Tidal distortion: Earth tides and Roche lobes

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(Supplement to Ch. 4 of *Astrophysics Processes* (AP) by Hale Bradt, Camb U. Press 2008)
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What we learn in this Supplement

Tidal distortion of an astronomical body usually arises from its **gravitational interaction** with a partner object in a **binary system**. The earth–moon system is an example. We calculate the **tidal forces** on the earth surface for **several special locations**, those on the earth–moon line and those on the band 90° from the moon zenith.

A general solution is obtained by writing the **perturbing potential of the moon's gravity and the centrifugal force** in the rotating coordinate system of the earth–moon binary orbit. From this we derive the tidal force at all positions on the earth surface and find the associated **changed positions of the equipotential surface** that would define the surface of a fluid earth or gaseous star. Small distortions yield an **oblate spheroid**, an ellipse rotated about its long axis, with its long dimension directed toward the moon. The earth's rapid 24-h rotation under the resultant non-uniform ocean heights leads to the rising and falling tides we observe.

The **tidal effect of the sun** is significant. Tidal forces are proportional to the moon (or sun) mass divided by the cube of the earth–moon (or earth-sun) separation. The solar tidal effect is thus about one-half that of the moon. It can significantly shift the location of the tidal bulges.

We extend our analysis to obtain the **Roche potential** for a binary stellar system. In this case we take into account the **gravitational potentials of *both* stars as well as the (fictitious) centrifugal potential**. The Roche potential gives a way to understand stellar shapes, limits to stellar expansion, and location of accretion flows in binary systems. It has validity insofar as both stars are synchronously rotating with the orbit and the orbits are circular. Close stellar binary systems often satisfy both these conditions.

Solutions to the problems are available to instructors on the password-protected CUP website for this text (see URL above).

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Useful values

Earth mass: 5.97×10^{24} kg
Moon mass: 7.35×10^{22} kg
Sun mass: 1.99×10^{30} kg
Earth radius (mean): 6371 km
Earth–moon separation (semi-major axis of relative orbit): 3.844×10^{8} m
Earth–moon separation range: $3.564–4.067 \times 10^{8}$ m
Earth–sun separation (semi-major axis of relative orbit ≡ 1 AU): 1.496×10^{11} m
Earth-sun separation range $1.471–1.521 \times 10^{11}$ m
Moon sidereal orbital period: 27.3 d

1 Introduction

Tidal interactions play an important role in the evolution of binary star systems, which are discussed in Section 4.5 of my textbook, *Astrophysics Processes* (AP; Cambridge, 2008), p. 159. They bring about circularization of orbits and synchronization of spin. They also enter into the determination of neutron-star and black-hole masses in binary systems; tidal distortions lead to modified light curves that depend on the system parameters.

A tidal problem of great interest to humankind is that of the ocean tides on the earth due to the gravitational pull of the moon and sun. For the simple two-body earth–moon problem, one can show easily with Newtonian physics that two tidal bulges form on the near and far side of the earth from the moon and that they are of equal height given that the radius of the earth is much less than the moon–earth separation. One also finds that the net tidal force is proportional to the mass and to the inverse cube of the separation between the earth and moon centers. This tells us that the tidal effect of the moon is about twice that of the sun. The solar influence can cause the ideal tidal bulge to shift by as much as 17° from the moon direction.

Herein, we first find directly the magnitude and direction of the tidal forces for the two-body (earth and moon) problem for selected locations on the earth before presenting the general case. Along the earth–moon line, the net tidal force is outward (from the earth center) on both the near and far sides of the earth from the moon. This tends to raise the ocean level at these locations. Along the band 90° from the moon's zenith ($\psi = 90°$), the net tidal force is inward, tending to lower the ocean level. These results quite simply demonstrate quantitatively the existence of the two tidal bulges and their equal sizes.

We then analyze the general case by writing the potentials of the moon and the centrifugal force in the rotating frame of reference of the moon-earth orbit and then expressing their sum in terms of earth centered coordinates. At the earth's surface, this potential perturbation modifies the potential due to earth gravity such that an initially unperturbed spherical equipotential surface takes on the shape of a prolate spheroid (an ellipse in rotation about its long axis). The negative gradient of the potential yields the tidal force at any position in the vicinity of the earth. This is used to map the net tidal force everywhere on the earth surface.

This development assumes that the gravitational force due to the earth itself is perfectly symmetric, i.e., it is that of a point source. In fact, the centrifugal distortions due to earth spin and the tidal effects calculated herein can create significant asymmetries in the earth's gravitational field. We do not address these effects here.

Finally, we introduce the role the sun plays in the earth tides and also the Roche potential which is useful in describing stellar binary systems.

Two excellent resources are referenced at the end of this Supplement, namely the NOAA document "Our Restless Tides" on their website and the textbook "Physics of the Earth, 4th

Edition" by Frank Stacey and Paul Davis (CUP 2008). Our development of the tidal potential follows that of Stacey and Davis.

If one is interested in the spin-orbit evolution of the earth–moon system through the tidal interactions, do investigate Problems 4.22–4.25 of my earlier book *Astronomy Methods* (CUP 2004), and also Stacey and Davis on "tidal friction". This topic is relevant to exoplanet studies.

I acknowledge helpful conversations with Dr. Nelson Caldwell of SAO/Harvard, Dr. Frank Stacey of of CSIRO Australia, Mr. R. E. White, Jr., publisher of the Eldridge Tide and Pilot Book, Dr. Alan Levine of MIT, and Prof. Saul Rappaport of MIT. Section 8 (Roche potential) follows course notes provided by Prof. Rappaport.

2 Binary system parameters

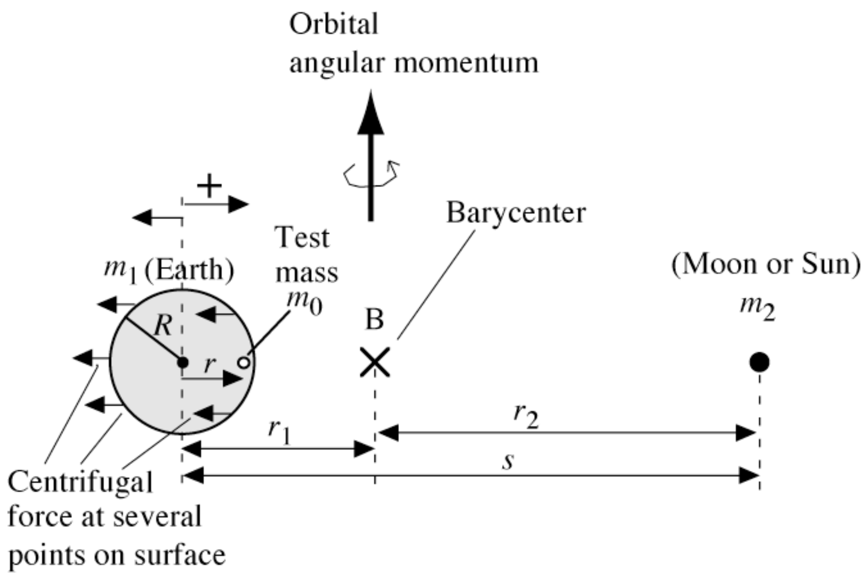


Figure 1. Geometry defining binary parameters and for determining the tidal forces on the earth–moon (or sun) line. The barycenter is locate. The barycenter B can be within the earth (as for the earth–moon system) or outside it (as for the earth–sun system).

Consider two bodies (Fig. 1), m_1 and m_2, orbiting their common center of mass (barycenter, B) with circular tracks. Let m_1 (the earth) have a finite size and radius R, whereas m_2 (moon or sun) may be taken as a point source because of its large distance, $s >> R$, and because we are not concerned here about the tidal forces on m_2.

