Worked solutions

Practice paper

1 The argument of the ratio is the difference in the arguments:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12} \text{ (Answer C)}$$

[1 mark]

2 Horizontal asymptote for a rational function with numerator and denominator of equal order is the ratio of the leading coefficients.

$y = 2$.

Vertical asymptotes are the roots of the denominator, but there are no real roots for $x^2 + 4 = 0$. (Answer A)

[1 mark]

3 **a** $w - z^* = (2 + \mathrm{i}) - (3 + 2\mathrm{i}) = -1 - \mathrm{i}$

[2 marks]

b $\frac{z}{w} = \frac{3-2\mathrm{i}}{2+\mathrm{i}} = \frac{3-2\mathrm{i}}{2+\mathrm{i}} \times \frac{2-\mathrm{i}}{2-\mathrm{i}} = \frac{4-7\mathrm{i}}{5}$

[3 marks]

4 **a** $\mathbf{a} = \begin{pmatrix} 2 \\ p-1 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} p \\ 2 \\ 2p-1 \end{pmatrix}$

so $\mathbf{a} \bullet \mathbf{b} = 2p + 2(p-1) - 2(2p-1) = 0$ for all values of p.

Therefore, **a** and **b** are perpendicular for all values of p.

[2 marks]

b $\mathbf{a}+\mathbf{b}=\begin{pmatrix}2+p\\p+1\\2p-3\end{pmatrix}=\begin{pmatrix}0\\2t\\14t\end{pmatrix}$ for some real t

Coefficient of **i**: $p = -2$

Coefficient of **j**: $t = -0.5$

Coefficient of **k**: $2p - 3 = -7 = 14t$ is consistent.

For $p = -2$, **a** + **b** is parallel to (2**j** + 14**k**)

[3 marks]

5 $\alpha+\beta+\gamma=0,\ \alpha\beta+\beta\gamma+\gamma\alpha=-\frac{5}{2}$ and $\alpha\beta\gamma=-3$

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}=\frac{\frac{5}{2}}{3}=\frac{5}{6}$$

[5 marks]

6 $$\sum_{r=1}^{n}(2r-1)^2=4\sum_{r=1}^{n}r^2-4\sum_{r=1}^{n}r+n=\frac{4}{6}n(n+1)(2n+1)-\frac{4}{2}n(n+1)+n$$

$$=\frac{1}{3}n\left(2(n+1)(2n+1)-6(n+1)+3\right)=\frac{1}{3}n\left(4n^2+6n+2-6n-6+3\right)$$

$$=\frac{1}{3}n\left(4n^2-1\right)=\frac{1}{3}n(2n-1)(2n+1)$$

[6 marks]

7 $V = \pi\int_{x=1}^{x=4} y^2 \, dx = \pi\int_{x=1}^{x=4} \left(x^{-\frac{1}{4}}+1\right)^2 dx = \pi\int_{x=1}^{x=4} x^{-\frac{1}{2}} + 2x^{-\frac{1}{4}} + 1 \, dx$

$$= \pi\left[2x^{\frac{1}{2}} + \frac{8}{3}x^{\frac{3}{4}} + x\right]_1^4 = \pi\left(\left(4+\frac{8}{3}\times 2^{\frac{3}{2}} + 4\right) - \left(2+\frac{8}{3}+1\right)\right)$$

$$= \pi\left(\frac{12+16\sqrt{2}+12-6-8-3}{3}\right) = \frac{7+16\sqrt{2}}{3}\pi$$

So $a = 7$, $b = 16$ and $c = 3$

[5 marks]

8 Direction vector of the first line is $\mathbf{d}_1 = 5\mathbf{i} - \mathbf{j} + \mathbf{k}$ which is not a multiple of the direction vector of the second line $\mathbf{d}_2 = 2\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$, so the two lines are not parallel or identical.

Suppose there is an intersection:

The parametric form of the first line is $x = 2 + 5t$, $y = -1 - t$, $z = 1 + t$

Substituting into the equation of the second line and multiplying through by 14:

$$7(3+5t) = 7(-2-t) = 2(t-1)$$

Subtracting $2t$ and adding 14 throughout: $33t + 35 = -9t = 12$

The right-hand side of this gives $t = -\frac{4}{3}$, which is not consistent with the left-hand side, so there is no consistent solution and the two lines do not intersect.

The lines are not parallel and do not intersect, so they are skew.

[5 marks]

9 **a** $\det \mathbf{A} = -9c - 2(c-1) = -11c + 2$

If $\mathbf{A}$ is singular, then $\det \mathbf{A} = 0$ so $c = \frac{2}{11}$

[3 marks]

b $\mathbf{A}^{-1} = \dfrac{1}{2-11c}\begin{pmatrix} -3 & 1-c \\ -2 & 3c \end{pmatrix}$

[2 marks]

10 a $|a| = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2$

a lies in the upper left-hand quadrant of the Argand plane, so $\dfrac{\pi}{2} < \arg(a) < \pi$

$\arg(a) = \pi + \arctan\left(\dfrac{1}{-\sqrt{3}}\right) = \dfrac{5\pi}{6}$

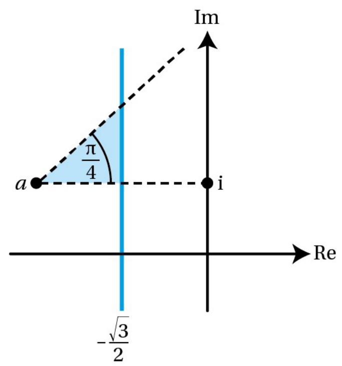
[2 marks]

b $|z-a| \le |z-\mathrm{i}|$:

The distance from a is no greater than the distance from i, so this locus consists of the points lying on the same side of the perpendicular bisector as i, including the bisector.

