Worked solutions

Practice paper 1

1 **a** $w - z^* = (2 + \text{i}) - (3 + 2\text{i}) = -1 - \text{i}$ **[2 marks]**

b $\frac{z}{w} = \frac{3-2\text{i}}{2+\text{i}} = \frac{3-2\text{i}}{2+\text{i}} \times \frac{2-\text{i}}{2-\text{i}} = \frac{4-7\text{i}}{5}$ **[3 marks]**

2 **a** $\mathbf{a} = \begin{pmatrix} 2 \\ p-1 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} p \\ 2 \\ 2p-1 \end{pmatrix}$.

So $\mathbf{a} \cdot \mathbf{b} = 2p + 2(p-1) - 2(2p-1) = 0$ for all values of p.

Therefore, **a** and **b** are perpendicular for all values of p. **[2 marks]**

b $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2+p \\ p+1 \\ 2p-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2t \\ 14t \end{pmatrix}$ for some real t.

Coefficient of **i**: $p = -2$

Coefficient of **j**: $t = -0.5$

Coefficient of **k**: $2p - 3 = -7 = 14t$ is consistent with the above.

For $p = -2$, $\mathbf{a} + \mathbf{b}$ is parallel to $(2\mathbf{j} + 14\mathbf{k})$. **[3 marks]**

3 $\alpha + \beta + \gamma = 0$, $\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{5}{2}$ and $\alpha\beta\gamma = -3$.

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma} = \frac{-\frac{5}{2}}{-3} = \frac{5}{6}$ **[5 marks]**

4 The direction vector of the first line is $\mathbf{d}_1 = 5\mathbf{i} - \mathbf{j} + \mathbf{k}$, which is not a multiple of the direction vector of the second line $\mathbf{d}_2 = 2\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$, so the two lines are neither parallel nor identical.

Suppose there is an intersection. The parametric form of the first line is:

$x = 2 + 5t$, $y = -1 - t$ and $z = 1 + t$.

Substituting into the equation of the second line and multiplying through by 14:

$7(3 + 5t) = 7(-2 - t) = 2(t - 1)$

Subtracting $2t$ and adding 14 throughout:

$33t + 35 = -9t = 12$

The right end of this gives $t = -\frac{4}{3}$, which is not consistent with the left end, so there is no consistent solution and the two lines do not intersect.

The lines are not parallel and do not intersect, so they are skew. **[5 marks]**

5 a $\det \mathbf{A} = -9c - 2(c-1) = -11c + 2$

If $\mathbf{A}$ is singular then $\det \mathbf{A} = 0$ so $c = \frac{2}{11}$. **[3 marks]**

b $\mathbf{A}^{-1} = \frac{1}{2-11c}\begin{pmatrix} -3 & 1-c \\ -2 & 3c \end{pmatrix}$ **[3 marks]**

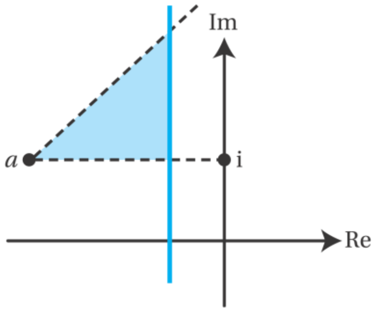
6 a $|a| = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2$

a lies in the upper left quadrant of the Argand plane so $\frac{\pi}{2} < \arg(a) < \pi$.

$\arg(a) = \pi + \arctan\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$ **[2 marks]**

b $|z-a| \le |z-\mathrm{i}|$: The distance from a is no greater than the distance from i, so this locus is the points lying on the far side of the perpendicular bisector from i, including the bisector.

$0 < \arg(z-a) < \frac{\pi}{4}$: The angle of a line connecting a to a point in the locus is strictly between 0° and 45° above the horizontal. a is not included in the locus.