The relation between the several distances in Fig. 1 are as follows, where r_1 is the distance from the barycenter to the center of m_1, r_2 the distance to the center of m_2, and s the separation of the centers. For now, we take these three distances to be positive definite quantities. From the definition of the barycenter,

$$r_1 = \frac{m_2}{m_1 + m_2} s = \frac{m_2}{m_T} s \qquad (1)$$

where $m_T = m_1 + m_2$ is the total mass and

$$s = r_1 + r_2 \qquad (2)$$

Kepler's third law, $Gm_T P^2 = 4\pi^2 s^3$, yields the angular orbital frequency ω (rad/s), where P is the orbital period and $\omega = 2\pi/P$,

$$\omega = \left(\frac{Gm_T}{s^3}\right)^{1/2} \tag{3}$$

These parameters r_1, r_2, s and ω are constant throughout the orbit for circular orbits.

In Fig. 1, we further specify a test mass m_0, an element of m_1, and define its position from the center of m_1 to be r. This coordinate defines an arbitrary position along the line connecting the masses with position zero at the center of m_1 and the positive direction to the right. The positions of interest will be $r = \pm R$, but for now we keep this position coordinate as a variable.

3 Force perturbations

Consider a distant observer in an inertial (non-accelerating) coordinate system viewing the earth–moon system from afar. The tidal bulges on the near and far sides of the earth (from the moon) would be seen by this observer to be the result of the decreasing gravitational force with increasing distance from the moon. A force F accelerates a mass m according to Newton's second law, $F = ma$.

Hence, the earth center (and all the earth to the extent it is a rigid body) would be accelerated toward the moon more strongly than the water on the earth's far side, thus tending to pull the earth away from that water. With appropriate current flows, this would create the far-side bulge of water. Similarly, the water on the near side of the earth is accelerated away from the earth to create the near-side bulge. The effect is therefore proportional to force differences, i.e., to the gradient of the force.

Another view is that of an observer at rest on the earth in the rotating frame of reference of the earth–moon system. (This is the view we take in our forthcoming calculation.) [This frame rotates slowly, only once in 27.3 days, and the acceleration of the earth due to its circular orbital motion about the earth–moon barycenter is toward the barycenter or to the right in Fig. 1.] Our observer thus finds that all objects appear (in the earth frame) to experience an outward (leftward) "inertial" or "centrifugal" force directed away from the barycenter (small arrows in Fig. 1) . This force is equal to the negative of the mass of the object, e.g., m_0, times the acceleration of the earth center, $-\omega^2 r_1$,

$$F_c = -\, m_0\, \omega^2 r_1 \tag{4}$$

This is not a real physical force but rather an artifact of the observer being in an rotating, and hence accelerating, frame of reference. It is sometimes called a "fictitious" force.

The centrifugal force has the same magnitude and direction for observers anywhere on the earth because the entire earth moves as a rigid body, and because, for this calculation, we assume the earth is not rotating (spinning) with respect to the distant stars. As this *non-spinning* earth is carried around its orbit, every mass element rotates in a circle of radius r_1 with orbital angular velocity ω. (The centers of rotation differ for each mass element.) All elements thus experience the same inertial force, that given in (4). We apply this "constant" force to test masses on the surface of the earth (small arrows in Fig. 1) where they can influence oceanic levels. This discussion is relevant to the force diagrams shown in "The Restless Tides" on the NOAA website.

The vector sum of the leftward centrifugal force and the rightward moon gravitational force on a test mass yields a net force at any point on the earth's surface. This vector sum is called the "tidal force". This perturbing force field will distort the overall shape of an otherwise spherical fluid earth or gaseous star.

This discussion ignores the effect of earth spin. The earth spins much faster than the lunar rotation rate (period 24 h vs. 27 d). Its effect is to create an equatorial bulge normal to the earth spin axis. While this would raise or lower the earth crust and oceans as a function of latitude, it would not give rise to any daily variation of ocean height at the fixed longitude and latitude of an observer. The bulge is symmetric about the spin axis, and the observer's daily motion is about that ssame axis.

4 Force calculations

As stated above, we first calculate the forces on the moon-earth line at the near and far (from the moon) surfaces of the earth, i.e., at moon zenith angles $\psi = 0°$ and 180°. Then, we find the force at the moon zenith angle $\psi = 90°$ (Fig. 2 below). Throughout these calculations, we assume that the size of m_2 is much less than the separation between m_1 and m_2, i.e., $R << s$. If our coordinate r is restricted to values of order R, then $r << s$. This assumption is appropriate for both the earth–moon and earth-sun problems. Since we desire only the perturbing forces, we do not include the earth's gravitational field.

Force on earth–moon line

The gravitational force F_g on test mass m_0 at arbitrary position $r << s$ due to m_2 is

$$F_g = \frac{Gm_0 m_2}{(s-r)^2} = \frac{Gm_0 m_2}{s^2\left(1-\frac{r}{s}\right)^2} \qquad (r << s)$$

$$\approx \frac{Gm_0 m_2}{s^2}\left(1+\frac{2r}{s}\right) \tag{5}$$

where we dropped second order terms in r/s because $r << s$, and applied the approximation $(1 - \varepsilon)^{-n} \approx 1 + n\varepsilon$ when $\varepsilon << 1$. Note that we calculate this force only on the line connecting the two stars and that it is positive for both positive and negative (but small) r as it should be; the pull of the moon is to the right for all positions on m_1. Recall that the positive direction is to the right in Fig. 1.

The centrifugal force is the same for all elements of m_1 and is given in (4). We eliminate ω with (3) and r_1 with (1) to find for our test mass m_0

$$F_c = -m_0 \frac{Gm_T}{s^3} \frac{m_2}{m_T} s = -\frac{Gm_0 m_2}{s^2} \tag{6}$$

The centrifugal force depends on m_2 and s, but not on position r in accord with our discussion above. Keep in mind that this (inertial) force exists only in the accelerating frame of the orbiting earth.

The total force is the sum of (5) and (6),

$$F_t = F_g + F_c \approx \frac{Gm_0 m_2}{s^2}\left(1 + \frac{2r}{s} - 1\right)$$

$$\Rightarrow \quad \approx \frac{2Gm_0 m_2 r}{s^3} \propto \frac{m_2 r}{s^3} \qquad (r << s) \qquad (7)$$

We find that the total force is proportional to the mass of the "other" gravitational body (moon or sun) and varies as the inverse cube of the separation distance s. At $r = 0$ (center of the earth), it is zero ($F_t = 0$) because the gravitational force exactly balances the centrifugal force there. As one moves away from the earth center, it deviates from zero.

On the "near" surface of m_1, we have $r = +R$, so the net force is

$$\Rightarrow \quad F_{t,n} \approx +\frac{2Gm_0 m_2 R}{s^3} \qquad (8)$$

At this surface, the centrifugal and gravitational forces are not in balance; the moon's gravity dominates slightly the centrifugal force. The net force is thus to the right. On the "far" surface of m_1, we have $r = -R$, and a net force of

$$\Rightarrow \quad F_{t,f} \approx -\frac{2Gm_0 m_2 R}{s^3} \qquad (9)$$

Again, there is a finite net force, but in this case, the centrifugal force dominates. The net force has the same magnitude as (8) but it is directed to the left.

The net tidal forces on the near and far sides of the earth (m_1) are thus equal in magnitude, given our approximation $R << s$, and both are in the outward direction from the earth center. This indicates that there are two tidal bulges of equal heights on the near and far sides of the earth.