$0 < \arg(z-a) < \dfrac{\pi}{4}$:

The angle of a line connecting a to a point in the locus is strictly between 0° and 45° above the horizontal. a is not included in the locus.



[4 marks]

11 a Substituting $r = \sqrt{x^2 + y^2}$ and $r\cos\theta = x$:

$$\sqrt{x^2+y^2} - x = 3 \text{ so } x^2 + y^2 = (3+x)^2 = 9 + 6x + x^2$$

$$y^2 = 6x + 9$$

[4 marks]

b The line equation rearranges to $r\cos\theta = \frac{4}{3}$

Substituting into the equation for C: $r - \frac{4}{3} = 3$

so at the intersection, $3r = \frac{4}{\cos\theta} = \frac{9}{1-\cos\theta}$

so $r = \frac{13}{3}$

[3 marks]

12 a
$$\ln(1+\sin x) = \ln\left(1 + x - \frac{1}{6}x^3 + \ldots\right)$$

$$= \left(x - \frac{1}{6}x^3 + \ldots\right) - \frac{1}{2}\left(x - \frac{1}{6}x^3 + \ldots\right)^2 + \frac{1}{3}\left(x - \frac{1}{6}x^3 + \ldots\right)^3 - \frac{1}{4}\left(x - \frac{1}{6}x^3 + \ldots\right)^4 + \ldots$$

$$= x - \frac{1}{2}x^2 + x^3\left(-\frac{1}{6} + \frac{1}{3}\right) + x^4\left(\frac{1}{6} - \frac{1}{4}\right) + \ldots$$

$$= x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \ldots$$

[6 marks]

b The expansion for $\sin x$ is valid for all x but the expansion for $\ln(1+y)$ is valid for $-1 < y \le 1$ so when $\sin x = -1$ the expansion will not converge.

$x = \frac{3\pi}{2}$ is an example value for which the series will not converge.

[2 marks]

13 **a** $3x^2 - 16y^2 = 48$

Dividing through by 48:

$$\frac{x^2}{16} - \frac{y^2}{3} = 1.$$

Using the fact that the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm\frac{b}{a}x$,

the asymptotes of H are $y = \pm\frac{\sqrt{3}}{4}x$.

[2 marks]

b **i** The line $y = mx + 1$ intersects H where:

$$3x^2 - 16(mx+1)^2 = 48$$
$$3x^2 - 16(m^2x^2 + 2mx + 1) = 48$$
$$3x^2 - 16m^2x^2 - 32mx - 16 = 48$$
$$(3 - 16m^2)x^2 - 32mx - 64 = 0$$

ii For the line $y = mx + 1$ to be tangent to H, the quadratic above must only have exactly one root, so the discriminant must be zero:

$$b^2 - 4ac = 0$$
$$(-32m)^2 - 4(3 - 16m^2)(-64) = 0$$
$$32m^2 + 8(3 - 16m^2) = 0$$
$$4m^2 + 3 - 16m^2 = 0$$
$$12m^2 = 3$$
$$m^2 = \frac{1}{4}$$
$$m = \pm\frac{1}{2}$$

[5 marks]

14 a Reflection in $z = 0$ is given by $\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

R followed by **M** is given by

$$\mathbf{MR} = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 3 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & -3 \\ 1 & 0 & 2 \end{pmatrix}$$

[3 marks]

b Position vector of the image point is $\begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & -3 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -13 \\ -12 \\ 7 \end{pmatrix}$

so the image point has coordinates (–13, –12, 7).

[1 mark]

c Require all points with position vector **u** such that **MRu** = **u**

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & -3 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ so } \begin{cases} x - 2y - 2z = x & (1) \\ -3z = y & (2) \\ x + 2z = z & (3) \end{cases}$$

(3) gives $x = -z$

(2) gives $y = -3z$.

Substituting into (1): $-z + 6z - 2z = -z$

So $4z = 0$, from which $x = y = z = 0$.

Therefore, the only invariant point under this transformation is the origin.

[3 marks]

15 Proposition: $9^n + 7^n - 2^n$ is divisible by 14 for all integers $n \geq 1$.

Base case: $n = 1$: $9^1 + 7^1 - 2^1 = 14$ so the proposition is true for $n = 1$.

Inductive step:

Assume the proposition is true for $n = k$: $9^k + 7^k - 2^k = 14a$, for some integer a.

Consider $n = k + 1$:

$$\begin{aligned} 9^{k+1} + 7^{k+1} - 2^{k+1} &= 9\left(9^k\right) + 7^{k+1} - 2^{k+1} \\ &= 9\left(14a - 7^k + 2^k\right) + 7^{k+1} - 2^{k+1} \quad \text{(using the assumption)} \\ &= 14(9a) - 9\left(7^k\right) + 9\left(2^k\right) + 7^{k+1} - 2^{k+1} \\ &= 14(9a) - 9\left(7^k\right) + 7\left(7^k\right) + 9\left(2^k\right) - 2\left(2^k\right) \\ &= 14(9a) - 2\left(7^k\right) + 7\left(2^k\right) \\ &= 14(9a) - 2 \times 7\left(7^{k-1}\right) + 7 \times 2\left(2^{k-1}\right) \\ &= 14\left(9a - 7^{k-1} + 2^{k-1}\right) \end{aligned}$$

So divisible by 14.

The statement is true for $n = 1$, and if it is true for $n = k$ then it is also true for $n = k + 1$. Therefore the statement is true for all $n \in \mathbb{Z}^+$, by induction.

[7 marks]