating to giving students structure and somewhere quiet to work, it is also worth considering which of the websites listed above can help. Parents might find the following websites useful in supporting revision and fostering Mathematical discussion at home (again there will be many others that you might wish to recommend):

- **BBC Bitesize** (bbc.co.uk).
- **Plus Maths** For fostering an environment of interest in this subject and knowledge of current developments (plus.maths.org).

In addition, parents can support students in their awareness of measurements and money through sharing information relating to bills and wages, and encouraging students to plan financially for things they want either through saving part-time job wages or pocket money. All these discussions increase students' awareness of the size of numbers and encourage better estimation/checking skills.

Parents can also be encouraged to take an interest in the mathematics their child is learning at school. There is not an expectation that they understand the mathematics and can help their child with independent work, but they should be encouraged to ask them what they have been doing or perhaps to get their child to explain their work to them. Reporting back to parents and carers throughout the year, as well as at particular reporting dates and parent evenings, can be really beneficial in encouraging students through an increased interest at home. Perhaps contact the parents and carers of the most successful students after every assessment point (this might be for different reasons including highest attaining and most improved) as well as contacting the parents and carers of students who are causes for concern. This method of praise is discreet, but having a 'well done' email sent home can become a goal for many students.

Preparing students for exams

The revision quizzes included in this Teacher's Resource are sets of cross-topic questions that you can use with students throughout the year or in the run up to assessment. This chapter lists further useful resources and ideas to help with your students' revision and exam preparation.

USING WEBSITES FOR REVISION

Some useful free websites for students to use when revising are listed below. There are of course many more good websites accessible online and an appropriate search often yields good results. At the time of writing much online material is still based on the pre-2015 GCSE programme of study so some adaptation might be needed when using them, but they are still worth using:

- **BBC Bitesize** The information on this website is often concise, pitched at the right level and considered by some students to be one of the most useful websites when revising (bbc.co.uk).
- **Emaths** Another good website for resources that can be used both inside and outside the classroom including targeted grade resources and past papers (emaths.co.uk).
- One of the most important things when encouraging students to revise independently using resources online is to reduce their options to a number of trusted sites and be specific about the types of activities they should be doing. Some of the material is not appropriate for students to be working through on their own and should instead be used by teachers for creating revision material.

MARKING WORK

Encouraging students to mark incorrect work that has been chosen to highlight common misconceptions and issues can be a really useful activity for several reasons. Firstly, it can help students identify common errors so they more aware of, and so able to avoid, possible pitfalls; secondly, it should encourage them to check their working at all times; and thirdly, it will promote the use of clear solutions with full working.

Creating these resources can be time consuming, but they can be used year on year as the misconceptions that lead students to make these errors rarely change. Work for students to mark can come from several sources including those listed here.

- A collection of work from a recently assessed piece of work, perhaps a homework or test. The challenge here is that you don't want your marking to be visible to students so solutions obtained from tests might work better, as the teacher's marking tends to only be in the margin for these assessments. The anonymity of the student can be hard to keep as they often either identify themselves or have a fairly good idea of who produced the work based on a process of elimination. This might not be a problem for a confident class, but with some classes this might not be a good idea.
- A 'mock' set of solutions that you have written. These might be in the form of Ms X's Homework or just a series of unnamed solutions for students to mark; the first option might be more motivating to students.
- A series of solutions that students have to mark and rank in order of how good they are. These can be used in conjunction with mark schemes where the challenge is for students to mark the solutions according to the mark scheme and rank the solutions from best to worst.
- These resources could be created collaboratively as a department and shared for use in lessons. It is good to write a range of possible answers to GCSE questions using the advice given by exam boards in examiner reports.

CREATING MODEL SOLUTIONS

Students can be encouraged to create model solutions to longer problems, particularly those set in context. Changing the way in which these problems are offered and the answers are presented can make a big difference to the way students approach the activity.

- Students can be offered a choice of problem to tackle. A good way to do this is to print the questions on different cards, without numbers. The cards can then be used to allocate problems randomly or students can select their own problems so as to give them more motivation to complete the work.

- Rather than writing solutions in exercise books, use plain A4 paper (or larger sheets). Using large pieces of sugar paper and thick pens mean solutions you wish to draw attention to for various reasons, for example because the students have annotated their working or made good use of diagrams, can easily be shared with the class.
- Encouraging students to work in pairs or small groups can be very motivating, particularly when the material is more challenging. It is good to make use of group work when students are working in lessons as these opportunities are not always available outside of lessons.
- An additional way to increase the challenge for solving longer problems is to place questions around the room (either on walls or tables) on large pieces of paper with plenty of space for working. Students work in teams or individually on each question for an allocated amount of time, aiming to get as far as they can in the time allowed. Once the time is up, students move on to the next question as directed and pick up the solution where the previous students left off. This teaches students about quality of written communication in their working as their peers have to be able to follow and continue their solution. In addition, it highlights the importance of checking, especially when they are jumping between questions in the exam when under stress.
- This task often reveals that parts of solutions are missed when students are not working on the problem from start to finish and stresses the importance of re-reading all questions and checking working at the end of the exam. Some students really enjoy this activity and others hate it, but all learn a huge amount from it, particularly about their preferred way of working in an exam.

REVISION ACTIVITIES AND GAMES

Revision activities and games are often very popular with students, particularly at times in the day when motivation is low. This is often because students struggling with motivation like the immediate feedback, which is possible with these activities.

- ‘Catchphrase’: Mathematics challenges based on the TV game. There are many examples of these online that can be used as templates. They are often better for shorter manipulation of number skill problems to allow a lot of questions to be answered in a short time.
- Relay challenges: These are very good for increasing the challenge in problems and encouraging motivation and speed. However, the solutions to these problems need to be easy to identify and prepared in advance so you can mark correct work with a glance and pass on to the next question. The lack of available teacher support encourages students to rely on each other in this task, so groups must be chosen wisely. This might not be appropriate for less able groups of students. See the **UKMT primary team challenge relay rounds** for examples of questions that work (ukmt.org.uk).
- Quizzes (sounds better than tests) can also be used successfully (rounds/quick fire/accuracy/speed).
- Starters and plenaries based on multiple-choice and true/false (sometimes referred to as ‘tick or trash’) problems can be used to recap simple concepts and formulae students should know. Laminated cards, mini-whiteboards or coloured pages in planners can be used to communicate students’ solutions visually for quick checking.
- ‘Noughts and crosses’ is also a very useful game for revising properties and definitions. The grid can be set up with a keyword or picture in each space. Students take it in turns to attempt to define the concept on the board. If successful they get to place their team’s symbol in the grid (a nought or a cross), if not it is the other team’s turn. The winning team is the one with a line of noughts or crosses (as in the traditional pen and paper game).
- Code-breaking activities can be very popular and motivating for students. However, the structure of these tasks needs to be carefully thought through to prevent students using language to solve the problems. One way to successfully disguise a hidden message in a word-processing programme is to use a font such as ‘wing dings’. Solutions to the questions can be matched to a symbol that can later be decoded.
- In a similar vein to the code-breaking tasks above, a series of padlock challenges are available to buy through the ATM in which the code reveals an additional question producing a final 4-digit code to unlock a metaphorical or physical combination padlock to receive a prize in class. These are a popular activity and rewards are very motivating for most challenges and tasks.
- For problems where students have to make a decision about what maths they need to apply, a series of activities can be created in which the first task is to categorise the problems by type of mathematics required. For example, this could be a series of right-angled triangle problems that students need to categorise into trigonometry, Pythagoras’

theorem, congruence and similarity problems. Additional revision could be included here by using a Venn diagram to sort the problems into those requiring just one or more than one technique.

USING EXAM QUESTIONS

Practising answering exam questions (from past papers, practice papers and specimen assessment material) allows students to experience the format of exams and the standard needed to attain particular grades. It is good for students to work on past papers and become familiar with the size and layout of the paper as well as the information pages at the front, as any small surprises over these details can make a big difference to students' confidence in exams.

The results of working through past papers can be used to determine areas of particular weakness in individuals or the whole class and allow for more tailored revision. Involving students in the marking of the questions also gives opportunities for them to learn what is required and so improve their exam technique, and also allows for both peer- and self-assessment. Using actual mark schemes can increase both teachers' and students' awareness of what things are important and what to look out for or avoid. It can be very useful to have some past papers, mark schemes and worked solutions available online via the school learning platform or website for students and parents to access at home.

- Exam questions can be sorted by difficulty to help students identify where they currently are and what they have to do to achieve their goals. This can be very motivating and help students see short-term progression, particularly if they focus on a particular topic.
- Questions can also be sorted by topic to create good end of topic homework or revision booklets in the run up to exams. Some teachers worry that students will recognise problems if they tackle them multiple times and so are reluctant to start students too early working on past paper questions. However, students often do not recognise the problems and gain much from repeating them, particularly when they are of lower ability.
- Students can work together on a set of questions, of varying style and difficulty, based around a particular topic. When they have attempted the questions, provide them with the relevant sections of the mark scheme and allow them to mark their own work to see the standard they have reached and increase awareness of any pitfalls to avoid in future.
- **Videos of worked solutions** to some past paper questions are available online. These were originally designed for A Level students, but some questions are accessible for the new GCSE specification. Videos like this can be created by you and your students; they might find it a very challenging task, but should gain a lot from it (examsolutions.net). There are some great applications available for tablets that can make creating these videos simpler, but they might still prove time consuming. The good news is that once they have been created they can be used each year with new cohorts.

EXAM TECHNIQUE

There is some good general advice that can be discussed with students regularly. It can be applied at all times, but can be particularly important in exam situations.

- If you are stuck on a question, start by writing down what you know as it might jog your memory. If you're not quite sure about the answer, it's still worth having a go, but don't waste too much time. If you're still having trouble, move on and come back to the question later.
- Read the question carefully; find the important words that give you information about what to do and look for key mathematical terms. For example, if the question asks you to sum you know you need to add and if the question describes a triangle as isosceles this tells you key information you might need.
- Think carefully before you start your answer: What calculations will you need? Will you use a mental method or are you allowed a calculator? Should you start with a rough estimate?
- Be careful with diagrams. Unless you are told that they are drawn accurately do not assume that they are, so work out lengths and angles using the information you have been given rather than measuring.
- Look at the number of marks available as this will give you a clue about how much to write. For example, if there are two marks for an explanation you will usually need to include two points (explanations don't normally need to be very long).

- Make sure that your answers are clear and mathematical and don't just repeat the question. You should explain your steps in a problem-solving question and not assume that something is 'obvious'. Make sure you include all the information that is necessary, for example if you are making a comparison you will need to refer to more than one piece of information.
- If you're asked to explain your answer or prove something, make sure you justify all your steps and if you're interpreting data, make sure that you explain how you reached your conclusion.
- Write down all the steps of your calculations because that way, even if you get the final answer wrong, you might get some marks for the correct methods. This means that when you use a calculator you should write down the calculation you are doing.
- Check that your answer is sensible by comparing it to a rough estimate. Make sure you've used the right units and that you've given your answer to the right level of accuracy.
- If you need to start your answer again, make it really clear what answer you would like to be marked by crossing out the work you think is wrong and leaving what you want the examiners to look at.

MAKE USE OF MARK SCHEMES AND EXAMINER REPORTS

Exam boards often provide a lot of information about how mark schemes are interpreted and reports following each exam covering general issues. These can be really useful sources of information about what examiners are looking for and how questions are marked.

REVISION TIPS

It might be helpful to give students a few tips when it comes to their revision. Here are a few ideas:

- Work in 30-minute bursts and take short breaks.
- Use notes to help you learn and remember key facts and methods.
- Write down anything you're unsure of and ask your teacher for help.
- Put up revision sheets in places where you'll often see them (bedroom walls, cupboard doors, etc.).
- Try to spot any bad habits and make sure you don't keep repeating them.
- Ask friends and family to help by asking you key facts, formulae and methods.

Revision quizzes: notes

Each of the following question sets (8 for Foundation and 8 for Higher) has been designed to last 40 minutes so that it can be used as a revision quiz within a single lesson. Alternatively, the questions may be set as additional practice during the course when students have covered the relevant topics.

Pages 313-346 contain mark schemes for the revision quizzes. Answers are not contained in the Student Book Answer Booklets so that you can set them under test conditions if you wish. Each quiz is worth a maximum of 40 marks.

To get students accustomed to the idea of calculator and non-calculator exam papers, a calculator may be used for even numbered quizzes, while odd numbered quizzes are non-calculator. This is indicated at the top of each question set.

Within each quiz, questions progress in terms of difficulty. A mix of AO1, AO2 and AO3 questions are included in each set, as indicated within the mark schemes.

The following abbreviations are used in the mark schemes:

- M = Method mark;
- A = Accuracy mark following on from a correct method (method can be implied);
- ft = Follow through mark (awarded for correct working based on an earlier error);
- oe = Or equivalent;
- AWRT = Answers which round to.

Formulae that students are not expected to memorise are given in the questions.

TOPICS

The following topics are covered in each set of questions.

	Foundation	Higher
Revision quiz 1	Number properties Improper fractions and mixed numbers Solving linear equations Area and perimeter of compound shapes Straight-line graphs Speed Bar charts Similar shapes	Ratio Solving linear equations Area and perimeter of compound shapes Straight-line graphs Fractions Similar shapes Indices Rearranging formulae Box plots
Revision quiz 2	Prime factorisation Multiples and factors Straight-line graphs Expanding brackets Circumference 3D shapes Congruency and proof Line graphs Best buy Volume Constructions Averages	Straight-line graphs Expanding brackets Circumference Best buy Averages Proportion Rounding Histogram Trigonometry and Pythagoras' theorem

	Foundation	Higher
Revision quiz 3	Lowest common multiples Multiplication and division Sketching graphs Simplifying expressions Fractions, decimals and percentages Area Inequalities on a number line Distance–time graphs Averages Probability	Sketching graphs Fractions, decimals and percentages Area Probability Proportion Problem solving with quadratics Vectors Surds
Revision quiz 4	Square numbers Using a calculator Rounding 2D and 3D shapes Factorising Percentages Area and perimeter Algebraic substitution Sampling Transformations Ratio	Using a calculator Rounding Factorising Percentages Area and perimeter Transformations Straight-line graphs Solving quadratics Cones
Revision quiz 5	Writing and using formulae Fractions and percentages Angles and polygons Pie charts Vector, mid-point and Pythagoras' theorem Solving quadratics – factorising Recurring decimals Converting between fractions and decimals Problem solving	Fractions and percentages Angles and polygons Vector, mid-point and Pythagoras' theorem Solving quadratics – factorising Recurring decimals Converting between fractions and decimals Sequences Graphs Weighted mean
Revision quiz 6	Highest common factor Indices Angles Proportion Solving inequalities Constructions Reverse percentages Exchange rates	Indices Angles Units of measurement Proportion Solving inequalities Reverse percentages Exchange rates
Revision quiz 7	Negative numbers Probability Rearranging formulae Ratio Vectors Scatter diagrams Simultaneous equations	Percentages Probability Rearranging formulae Volume and surface area Vectors Simultaneous equations Speed–time graph

	Foundation	Higher
Revision quiz 8	Using a calculator Expanding and factorising Bearings Perimeter and area Ratio Transformations Mean Compound interest Pythagoras Best buys Right-angled trigonometry	Expanding and factorising Mean Compound interest Trigonometry Maximum and minimum values Reverse percentages Iteration Simultaneous equations Pythagoras



- 1 Using the following list of numbers:

24 32 17 25 38 18 64 1

Identify a number that is:

a a square number. **b** a prime number. **c** a cube number. [3]

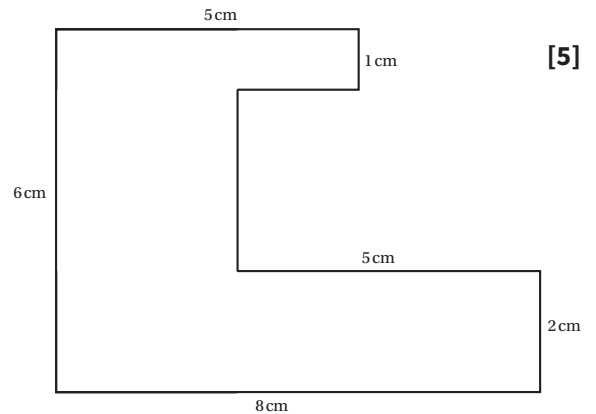
- 2 **a** Write $\frac{17}{5}$ as a mixed number. **b** Write $3\frac{2}{7}$ as an improper fraction. [2]

- 3 Solve:

a $3x - 12 = 42$ **b** $16 = 5x - 24$ **c** $12 + 3x = 7x - 24$ [4]

- 4 For the shape on the right, find:

a the area. **b** the perimeter. [5]



[3]

[2]

- 5 Draw the line $y = 3x - 4$, for values of x between -2 and 4 .

- 6 Sketch the plan and elevation of a square based pyramid.

- 7 Find:

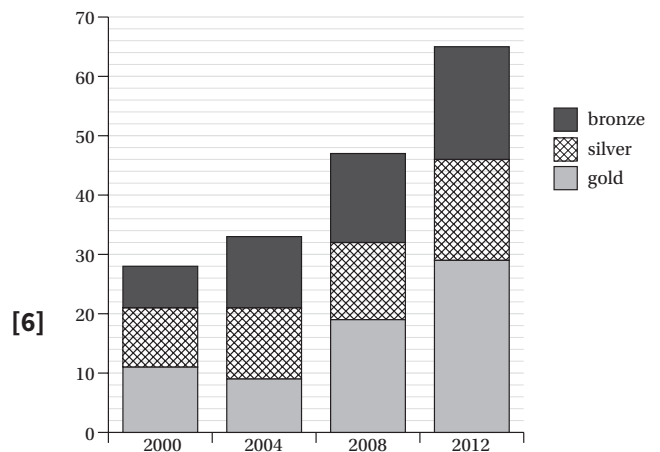
a $\frac{2}{15} + \frac{3}{10}$ **b** $1\frac{2}{5} - \frac{2}{3}$

c $\frac{6}{7} \times \frac{3}{8}$ **d** $\frac{14}{15} \div \frac{7}{10}$ [8]

- 8 A car travels 24 km in 20 minutes. At what speed is it travelling? [2]

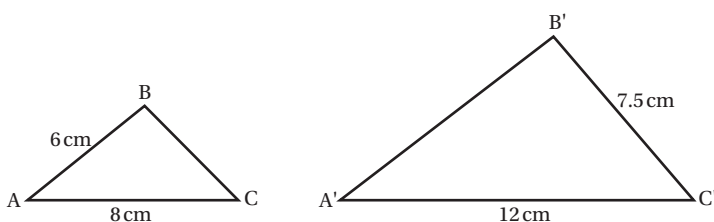
- 9 Here is a composite bar chart showing the Olympic medals won by Great Britain in the last four Olympic Games.