Force with moon on horizon

Consider now the upper and lower positions on the earth's surface in Fig. 2, where the moon is at zenith angle $\psi \approx 90°$. The gravitational force there is typical of the entire circumferential band with this zenith angle. The horizontal component of the gravitational force very closely cancels the centrifugal force, but there is a vertical component F_v of the moon's gravity (Fig. 2) that tends to lower the oceans. The centrifugal force has no vertical component to offset this.

From the geometry of Fig. 2, we have

$$\frac{R}{b} = -\frac{F_v}{F_g} \qquad (10)$$

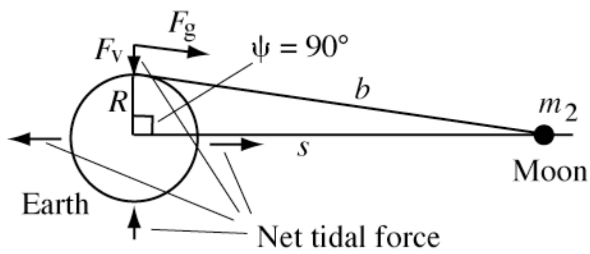


Figure 2. Geometry for calculating the force on the earth's surface at moon zenith angle $\psi \approx 90°$. The horizontal component of the moon's gravity closely cancels the centrifugal force. The vertical component F_V is thus the net tidal force along the surface band with $\psi = 90°$. The net tidal force is shown for three other locations.

$$F_v = -F_g \frac{R}{b} = -\frac{Gm_2 m_0}{b^2}\frac{R}{b} = -\frac{Gm_2 m_0 R}{\left(R^2+s^2\right)^{3/2}} = -\frac{Gm_2 m_0 R}{s^3\left(1+\frac{R^2}{s^2}\right)^{3/2}}$$

$$\Rightarrow \qquad \approx -\frac{Gm_2 m_0 R}{s^3} \tag{11}$$

where we invoked our assumption that $R << s$ and dropped terms in R^2/s^2.

We find a tide-lowering force of magnitude one-half that of the lifting force on the earth–moon line (8). For an earth centered spherical coordinate system, the tidal force is positive (outward) along the earth–moon line and negative along the band 90° from the moon ($\psi = 90°$) as shown in Fig. 2. This leads to tidal bulges along the earth–moon line and tidal depressions along the band normal to it.

These tidal forces, (8), (9) and (11), are much less than the s^{-2} gravitational force of the moon, by a factor of $R/s = 0.017$, because the moon's gravitational force is largely (but not exactly) cancelled by the centrifugal force at the earth's surfaces. The s^{-3} dependence in these expressions is that of the gradient of the s^{-2} gravitational force.

5 Potential perturbation

The magnitude of the distortions (in meters) and the overall distorted shape follows from a more general analysis in terms of potentials.

Rotating frame of reference

The most general approach to the tidal forces is through the potential. Consider, again, the frame of reference that rotates with the orbit. Both bodies, earth and moon, will be stationary in this frame if their orbits in inertial space are circular. Further, in contrast to the earlier case, we now take the earth to be synchronously rotating with the orbit, so that the same face of the earth always faces the moon. All mass elements of the earth are thus stationary in the rotating frame. In the inertial frame, which is fixed to the distant stars, though, the earth makes one rotation per orbit.

We again are interested only in perturbations to the earth's gravitational potential and assume the latter to be spherically symmetric. One need therefore consider, for now, only the gravitational potential of the moon together with the centrifugal potential. Since all elements of the earth are stationary in this frame, the velocity-dependent coriolis force is zero for all mass elements of the earth.

The centrifugal force (per unit mass) is $+\omega^2 \ell$, where ℓ is the distance from the rotation axis (barycenter) to the test mass m_0 at point P (Fig. 3a) and the plus sign indicates an outward (radial) force. Integrate this with the barycenter as the reference position for zero potential to find that the associated centrifugal potential function is $-\omega^2 \ell^2/2$.

Similarly, integrate the gravitational force of m_2 (moon) with infinity as the reference point for zero potential to find the potential at point P to be $-Gm_2/b$, where b is the distance from m_2 to P. Both of these potentials are sketched in Fig. 3b for positions along the earth–moon line.

Calculation of potential

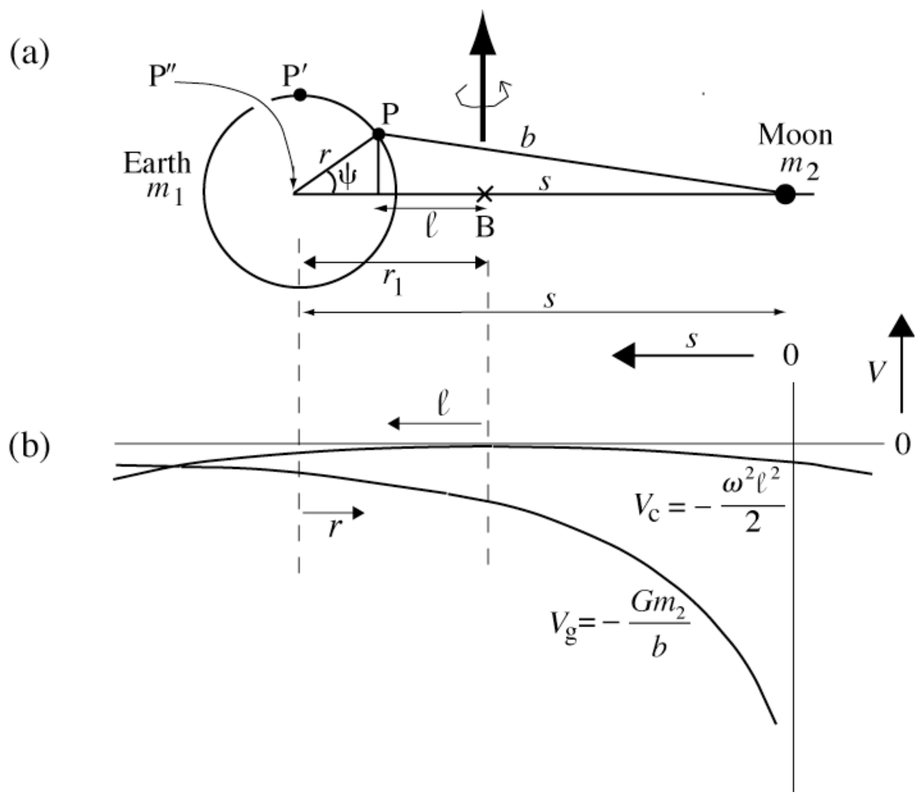


Figure 3: (a) Geometry for a position P lying in plane which includes the earth–moon line and the axis of rotation for the orbital motion. (b) Sketched potentials of the contributing centrifugal and lunar gravitational forces in this plane along the earth–moon line. At the earth center, $r = 0$, the *slopes* of the two functions are equal in magnitude and of opposite signs, indicating that the centrifugal and gravitational forces cancel exactly at the earth center. Inward of this (to the right), the greater slope of V_g tells us that its force dominates the centrifugal force. Outward, the centrifugal slope is the greater of the two, so centrifugal force dominates. Points P′ and P″ lie on the surface of the earth. The latter is above the plane of the page (toward the reader).

A total perturbing potential V_p at the position of the earth due to these two effects may be taken to be their sum,

$$V_p = -\frac{Gm_2}{b} - \frac{\omega^2 \ell^2}{2} \tag{12}$$

We do not include the gravitational field of the earth because, as noted, we are interested only in forces that perturb the spherically symmetric (ideal) potential of earth's gravity.

This expression gives V_p at any position in three dimensional space. We are interested only in positions in the vicinity of the earth and in particular for positions on the earth's surface. Our goal is to express the perturbing potential V_p in the earth vicinity in terms of polar coordinates with origin at the earth center.

