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CONDITIONAL FEYNMAN INTEGRAL AND SCHRÖDINGER INTEGRAL EQUATION ON A FUNCTION SPACE

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Abstract

Let $C^r[0, t]$ be the function space of the vector-valued continuous paths $x : [0, t] \to \mathbb{R}^r$ and define $X_t : C^r[0, t] \to \mathbb{R}^{(n+1)r}$ by $X_t(x) = (x(0), x(t_1), \ldots, x(t_n))$, where $0 < t_1 < \cdots < t_n = t$. In this paper, using a simple formula for the conditional expectations of the functions on $C^r[0, t]$ given X_t, we evaluate the conditional analytic Feynman integral $E^{anf_q}[F_t|X_t]$ of F_t given by

$$F_t(x) = \exp\left\{\int_0^t \theta(s, x(s))\, ds\right\} \quad \text{for } x \in C^r[0, t],$$

where $\theta(s, \cdot)$ are the Fourier–Stieltjes transforms of the complex Borel measures on $\mathbb{R}^r$, and provide an inversion formula for $E^{anf_q}[F_t|X_t]$. Then we present an existence theorem for the solution of an integral equation including the integral equation which is formally equivalent to the Schrödinger differential equation. We show that the solution can be expressed by $E^{anf_q}[F_t|X_t]$ and a probability distribution on $\mathbb{R}^r$ when $X_t(x) = (x(0), x(t))$.

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1. Introduction

We consider the Schrödinger differential equation in quantum mechanics,

$$\frac{i}{2q}\frac{\partial^2}{\partial t^2}\Gamma(t, \vec{\xi}_1) - \frac{\partial}{\partial t}\Gamma(t, \vec{\xi}_1) + \theta(t, \vec{\xi}_1)\Gamma(t, \vec{\xi}_1) = 0, \tag{1.1}$$

for $(t, \vec{\xi}_1) \in (0, \infty) \times \mathbb{R}^r$, where $q \in \mathbb{R} - \{0\}$ and θ is the time-dependent potential, and the initial state of the particle is given by

$$\lim_{t\to 0^+} \Gamma(t, \vec{\xi}_1) = \psi(\vec{\xi}_1) \quad \text{for } \vec{\xi}_1 \in \mathbb{R}^r. \tag{1.2}$$

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Let $C_0^r[0, t]$ be the r-dimensional Wiener space. It is well known that the space $C_0^r[0, t]$ is equipped with the Wiener measure, which is a probability measure. On the Wiener space, several solutions of an integral equation which is formally equivalent to (1.1) with the initial condition (1.2) were presented in [2, 3, 5–7, 9]. In [2, 3], Cameron and Storvick showed that the solution can be expressed as the analytic Feynman integral of the functional of the form

$$\exp\left\{\int_0^t \theta(s, x(s))\, ds\right\}\psi(x(t) + \xi_1) \tag{1.3}$$

when $r = 1$. In [9], Johnson and Skoug extended the result of [2] to arbitrary dimension $r \in \mathbb{N}$, and in their proof they were able to avoid the dependence on the use of the machinery from [1]. In [10], Park and Skoug derived a simple formula for the conditional Wiener integrals of the functions on $C_0^1[0, t]$ with the conditioning function $X : C_0^1[0, t] \to \mathbb{R}^n$ given by $X(x) = (x(t_1), \ldots, x(t_n))$, where $0 < t_1 < \cdots < t_n = t$. In [7], Chung and Skoug proved that the solution can be expressed as the conditional analytic Feynman integral of the functional given by (1.3) using the conditioning function X on $C_0^r[0, t]$, when $n = 1$.

On the other hand, let $C[0, t]$ denote the space of the real-valued continuous functions on the interval $[0, t]$. Ryu and Im introduced a probability measure w_φ on $(C[0, t], \mathcal{B}(C[0, t]))$, where $\mathcal{B}(C[0, t])$ denotes the Borel σ-algebra on $C[0, t]$ and φ is a probability distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ [8, 11]. This measure space is a generalization of the Wiener space $C_0^1[0, t]$. In the Wiener space, every path x starts at the origin, that is, $x(0) = 0$. If the paths x start at any points, that is, if $x \in C[0, t]$, certain theories on the space cannot hold or some of them must be modified. Fortunately, in [4], the author was able to derive a simple formula for the conditional w_φ-integral of the functions on $C[0, t]$ with the vector-valued conditioning function $X_t : C[0, t] \to \mathbb{R}^{n+1}$ given by $X_t(x) = (x(0), x(t_1), \ldots, x(t_n))$. This formula expresses the conditional w_φ-integral directly in terms of the nonconditional w_φ-integral.