- a** How many medals were won in 2004?
b How many silver medals were won in 2008?
c How many gold medals were won in total in the last four Olympic Games?
d What is the mean number of medals won in the last four Olympic Games?



[6]

- 10 The following two triangles are similar.

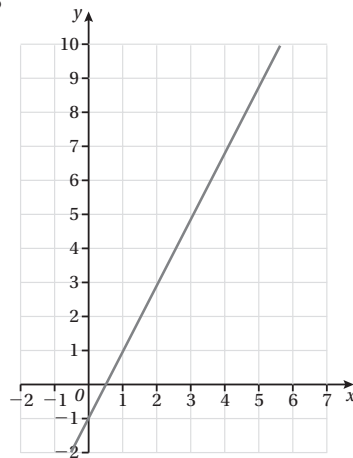


- a** What does it mean to be mathematically similar?
b Find the length of $A'B'$.
c Find the length of BC . [5]

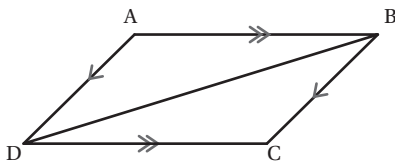
Total marks: 40



- 1 Find the prime factorisation of 630. Write your answer using indices. [3]
- 2 List: [3]
- a the first five multiples of 7. b all the factors of 36.
- 3 What is the equation of this line? [2]



- 4 Expand and simplify where possible: [5]
- a $3(x + 4)$ b $2x(x - 5)$ c $5(x - 4) + 3(x + 10)$
- 5 The diameter of a sports car tyre is 44 cm. What is the tyre's circumference? [3]
- 6 How many of the following does a cuboid have: [3]
- a faces. b vertices. c edges.
- 7 ABCD is a parallelogram. Prove that triangles ABD and BCD are congruent. [3]



- 8 The table below shows the rainfall in Salisbury last year.

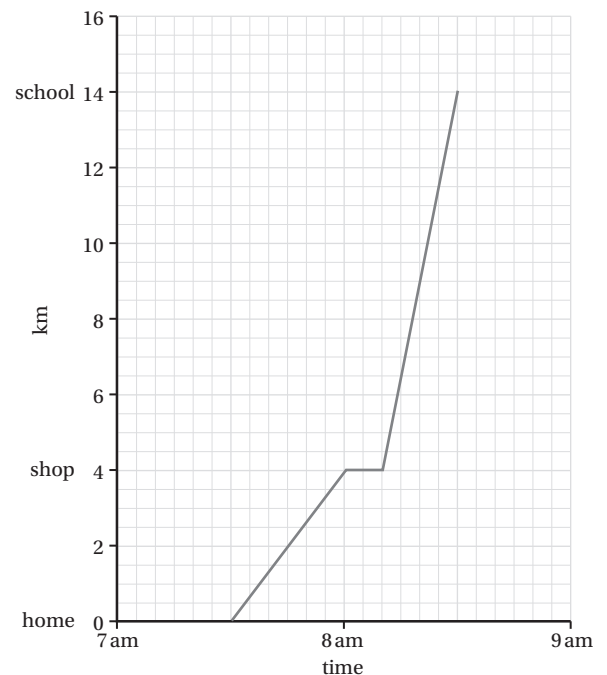
Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
mm	75	60	50	57	55	45	52	43	55	88	89	86

- Plot a line graph for this information. [4]
- 9 Crisps are sold in multipacks. You can buy 12 packets for £3.79. [4]
- Jenny buys 4 multipacks.
- She sells 30 packets of crisps for 42p each, gives 8 packets for free to her friends and sells the rest for 50p a packet.
- How much profit does Jenny make?
- 10 A shoe box measures 30 cm by 20 cm by 15 cm. [4]
- The shoe boxes need to be packed into large crates for transportation.
- The crates measure 1.2 m by 0.8 m by 0.75 m.
- How many shoe boxes will fit in one crate? Show your workings clearly.
- 11 Using a ruler and compasses, construct an equilateral triangle with side length 6 cm. [3]
- 12 The mean of 5 numbers is 6.5. [3]
- When a sixth number is added, the mean increases to 7.
- What is the sixth number?

Total marks: 40



- 1 Two buses leave the station at midday.
Bus 4R returns every 12 minutes.
Bus 5 returns every 16 minutes.
When will they both next be at the station together? [3]
- 2 Find:
a 34×56 **b** $552 \div 24$ [4]
- 3 Draw the graph of $y = x^2 - 3$ for values of x between -3 and 3 . [5]
- 4 Simplify:
a $3e + 2f - e + 4f$ **b** $a + 3a^2 + 4a - a^2$ [3]
- 5 Write:
a 0.06 as a percentage. **b** 56% as a fraction in its simplest form.
c $\frac{2}{5}$ as a decimal. **d** 1.45 as a percentage. [4]
- 6 The area of a triangular sail is 9 m^2 .
The height of the sail is 6 m.
What is the base length? [2]
- 7 For each of the following, draw a number line and mark on the inequalities.
a $f < 3$ **b** $-1 \geq a$ **c** $-3 \leq g < 2$ [4]
- 8 The diagram on the right shows Jenny's journey to school.
a How far had she travelled 15 minutes after leaving her house?
b How long did she stop at the shop for?
c How fast did she travel from the shop to school? [4]
- 9 A class of ten students has the following set of test scores. (The test was out of 50.)
37 45 37 40 41 38 41 41 40 35
(sum of these scores = 395)
Find:
a the modal test score.
b the median test score.
c the mean test score.
d the range of the test scores.
Another class has a mean test score of 45 and a score range of 20.
e Compare the results of the two classes. [7]
- 10 Talfiq flips a coin and rolls an octahedral dice (numbered 1–8).
a List all the possible combinations of his results.
b What is the probability that he gets tails and an even number? [4]



Total marks: 40



1 List the first six square numbers. [2]

2 Type the following calculation into your calculator.

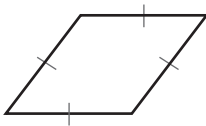
$$\frac{32 - 4.5^2}{1.3 + \sqrt{3.92}}$$

a Write down all the digits on your calculator display.

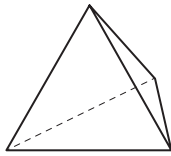
b Round down your answer to 2 decimal places. [2]

3 Name each of the shapes.

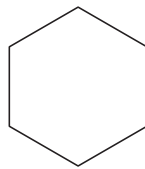
a



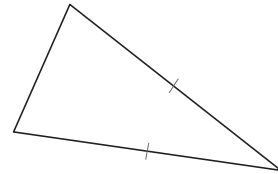
b



c



d



4 Factorise:

a $3x + 12$

b $4xy - 8y^2$

c $12a^2b + 18ab^2$ [5]

5 **a** Find 12.5% of £230.

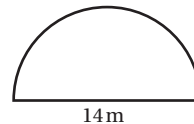
b Increase £480 by 18%.

c Decrease \$60 by 5.4%. [6]

6 Calculate the following for this shape:

a area.

b perimeter. [6]



7 The formula for the speed, v , of an object is:

$$v = u + at$$

where u is the initial speed, a , is the acceleration and t is the time travelling.

A runner is travelling at 4 m/s (metres per second).

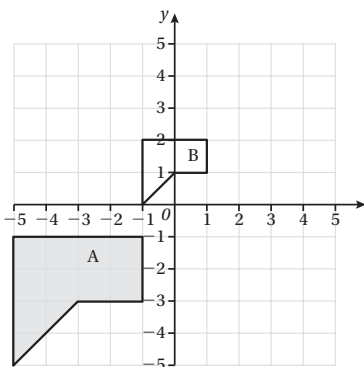
She accelerates at a rate of 0.5 m/s².

How fast is she travelling after three seconds? [3]

8 A restaurant manager wants to survey how often people eat out. She asks all of her customers one Friday night how often they eat at a restaurant.

Would this be a sensible sample? Give reasons for your answer. [2]

9 Describe the transformation from shape A to shape B. [3]



10 Lottery winnings are shared out in the ratio of 4 : 5 between two brothers, John and David. David receives £125 000.

What were the total winnings? [3]

11 Julie is comparing her test results in different lessons.

In science she scores 45 out of 52.

In maths she scores 50 out of 60.

In which test did she perform better? Explain your answer. [4]

Total marks: 40



- 1** Find the value of:
a $\sqrt{81}$ **b** 11^2 **c** $\sqrt[3]{64}$ **d** 3^5 [4]
- 2** Jimmy delivers newspapers. Every Saturday he works he gets paid £5.50. In addition he gets paid 20p per newspaper he delivers.
a Write a formula for the total amount [T], in pounds, that Jimmy gets paid if he delivers n newspapers.
b How much does Jimmy get paid if he delivers 15 newspapers?
c Last week Jimmy earned £9.70. How many newspapers did he deliver? [4]
- 3** Find:
a $\frac{2}{3}$ of 24 **b** 35% of 480 kg **c** 125% of \$64 [6]
- 4** Write:
a the sum of the internal angles of an octagon.
b the size of an external angle of an equilateral triangle.
c the size of an internal angle of a regular hexagon. [5]
- 5** Draw a pie chart to display the following information.
- | Method of transport | Number of students |
|---------------------|--------------------|
| Walk | 57 |
| Bike/scooter | 39 |
| Bus | 10 |
| Car | 14 |
| Train | 0 |
- [4]
- 6** Point A has co-ordinates (3, -1).
 Point B has co-ordinates (7, -4).
 Find:
a the vector from point B to point A.
b the midpoint of line segment AB.
c the length of line segment AB [6]
- 7** Solve by factorising $x^2 + 6x - 16 = 0$ [4]
- 8** Write:
a 0.36 as a fraction in its simplest form.
b $\frac{5}{8}$ as a decimal. [4]
- 9** A rectangular lawn measures 16 m by 30 m.
 Lawn feed costs £4.50 per kilogram. A kilogram will cover 20 m² of lawn.
 How much will it cost to feed the entire lawn? [3]

Total marks: 40

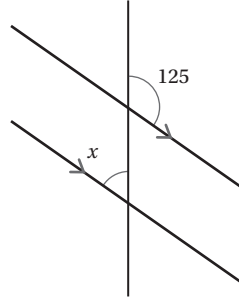


1 What is the highest common factor of 98 and 168? [2]

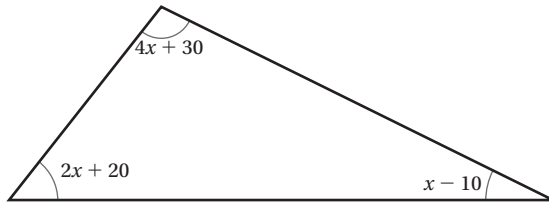
2 Simplify:

a $b^5 \times b^3$ **b** $\frac{s^2}{s^{-4}}$ **c** $3ab^2 \times 5ab$ **d** $\frac{12s^2t}{4st}$ [6]

3 Find the size of angle x . Explain your answer. [3]



4 Construct and solve an equation to find the value of x in the diagram.



5 The following recipe makes eight cupcakes.

- 2 eggs
- 50 g caster sugar
- 60 g self-raising flour
- 50 g butter
- 1 tsp vanilla essence

- a** How much of each ingredient is needed to make 12 cupcakes?
b I have 200 g butter, 150 g self-raising flour, 6 eggs, 150 g caster sugar and 10 tsp of vanilla essence. What is the maximum number of cupcakes I can make? [5]

6 Solve:

- a** $3y + 11 \geq 5y - 7$
b $3 < 4x - 9 \leq 23$
c What integer values of x solve the inequality in part **b**? [6]

7 Construct a triangle with side lengths 3 cm, 5 cm and 6 cm. [2]

- 8 a** In the last year, house prices have gone up by 9% on average.
 A house now costs £196 200.
 What was its price last year?

- b** A car has depreciated in value by 12%. It now costs £5720.
 What was its original price? [5]

9 In January, the exchange rate between pounds and euros was £1 = €1.26.

- a** A jumper costs €35.
 How many pounds is this?
b A coat costs £40.
 How many euros is this?
c A pair of trainers costs £75 or €90.
 Which is the better deal? Show your working. [7]

Total marks: 40



- 1 The table shows the record minimum temperatures for various places across the world.

Edinburgh	Moscow	Miami	Lima	Ottawa	Rome
-15°C	-42°C	2°C	9°C	-36°C	-11°C

- a What is the range of the temperatures in the table?
b What is the median temperature?
c Barcelona's minimum temperature is 20 degrees warmer than Edinburgh's.
What is this temperature?

[4]

- 2 A bag contains 3 blue and 7 red counters.

A counter is picked and replaced. A second counter is picked and replaced.

- a What is the probability that the first counter is blue?
b What is the probability that the second counter is red?
c What is the probability that the two counters picked are the same colour?

[5]

- 3 Make c the subject of the formula:

$$a = bc^2 + f$$

[3]

- 4 Consider the following number sequence:

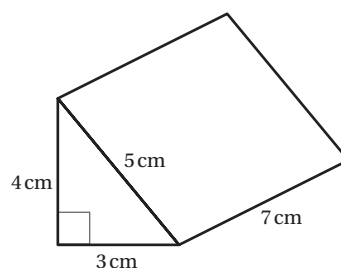
6, 13, 20, 27, 34, ...

- a What are the next two numbers in the sequence?
b What is the n th term of the sequence?
c What is the 21st number in the sequence?
d After how many numbers will the sequence equal 97?

[5]

- 5 For the triangular prism, find:

- a the volume.
b the surface area.
c Convert your answer to part b into mm^2



[7]

- 6 For the numbers 1 to 20, what is the ratio of prime numbers to non-prime numbers?

Write the ratio in its simplest form.

[2]

- 7 If $a = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $b = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, find:

- a $a + b$ b $3b$ c $2b - a$ d a vector parallel to a

[4]

- 8 The table shows the temperature and a shop's ice cream sales.

Temp ($^{\circ}\text{C}$)	32	24	18	11	22	10	15	20	27	17	19	30	21	22	12
Sales (£)	95	82	65	42	78	42	50	70	80	66	68	100	75	75	45

- a Draw a scatter graph of this information and comment on your results.
b How much would the owner expect to earn if the temperature was 25°C ?

[6]

- 9 Solve $3x - 2y = 1$ and $4x - 3y = 0$

[4]

Total marks: 40



- 1 Calculate:

$$\frac{3.3 \times 10^{-2}}{1.6 \times 10^5}$$

Write your answer in normal form to two significant figures.

[2]

- 2 a Expand and simplify $(x + 3)(x - 4)$

b Factorise $x^2 - 8x + 12$

[4]

- 3 The bearing of Exeter from Northampton is 230° .

What is the bearing of Northampton from Exeter?

[2]

- 4 For the shape on the right, find:

a the perimeter.

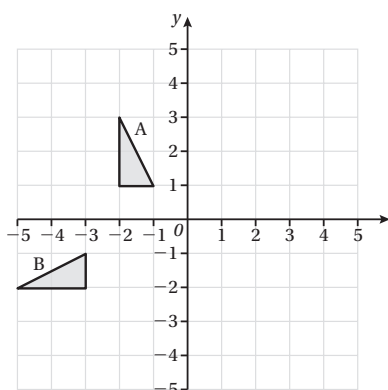
b the area.

[8]

[2]

- 5 Share 24 in the ratio 3 : 5.

6



Describe the single transformation that maps shape A onto shape B.

[4]

- 7 The table shows the heights of tomato seedlings at a nursery.

Height, h (mm)	Frequency
$0 \leq h < 10$	8
$10 \leq h < 30$	9
$30 \leq h < 40$	24
$40 \leq h < 60$	18
$60 \leq h < 100$	12

Find the mean height of a tomato seedling.

[4]

- 8 £120 is invested in an account that gives 3.5% compound interest per year.

How much will be in the account after 3 years?

[3]

- 9 A rectangle measures 12 cm by 16 cm.

What is the length of its diagonal?

[3]

- 10 Which of the following deals is the best offer? Explain your answer.

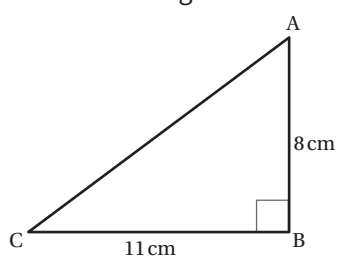
a 6 tins of beans for £2.45

b 9 tins of beans for £3.80

c 4 tins of beans for £1.52

[4]

- 11 Find the size of angle BCA to the nearest degree.



[4]

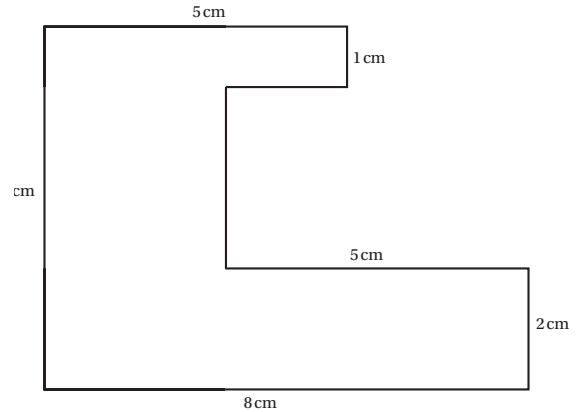
Total marks: 40



- 1** Richard, Sabah and Tariq have £135 to split between them.
Richard gets half the amount of Tariq.
Sabah gets three times more than Tariq.
How much do they each receive? [3]

2 Solve: **a** $16 = 5x - 24$ **b** $12 + 3x = 7x - 24$ [3]

- 3** For the shape on the right, find:
a the area.
b the perimeter. [5]



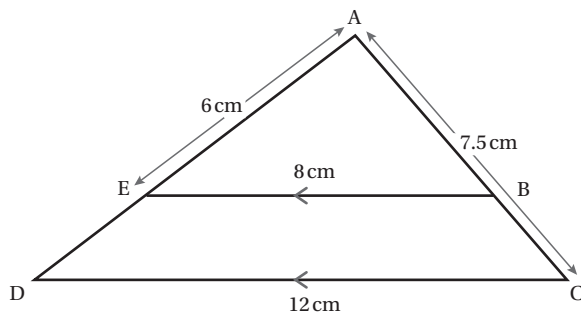
- 4** Find the equation of the line passing through the points (3, 5) and (5, 11). [4]

5 Find:

a $\frac{2}{15} + \frac{3}{10}$ **b** $1\frac{2}{5} - \frac{2}{3}$
c $\frac{6}{7} \times \frac{3}{8}$ **d** $\frac{14}{15} \div \frac{7}{10}$ [8]

6 In the following diagram: [5]

- a** find the length of AD.
b find the length of CB.
c Angle EDC is 40° . What is angle AEB? Give a reason for your answer.