Figure 3 shows the geometry in the plane that contains the earth–moon line and the rotation axis. It is necessary, though, to consider positions off this plane because the distance ℓ from the rotation axis depends on the position components of P both in and out of the plane. Compare, for example, point P′ at the top of the earth in Fig. 3a and point P″ on the earth surface directly in front of the earth center. The latter point will be further from the rotation axis than the former and hence will have a lower centrifugal potential. Both of these points are at the same distance from m_2 and hence have the same gravitational potential.

Figure 4 provides the geometry needed for our calculation. It considers a point P on the earth surface below the plane of the page at azimuthal angle λ measured in the plane of the orbit from the moon direction and polar angle θ measured from the orbital pole. The angle ψ is the angle of P relative to the moon direction. It is a function of θ and λ.

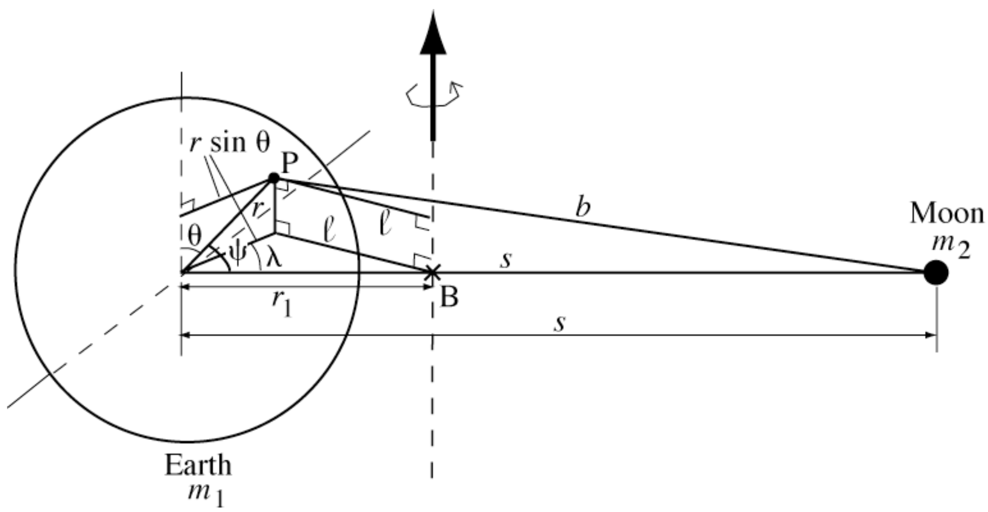


Figure 4. Three-dimensional geometry for determining the potential perturbation at point P on the earth's surface in terms of the earth-centered coordinates, r, θ, and λ, or r, θ and ψ. The azimuthal angle λ is measured in the plane of the orbit, and the polar angle θ is measured from the orbital pole. The angle ψ is the angle of point P measured from the direction of m_2 (moon). The barycenter B is shown outside the earth; for the earth–moon system it is a bit inside the earth surface. The distance from P to the rotation axis is ℓ and to the moon is b. The earth center is distant r_1 from the barycenter and distant s from the moon. Adapted from Stacey and Davis, ibid.

We proceed to rewrite (12) in terms of the earth-centered coordinates, r, ψ, and θ, and the system parameters, specifically the masses and separation s while again taking $r/s \ll 1$. From the geometry of Fig. 4, we write the law of cosines for the large triangle with side b and opposing angle ψ and solve for b^{-1},

$$b^2 = r^2 + s^2 - 2rs\cos\psi$$
$$\frac{1}{b} = \frac{1}{s}\left(1 + \frac{r^2}{s^2} - \frac{2r}{s}\cos\psi\right)^{-1/2}$$
$$\approx \frac{1}{s}\left(1 - \frac{1}{2}\frac{r^2}{s^2} + \frac{r}{s}\cos\psi + \frac{3}{2}\frac{r^2}{s^2}\cos^2\psi + ...\right) \qquad (13)$$

The latter expression is expanded in terms of the small quantity r/s with the binomial expansion, while retaining all first and second order terms, i.e., those in r/s and r^2/s^2.

To obtain the distance ℓ, we apply the law of cosines to the smaller triangle in the orbital plane with side ℓ and opposing angle λ and solve for ℓ^2. We make no approximations in this case,

$$\ell^2 = r_1^2 + (r\sin\theta)^2 - 2r_1 r\sin\theta\cos\lambda$$
$$\ell^2 = \left(\frac{m_2}{m_\mathrm{T}}s\right)^2 + r^2\sin^2\theta - 2\frac{m_2}{m_\mathrm{T}}s\ r\cos\psi \qquad (14)$$

Here we have invoked (1) for r_1 and the trigonometric identity $\sin\theta\cos\lambda = \cos\psi$. (The projection of r onto the earth–moon line is its projection onto the orbital plane which is then projected onto the earth–moon line.)

Substitute (13) and (14) into (12) and use $\omega^2 = Gm_\mathrm{T}/s^3$ from (3) as needed to find (Problem 1a) that

$$\Rightarrow \qquad V_\mathrm{p} = -\frac{Gm_2}{s}\left(1 + \frac{m_2}{2m_\mathrm{T}}\right) - \frac{Gm_2 r^2}{s^3}\left(\frac{3}{2}\cos^2\psi - \frac{1}{2}\right) - \frac{1}{2}\omega^2 r^2\sin^2\theta \qquad (15)$$

This is the result we desire.

The first term on the right (a product) is the sum of the two potentials (12) at the center of the earth. It is merely the sum of the two curves shown in Fig. 3 at that position. It is a constant adjustment to the potential at the earth center; it is independent of time or of position r, θ, λ. It has no tidal effect. The other two terms of (15) are zero at the earth center ($r = 0$); however, at positions removed from the earth center, they become finite and contribute perturbations.

The final term is due to a centrifugal force arising from the spin of the earth in our calculation; recall that our model has the earth rotating synchronously with the orbit. The form of this term is similar to that of the last term of (12) but with the distance ℓ being replaced by $r\sin\theta$, the distance of point P from the earth spin axis. This potential term is symmetric about an axis normal to the orbit plane and passing through the earth center, and leads to an "equatorial" bulge in the orbital plane. This is expected from the once-per-orbit earth spin of our model.

An earth observer standing on our hypothetical synchronously rotating earth would see no falling or rising tides (due to this term) because the bulge height is symmetric about the spin axis (the orbital pole), and the observer's position rotates about that same axis. In fact the earth rotates much more rapidly than once per lunar orbit and its axis is tilted roughly 23° from the moon orbital plane. An additional angular momentum may be added vectorially to that of the synchronous rotation of our model to obtain the actual angular momentum. The resultant equatorial bulge is normal to the resultant spin axis as is the observer's motion. Hence, again, there is no tidal effect. We thus dismiss the third term of (15) as not giving rise to daily tidal effects.

The central term on the right is the tidal potential we seek. We rewrite it here with the symbol V.

$$\Rightarrow \qquad V(r,\psi) = -\frac{Gm_2 r^2}{2s^3}\left(3\cos^2\psi - 1\right) \tag{16}$$

At the fixed radius of the earth, $r = R$, this part of the potential perturbation is a function of ψ alone; there is no dependence on λ. The potential perturbation is thus symmetric about the earth–moon line. In addition the $\cos^2\psi$ dependence shows potential minima at the near and far sides of the earth, at $\psi = 0°$ and 180°. At these angles, the r^2 dependence indicates the potential is decreasing with distance. The negative gradient of the potential and hence the tidal force are thus positive or outwards. This again indicates that there are tidal bulges on the near and far sides of the earth from the moon.