In this paper, using the simple formula with the conditioning function X_t on $C^r[0, t]$, the product space of $C[0, t]$, we evaluate the conditional analytic Feynman w_φ^r-integral $E^{anf_q}[F_t|X_t]$ of F_t of the form

$$F_t(x) = \exp\left\{\int_0^t \theta(s, x(s))\, ds\right\} \quad \text{for } x \in C^r[0, t],$$

where $\theta(s, \cdot)$ are the Fourier–Stieltjes transforms of the complex Borel measures on $\mathbb{R}^r$, and provide an inversion formula for $E^{anf_q}[F_t|X_t]$. Then we present an existence theorem for the solution of the integral equation

$$\begin{aligned} H(t, \vec{\xi}_1, -iq) = {}& \left(\frac{q}{2\pi i t}\right)^{r/2} \int_{\mathbb{R}^r} \exp\left\{\frac{iq}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} d\varphi^r(\vec{\xi}_0) + \int_0^t \left[\frac{q}{2\pi i(t-s)}\right]^{r/2} \\ & \times \overline{\int}_{\mathbb{R}^r} \theta(s, \vec{u}) H(s, \vec{u}, -iq) \exp\left\{\frac{iq}{2(t-s)}\|\vec{\xi}_1 - \vec{u}\|_2^2\right\} d\vec{u}\, ds, \end{aligned}$$

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which includes an integral equation formally equivalent to (1.1). We show that the solution can be expressed as the function of the form

$$H(t, \vec{\xi}_1, -iq) = \left(\frac{q}{2\pi i t}\right)^{r/2} \int_{\mathbb{R}^r} \exp\left\{\frac{iq}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} E^{anf_q}[F_t|X_t](\vec{\xi}_0, \vec{\xi}_1)\, d\varphi^r(\vec{\xi}_0),$$

where $X_t : C^r[0, t] \to \mathbb{R}^{2r}$ is given by $X_t(x) = (x(0), x(t))$, in particular. Then [7, Theorem 6], can be obtained from our result if φ^r is the Dirac measure concentrated at 0.

2. The analogue of the Wiener space

Throughout this paper, let $\mathbb{C}$ and $\mathbb{C}_+$ denote the set of the complex numbers and the set of the complex numbers with positive real parts, respectively. We begin by introducing the probability measure w_φ on $(C[0, t], \mathcal{B}(C[0, t]))$.

For a positive real t, let $C = C[0, t]$ be the space of all real-valued continuous functions on the closed interval $[0, t]$ with the supremum norm. For

$$\vec{t} = (t_0, t_1, \ldots, t_n) \quad \text{with } 0 = t_0 < t_1 < \cdots < t_n \le t,$$

let $J_{\vec{t}} : C[0, t] \to \mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \ldots, x(t_n)).$$

For $B_j (j = 0, 1, \ldots, n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0, t]$ is called an interval; let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we let

$$m_\varphi\left(J_{\vec{t}}^{-1}\left(\prod_{j=0}^n B_j\right)\right) = \int_{B_0} \int_{\prod_{j=1}^n B_j} W_n(\vec{t}; u_0, u_1, \ldots, u_n)\, d(u_1, \ldots, u_n)\, d\varphi(u_0),$$

where

$$W_n(\vec{t}; u_0, u_1, \ldots, u_n) = \left[\prod_{j=1}^n \frac{1}{2\pi(t_j - t_{j-1})}\right]^{1/2} \exp\left\{-\frac{1}{2}\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\}.$$

$\mathcal{B}(C[0, t])$, the Borel σ-algebra of $C[0, t]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $(C[0, t], \mathcal{B}(C[0, t]))$ such that $w_\varphi(I) = m_\varphi(I)$ for all I in $\mathcal{I}$. This measure w_φ is called an analogue of the Wiener measure associated with the probability measure φ [8, 11]. Let r be a positive integer and $C^r = C^r[0, t]$ be the product space of $C[0, t]$ with the product measure w_φ^r. Since $C[0, t]$ is a separable Banach space, we have $\mathcal{B}(C^r[0, t]) = \prod_{j=1}^r \mathcal{B}(C[0, t])$. This probability measure space $(C^r[0, t], \mathcal{B}(C^r[0, t]), w_\varphi^r)$ is called an analogue of r-dimensional Wiener space.

LEMMA 2.1 [8, Lemma 2.1]. *If $f:\mathbb{R}^{n+1}\to\mathbb{C}$ is a Borel measurable function, then*

$$\int_C f(x(t_0), x(t_1), \ldots, x(t_n))\, dw_\varphi(x) \stackrel{*}{=} \int_{\mathbb{R}}\int_{\mathbb{R}^n} f(u_0, u_1, \ldots, u_n) W_n(\vec{t}; u_0, u_1, \ldots, u_n)\, d(u_1, \ldots, u_n)\, d\varphi(u_0),$$

where $\stackrel{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Let $\{e_k \mid k = 1, 2, \ldots\}$ be a complete orthonormal subset of $L_2[0, t]$ such that each e_k is of bounded variation. For f in $L_2[0, t]$ and x in $C[0, t]$, we let

$$(f, x) = \lim_{n\to\infty} \sum_{k=1}^{n} \int_0^t \langle f, e_k\rangle e_k(s)\, dx(s)$$

if the limit exists. Here $\langle\cdot,\cdot\rangle$ denotes the inner product on $L_2[0, t]$. (f, x) is called the Paley–Wiener–Zygmund integral of f according to x. Note that $\langle\cdot,\cdot\rangle$ also denotes the inner product on the Euclidean space if there is no danger of confusion.