7 Evaluate:

a 8^0 **b** 7^{-2} **c** $27^{\frac{2}{3}}$ **d** $\left(\frac{16}{25}\right)^{-\frac{1}{2}}$ [6]

8 Rearrange the formula $aw + b = w - h$ to make w the subject. [3]

9 80 workers in a factory were asked their ages.

The eldest worker was 58, the youngest was 19.

The median age was 33, the lower quartile of ages was 28 and the interquartile range was 12.

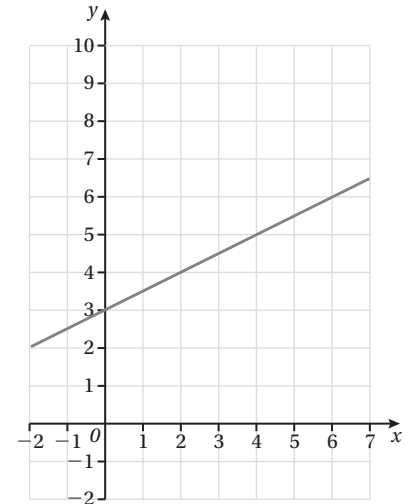
Represent this information on a box plot. [3]

Total marks: 40



1 What is the equation of this line?

[2]



2 Expand and simplify where possible:

a $3(x + 4)$ **b** $2x(x - 5)$ **c** $5(x - 4) + 3(x + 10)$

3 The diameter of a sports car tyre is 44 cm. How many full rotations does the tyre make if the car travels 1.2 km?

[5]

4 Crisps are sold in multipacks. You can buy 12 packets for £3.79. Jenny buys 4 multipacks. She sells 30 packets of crisps for 42p each. She gives 8 packets for free to her friends and sells the rest for 50p a packet. What percentage profit does Jenny make?

[4]

5 The mean of 5 numbers is 6.5. When a 6th number is added the mean increases to 7. What is the sixth number?

[5]

[3]

6 The distance, d , an object falls is directly proportional to the square of the time, t , it is falling for. A rock falls from a cliff edge. After 3 seconds it has travelled 44.1 metres.

a Find a formula linking d and t .

b How far will the rock have fallen after 5 seconds?

c The White Cliffs of Dover reach a maximum height of 110 m. How long would it take a rock to fall from the top to the bottom? Give your answer correct to 2 decimal places.

[9]

7 To the nearest metre, a football pitch has a length of 98 m and a width of 42 m. Calculate the upper bound of the area of the pitch.

[2]

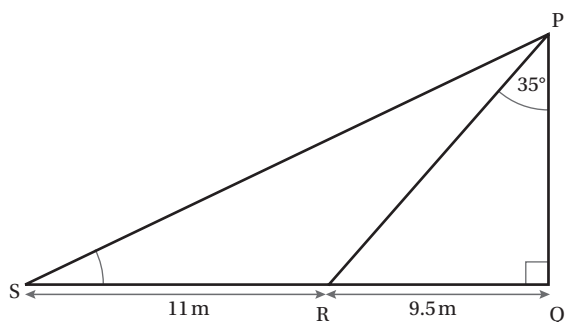
8 The table below shows the heights of tomato seedlings at a nursery. Draw a histogram to represent this data.

[5]

Height, h (mm)	Frequency
$0 \leq h < 10$	8
$10 \leq h < 30$	9
$30 \leq h < 40$	16
$40 \leq h < 60$	18
$60 \leq h < 100$	12

9 Find angle PSR. Give your answer to the nearest degree.

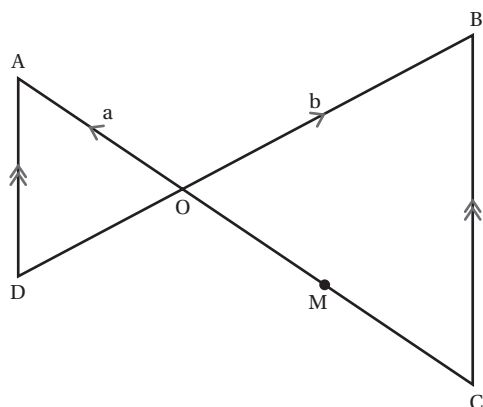
[5]



Total marks: 40



- 1 Draw the graph of $y = x^2 - 2x - 3$ for x between -2 and 4. [5]
- 2 a Write 56% as a fraction in its simplest form.
b Write $\frac{3}{16}$ as a decimal. c Write 1.45 as a percentage. [3]
- 3 The area of a triangular sail is 9 m^2 . The height of the sail is 6 m. What is the base length? [2]
- 4 Talfiq flips a coin and rolls an octahedral dice (numbered 1–8).
a List all the possible combinations of his results.
b What is the probability that he gets tails and an even number or an odd number? [4]
- 5 The variable s is inversely proportion to d . When $s = 4$, $d = 0.5$.
a Find a formula linking s and d .
b When $d = 0.2$, what is s ?
c What is d when $s = 0.05$? [8]
- 6 A rectangle has an area of $2x^2 + 5x + 2$. The rectangles perimeter is 42 cm. Find the dimensions of the rectangle. [7]
- 7 In the diagram $2AO = OC$, vector **a** describes the journey from O to A and vector **b** the journey from O to B. M is the midpoint of OC.

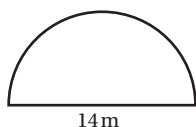


- a Give the vector for OC.
 - b Give the vector for CB.
 - c Give the vector for MD.
- A line segment is drawn from A to B and then continues to point E such that $AB : AE$ is 3 : 4
- d Find the vector from M to E. [7]
- 8 Simplify each of the following:
a $\sqrt{8}$ b $\sqrt{28} \times \sqrt{63}$ c $(1+\sqrt{2})(2+\sqrt{2})$ [4]

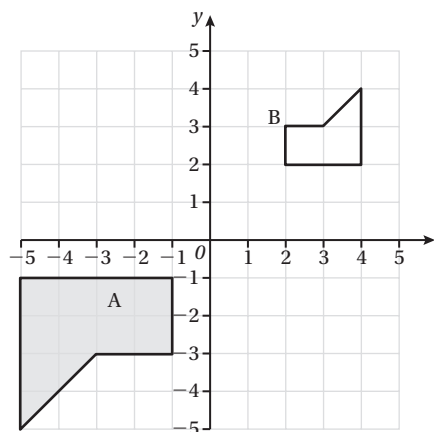
Total marks: 40



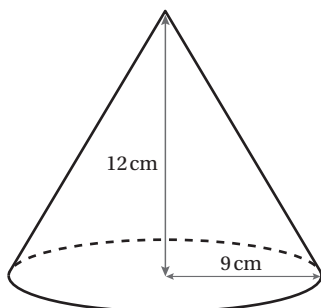
- 1 Type the following calculation into your calculator $\frac{32 - 4.5^2}{1.3 + \sqrt{3.92}}$
 - a Write down all the digits on your calculator display.
 - b Round down your answer to 2 decimal places. [2]
- 2 Factorise: a $4xy - 8y^2$ b $12a^2b + 18ab^2$ c $x^2 - 16$ [6]
- 3 a Find 12.5% of £230. [6]
 - b Increase £480 by 18%.
 - c Decrease \$60 by 5.4%.
- 4 For the following shape, calculate:
 - a the area.
 - b the perimeter. [6]



- 5 Describe the transformation from shape A to shape B. [3]



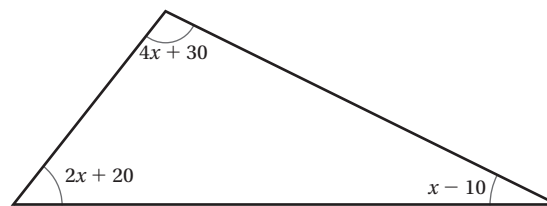
- 6 a Find the equation, $y = mx + c$, of the line that is parallel to $y = 10 - 3x$ and goes through the point (2, -4). [4]
 - b Give the equation of any line parallel to $y = 10 - 3x$. [5]
- 7 Solve $x^2 - 4x - 11 = 0$, giving your answers to 3 significant figures. [5]
- 8 The sloping face of a cone is made from a sector of a circle. Find the angle of the sector used to produce the cone below. [8]



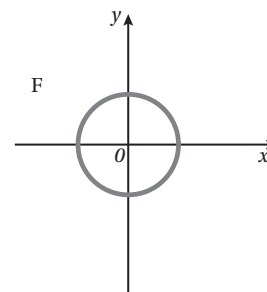
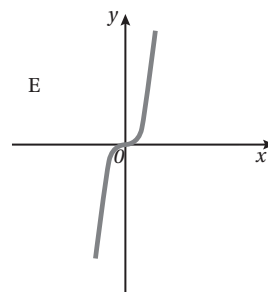
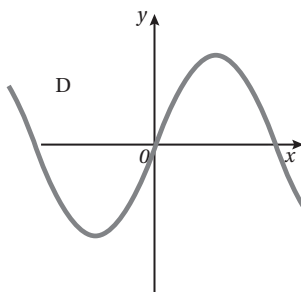
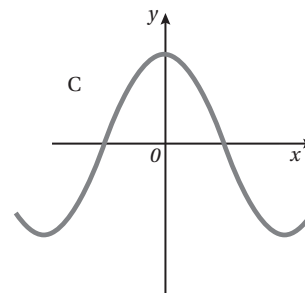
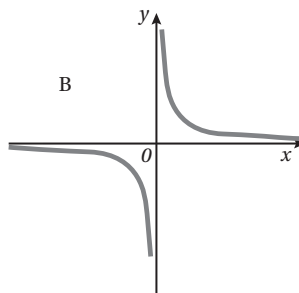
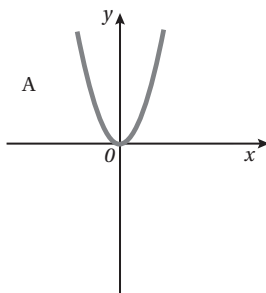
Total marks: 40



- 1 **a** Find $\frac{2}{3}$ of 24. **b** Find 35% of 480 kg. **c** Find 125% of \$64. [6]
- 2 **a** What is the sum of the interior angles of an octagon?
b What is the size of an exterior angle of an equilateral triangle?
c What is the size of an interior angle of a regular hexagon? [5]
- 3 Point A has coordinates (3, -1) and point B has coordinates (7, -4). [6]
a Find the vector from point B to point A.
b Find the mid-point of line segment AB.
c Find the length of line segment AB.
- 4 Solve:
 $x^2 + 6x - 16 = 0$ [4]
- 5 Construct and solve an equation to find the value of largest angle in the diagram below. [5]



- 6 Prove that $0.\dot{3}\dot{6} = \frac{4}{11}$ [3]
- 7 For the sequence 4, 10, 18, 28, 40, ...:
a find the next two terms. [2]
b find the n th term. [3]
- 8 Match each of the following graphs with their equation.



$x^2 + y^2 = 25$ $y = \cos(x)$ $y = x^2$ $y = \sin(x)$ $y = x^3$ $xy = 4$

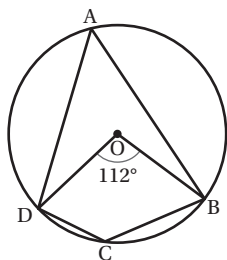
- 9 The average allowance given to 14 boys is £22.50. The average for the whole class of 14 boys and 16 girls is £28.10. What was the average for the girls? [3]

Total marks: 40



1 Simplify: **a** $\frac{s^2}{s^{-4}}$ **b** $3ab^2 \times 5ab$ **c** $\frac{12s^2t}{4st}$ **d** $(2x^3)^4$ [7]

2 In the diagram below, find the size of angle BCD. Give reasons for your answer. [3]

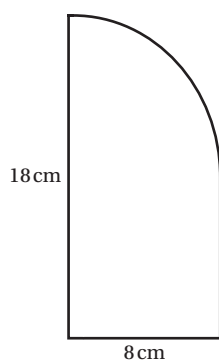


3 Convert 1.7 m^3 into cm^3 . [2]

4 For the following shape, find:

a the perimeter.

b the area. [8]



5 **a** Solve: $3 < 4x - 9 \leq 23$
b What integer values of x solve the inequality above?
c Show on a coordinate grid ($-3 \leq x \leq 5$) the area that satisfies the following inequalities:
 $-2 < x \leq 4$, $y \geq 0$ and $x + y \leq 6$ [9]

6 **a** House prices have gone up in the last year on average by 9%. A house now costs £196 200. What was its price last year?
b A car has depreciated in value by 12%. It now costs £5720. What was its original price? [5]

7 The exchange rate between pounds and euros in January was £1 = €1.26.
 A catering company can buy 1.5 kg of asparagus in the UK for £16 or 800 g for €10.50 in France. Which is the better deal? Show your working clearly. [6]

Total marks: 40



1 Jenny buys 24 cans of cola. Each pack of 6 cans costs £3.75. She drinks 4 cans and sells the rest. What cost should she charge if she wants to make a profit of 20%? **[5]**

2 A bag contains 3 blue and 7 red counters. A counter is picked and not replaced. A second counter is then picked.

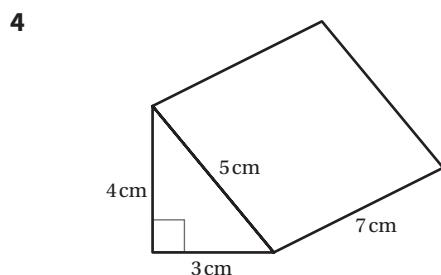
a What is the probability that the first counter is green?

b Given that the first counter was blue what is the probability that the second counter is red?

c Draw a tree diagram to represent this problem.

d What is the probability that the two counters picked are different colours? **[8]**

3 Make c the subject of this formula: $a = bc^2 + f$ **[3]**



a Find the volume of this triangular prism.

b Find the surface area of this triangular prism.

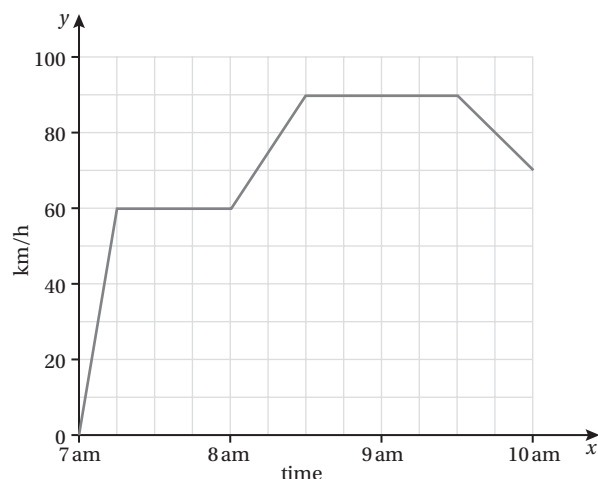
c Convert your answer to part b into mm^2 . **[8]**

5 If $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, find:

a $\mathbf{a} + \mathbf{b}$ **b** $3\mathbf{b}$ **c** $2\mathbf{b} - \mathbf{a}$ **d** a vector parallel to $\mathbf{a}$ **[4]**

6 Solve: $3x - 2y = 1$ and $4x - 3y = 0$ simultaneously. **[4]**

7 The graph below is a speed–time graph for a long distance freight train.



a What was the train's speed at 9 am?

b What was the train's rate of acceleration from 7 am to 7.15 am?

c How far did the train travel between 9.30 am and 10 am? **[8]**

Total marks: 40



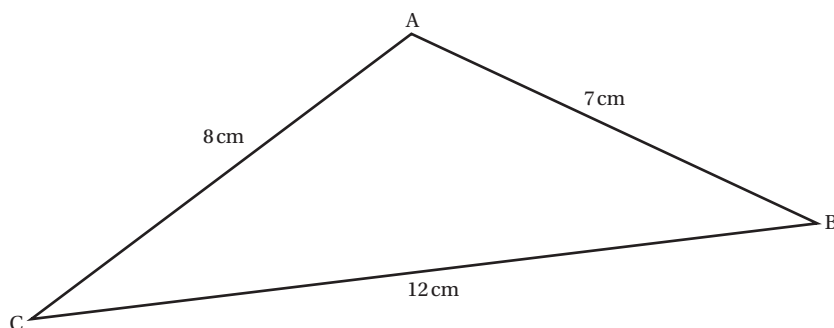
- 1 **a** Expand and simplify $(x + 3)(x - 4)$ **b** Factorise $x^2 - 8x + 12$ **c** Simplify $\frac{48x^3y^{-2}}{16xy^{-4}}$ [6]

- 2 The table below shows the heights of tomato seedlings at a nursery. Find an estimate for the mean height of a tomato seedling. [4]

Height, h (mm)	Frequency
$0 \leq h < 10$	8
$10 \leq h < 30$	9
$30 \leq h < 40$	24
$40 \leq h < 60$	18
$60 \leq h < 100$	12

- 3 £120 is invested in an account that gives 3.5% compound interest per year. How much will be in the account after 3 years? [3]

- 4 Find the size of angle ABC to the nearest degree.