Forces revisited

One can obtain the tidal forces quantitatively in three dimensional space directly from the potential (16). Recall that the negative gradient of the potential yields the force per unit mass. In our case, it is the perturbing force due to the moon's gravity and the orbital centrifugal force. A decreasing potential indicates a force in the direction of steepest decline. Recall that, in polar coordinates, one can write the gradient as

$$\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{u}}_{\mathrm{r}} + \frac{1}{r}\frac{\partial f}{\partial \psi}\hat{\boldsymbol{u}}_{\psi} \tag{17}$$

where the hatted symbols are unit vectors.

We work here only in the plane of the page (Fig. 3a) because the tidal function V (16) is symmetric about the earth–moon line; a rotation about the earth–moon line yields the same potential at a given ψ. Hence, the gradient of the potential is constrained to lie in the plane of the paper. Our result for the tidal force in this plane will apply to any other plane that contains the earth–moon line.

Apply (17) to (16) to obtain the tidal force vector at the radius R from earth center as a function of ψ (Problem 1b),

$$\frac{\boldsymbol{F}}{m_0} = -\nabla V \underset{r\to R}{=} \frac{Gm_2 R}{s^3}\left[\left(3\cos^2\psi - 1\right)\hat{\boldsymbol{u}}_{\mathrm{r}} - \frac{3}{2}\sin 2\psi\ \hat{\boldsymbol{u}}_{\psi}\right] \tag{18}$$

One readily finds that this yields the results (8), (9) and (11) obtained above for $\psi = 0$ and 90° respectively.

Figure 5 illustrates quantitatively the directions and relative magnitudes of this tidal perturbing force as a function of ψ. Note that it tends to squeeze at the top and bottom and to elongate at the left and right. At $\cos^2\psi = 1/3$ ($\psi = 54.7°$), the radial component is zero, and the azimuthal component is negative with value $-(3/2)(\sin 2\psi) = -1.4$. The latter magnitude may be compared to the magnitude at $\psi = 0°$, namely 2.0.

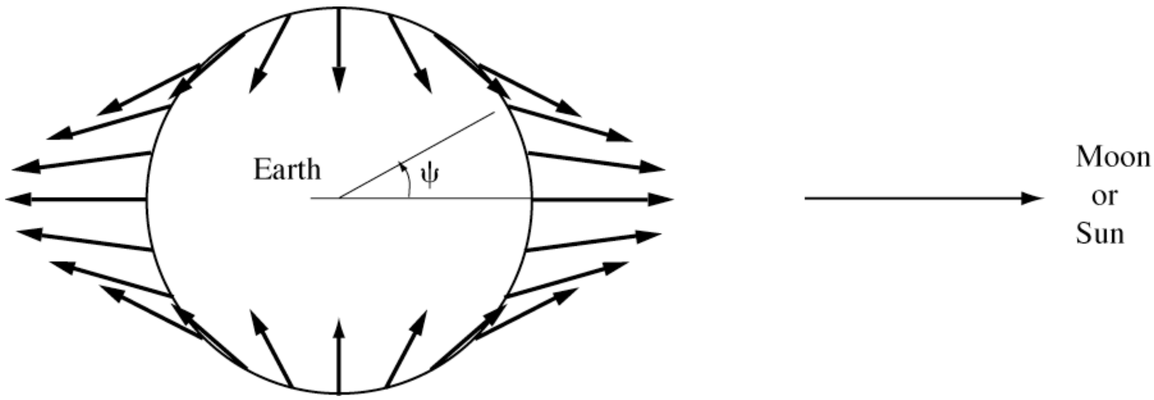


Figure 5. Tidal force vectors at earth's surface at 15° increments of the zenith angle ψ, from (18). Inspired by Stacey and Davis. The line lengths (up to the bases of the arrowheads) are proportional to the magnitudes.

6 Prolate spheroid

Potential distortion

The potential (16) is a tiny perturbation to the gravitational potential of the earth itself at $r = R$, namely $V_E = -Gm_1/R$. In Fig. 6, this potential is shown with and without the effect of a perturbation such as (16). Without the perturbation, a fluid earth or a gaseous star would ideally be spherical; it would have radius R at both $\psi = 0°$ and at 90° (Fig. 6a).

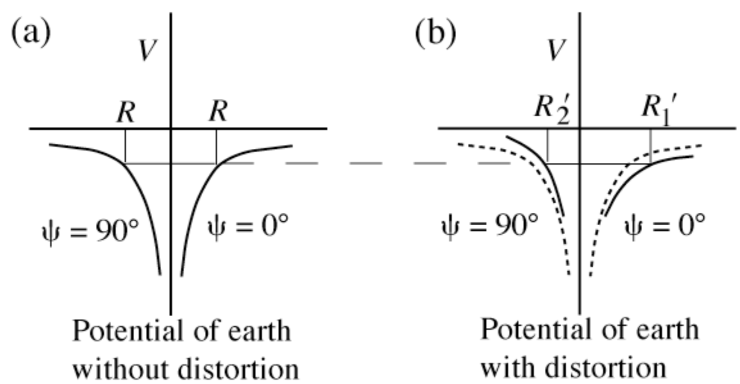


Figure 6. Potential at two angular locations due to earth's gravity (a) without perturbation and (b) without perturbation (dashed lines) and with perturbation (solid shorter lines). The perturbation is due to the moon's gravity and orbital motion.

The perturbed potential shown in Fig. 6b (solid lines) is drawn in accord with (16); it increases the potential at $\psi = 90°$ and decreases it at $\psi = 0°$. The position of a given potential value is thus moved to the right in both cases. At $\psi = 0°$, a previously unperturbed value of the potential extends to the larger radius $R_1' > R$ while at $\psi = 90°$, it moves into the radius $R_2' < R$. As the object (a gaseous star or the water level on earth) adjusts to a single potential level, the result is a shape that is asymmetric, such that in three dimensions it is suggestive of a prolate spheroid.

Equipotential displacement

The potential (16) is a small perturbation at position r,ψ to that of earth gravity. At the earth surface, $r=R$, it could be viewed as a differential quantity, $V \to \mathrm{d}V$. In general, a change of potential is the negative of the work done on unit mass by a force during a displacement d*r*. In our case, that force is the gravitational force of the earth's mass m_1 on a unit test mass at the earth's surface, namely,

$$g \equiv F / m_0 = \frac{Gm_1}{R^2} \qquad (19)$$

The distance associated with the small perturbing potential d*V* (16) at the earth surface is thus, from (16),

$$\mathrm{d}r = -\frac{\mathrm{d}V}{g} = -\frac{-Gm_2R^2}{2s^3}\left(3\cos^2\psi - 1\right)\frac{R^2}{Gm_1}$$

$$\Rightarrow \qquad = +\frac{m_2}{m_1}\left(\frac{R}{s}\right)^3 \frac{R}{2}\left(3\cos^2\psi - 1\right) \qquad (20)$$

Substitute in values for the earth (m_2) – moon (m_1) system to find

$$\mathrm{d}r = 5.6\times 10^{-8}\,\frac{R}{2}\left(3\cos^2\psi - 1\right)$$
$$= 0.179\left(3\cos^2\psi - 1\right)\ \mathrm{m}$$

At $\psi = 0°$, the sub-lunar position, the equipotential is raised 0.36 m and, at $\psi = 90°$, it is lowered 0.18 m, giving a peak to peak excursion of 0.54 m. This does not take into account tidal deformation of the earth itself nor the effect of land masses on tidal flows, but this value is of the same order of magnitude as the observed tides.