[4 marks]

7 Proposition: $11^n - 4^n$ is a multiple of 7 for all integers $n \geq 1$.

Base case: $11^1 - 4^1 = 7 = 7 \times 1$ so the proposition is true for $n = 1$.

Inductive step:

Assume that the proposition is true for $n = k$ so that $11^k - 4^k = 7A$ for some integer A.

Working towards: $11^{k+1} - 4^{k+1} = 7B$ for some integer B.

$$\begin{aligned} 11^{k+1} - 4^{k+1} &= 11\left(11^k - 4^k\right) + 7 \times 4^k \\ &= 11 \times 7A + 7 \times 4^k \quad \text{(using } 11^k - 4^k = 7A\text{)} \\ &= 7B, \text{ where } B = 11A + 4^k \text{ is an integer.} \end{aligned}$$

So the proposition is true for $n = k + 1$.

Conclusion:

The proposition is true for $n = 1$, and if true for $n = k$ it is also true for $n = k + 1$.

Hence, the proposition is true for all integers $n \geq 1$, by the principle of mathematical induction. **[7 marks]**

8 **a** Let $x = u - 3$

$$(u-3)^3 + 9(u-3)^2 + 31(u-3) + 39 = 0$$

$$u^3 - 9u^2 + 27u - 27 + 9u^2 - 54u + 81 + 31u - 93 + 39 = 0$$

$$u^3 + 4u = 0$$

[2 marks]

b $u(u^2 + 4) = 0$

$u = 0$ or $\pm 2\text{i}$

So the solutions to the original problem are $x = -3$ or $-3 \pm 2\text{i}$. **[5 marks]**

9 **a** Expanding about the first column:

$$\det \mathbf{M} = \begin{vmatrix} 0 & 3 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 0 & 3 \end{vmatrix} = 0 + -6 = -6$$

[1 mark]

b Reflection in $z = 0$ is given by $\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

$\mathbf{R}$ followed by $\mathbf{M}$ is given by $\mathbf{MR} = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 3 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.

[3 marks]

c Position vector of the image point is $\begin{pmatrix}1 & -2 & -2\\0 & 0 & -3\\1 & 0 & 2\end{pmatrix}\begin{pmatrix}-1\\2\\4\end{pmatrix}=\begin{pmatrix}-13\\-12\\7\end{pmatrix}$,

so the image point has coordinates (–13, –12, 7). **[1 mark]**

d Original cube has volume $3^3 = 27$.

det $\mathbf{RM} = -\det \mathbf{M} = 6$ so the image shape has volume $27 \times 6 = 162$. **[2 marks]**

10 At Q, $\begin{pmatrix}-2\\2\\7\end{pmatrix}+\lambda\begin{pmatrix}-1\\0\\2\end{pmatrix}=\begin{pmatrix}5\\4\\7\end{pmatrix}+\mu\begin{pmatrix}3\\1\\1\end{pmatrix}$

$$\begin{cases}-2-\lambda=5+3\mu & (1)\\ 2=4+\mu & (2)\\ 7+2\lambda=7+\mu & (3)\end{cases}$$

(2): $\mu = -2$

(3) – (2): $5 + 2\lambda = 3$ so $\lambda = -1$.

(1): $-2 - \lambda = -1 = 5 + 3\mu$ is consistent with the above, so the two lines do intersect where $\mu = -2$ and $\lambda = -1$, so Q has coordinates (–1, 2, 5).

The vector product of the two directions is $\mathbf{n}=\begin{pmatrix}-1\\0\\2\end{pmatrix}\times\begin{pmatrix}3\\1\\1\end{pmatrix}=\begin{pmatrix}-2\\7\\-1\end{pmatrix}$ so the line through Q with direction $\mathbf{n}$ has vector equation $\mathbf{r}=\begin{pmatrix}-1\\2\\5\end{pmatrix}+\nu\begin{pmatrix}-2\\7\\-1\end{pmatrix}$. **[7 marks]**