- 5 Using the fact that $a = 2.4$ (1 decimal place), $b = 12.0$ (1 decimal place) and $c = 120$ (2 significant figures), find the maximum and minimum values of $\frac{3c}{b-a}$. Show your working clearly. [5]

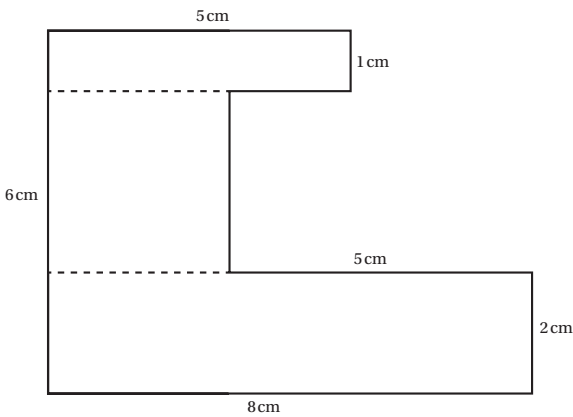
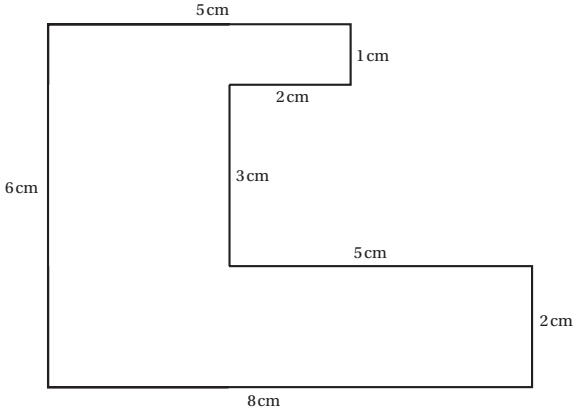
- 6 The starting price of a china doll at an auction is £35. It eventually sells for £43.50. What percentage increase is this? [3]

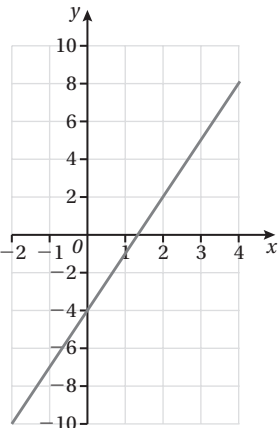
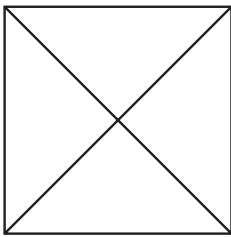
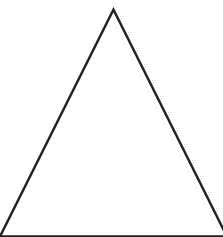
- 7 Using the iteration formula $x_{n+1} = 7 - \frac{1}{x_n}$ and the starting value $x_1 = 1$ carry out the first four iterations and suggest a suitable solution to the equation $x = 7 - \frac{1}{x}$. [5]

- 8 Find the length of the chord between the points at which the line $y = x + 2$ meets the circle $x^2 + y^2 = 100$. [8]

Total marks: 40

Mark scheme

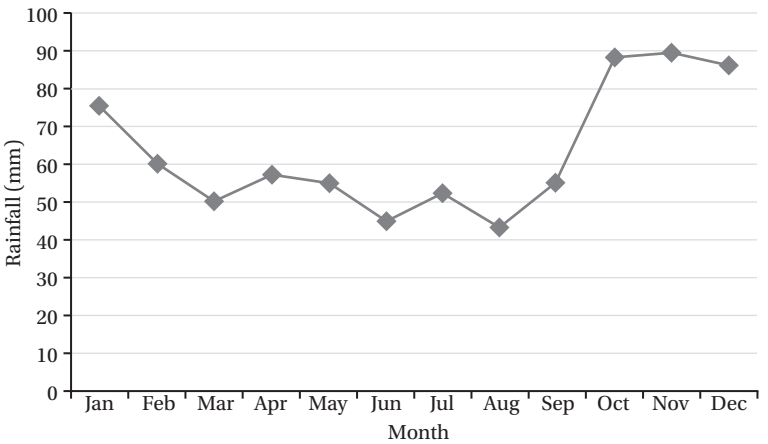
Question	Answer	AO	Marks	Notes
1 a	Any of 1, 25 or 64	AO1	A1	
1 b	17	AO1	A1	
1 c	Any of 1 or 64	AO1	A1	
2 a	$3\frac{2}{5}$	AO1	A1	
2 b	$\frac{23}{7}$	AO1	A1	
3 a	$x = 18$	AO1	A1	
3 b	$x = 8$	AO1	A1	
3 c	$12 + 3x = 7x - 24$ $12 = 4x - 24$ $36 = 4x$ $9 = x$	AO2	A1	
	$x = 9$	AO2	A1	
4 a		AO3	M1	for splitting shape (any split)
	$5 \times 1 = 5$ $3 \times 3 = 9$ $2 \times 8 = 16$ Total = 30 cm^2	AO3	M1	attempted areas of separate parts
	Area = 30 cm^2	AO3	A1	
4 b		AO3	M1	find missing lengths
	$5 + 1 + 2 + 3 + 5 + 2 + 8 + 6$ = 32 cm Perimeter = 32 cm	AO3	A1	

5		AO2	M2 A1	M1 correctly drawn axis M1 3 or more correctly plotted points
6	Plan 	AO1	A1	
	Elevation 	AO1	A1	
7 a	$\frac{2}{15} + \frac{3}{10}$ $= \frac{4}{30} + \frac{9}{30}$ $= \frac{13}{30}$	AO1	M1	common denominator
	$\frac{13}{30}$	AO1	A1	
7 b	$1\frac{2}{5} - \frac{2}{3}$ $= \frac{7}{5} - \frac{2}{3}$ $= \frac{21}{15} - \frac{10}{15}$ $= \frac{11}{15}$	AO2	M1	change mixed number to improper fraction
	$\frac{11}{15}$	AO2	A1	

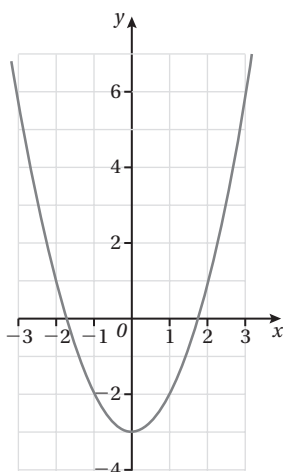
7 c	$\frac{6}{7} \times \frac{3}{8}$ $= \frac{18}{56}$ $= \frac{9}{28}$	AO1	M1	multiply numerators and denominators
	$\frac{9}{28}$	AO1	A1	simplified answer
7 d	$\frac{14}{15} \div \frac{7}{10}$ $= \frac{14}{15} \times \frac{10}{7}$ $= \frac{2}{3} \times \frac{2}{1}$ $= \frac{4}{3}$ $= 1\frac{1}{3}$	AO1	M1	'flip' and multiply
	$1\frac{1}{3}$	AO1	A1	answer as a mixed number
8	72 km/h	AO3	A2	1 mark for 72 1 mark for km/h
9 a	33	AO2	A1	
9 b	13	AO2	A1	
9 c	68	AO3	M1 A1	M1 attempt to add totals
9 d	$28 + 33 + 47 + 68 = 173$ $173 \div 4 = 43.25$ Mean is 43.25	AO2	A1	totals
		AO2	A1	
10 a	The shapes are enlargements of each other. Angles are the same, ratio of the sides are the same.	AO2	A1	one of two given definitions sufficient
10 b	$12 \div 8 = 1.5$ So scale factor small to big is $\times 1.5$ $6 \times 1.5 = 9$ cm	AO2	M1	
	9 cm	AO2	A1	
10 c	$8 \div 12 = 2/3$ $7.5 \times 2/3 = 5$ cm	AO2	M1	
	5 cm	AO2	A1	

Mark scheme

Question	Answer	AO	Marks	Notes
1	630 $= 2 \times 315$ $= 2 \times 3 \times 105$ $= 2 \times 3 \times 3 \times 35$ $= 2 \times 3 \times 3 \times 5 \times 7$	AO1	A2	
	$2 \times 3^2 \times 5 \times 7$	AO1	A1	
2 a	7, 14, 21, 28, 35	AO1	A1	
2 b	$36 = 1 \times 36$ $36 = 2 \times 18$ $36 = 3 \times 12$ $36 = 4 \times 9$ $36 = 6 \times 6$	AO1	A1	
	1, 2, 3, 4, 6, 9, 12, 18, 36	AO1	A1	
3	$y = 2x - 1$	AO2	A2	A1 for correct gradient A1 for correct y intercept
4 a	$3x + 12$	AO1	A1	
4 b	$2x^2 - 10x$	AO1	A2	A1 for $2x^2$ A1 for $-10x$
4 c	$5x - 20 + 3x + 30$ $8x + 10$	AO1	A1	brackets expanded correctly
	$8x + 10$	AO1	A1	answer simplified correctly
5	$C = \pi d$ $= \pi \times 44$	AO2	A1	
	$= 138.230077$	AO2	A1	
	138.2 cm	AO2	A1	
6 a	6	AO1	A1	
6 b	8	AO1	A1	
6 c	12	AO1	A1	
7	Triangles share side BD	AO2	M1	
	Parallelogram so opposite sides are the same length $AB = DC$ $AD = BC$	AO2	M1	
	By SSS triangles are congruent	AO2	M1	

8	 Rainfall (mm) Month	AO2	M1 A3	
9	Buys $£3.79 \times 4 = £15.16$ $12 \times 4 = 48$ packets bought	AO3	M1	
	Earns $30 \times £0.42 = £12.60$ $10 \times £0.50 = £5$	AO3	A1	
	Total = £17.60	AO3	A1	
	Profit $£17.60 - £15.16$ $£2.44$	AO3	A1	
10	$1.2 \text{ m} \div 0.3 = 4$	AO3	M1	
	$0.8 \text{ m} \div 0.2 = 4$	AO3	A1	
	$0.75 \text{ m} \div 0.15 = 5$	AO3	M1	
	$4 \times 4 \times 5 = 80$	AO3	A1	
11	Check students' drawings	AO1	A3	
12	$5 \times 6.5 = 32.5$	AO2	M1	
	$6 \times 7 = 42$	AO2	A1	
	$42 - 32.5 = 9.5$	AO2	A1	

Mark scheme

Question	Answer	AO	Marks	Notes																		
1	Bus 4R 12.00, 12.12, 12.24, 12.36, 12.48, 13.00 Bus 5 12.00, 12.16, 12.32, 12.48	AO3	M1 A1	M1 listing multiples A1 correct multiples																		
	12.48	AO3	A1																			
2 a	1904	AO1	M1 A1	M1 attempted valid method A1 for correct answer																		
2 b	23	AO1	M1 A1	M1 attempted valid method A1 for correct answer																		
3	<table border="1"><tr><td>x</td><td>-3</td><td>-2</td><td>-1</td><td>0</td></tr><tr><td>y</td><td>6</td><td>1</td><td>-2</td><td>-3</td></tr></table> <table border="1"><tr><td>x</td><td>1</td><td>2</td><td>3</td></tr><tr><td>y</td><td>-2</td><td>1</td><td>6</td></tr></table> (-3, 6), (-2, 1), (-1,-2), (0,-3), (1,-2), (2, 1), (3, 6)	x	-3	-2	-1	0	y	6	1	-2	-3	x	1	2	3	y	-2	1	6	AO2	M1 A2	M1 table or similar drawn A1 correct coordinates (0,-3), (1,-2), (2, 1), (3, 6) A1 correct co-ordinates (-3, 6), (-2, 1), (-1,-2)
	x	-3	-2	-1	0																	
y	6	1	-2	-3																		
x	1	2	3																			
y	-2	1	6																			
	AO2	M1 A1	M1 axis drawn correctly A1 points correctly plotted and joined with a smooth curve																			
4 a	$2e + 6f$	AO1	A1																			
4 b	$2a^2 + 5a$	AO1	A2	A1 for $2a^2$ A1 for $+ 5a$																		
5 a	6%	AO1	A1																			
5 b	$\frac{14}{25}$	AO2	A1	do not accept unsimplified versions																		
5 c	0.4	AO2	A1																			
5 d	145%	AO1	A1																			
6	Area = $\frac{1}{2}$ base $\times$ height $9 = \frac{1}{2}b \times 6$ $9 = 3b$ $b = 3$	AO2	M1	use of formula or similar																		
	3 m	AO2	A1																			

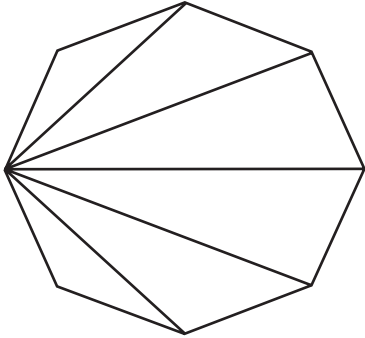
7 a		AO1	1	A1
7 b		AO1	1	A1
7 c		AO1	2	A1 -3 and 2 marked A1 correct filled/unfilled circles
8 a	2 km	AO2	A1	
8 b	10 mins	AO2	A1	
8 c	20 mins = 10 km 60 mins = 30 km	AO3	M1	
	30 km/h	AO3	A1	
9 a	41	AO1	A1	
9 b	35 37 37 38 40 40 41 41 41 45	AO2	M1	
	40	AO2	A1	
9 c	39.5	AO2	A1	
9 d	10	AO1	A1	
9 e	They had a higher mean – so on average did better but their range was a lot larger meaning they were much more inconsistent.	AO3	A2	A1 for comparing mean with explanation A1 for comparing range with explanation
10 a	H1 H2 H3 H4 H5 H6 H7 H8 T1 T2 T3 T4 T5 T6 T7 T8	AO1	A2	-1 for every forgotten combination
10 b	H1 H2 H3 H4 H5 H6 H7 H8 T1 T2 T3 T4 T5 T6 T7 T8	AO2	A1	
	$\frac{1}{4}$	AO2	A1	

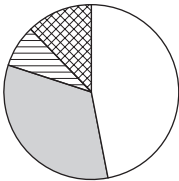
Mark scheme

Question	Answer	AO	Marks	Notes
1	1 4 9 16 25 36	AO1	A2	–1 for every mistake
2 a	3.582427399568..	AO1	A1	AWRT
2 b	3.58	AO1	A1	ft
3 a	Rhombus	AO1	A1	
3 b	Triangular based pyramid	AO1	A1	accept tetrahedron
3 c	Regular hexagon	AO1	A1	
3 d	Isosceles triangle	AO1	A1	
4 a	$3(x + 4)$	AO1	A1	
4 b	$4y(x - 2y)$	AO1	A2	A1 for $4y(x...)$ A1 for $- 2y$
4 c	$6ab(2a + 3b)$	AO1	A2	A1 for $6ab(2a$ A1 for $+3b$
5 a	0.125×230 or $10\% + 2.5\%$	AO1	M1	
	£28.75	AO1	A1	
5 b	1.18×480 or find 18% and add on	AO1	M1	
	£566.40	AO1	A1	
5 c	0.946×60 or find 5.4% and subtract	AO2	M1	
	£56.76	AO2	A1	
6 a	$\pi \times r^2 \div 2$	AO2	M1	
	153.9	AO2	A1	AWRT
	77 m ²	AO2	A1	AWRT
6 b	$\pi \times d \div 2$	AO2	M1	M1
	43.98	AO2	A1	AWRT
	36 m	AO2	A1	AWRT
7	$v = u + at$ $u = 4$ $a = 0.5$ $t = 3$	AO2	M1	attempt at substitution
	$v = 4 + 0.5 \times 3$	AO2	M1	
	5.5 m/s	AO2	A1	
8	Not a sensible sample because these people obviously eat out as they are eating at the restaurant.	AO3	A2	
9	Enlargement	AO2	A1	
	Scale factor 0.5	AO2	A1	
	Centre (3, 5)	AO2	A1	
10	1 part = 25 000	AO2	A1	
	9 parts = 225 000	AO2	M1 A1	M1 9 parts in total A1 £225 000

11	$\frac{45}{52} = 86.5\%$ $\frac{50}{60} = 83.3\%$	AO3	M1 A1	M1 writing marks for one test in fraction format A1 writing marks for one test in percentage (or decimal)
	Julie did better in her science test as if you change the scores into percentages her percentage in science was 86.5% and maths 83.3%	AO3	A2	A1 for science A1 for explanation of comparing percentages

Mark scheme

Question	Answer	AO	Marks	Notes
1 a	9	AO1	A1	
1 b	121	AO1	A1	
1 c	4	AO1	A1	
1 d	243	AO1	A1	
2 a	$T = 5.5 + 0.2n$	AO1	A1	
2 b	£8.50	AO2	A1	
2 c	9.7 – 5.5	AO2	M1	
	21	AO2	A1	
3 a	8	AO1	A1	
	16	AO1	A1	
3 b	10% = 48 20% = 96 30% = 144 5% = 24 35% = 168	AO1	M1	M1 for finding 10% and another percentage such as 5%, 20%, or 30%
	168 kg	AO1	A1	
3 c	25% = 16	AO1	A1	
	\$80	AO1	A1	
4 a	 $6 \times 180 = 1080^\circ$ or $(n - 2) \times 180$ where $n = 8$ or $360/8 = 45^\circ$ external $180 - 45 = 135^\circ$ internal 135×8	AO2	M1	attempt at a valid method
	1080°	AO2	A1	
4 b	120°	AO1	A1	
4 c	$360 \div 6 = 60^\circ$ external $180 - 60 = 120^\circ$	AO2	M1	
	120°	AO2	A1	

5	 Sum = 120 $360 \div 120 = 3^\circ$ per person Walk $57 \times 3 = 171^\circ$ Bike $39 \times 3 = 117^\circ$ Bus $10 \times 3 = 30^\circ$ Car $14 \times 3 = 42^\circ$	AO2	A4	A2 for angles correctly calculated A1 for pie chart accurately drawn A1 for sectors labelled
6 a	$\begin{pmatrix} -4 \\ 3 \end{pmatrix}$	AO1	A1	
6 b	$(3 + 7)/2 = 5$ $(-1 + -4)/2 = -2.5$	AO1	M1	valid method used
	(5, -2.5)	AO1	A1	
6 c	$4^2 + 3^2 = 25$ $\sqrt{25} = 5$	AO3	M1 A1	M1 using Pythagoras A1 25
	5	AO3	A1	
7	$(x + 8)(x - 2) = 0$	AO1	A3	A1 $(x +) (x -) = 0$ A1 ± 2 and ± 8 A1 $(x + 8) (x - 2) = 0$
	$x = -8$ or $x = 2$	AO1	A1	
8 a	$\frac{36}{99} = \frac{4}{11}$	AO2	A1	
	$\frac{4}{11}$	AO2	A1	
8 b	$5 \div 8$ $\begin{array}{r} 0.625 \\ 8 \overline{) 5.000} \\ \underline{40} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array}$	AO1	M1	
	0.625	AO1	A1	
9	$16 \times 30 = 480 \text{ m}^2$	AO3	A1	
	$480 \div 20 = 24$	AO3	A1	
	$24 \times 4.5 =$ $24 \times 4 = 96$ $24 \times 0.5 = 12$ Total = 108 £108	AO3	A1	