Elliptical shape

The resultant distorted surface is shown highly exaggerated in Fig. 7. It has the appearance of a prolate spheroid, a solid formed by the rotation of an ellipse rotated about its major axis. We now show that indeed it is a prolate spheroid if the disturbance is small relative to the mean radius.

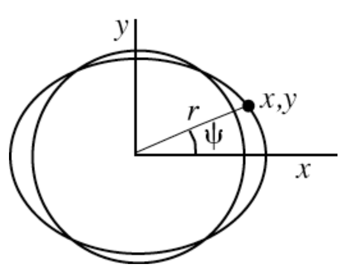


Figure 7. Exaggerated tidal distortion of a fluid earth or a gaseous star.

Write the equation of an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{21}$$

where *a* and *b* are the semimajor and semiminor axes respectively. Convert it to polar coordinates where *r* is measured *from the center of the figure*, i.e., not from a focus. We have

$$\begin{aligned} x &= r\cos\psi \\ y &= r\sin\psi \\ r^2 &= x^2 + y^2 \end{aligned} \tag{22}$$

Eliminate *y* from the latter equation with (21), invoke $x = r\cos\psi$ and solve for *r* to obtain

$$r^2 = \frac{b^2}{1 - \left(1 - \dfrac{b^2}{a^2}\right)\cos^2\psi} \tag{23}$$

For small disturbances, we write $b = a - \varepsilon$ where $\varepsilon/a \ll 1$. Expand this while dropping terms in ε^2/a^2 and retaining those in ε/a to find

$$r \approx a\left[1 + \frac{\varepsilon}{a}\left(\cos^2\psi - 1\right)\right] \tag{24}$$

Rewrite this expression in terms of the unperturbed radius *R*. From (20), we know that the radial equipotential perturbation is zero when $\cos^2\psi = 1/3$ and hence $r = R$. At this angle, (24) becomes

$$R = a\left(1 - \frac{2}{3}\frac{\varepsilon}{a}\right) \tag{25}$$

Eliminate the leading *a* in (24) with (25) to obtain, after again dropping second order terms,

$$\Rightarrow \qquad r = R\left[1 + \frac{\varepsilon}{a}\left(\cos^2\psi - \frac{1}{3}\right)\right] = R\left[1 + \frac{a-b}{3a}\left(3\cos^2\psi - 1\right)\right] \tag{26}$$

This is the equation of an ellipse in polar coordinates for small disturbances, with the origin at the center. It has the form of our perturbation (20) plus the constant *R,* the radius of the original sphere. We have thus shown that the distorted shape implied by our derived result (20) is an ellipse. It follows in turn that the tidally distorted solid body is a prolate spheroid, but only for small distortions.

Close binary stellar systems can have large distortions that depart markedly from a spheroid. See the discussionj of Roche lobes in Section 8 below and also in AP, Section 4.5.

7 Tidal effect of the sun

The tidal force is proportional to m_2/s^3 according to (16). The ratio of the tidal force of the moon to that of the sun is thus,

$$\Rightarrow \quad \frac{F_\mathrm{m}}{F_\mathrm{s}} = \frac{m_\mathrm{m}}{s_\mathrm{m}^3}\frac{s_\mathrm{s}^3}{m_\mathrm{s}} = 2.2 \qquad (27)$$

where we used the masses and semi-major axes given above after the Table of Contents. The moon is about 390 times closer than the sun, but the sun is 27.1×10^6 times more massive. The tidal forces of the two bodies are thus comparable but the moon is dominant by a factor of 2.2 for these separations.

When the moon is least effective (at apogee) and the sun most effective (earth at perihelion), this ratio is reduced to 1.75. The interaction of the solar and lunar tides lead to an angular displacement of the tidal bulge from the sub-lunar point by as much as 17° (great circle angle; see Problems 3 and 4). This can lead to tidal advances or delays of one hour or considerably more (depending on observer latitude and moon declination) from the times of moon meridian transits. When the sun is least effective, at lunar perigee and earth's aphelion, the moon is dominant by a factor of 2.9. In other words the solar effect ranges from about 35% to 57% that of the moon.

The interaction of the earth land masses with the oceans gives rise to *tidal friction*. The rapid earth rotation (once per day) drags the tidal bulge away from the sub-lunar position by about 3 degrees; here we neglect the solar effect. The interaction of the displaced bulge with the moon gradually raises the moon to higher and higher orbits and the tidal "friction" slows the spin of the earth. This evolution of the earth–moon orbit and earth spin is another fascinating topic.

Tidal times and heights are affected greatly by land masses which impede tidal currents, by traveling ocean waves that reflect from, and sometimes resonate between, land masses, and underwater typography that affects wave speeds.

As I have been working on this document, a total solar eclipse occurred in Asia on 2009 July 22. It was advertised as having one of the longest possible periods of totality, well over 6 minutes. Around the same time, tides were very high. This would be expected for two reasons. (1) The moon was perfectly aligned with the sun which yields "Spring Tides"; see Problem 3. (2) The moon was at perigee which yields significantly larger tides because it is close to the earth. Also at perigee, the moon has a significantly larger angular size which allows it to cover the sun for a longer time and thus extend the duration of totality. In July, the sun is farthest from the earth (i.e., the earth is at aphelion) which contributes to the lengthening of totality but decreases its tidal effect – but this a relatively small effect.) For more on eclipses see my textbook, *Astronomy Methods*, Section 4.4.

8 Roche potential

In the case of two stars in a binary orbit, it is useful to construct a potential for the *entire* system, again in the rotating frame of reference. In this case, we include the gravitational potential of *both* stars as well as the centrifugal inertial (fictitious) potential. We again assume the stars are synchronously spinning with the orbit and the orbits are circular. (The tidal interactions in a close binary stellar system often bring about these conditions.) Hence, all particles are motionless in this rotating frame of reference, and there are no coriolis forces. This potential is sometimes called a *pseudopotential* because it contains the (inertial) centrifugal term.

The approximations made above for the tides are no longer appropriate because we wish the potential function to be valid at all positions in the vicinity of the two stars, not just at the location of one of them. We further assume that the individual potential of each of the two masses is spherically symmetric with a $1/r$ dependence. Corrections due to possible asymmetries in the mass distributions of the individual stars are not taken into account.

This development approaches the Roche lobes from the perspective of potentials rather than from forces as presented in AP, Section 4.5. We take the origin of the coordinate system to be at the barycenter of the system, not at star #1 as in AP (Fig. 4.16a and the associated discussion).

Set up a cartesian coordinate system with the origin at the barycenter B and with the x,y plane being the plane of the orbit (Fig. 8).

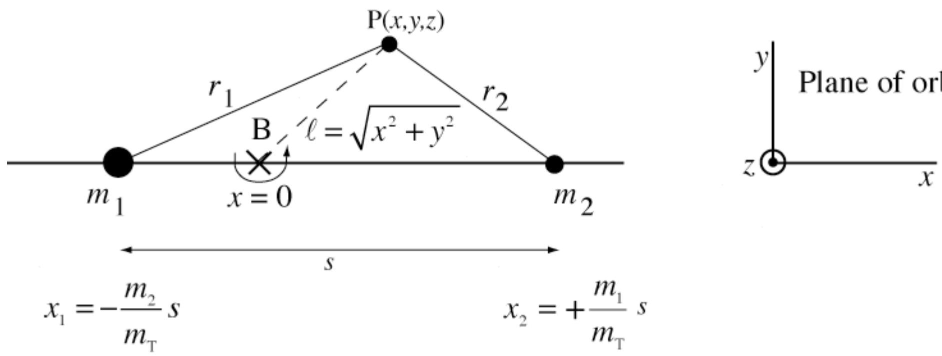


Figure 8. Coordinate system for calculating the Roche potential in three dimensions at arbitrary point P. The x,y plane is the plane of the orbit. The barycenter B is the rotation axis and ℓ is the distance from it to point P. In the frame of reference rotating with the orbit, the two masses are stationary.