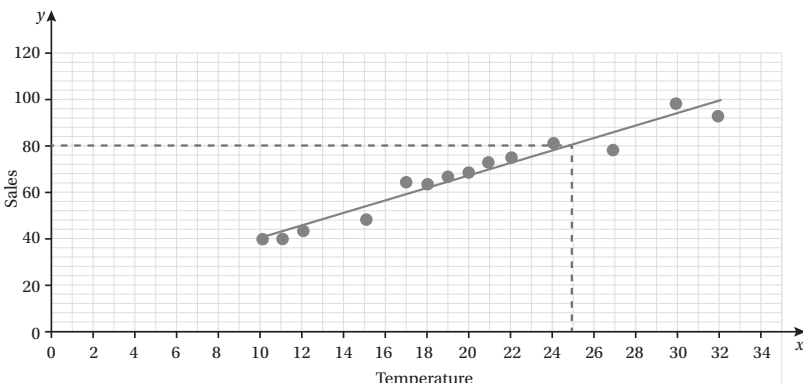
Mark scheme

Question	Answer	AO	Marks	Notes
1	Factors of 98 1, 98, 2, 49, 7, 14 Of these 14 is the highest factor of 168	AO1	M1	listing factors using prime factorisation
	14	AO1	A1	
2 a	b^8	AO1	A1	
2 b	S^6	AO1	A1	
2 c	$15a^2b^3$	AO1	A2	A1 for $15a^2$ A1 for b^3
2 d	$3s$	AO1	A2	A1 for 3 A1 for s
3	55°	AO2	A1	
	Corresponding angles are equal	AO2	A1	
	angles on a straight line add up to 180°	AO2	A1	
4	Angles in a triangle add up to 180°	AO3	M1	set total to 180°
	$4x + 30 + x - 10 + 2x + 20 = 180$ $7x + 40 = 180$ $7x = 140$ $x = 20$	AO3	M2	M1 for attempt to add expressions M1 for attempt to solve equation
	$x = 20^\circ$	AO3	A1	
5 a	3 eggs 75 g sugar 90 g flour 75 g butter 1½ tsp vanilla essence	AO3	A2	– 1 for each mistake
5 b	6 eggs – 3 batches 24 cakes 150 g sugar – 3 batches 24 cakes 150 g flour – 2.5 batches 20 cakes 200 g butter – 4 batches 32 cakes	AO3	M2	correctly found number of cakes for 2 ingredients
	20 cakes	AO3	A1	
6 a	$3y + 11 \geq 5y - 7$ $11 \geq 2y - 7$ $18 \geq 2y$	AO1	M1	collect ys or constants
	$9 \geq y$	AO1	A1	
6 b	$3 < 4x - 9 \leq 23$ $12 < 4x \leq 32$ $3 < x \leq 8$	AO1	M1	collect constants
	$3 < x \leq 8$	AO1	A1	
6 c	4, 5, 6, 7, 8	AO1	A2	– 1 for each mistake
7	Students' own drawings	AO1	A2	
8 a	$196\,200/109 \times 100$	AO3	M1	
	£180 000	AO3	A1	
8 b	$5720/88 \times 100$	AO3	M2	M1 for 88 or 0.88 M1 for $\div 88$ or 0.88
	£6500	AO3	A1	
9 a	$35/1.26$	AO2	M1	
	£27.78	AO2	A1	

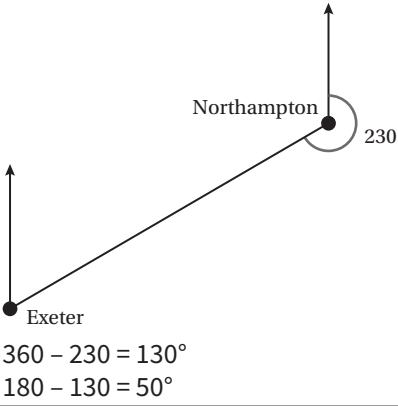
9 b	40×1.26	AO2	A1	
	€50.40	AO2	A1	
9 c	$75 \times 1.26 = \text{€}94.50$ Compared to €90 Or $90/1.26 = \text{£}71.43$ Compared to £75	AO3	M2	M1 for attempt to change euros to pounds or vice versa M1 for €90.40 or £71.43
	Better deal to pay in euros	AO3	A1	

Mark scheme

Question	Answer	AO	Marks	Notes
1 a	51 °C	AO1	A1	
1 b	$(-15 + -11)/2$	AO1	M1	identified -15 and -11 as middle pair
	-13 °C	AO1	A1	
1 c	5 °C	AO3	A1	
2 a	0	AO1	A1	
2 b	7/10	AO1	A1	oe
2 c	Red and red $\frac{7}{10} \times \frac{7}{10} = \frac{49}{100}$ Blue and blue $\frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$ Total $\frac{49}{100} + \frac{9}{100} = \frac{58}{100}$ $= \frac{29}{50}$	AO2	M2	M1 for one multiplication attempted M1 for 2 multiplications added
	$\frac{58}{100}$	AO2	A1	
3	$a - f = bc^2$	AO1	M1	
	$\frac{a - f}{b} = c^2$	AO1	M1	
	$\sqrt{\frac{a - f}{b}} = c$	AO1	A1	
4 a	41, 48	AO1	A1	
4 b	$7n - 1$	AO1	A1	
4 c	146	AO1	A1	
4 d	$97 + 1$	AO2	M1	
	14	AO2	A1	
5 a	$\frac{1}{2}(4 \times 3) \times 7$	AO2	M1	
	42 cm ³	AO2	A2	A1 for 42 A1 for cm ³
5 b	$\frac{1}{2}(4 \times 3) \times 2 = 12$	AO2	M1	breaking shape up into faces
	two correctly calculated faces	AO2	A1	
	$65 \times 7 = 35$ $4 \times 7 = 28$ $3 \times 7 = 21$ 9 cm ²	AO2	A1	

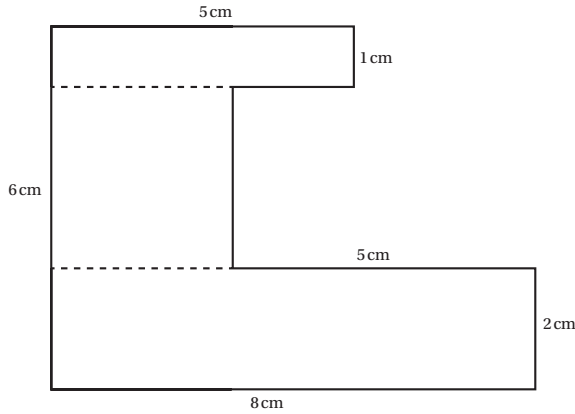
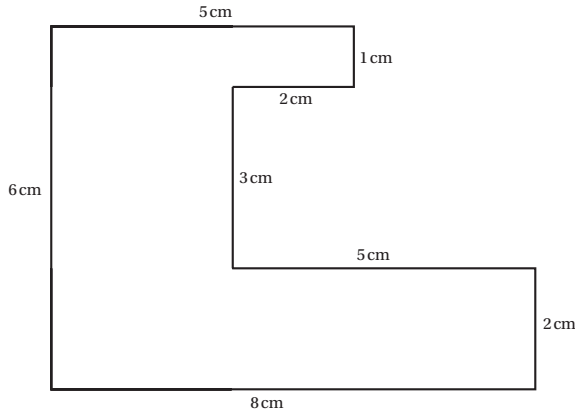
5 c	9600 mm ²	AO2	A1	
6	8 : 12	AO1	A1	
	2 : 3	AO1	A1	
7 a	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$	AO1	A1	
7 b	$\begin{pmatrix} -9 \\ 6 \end{pmatrix}$	AO1	A1	
7 c	$\begin{pmatrix} -8 \\ 5 \end{pmatrix}$	AO1	A1	
7 d	$\begin{pmatrix} 2k \\ -k \end{pmatrix}$	AO2	A1	for any value of k
8 a		AO3	M2 A2	M1 sensible scale drawn M1 scale labelled A2 points plotted – 1 for each mistake (max 2)
8 b		AO3	M2	M1 line of best fit drawn M1 answer read off line of best fit
9	$3x - 2y = 1$ ($\times 3$) $4x - 3y = 0$ ($\times 2$) $9x - 6y = 3$ (1) $8x - 6y = 0$ (2) $(1) - (2)$ $x = 3$ Sub into $3x - 2y = 1$ $9 - 2y = 1$ $2y = 8$ $y = 4$	AO2	M2	M1 for multiply LHS through to get same x or y coefficient M1 for subtract equations
	$x = 3$ $y = 4$	AO2	A2	A1 for $x = 3$ A1 for $y = 4$

Mark scheme

Question	Answer	AO	Marks	Notes
1	0.00000021	AO1	A2	A1 AWRT 0.000 000 2 A1 0.000 000 21
2 a	$x^2 + 3x - 4x - 12$	AO1	A1	
	$x^2 - x - 12$	AO1	A1	
2 b	$(x - 6)(x - 2)$	AO1	A2	A1 ± 6 and ± 2 A1 $(x - 6)(x - 2)$
3	 Northampton Exeter 230 $360 - 230 = 130^\circ$ $180 - 130 = 50^\circ$ 050°	AO3	M1	attempt at sketch
		AO3	A1	
4 a	8 + 10 + 18 + quarter circle	AO2	M1	identifies quarter circle
	Quarter circle radius = 8 $C = \pi d$ $C = \pi \times 16$	AO2	M1	$\pi \times d \div 4$
	Quarter so $\div 4$ $(\pi \times 16) \div 4 = 12.566706$	AO2	A1	AWRT 12.6
	$8 + 10 + 18 + (\pi \times 16) \div 4$ $= 48.5663706$ 48.6 cm	AO2	A1	AWRT 48.6
4 b	8×10 + quarter circle	AO2	M1	identifies quarter circle
	Quarter circle radius = 8 $A = \pi r^2$ $A = \pi \times 64$ Quarter so $\div 4$	AO2	M1	$\pi \times r^2 \div 4$
	$\pi \times 64 \div 4 = 50.26548$	AO2	A1	AWRT 50.3
	$8 \times 10 + \pi \times 64 \div 4 = 130.26548$ 130.3 cm ²	AO2	A1	AWRT 130.3
5	3 + 5 = 8 24 $\div$ 8 = 3 3 $\times$ 3 = 9 5 $\times$ 3 = 15	AO1	M1	identifies 8 parts or that 1 part = 3
	9:15	AO1	A1	
6	reflection	AO1	A1	
	90°	AO1	A1	
	anticlockwise	AO1	A1	
	around (-1, -1)	AO1	A1	

7	h fre MP MP × fre				AO2	M2	M1 mid points M1 mid points × frequencies
	H	fre	MP	MP × fre			
	0 ≤ h < 10	8	5 40				
	10 ≤ h < 30	9	20	180			
	30 ≤ h < 40	24	35	840			
	40 ≤ h < 60	18	50	900			
	60 ≤ h < 100	12	80	960			
	Sum = 2920				AO2	A1	
	2920 ÷ 71 = 41.1				AO2	A1	41.1
8	120 × 1.035 ³				AO2	m2	M1 × 1.035 M1 × 1.035 × 1.035 × 1.035 or × 1.035 ³ give for incorrect percentage
	£133.05				AO2	A1	
9	12 ² +16 ² = 144 + 256				AO2	M1	use of Pythagoras
	= 400				AO2	A1	
	√400 = 20 20 cm				AO2	A1	20
10	2.45 ÷ 6 = 0.408333 3.80 ÷ 9 = 0.4222222 1.52 ÷ 4 = 0.38				AO3	M1 A1	M1 attempt to try and find price per tin or to scale up purchase to a common number of tins A1 price per tin for the three
	Packs of 4 are the best offer Pack of 6 41p per tin Pack of 9 42p per tin Pack of 4 38p per tin				AO3	A2	A1 correct choice A1 explanation
11	Tan(θ) = opp/adj				AO1	A1	identifies tan as correct trigonometric function to use
	Tan(θ) = 8/11					A1	
	θ= tan ⁻¹ (8/11) θ= 36.027					M1	uses an inverse trig function
	36°				AO1	A1	

Mark scheme

Question	Answer	AO	Marks	Notes
1	Ratio of 1 : 6 : 2	AO1	M1	for ratio seen
	$135 \div 9 = 15$	AO1	M1	
	Richard £15 Sabah £90 Tariq £30	AO1	A1	
2 a	$x = 8$	AO1	A1	
2 b	$12 + 3x = 7x - 24$ $12 = 4x - 24$ $36 = 4x$ $9 = x$	AO2	A2	
	$x = 9$	AO2		
3 a		AO3	M1	for splitting shape (any split)
	$5 \times 1 = 5$ $3 \times 3 = 9$ $2 \times 8 = 16$	AO3	M1	
	Area = 30 cm^2	AO3	A1	
3 b		AO3	M1	find missing lengths
	$5 + 1 + 2 + 3 + 5 + 2 + 8 + 6$ Perimeter = 32 cm	AO3	A1	
4	Gradient = $(11 - 5) \div (5 - 3)$	AO3	M1	for $(11 - 5)$ and $(5 - 3)$
	Gradient = 3	AO3	A1	
	Sub in to find c $11 = 3 \times 5 + c$ $c = -4$	AO3	M1	
	$y = 3x - 4$	AO3	A1	

5 a	$\frac{2}{15} + \frac{3}{10}$ $= \frac{4}{30} + \frac{9}{30}$ $= \frac{13}{30}$	AO1	M1	common denominator
	$\frac{13}{30}$	AO1	A1	
5 b	$1\frac{2}{5} - \frac{2}{3}$ $= \frac{7}{5} - \frac{2}{3}$ $= \frac{21}{15} - \frac{10}{15}$ $= \frac{11}{15}$	AO2	M1	change mixed number to improper fraction
	$\frac{11}{15}$	AO2	A1	
5 c	$\frac{6}{7} \times \frac{3}{8}$ $= \frac{18}{56}$ $= \frac{9}{28}$	AO1	M1	multiply numerators and denominators
	$\frac{9}{28}$	AO1	A1	simplified answer
5 d	$\frac{14}{15} \div \frac{7}{10}$ $= \frac{14}{15} \times \frac{10}{7}$ $= \frac{2}{3} \times \frac{2}{1}$ $= \frac{4}{3}$ $= 1\frac{1}{3}$	AO1	M1	'flip' and multiply
	$1\frac{1}{3}$	AO1	A1	answer as a mixed number
6 a	$12 \div 8$	AO2	M1	
	9 cm	AO2	A1	
6 b	$8 \div 12$	AO2	M1	
	2.5 cm	AO2	A1	
6 c	40° ABE and ACD are similar hence AEB and EDC are equal or corresponding angles are equal	AO2	A1	must include reason
7 a	1	AO1	A1	
7 b	1/49	AO1	A1	

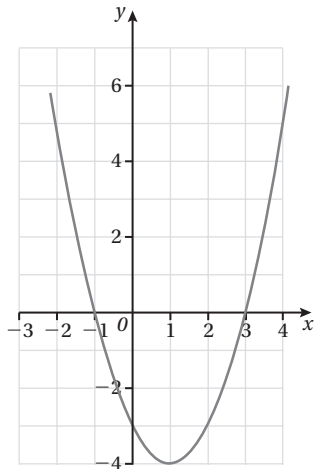
7 c	$\sqrt[3]{\quad}$	AO2	M1	
	9	AO2	A1	
7 d	$\left(\frac{16}{25}\right)^{\frac{1}{2}}$ $\left(\frac{25}{16}\right)^{\frac{1}{2}}$ $\frac{5}{4}$	AO2	M1	reciprocate
	$\frac{5}{4}$ or $\frac{1}{4}$	AO2	A1	
8	$aw + b = w - h$ $b + h = w - aw$	AO2	M1	attempt to collect w only on one side
	$b + h = w(1 - a)$	AO2	M1	factorise w
	$w = \frac{b + h}{1 - a}$	AO2	A1	
9	Upper quartile = $28 + 12 = 40$	AO2	M1	
	Accurately drawn box plot Lowest = 19 Lower quartile = 28 Median = 33 Upper quartile = 40 Highest = 58	AO2	M1	for box plot seen with 3 or 4 values in correct place
	Box plot seen with all values correct	AO2	A1	

Mark scheme

Question	Answer	AO	Marks	Notes
1	$y = \frac{x}{2} + 3$	AO2	A2	A1 for correct gradient A1 for correct y intercept
2 a	$3x + 12$	AO1	A1	
2 b	$2x^2 - 10x$	AO1	A2	1 mark for $2x^2$ 1 mark for $- 10x$
2 c	$5x - 20 + 3x + 30$	AO1	A1	brackets expanded correctly
	$8x + 10$	AO1	A1	answer simplified correctly
3	$\pi \times 44$	AO2	A1	
	AWRT 138.2 cm	AO2	A1	
	120 000 cm	AO2	A1	
	868	AO2	A1	
4	3.79×4	AO3	M1	
	17.60	AO3	A1	
	profit £2.44	AO3	A1	
	profit/ 15.16×100	AO3	M1	
	16%	AO3	A1	
5	5×6.5	AO2	M1	
	42	AO2	A1	
	9.5	AO2	A1	
6 a	$d = kt^2$	AO3	A1	
	$44.1 = k \times 9$	AO3	M1	
	4.9	AO3	A1	
	$d = 4.9t^2$	AO3	A1	
6 b	$D = 4.9 \times 5^2$	AO2	M1	
	122.5 m		A1	
6 c	$110 = 4.9t^2$	AO2	M1	
	$t = \sqrt{(110/4.9)}$	AO2	M1	
	4.74 secs	AO2	A1	
7	98.5×42.5	AO1	M1	
	4186.25 m^2	AO1	A1	
8	F.d 0.8 0.45 1.6 0.9 0.3 histogram	AO2	M1 A4	M1 axis drawn and labelled A2 frequency densities found (–1 for each error, max –2) A1 bars drawn in correct widths (ignore heights) A1 bars correct heights

9	$\tan(35) = 9.5/PQ$ $PQ = 9.5/\tan(35)$ $PQ = 13.567\ 406$	AO2	M1	finding PQ
	$9.5/\tan(35)$ or AWRT 13.6	AO2	A1	
	$\tan(a) = PQ/SQ$	AO2	M1	
	$\tan(a) = \text{AWRT } 0.66$	AO2	A1	
	AWRT 33°	AO2	A1	

Mark scheme

Question	Answer	AO	Marks	Notes																		
1	<table><tr><td>x</td><td>-2</td><td>-1</td><td>0</td><td>1</td></tr><tr><td>y</td><td>5</td><td>0</td><td>-3</td><td>-4</td></tr></table> <table><tr><td>x</td><td>2</td><td>3</td><td>4</td></tr><tr><td>y</td><td>-3</td><td>0</td><td>5</td></tr></table> (-2, 5), (-1, 0), (0,-3), (1,-4), (2,-3), (3, 0), (4, 5)	x	-2	-1	0	1	y	5	0	-3	-4	x	2	3	4	y	-3	0	5	AO2	M1 A2	M1 table or similar drawn A1 correct coordinates (1,-4), (2,-3), (3, 0), (4, 5) A1 correct coordinates (-2, 5), (-1, 0), (0,-3)
	x	-2	-1	0	1																	
y	5	0	-3	-4																		
x	2	3	4																			
y	-3	0	5																			
	AO2	M1 A1	M1 axis drawn correctly A1 points correctly plotted and joined with a smooth curve																			
2 a	$\frac{14}{25}$	AO2	A1	do not accept unsimplified versions																		
2 b	0.1875	AO2	A1																			
2 c	145%	AO1	A1																			
3	Area = $\frac{1}{2}$ base $\times$ height $9 = \frac{1}{2}b \times 6$ $9 = 3b$ $b = 3$	AO2	M1	use of formula or similar																		
	3 m	AO2	A1																			
4 a	H1 H2 H3 H4 H5 H6 H7 H8 T1 T2 T3 T4 T5 T6 T7 T8	AO1	A2	-1 for every forgotten combination																		
4 b	H1 H2 H3 H4 H5 H6 H7 H8 T1 T2 T3 T4 T5 T6 T7 T8	AO2	A1																			
	$\frac{1}{2}$	AO2	A1																			
5 a	$s = k/d$	AO2	A1																			
	$s = 2/d$	AO2	M1	substitutes values into formula above																		
	2	AO2	A1																			
	$s = 2/d$	AO2	A1																			

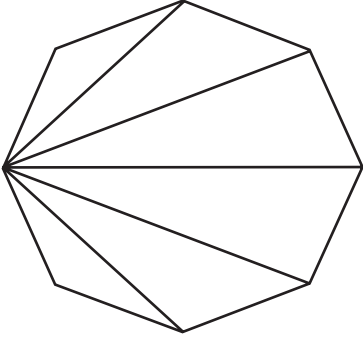
5 b	$s = 2/0.2$	AO2	M1	substitutes values into own formula
	10	AO2	A1	
5 c	$0.05 = 2/d$ $d = 2/0.05$	AO2	M1	substitutes values into own formula
	40	AO2	A1	
6		AO3	M1	attempt to factorise
	$(x + 2)(2x + 1)$	AO3	A1	
	adds side lengths	AO3	M1	
	sides lengths total $6x + 6$	AO3	A1	
	sets perimeter = 42	AO3	M1	
	$x = 6$	AO3	A1	
	13 cm by 8 cm	AO3	A1	
7 a	$-2a$	AO1	A1	
	$2a + b$	AO2	A1	
	$a - \frac{1}{2}b$	AO2	A2	
	$\overrightarrow{MB} = a + b$ $\overrightarrow{BE} = 1/3 AB$ $\overrightarrow{BE} = 1/3 (b - a)$ $\overrightarrow{ME} = \overrightarrow{MB} + \overrightarrow{BE}$ $= a + b + 1/3 (b - a)$ $= 2/3a + 4/3b$	AO3	M2	
	$ME = 2/3a + 4/3b$	AO3	A1	
8 a	$2\sqrt{2}$	AO1	A1	
8 b	42	AO2	A1	
8 c	$2 + 2\sqrt{2} + \sqrt{2} + \sqrt{4}$	AO2	A1	two parts of $2 + 2\sqrt{2} + \sqrt{2} + \sqrt{4}$
	$4 + 3\sqrt{2}$	AO2	A1	

Mark scheme

Question	Answer	AO	Marks	Notes
1 a	3.582 427 399 568..	AO1	A1	AWRT
1 b	3.58	AO1	A1	ft
2 a	$4y(x - 2y)$	AO1	A2	A1 for $4y(x...)$ A1 for $-2y$
2 b	$6ab(2a + 3b)$	AO1	A2	A1 for $6ab(2a$ A1 for $+3b)$
2 c	$(x - 4)(x + 4)$	AO1	A2	A1 for $(x - a)(x + a)$
3 a	0.125×230 or $10\% + 2.5\%$	AO1	M1	
	£28.75	AO1	A1	
3 b	1.18×480 or find 18% and add on	AO1	M1	
	£566.40	AO1	A1	
3 c	0.946×60 or find 5.4% and subtract	AO2	M1	
	£56.76	AO2	A1	
4 a	$\pi \times r^2 \div 2$	AO2	M1	
	153.9	AO2	A1	AWRT
	77 m^2	AO2	A1	AWRT
4 b	$\pi \times d \div 2$	AO2	M1	
	43.98	AO2	A1	AWRT
	36 m	AO2	A1	AWRT
5	Enlargement	AO2	A1	
	Scale factor -0.5	AO2	A1	
	Centre (1, 1)	AO2	A1	
6 a	Gradient = -3	AO2	M1	
	$-4 = -3 - 2 + c$	AO2	M1	
	$y = 2 - 3x$	AO2	A1	
6 b	$\frac{1}{3}$	AO1	A1	
7	5.87 or -1.87	AO2	M1 A4	M1 substitution into quadratic formula A1 4 and 16 seen or implied $\text{A1 } \frac{4 + \sqrt{60}}{2}$ $\text{A1 } \frac{4 - \sqrt{60}}{2}$ A1 5.87 and -1.87

8	Cones circumference $= 2\pi r$ $= 2\pi \times 9$	AO3	M1	
	56.549	AO3	A1	AWRT
	Use of Pythagoras	AO3	M1	
	15 cm	AO3	A1	
	Circumference of sector's full circle $2\pi r$ $= 2\pi \times 15$	AO3	M1	
	94.248	AO3	A1	AWRT
	Circumference of cone $\div$ sector's full circle circumference	AO3	M1	
	216°	AO3	A1	

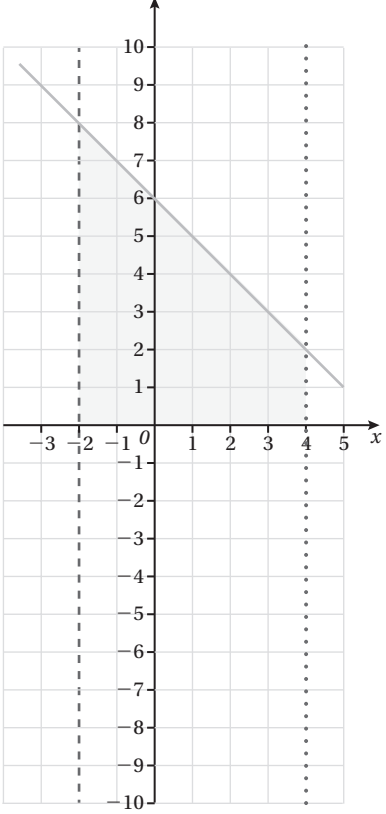
Mark scheme

Question	Answer	AO	Marks	Notes
1 a	8	AO1	A1	
	16	AO1	A1	
1 b	10% = 48 20% = 96 30% = 144 5% = 24 35% = 168	AO1	M1	finding 10% and another percentage such as 5%, 20%, or 30%
	168 kg	AO1	A1	
1 c	25% = 16	AO1	A1	
	\$80	AO1	A1	
2 a	 $6 \times 180 = 1080^\circ$ or $(n - 2) \times 180$ where $n = 8$ or $360/8 = 45^\circ$ external $180 - 45 = 135^\circ$ internal 135×8	AO2	M1	attempt at a valid method
	1080°	AO2	A1	
2 b	120°	AO2	A1	
2 c	$360 \div 6 = 60^\circ$ external $180 - 60 = 120^\circ$	AO2	M1	
	120°	AO2	A1	
3 a	$3 - 7$ $-1 - -4$	AO1	A1	
3 b	$(3 + 7)/2 = 5$ $(-1 + -4)/2 = -2.5$	AO2	M1	valid method used
	(5, -2.5)	AO2	A1	
3 c	$4^2 + 3^2 = 25$	AO3	M1	using Pythagoras
	25	AO3	A1	
	5	AO3	A1	
4	$(x +) (x -) = 0$	AO1	A1	
	± 2 and ± 8	AO1	A1	
	$(x + 8) (x - 2) = 0$	AO1	A1	
	$x = -8$ or $x = 2$	AO1	A1	

5	$4x + 30 + x - 10 + 2x + 20$	AO3	M1	attempt to add expressions
	$7x + 40 = 180$	AO3	M1	set total to 180°
	$7x = 140$ $x = 20$	AO3	M1	attempt to solve equation
	$x = 20^\circ$	AO3	A1	
	110°	AO3	A1	
6	$x = 0.36$ $100x = 36.36$ $99x = 0.36$ $x = \frac{36}{99}$ $\frac{36}{99} = \frac{4}{11}$	AO2	A3	
7 a	54, 70	AO1	A2	
7 b	$n^2 + 3n$	AO2	M2 A1	M1 for attempt to find 2nd difference M1 for comparison to n^2
8	A $y = x^2$ B $xy = 4$ C $y = \cos(X)$ D $y = \sin(X)$ E $y = x^3$ F $x^2 + y^2 = 25$	AO1	A3	–1 for every mistake or omission, max –3
9	843	AO3	A1	
	528	AO3	A1	
	£33	AO3	A1	

Mark scheme

Question	Answer	AO	Marks	Notes
1 a	s^6	AO1	A1	
1 b	$15a^2 b^3$	AO1	A2	A1 for $15a^2$, A1 for b^3
1 c	$3s$	AO1	A2	A1 for 3, A1 for s
1 d	$16x^{12}$	AO1	A2	A1 for 16, A1 for x^{12}
2	Angle at the centre is double the angle at the circumference	AO2	M1	
	Opposite angles in a cyclic quadrilateral (ABCD) sum to 180°	AO2	M1	
	124°	AO2	A1	
3	1 000 000	AO1	A1	
	$1\,700\,000\text{ cm}^3$	AO1	A1	
4 a	$8 + 10 + 18 + \text{quarter circle}$	AO2	M1	
	$\pi \times d \div 4$	AO2	M1	
	12.6	AO2	A1	AWRT
	48.6 cm	AO2	A1	AWRT
4 b	$8 \times 10 + \text{quarter circle}$	AO2	M1	
	$\pi \times r^2 \div 4$	AO2	M1	
	50.26548	AO2	A1	AWRT
	130.3 cm^2	AO2	A1	AWRT
5 a	$3 < 4x - 9 \leq 23$ $12 < 4x \leq 32$ $3 < x \leq 8$	AO1	M1	collect constants
	$3 < x \leq 8$	AO1	A1	
5 b	4, 5, 6, 7, 8	AO1	A2	-1 for each mistake

5 c		AO2	A5	A1 for line $x = -2$ shown dashed A1 for line $x = 4$ shown dotted A1 for line $y = 0$ shown solid A1 for line $x + y = 6$ shown solid A1 for trapezium identified
6 a	÷ 109 or 1.09 £180 000	AO3	M1	
6 b	88 or 0.88 ÷ 88 or 0.88 £6500	AO3	M1	
7	1500 g = £16 800 g = €10.50 100 g = £1.0666666 100 g = €1.3125	AO3	M1 A1	M1 scaled up weights A1 scaled up correctly
	£1.066666 × 1.26 = €1.344	AO3	M1 A1	M1 converted one cost into other currency A1 converted correctly
	Better deal to buy in France. In UK £1.07 for 100 g, which in euros is €1.34.	AO3	A2	A1 for choosing France A1 for explaining reason

Mark scheme

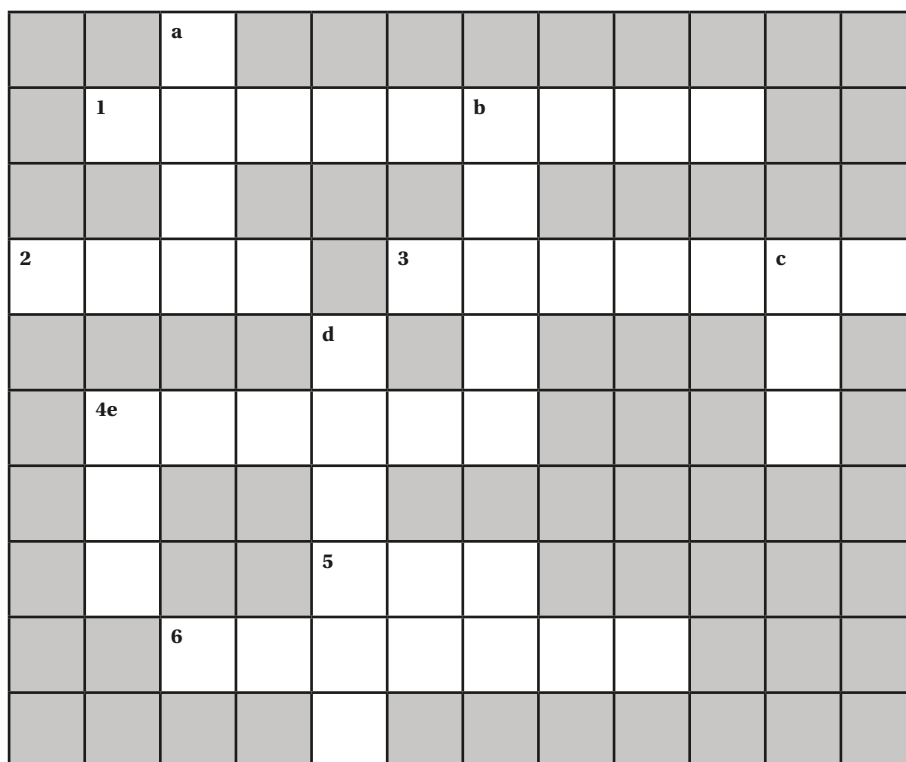
Question	Answer	AO	Marks	Notes
1	£15 spent	AO3	A1	
	20% of cost spent	AO3	M1	
	$15 + 3 = £18$	AO3	A1	
	Wanted income $\div 20$	AO3	M1	
	90p	AO3	A1	
2 a	0	AO1	A1	
2 b	7/9	AO1	A1	oe
2 c		AO2	M1 A2	M1 correct number of branches A1 probabilities on branches A1 colours at ends of branches
2 d	Red and then blue Blue and then red Total	AO2	A3	
3	$a - f = bc^2$	AO1	M1	
	$\frac{a - f}{b} = c^2$	AO1	M1	
	$\sqrt{\frac{a - f}{b}} = c$	AO1	A1	
4 a	$\frac{1}{2}(4 \times 3) \times 7$	AO2	M1	
	42 cm ³	AO2	A2	
4 b	$5 \times 7 = 35$ $4 \times 7 = 28$ $3 \times 7 = 21$	AO2	M1	breaking shape up into faces
	Hypotenuse of triangular cross section	AO2	M1	using Pythagoras' theorem to find missing edge
	Two correctly calculated faces	AO2	A1	
	96 cm ²	AO2	A1	
4 c	9600 mm ²	AO2	A1	
5 a	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$	AO1	A1	
5 b	$\begin{pmatrix} -9 \\ 6 \end{pmatrix}$	AO1	A1	
5 c	$\begin{pmatrix} -8 \\ 5 \end{pmatrix}$	AO1	A1	

5 d	$\begin{pmatrix} 2k \\ -k \end{pmatrix}$	AO2	A1	for any value of k
6	$3x - 2y = 1$ ($\times 3$) $4x - 3y = 0$ ($\times 2$) $9x - 6y = 3$ (1) $8x - 6y = 0$ (2)	AO2	M1	multiply through to get same x or y coefficient
	$(1) - (2)$ $x = 3$	AO2	M1	subtract equations
	$x = 3$	AO2	A1	
	$y = 4$	AO2	A1	
7 a	90 km/h	AO1	A1	
7 b	60 km in 0.25 h	AO2	M1	
	$60 \div 0.25$	AO2	M1	
	240 km/h ²	AO2	A1	
7 c	Area under graph	AO3	M1	
	$\frac{1}{2} h(a + b)$	AO3	M1	
	$\frac{1}{2} \times 0.5 (90 + 70)$	AO3	A1	
	40 km	AO3	A1	