The positions of the two masses are given in the figure; see (1) above.

This potential function can thus be written as, following that given in (12),

$$V = -\frac{Gm_1}{r_1} - \frac{Gm_2}{r_2} - \frac{\omega^2 \ell^2}{2} \qquad (28)$$

where the last term is the centrifugal term. Recall that $\omega^2 = Gm_T/s^3$ (3). Since ℓ is measured radially from the spin axis in the x,y plane, one has $\ell^2 = x^2 + y^2$. For a position P at arbitrary x,y,z, the potential may be written as

$$V(x,y,z) = -\frac{Gm_T}{s}\left[\frac{m_1 s}{m_T\sqrt{(x-x_1)^2 + y^2 + z^2}} + \frac{m_2 s}{m_T\sqrt{(x-x_2)^2 + y^2 + z^2}} + \frac{x^2 + y^2}{2}\right] \qquad (29)$$

where $m_T = m_1 + m_2$. This is called the *Roche potential.* Define the dimensionless lengths: $X \equiv x/s$, $Y \equiv y/s$, $Z \equiv z/s$ and introduce the ratio of masses, $q \equiv m_1/m_2$,

$$\Rightarrow \qquad V(x,y,z) = -V_0 \left[\frac{q}{(1+q)\sqrt{(X-X_1)^2+Y^2+Z^2}} + \frac{1}{(1+q)\sqrt{(X-X_2)^2+Y^2+Z^2}} + \frac{X^2+Y^2}{2} \right] \qquad (30)$$

where $V_0 = Gm_T/s$ is a constant associated with a particular binary system. Note that the positions X_1 and X_2 may be written as, from the expressions in Fig. 8,

$$X_1 = \frac{x_1}{s} = -\frac{m_2}{m_T} = -\frac{1}{1+q}$$
$$X_2 = \frac{x_2}{s} = \frac{m_1}{m_T} = \frac{q}{1+q} \qquad (31)$$

A plot of V in the orbital plane, in perspective, is shown in Fig. 9 for two unequal masses. One can see the potential wells of the two stars, one more pronounced than the other and the fall off at large distances driven by the centrifugal term.

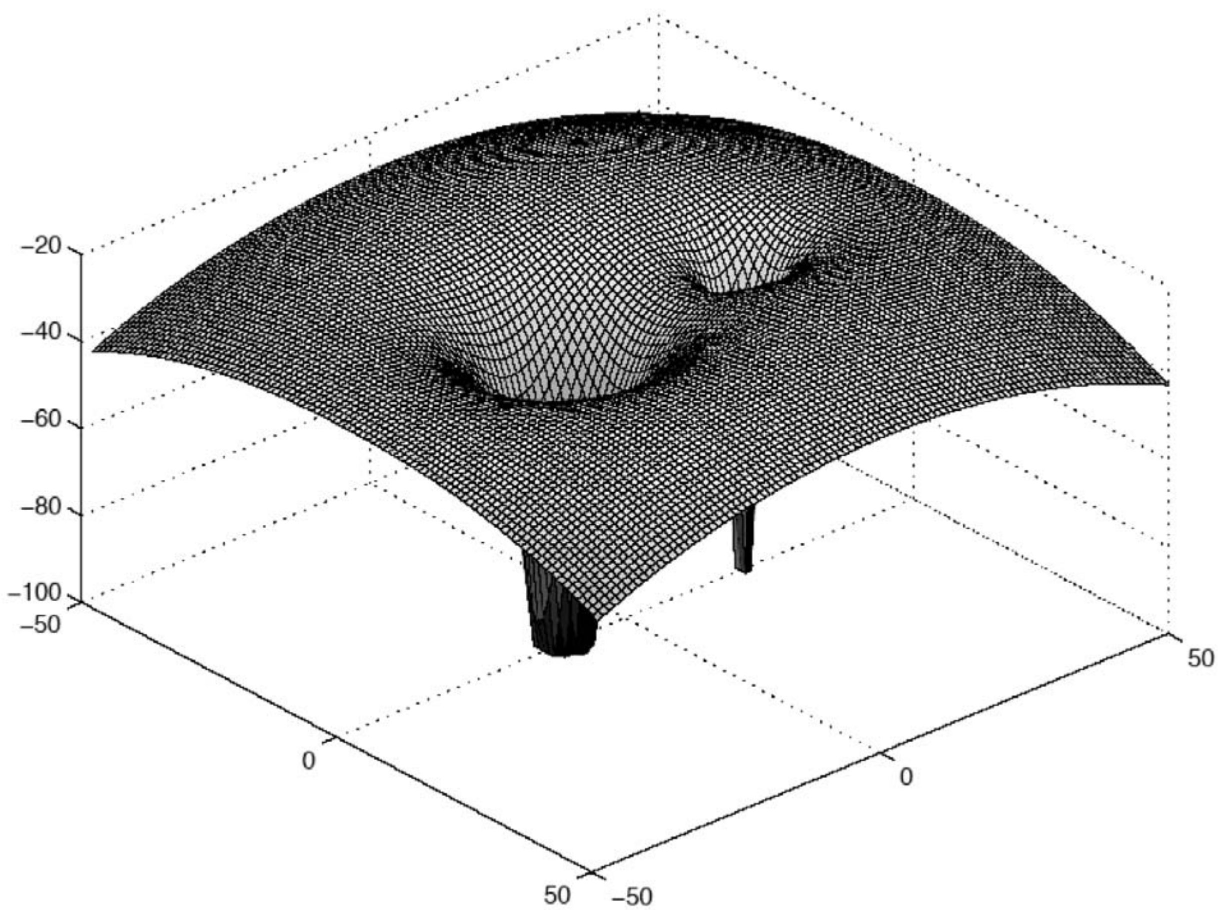


Figure 9. Perspective plot of Roche potential for a binary stellar system. Courtesy S. Rappaport.

This dimensionless form of the Roche potential (30) has the following features, to quote course notes of S. Rappaport:

"(i) to within simple overall scale factors (i.e., potential in units of V_0 and lengths in units of binary separation), the shape of the equipotential surfaces depend *only* on the mass ratio $q = m_1/m_2$.

"(ii) very near either mass, the potential behaves simply as $1/r$, as expected for an isolated point mass.

"(iii) very far from the center of mass of the binary, the dominant part of the potential is just the centrifugal repulsion term.

"(iv) all five of the Lagrange points [where there is no force on a stationary test mass] are implicitly contained in $V(x,y,z)$.

"(v) equipotential surfaces of V describe the approximate shapes of stars of finite size whose centers are located at x_1 and x_2. "

The locations of the five Lagrange points are shown in Fig. 10. Their locations may be obtained from the potential function (30); see Problem 6. They lie in the x,y plane (Problem 5), three of them are saddle points (L1, L2, and L3) and two are true maxima (L4 and L5).

The contours shown in Fig. 10 are equipotentials of the Roche potential (30). With the exception of the outer one, each passes through a Lagrange point. The "figure 8" contour passes through L1 and is called the *Roche Lobe*. As described in Section 4.5 of AP, this is the location where accretion of gas from one star to the other first takes place if the first star expands (in its evolutionary development) to fill its Roche lobe.

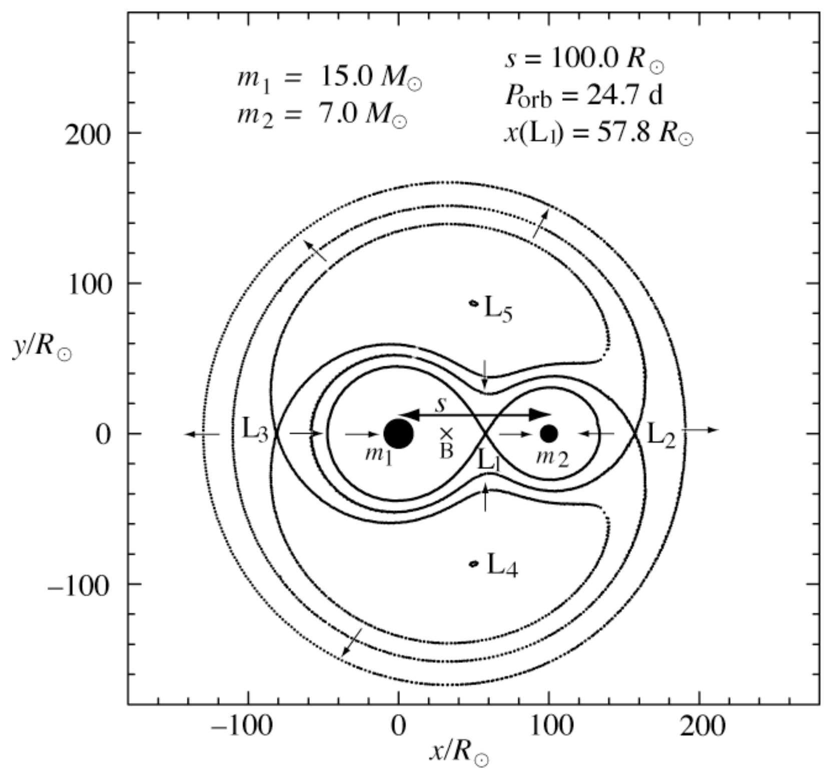


Figure 10. Equipotential contours for a binary system with stars of 15.0 and 7.0 $M_\odot$. Three of the contours pass through a Lagrange point. Those for L4 and L5 surround the maximum point. [Adapted from T. Tauris and E. van den Heuvel in *Compact Stellar X-ray Sources,* eds. W. Lewin and M. van der Klis, Cambridge 2006, Fig. 16.9,]

Further reading:

"The Restless Tides", NOAA website: http://tidesandcurrents.noaa.gov/restles1.html

Frank D. Stacey and Paul M. Davis, *Physics of the Earth,* Fourth edit., Cambridge Univ. Press, 2008.

Hale Bradt, *Astronomy Methods*, CUP, 2004. Problems 4.22–4.25, regarding the effect of tidal friction on the lunar orbit and earth spin and Section 4.4 for a description of total solar eclipses. See also Stacey and Davis on tidal friction.

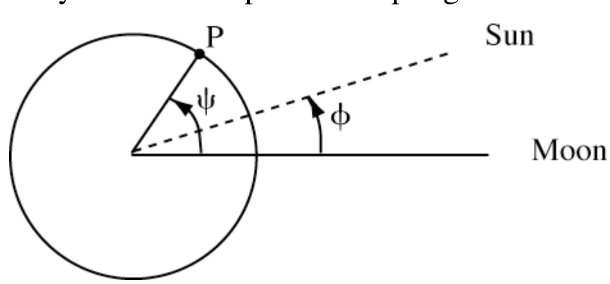
Hale Bradt, *Astrophysical Processes (AP)*, CUP, 2008, Section 1.5 on binary star dynamics and Section 4.5 for pseudopotentials in binary stellar systems.

Problems

Problem 1. (a) Verify that the perturbing tidal potential (12) may be written as (15), given the approximation $r/s \ll 1$. (b) Take the gradient of (16) to obtain the force (18); demonstrate that the force diagram (Fig. 6) is quantitatively correct.

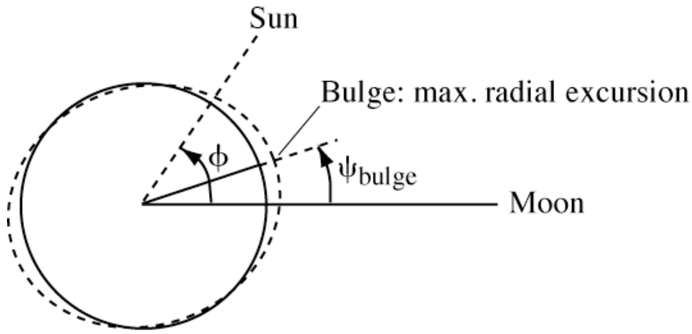
Problem 2. We demonstrated (immediately below Fig. 3) that the potential V_p (12) is not symmetric about the earth–moon line. However, just below (16), we argued that the potential V is symmetric about the earth–moon line. Are these arguments self consistent? Does the asymmetry of (12) appear in its rewritten version (15)? If so, how so?

Problem 3. Create tidal force diagrams like that of Fig. 5 but include the effect of the sun under the assumption that it has exactly 1/2 the effect of the moon. Let it lie in the plane of the paper at an angle ϕ from the moon. See Sketch below. First, re-create Fig. 5 (no sun) and then do the combined effect for the sun at $\phi = 0°$ (on the earth–moon line), at $\phi = 60°$ and at $\phi = 90°$. Maintain the same scale factor for the vectors for all four diagrams so the force magnitudes can be compared from diagram to diagram. Begin by summing potentials for the moon given in (16) and the sun (1/2 that of (16)) and then taking the negative gradient of the sum to get the combined tidal force vector (both radial and azimuthal components) at each angular position ψ on the earth's surface (at radius $r = R$). For plotting purposes, determine the amplitude of each such vector and its direction relative to the horizontal. Which of your results represents "Spring Tides" and "Neap Tides"?



Problem 4. Consider the combined effect of the sun and moon with the sun's effect being a fraction f of that of the moon. Let the sun be at great-circle angle ϕ from the moon. (f is approximately 1/2). See Sketch below. (a) Calculate the angle $\psi_{\rm bulge}$ of the maximum tidal radial displacement dr as a function of ϕ and f. Begin with the expression (20) for dr. (b) Find the solar offset angle ϕ that gives the maximum angular offset $\psi_{\rm bulge,max}$ of the bulge from the moon direction as a function of f. (c) Use your result to determine the maximum

possible angular offset $\psi_{bulge,max}$ of high tide from the moon direction if the sun is one half as effective as the moon ($f = 1/2$). Give the associated solar angle ϕ. (d) Repeat (c) for the sun being maximally effective relative to the moon ($f = 1/1/75 = 0.571$; see text). [Ans. $2\psi_{bulge} = \tan^{-1}[f \sin 2\phi/(1 + f \cos 2\phi)]$; $\cos 2\phi = -f$; 15°, 60°; 17.4°, 62.4°]



Problem 5. How does the potential (30) behave, qualitatively, as a function of z at the positions of the five Lagrange points L1–L5? What does this imply about matter flow through L1, for example. No calculation is required.

Problem 6. (a) Use the Roche potential (30) in a plotting program, e.g. Mathematica, to make a graphical presentation similar to Fig. 9 for $m_1 = 15.0\ M_\odot$ and $m_2 = 7.0\ M_\odot$. (b) Make a plot of contours of V/V_0 (30), including those that pass through the Lagrange points. You will first have to calculate the locations of the three Lagrange points., L1, L2, and L3 with a maximization along the x axis and then find the potential at those positions. Perhaps you can demonstrate that the locations of L4 and L5 are at the points that form an equilateral triangle with the two masses. Again, make use of a program such as Mathematica. Your result should match that of Fig. 10.

END