Mark scheme

Question	Answer	AO	Marks	Notes																				
1 a	$x^2 + 3x - 4x - 12$	AO1	A1																					
	$x^2 - x - 12$	AO1	A1																					
1 b	± 6 and ± 2	AO1	A1																					
	$(x - 6)(x - 2)$	AO1	A1																					
1 c	$\frac{18x^3y^{-2}}{48xy^{-4}}$ $= 3x^2y^2$	AO1	A2	A1 for 3 A1 for x^2y^2																				
2	<table><tr><th>h</th><th>Fre</th><th>MP</th><th>MP x fre</th></tr><tr><td>$0 \leq h < 10$</td><td>8</td><td>5</td><td>40</td></tr><tr><td>$10 \leq h < 30$</td><td>9</td><td>20</td><td>180</td></tr><tr><td>$30 \leq h < 40$</td><td>24</td><td>35</td><td>840</td></tr><tr><td>$60 \leq h < 100$</td><td>12</td><td>80</td><td>960</td></tr></table>	h	Fre	MP	MP x fre	$0 \leq h < 10$	8	5	40	$10 \leq h < 30$	9	20	180	$30 \leq h < 40$	24	35	840	$60 \leq h < 100$	12	80	960	AO2	M2	M1 for mid points M1 for mid points $\times$ frequencies
	h	Fre	MP	MP x fre																				
	$0 \leq h < 10$	8	5	40																				
	$10 \leq h < 30$	9	20	180																				
$30 \leq h < 40$	24	35	840																					
$60 \leq h < 100$	12	80	960																					
	2920	AO2	A1																					
	41.1	AO2	A1																					
3	120×1.035^3	AO2	M2	M1 $\times 1.035$ M1 $\times 1.035 \times 1.035 \times 1.035$ or $\times 1.035^3$ give for incorrect percentage																				
	£133.05	AO2	A1																					
4	$a^2 = b^2 + c^2 - 2bccos(A)$	AO2	M1	cosine rule																				
	$a = 8$ $b = 7$ $c = 12$ $64 = 49 + 144 - 2 \times 7 \times 12cos(A)$	AO2	A1	substituted values correctly																				
	193 and 168	AO2	A1																					
	$cos(A) = 129/168$	AO2	M1																					
	$A = cos^{-1}(129/168)$	AO2	M1																					
	40°	AO2	A1	AWRT																				
5	$a = 2.35 - 2.45$ $b = 11.95 - 12.05$ $c = 115 - 125$	AO2	A1	shows range of values for at least one of the variable																				
	Max = $3 \times 125/(11.95 - 2.45)$ Min = $3 \times 115/(12.05 - 2.35)$	AO2	M2	correctly chosen large/small values chosen for max and min																				
	Max = 39.473 684 21	AO2	A1																					
	Min = 35.567 0103	AO2	A1																					
6	$43.5 - 35 = 8.5$	AO1	M1																					
	8.5/35	AO1	M1																					
	24%	AO1	A1																					

7	6	AO2	A1	
	Attempt to use 6 as next value of x	AO2	M1	
	6.833 33	AO2	A1	
	6.854 093	AO2	A1	
	Suggest 6.85	AO2	A1	
8	$x^2 + y^2 = 100$ $y = x + 2$ $x^2 + (x + 2)^2 = 100$	AO3	M1	substitute in $x + 1$
	$x^2 + x^2 + 4x + 4 = 100$	AO3	A1	
	$x^2 + 2x - 48 = 0$	AO3	M1	
	$(x + 8)(x - 6) = 0$	AO3	A1	
	$x = -8$ or $x = 6$	AO3	A1	
	$(-8, -6)$ and $(6, 8)$	AO3	A1	
	Pythagoras	AO3	M1	use of Pythagoras
	19.8 units	AO3	A1	AWRT



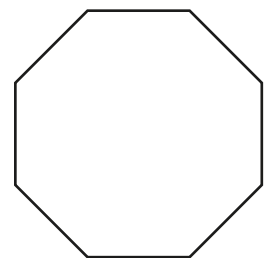
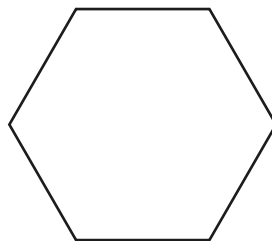
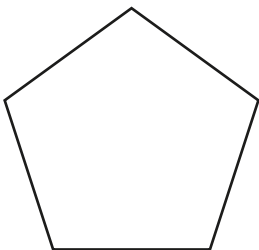
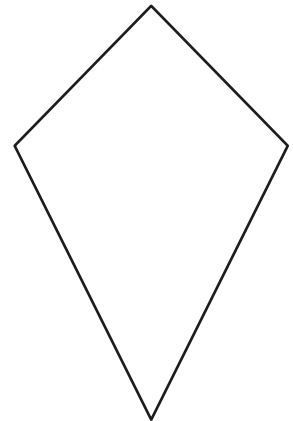
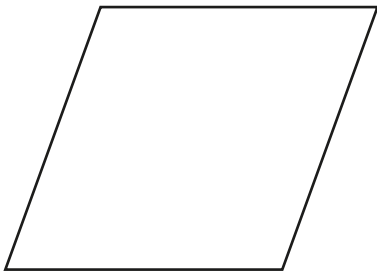
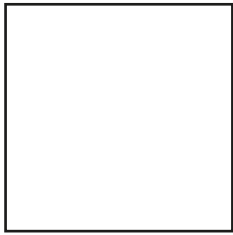
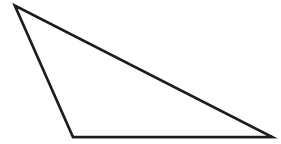
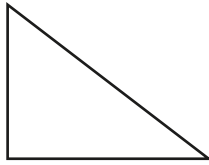
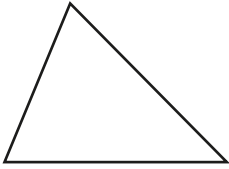
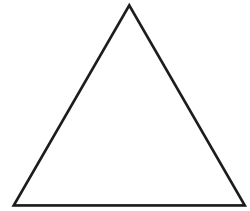
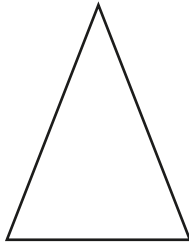
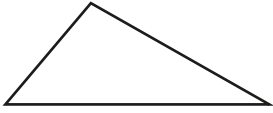
Clues

Across

- 1 The times tables are examples of these.
- 2 Whole numbers divisible by 2.
- 3 Another word for factor.
- 4 Numbers in the sequence 1, 4, 9, 16, ...
- 5 An even prime number.
- 6 The result of a multiplication.

Down

- a $n \times n \times n$ is the __ of n .
- b Numbers with only two factors.
- c Whole numbers that are not exactly divisible by 2.
- d Number that divides into another with no remainder.
- e HCF of 12 and 18.



Angle facts

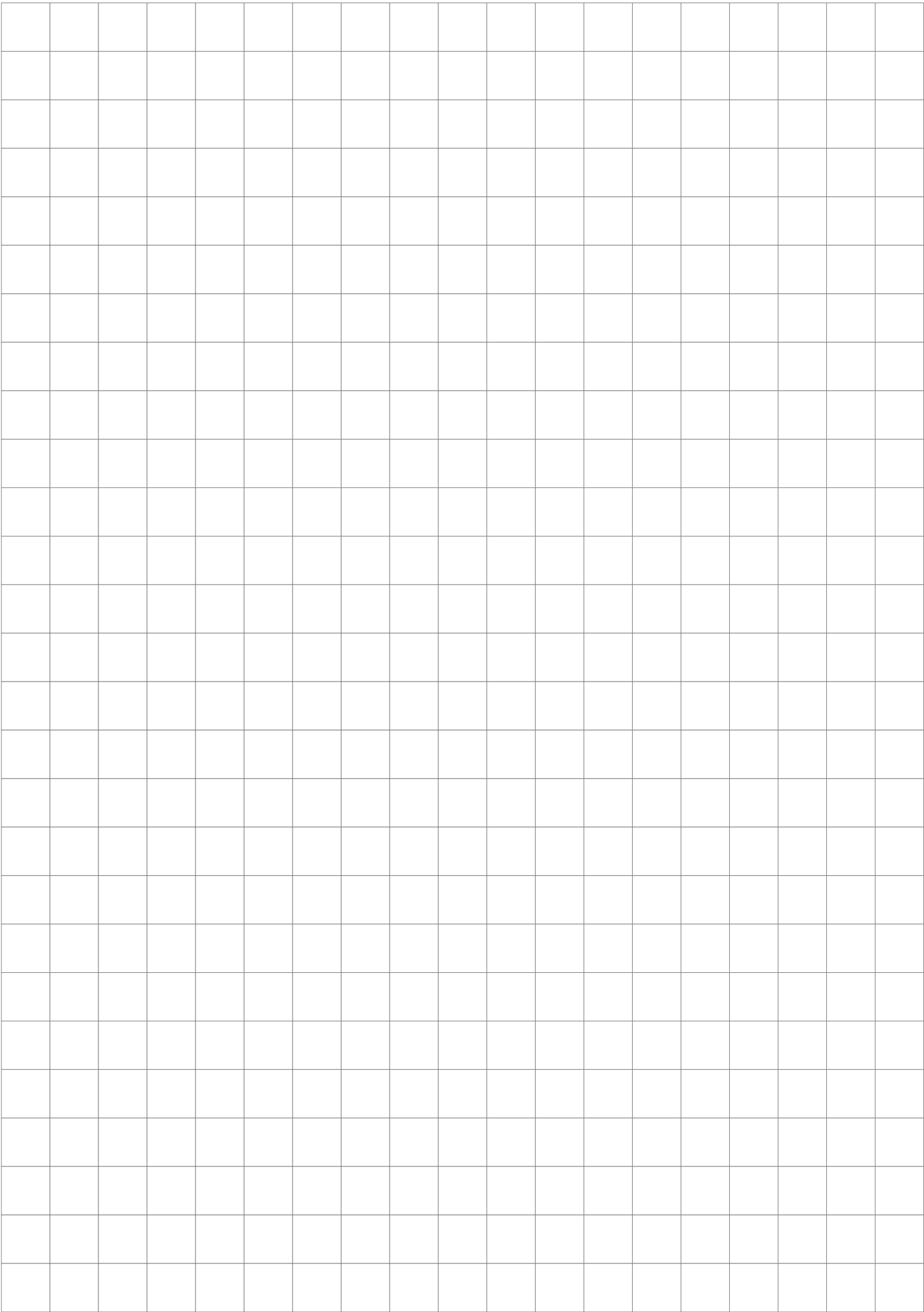
- Angles round a point add up to 360° .
- Angles on a straight line add up to 180° .
- Vertically opposite angles are equal.
- Corresponding angles are equal.
- Alternate angles are equal.
- Co-interior angles add up to 180° .
- Angles in a triangle add up to 180° .
- All angles in an equilateral triangle are 60° .
- The base angles in an isosceles triangle are equal.
- The exterior angle of a triangle is equal to the sum of the two opposite interior angles.
- Interior angles in a polygon equal $(n - 2) \times 180^\circ$ where n is the number of sides.
- Exterior angles of any polygon add up to 360° . For a regular polygon, the size of one exterior angle = $\frac{360}{n}$ where n is the number of sides.

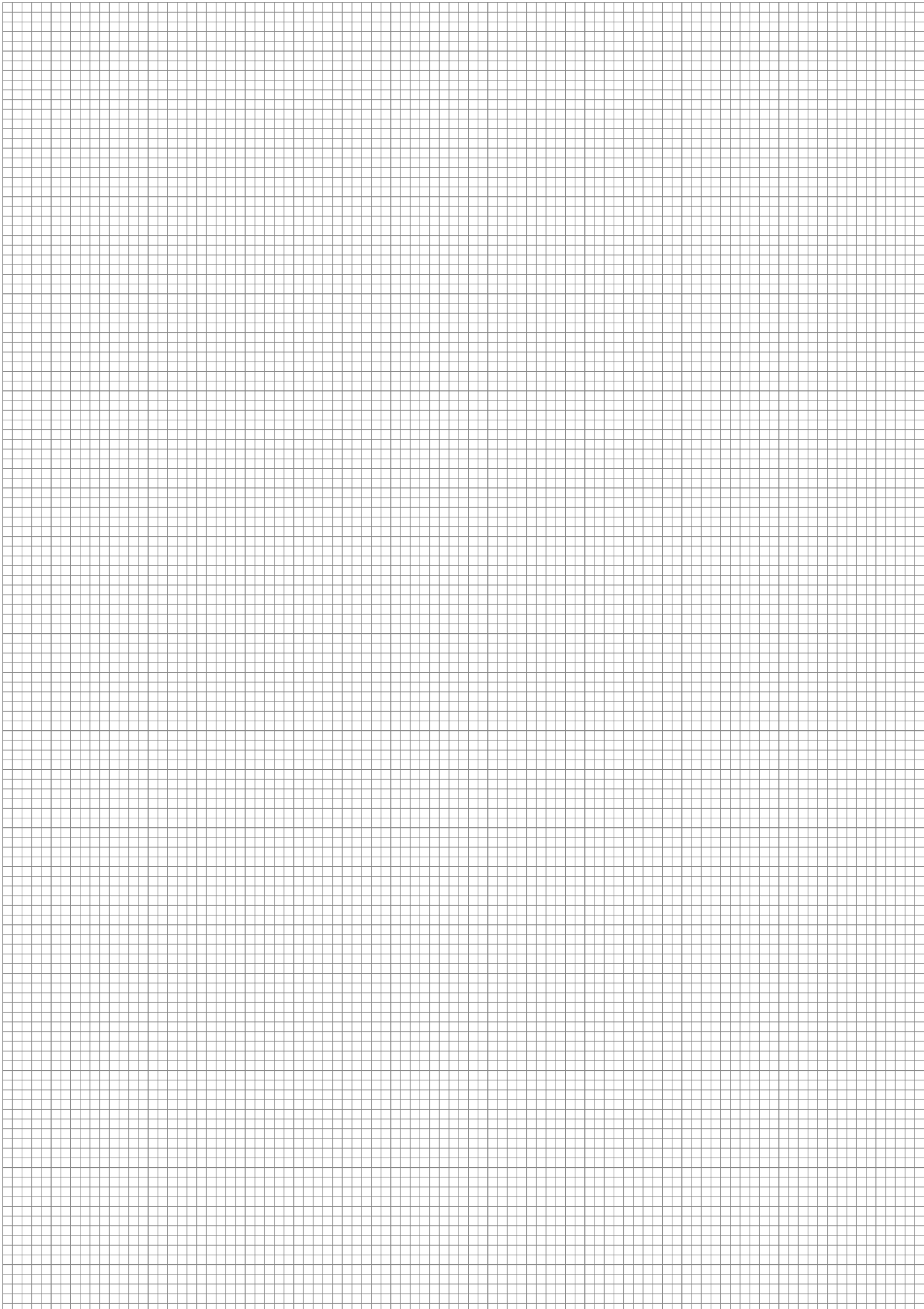
Required formulae – Foundation

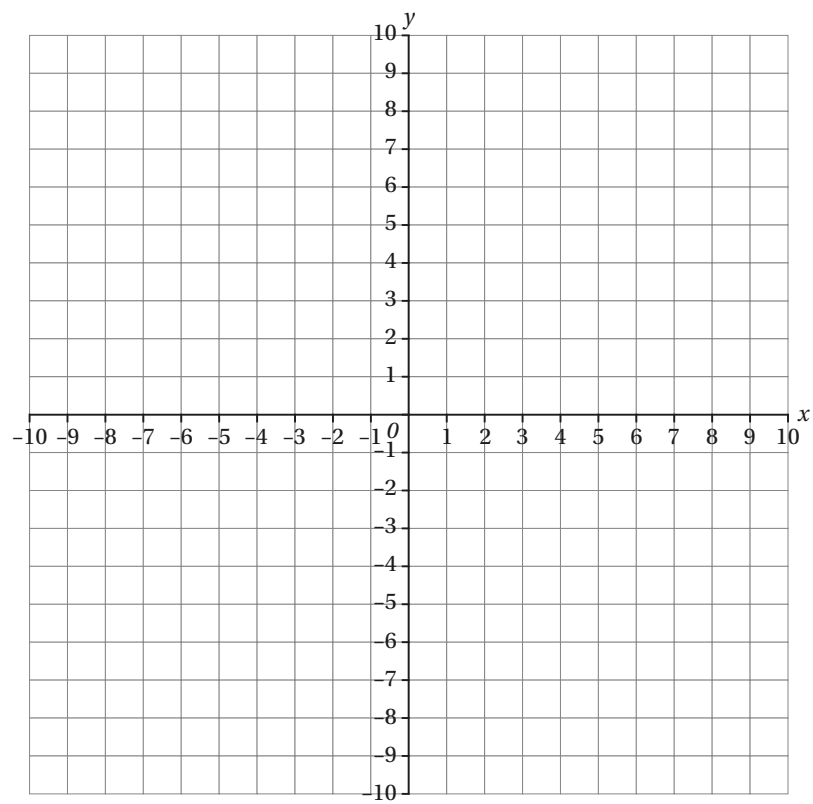
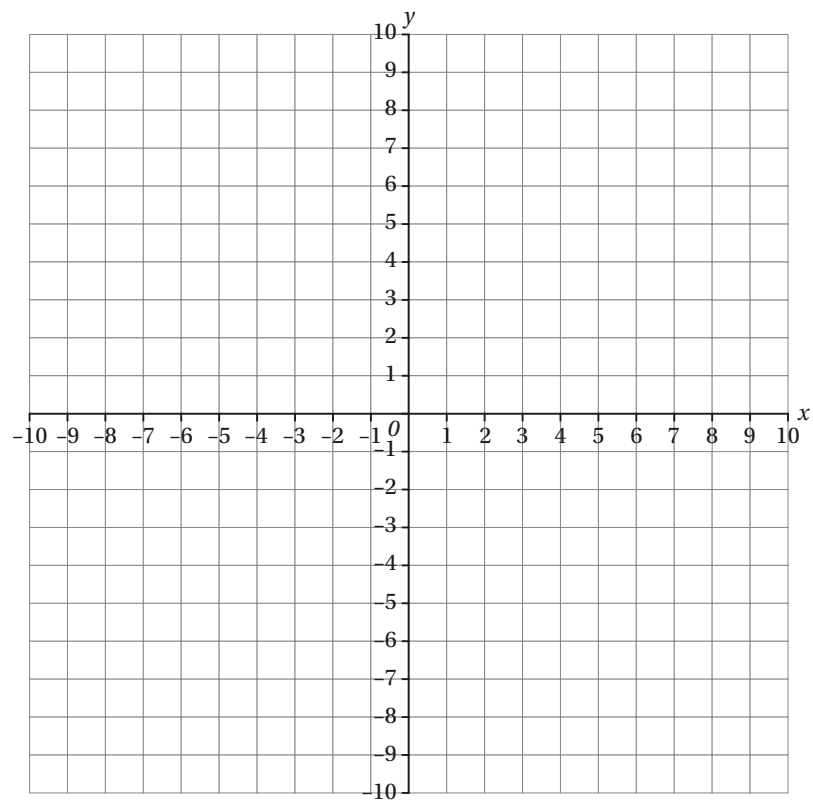
Circumference of a circle	$2\pi r$ or πd
Area of a circle	πr^2
Pythagoras' theorem	$a^2 + b^2 = c^2$
Trigonometry	$\sin \theta = \frac{\text{opp}}{\text{hyp}}$ $\cos \theta = \frac{\text{adj}}{\text{hyp}}$ $\tan \theta = \frac{\text{opp}}{\text{adj}}$

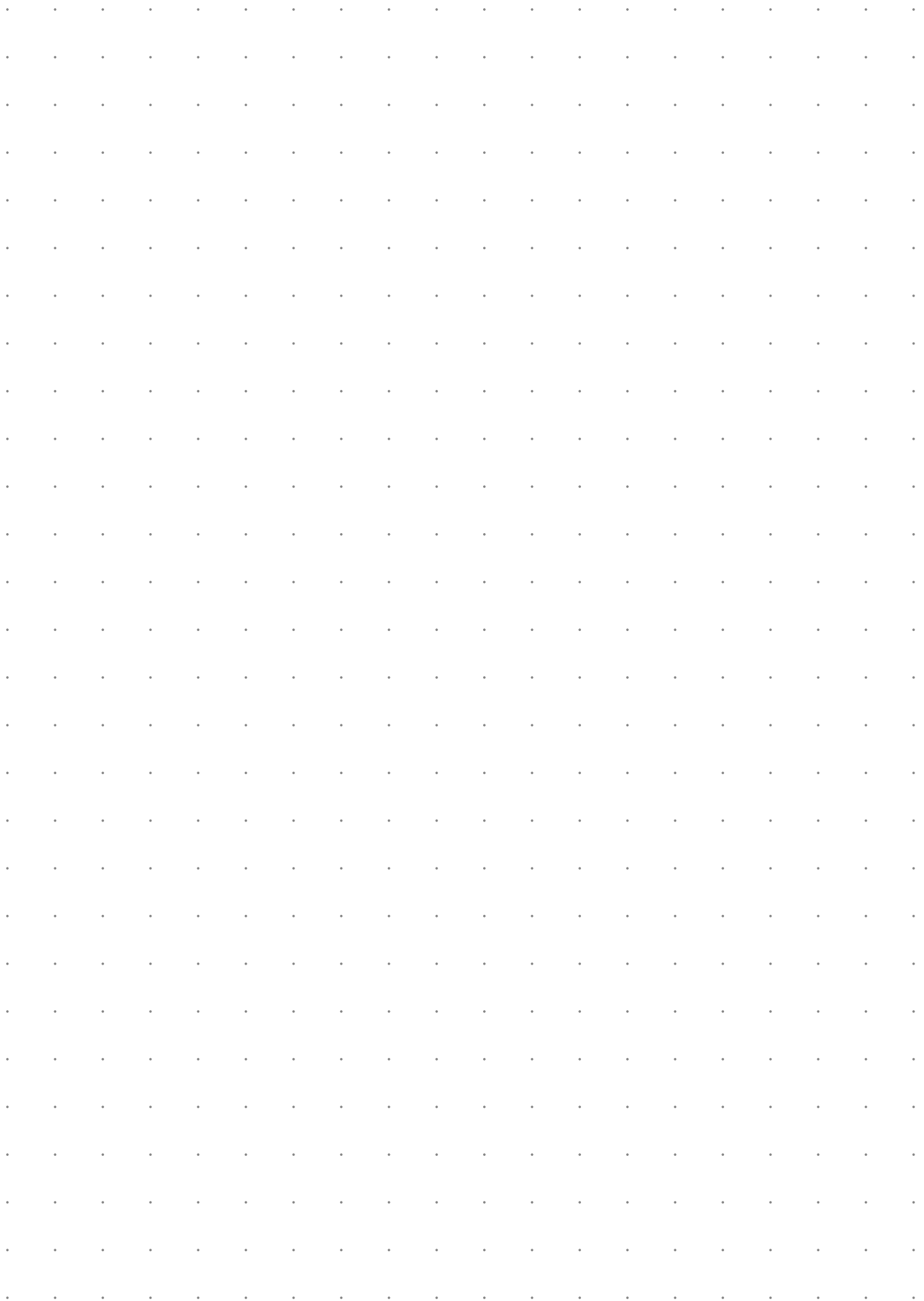
Required formulae – Higher

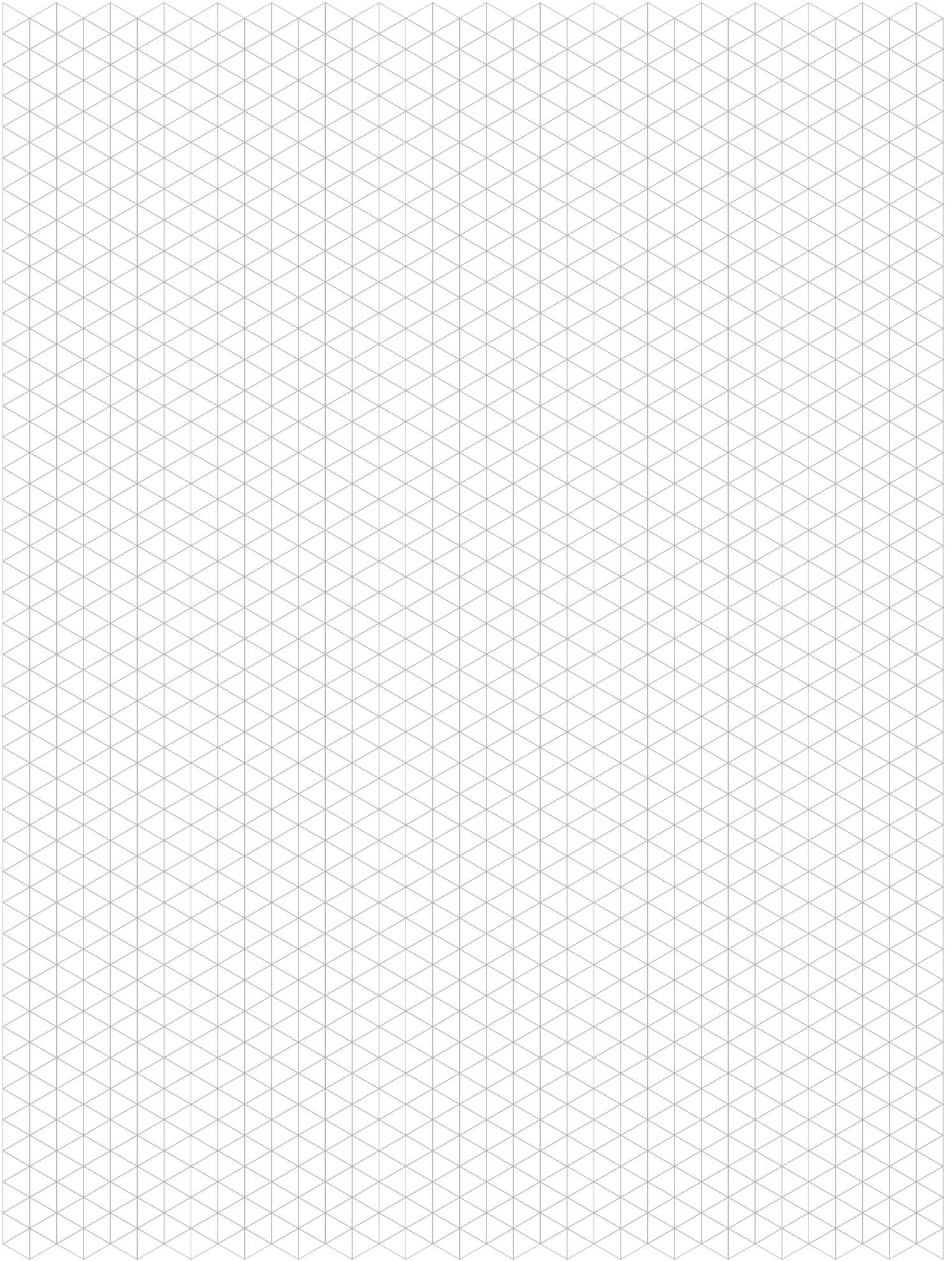
Circumference of a circle	$2\pi r$ or πd
Area of a circle	πr^2
Pythagoras' theorem	$a^2 + b^2 = c^2$
Trigonometry	$\sin \theta = \frac{\text{opp}}{\text{hyp}}$ $\cos \theta = \frac{\text{adj}}{\text{hyp}}$ $\tan \theta = \frac{\text{opp}}{\text{adj}}$
Quadratic formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
Cosine rule	$a^2 = b^2 + c^2 - 2bc \cos A$
Area of a triangle	$\frac{1}{2} ab \sin C$

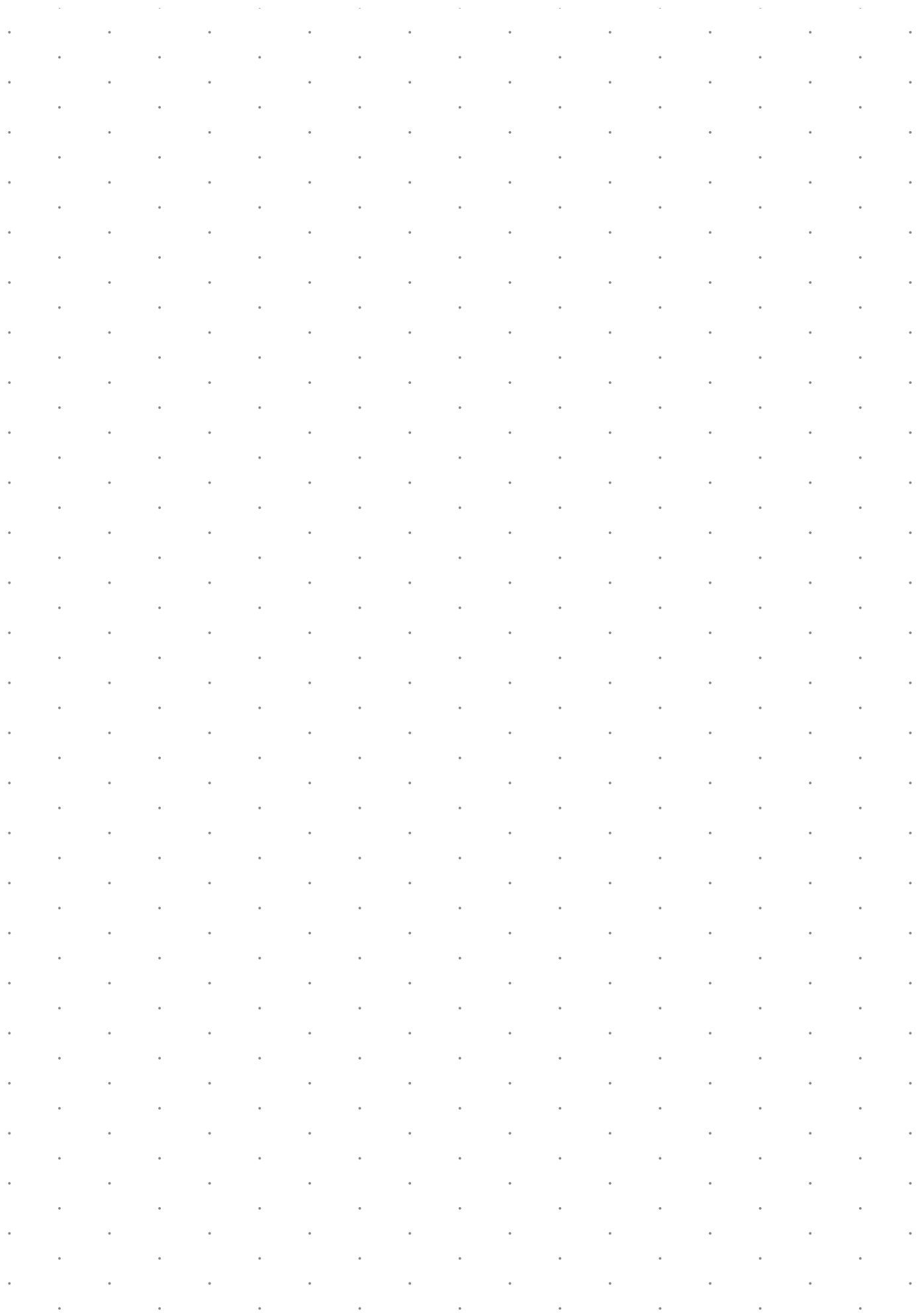




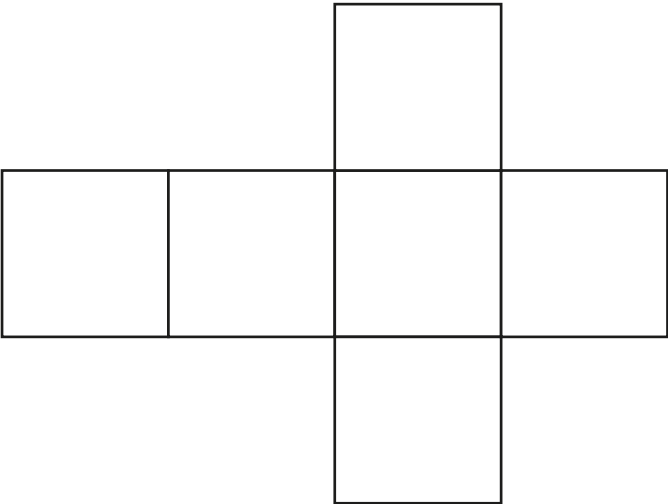




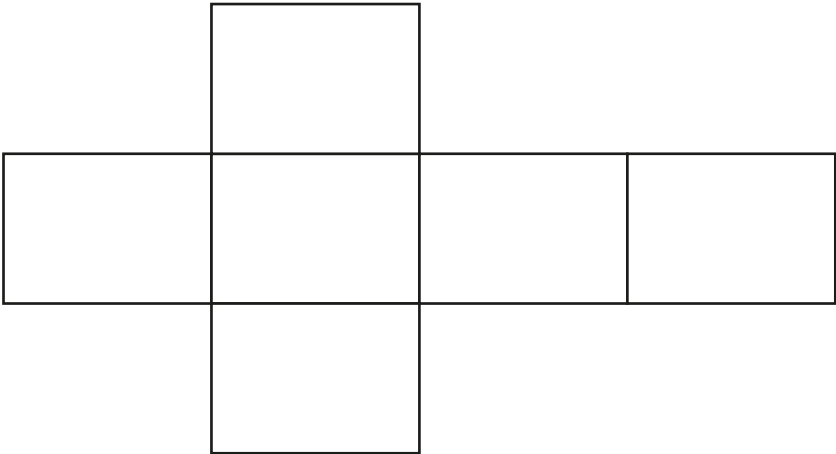




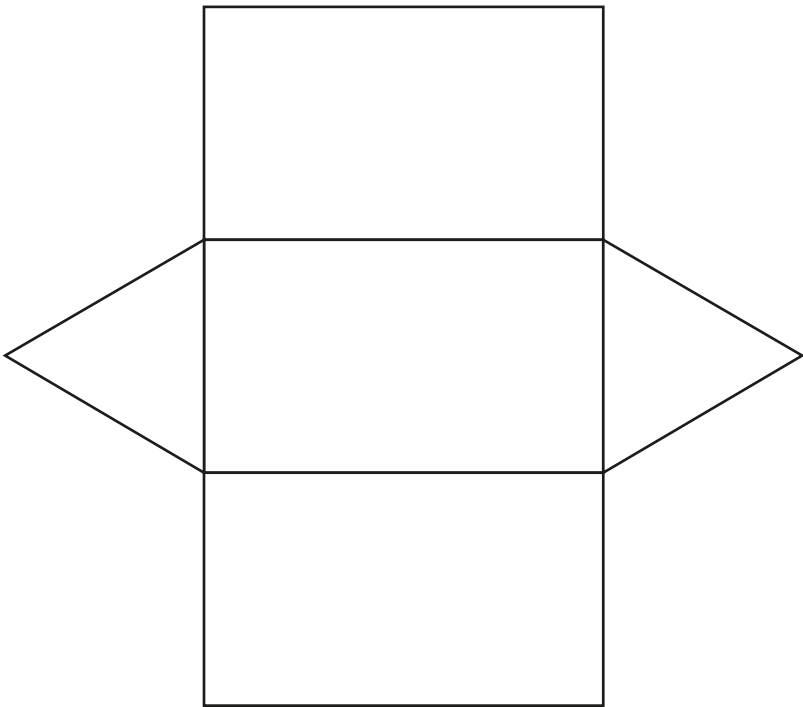
Cube



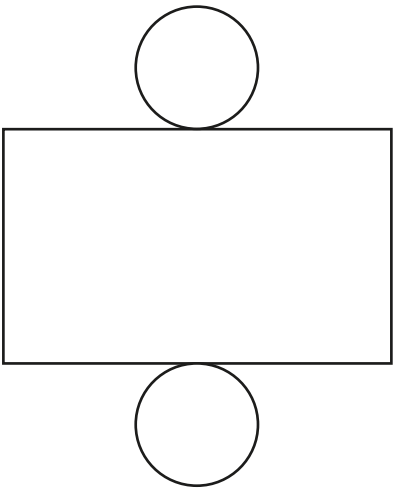
Rectangular prism



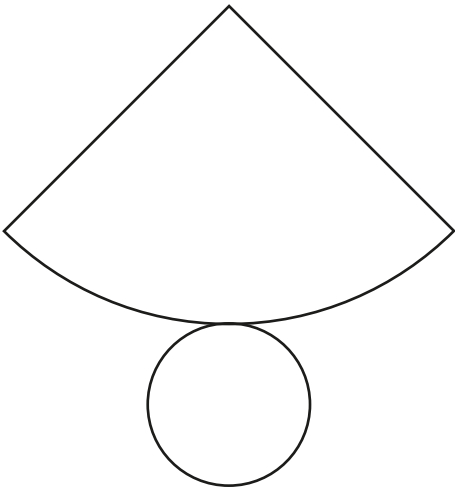
Triangular prism



Cylinder



Cone



Sphere