Applying [8, Theorem 3.5], we can easily prove the following lemma.

LEMMA 2.2. *Let $\{h_1, h_2, \ldots, h_n\}$ be an orthonormal system of $L_2[0, t]$. For $i = 1, 2, \ldots, n$, let $Z_i(x) = (h_i, x)$ for $x \in C[0, t]$. Then $Z_1, Z_2, \ldots, Z_n$ are independent and each Z_i has the standard normal distribution. Moreover, if $f:\mathbb{R}^n\to\mathbb{R}$ is Borel measurable, then*

$$\int_C f(Z_1(x), Z_2(x), \ldots, Z_n(x))\, dw_\varphi(x) \stackrel{*}{=} \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(u_1, u_2, \ldots, u_n) \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} u_j^2\right\} d(u_1, u_2, \ldots, u_n),$$

where $\stackrel{*}{=}$ means that if either side exists, then both sides exist and they are equal.

The following lemma is needed to prove Theorem 4.1.

LEMMA 2.3. *Let $0 < u < s < t$ and*

$$Y_{u,s}(x) = x(u) - x(0) - \frac{u}{s}(x(s) - x(0)), \qquad Y_{s,t}(x) = x(s) - x(0) - \frac{s}{t}(x(t) - x(0)),$$

for $x \in C^r[0, t]$. Then $Y_{u,s}$ is normally distributed with mean vector 0 and covariance matrix $(u/s)(s-u)I_r$, where I_r is the r-dimensional identity matrix. Moreover, if g is a Borel measurable function on $\mathbb{R}^{2r}$ such that $g(Y_{u,s}(x), Y_{s,t}(x))$ is integrable on C^r, then the function $g(Y_{u,s}(x), Y_{s,t}(y))$ is integrable on $C^r[0, t]\times C^r[0, t]$ and

$$\int_{C^r} g(Y_{u,s}(x), Y_{s,t}(x))\, dw_\varphi^r(x) = \int_{C^r}\int_{C^r} g(Y_{u,s}(x), Y_{s,t}(y))\, dw_\varphi^r(x)\, dw_\varphi^r(y) = \int_{C^r}\int_{C^r} g(Y_{u,s}(x), Y_{s,t}(y))\, dw_\varphi^r(y) dw_\varphi^r(x).$$

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PROOF. Let B be any Borel subset of $\mathbb{R}^r$. Then, by the change of variable theorem and Lemma 2.1,

$$
\begin{aligned}
w_\varphi^r(Y_{u,s}\in B) &= \int_{C^r} \chi_B\Big(x(u)-x(0)-\frac{u}{s}(x(s)-x(0))\Big)\,dw_\varphi^r(x)\\
&= \Big[\frac{1}{(2\pi)^2u(s-u)}\Big]^{r/2}\int_{\mathbb{R}^{2r}}\chi_B\Big(\vec v_1-\frac{u}{s}\vec v_2\Big)\exp\Big\{-\frac{1}{2u}\|\vec v_1\|_2^2\\
&\quad -\frac{1}{2(s-u)}\|\vec v_2-\vec v_1\|_2^2\Big\}\,d(\vec v_1,\vec v_2),
\end{aligned}
$$

where χ_B denotes the indicator function of B. Let $\vec u=\vec v_1-(u/s)\vec v_2$. Then, by Fubini's theorem and the change of variable theorem again,

$$
\begin{aligned}
w_\varphi^r(Y_{u,s}\in B) &= \Big[\frac{1}{(2\pi)^2u(s-u)}\Big]^{r/2}\Big(\frac{s}{u}\Big)^r\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\chi_B(\vec u)\exp\Big\{-\frac{1}{2u}\|\vec v_1\|_2^2\\
&\quad -\frac{1}{2(s-u)}\Big\|\frac{s-u}{u}\vec v_1-\frac{s}{u}\vec u\Big\|_2^2\Big\}\,d\vec v_1\,d\vec u\\
&= \Big[\frac{1}{(2\pi)^2u(s-u)}\Big]^{r/2}\Big(\frac{s}{u}\Big)^r\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\chi_B(\vec u)\exp\Big\{-\frac{s}{2u^2}\|\vec v_1-\vec u\|_2^2\\
&\quad -\frac{s\|\vec u\|_2^2}{2u(s-u)}\Big\}\,d\vec v_1\,d\vec u\\
&= \Big[\frac{s}{2\pi u(s-u)}\Big]^{r/2}\int_{\mathbb{R}^r}\chi_B(\vec u)\exp\Big\{-\frac{s\|\vec u\|_2^2}{2u(s-u)}\Big\}\,d\vec u,
\end{aligned}
$$

which proves the first part of the theorem.

To prove the second part of the theorem it suffices to prove the theorem for the case $g=\chi_B$, where B is any Borel subset of $\mathbb{R}^{2r}$. Now, by Lemma 2.1,

$$
\begin{aligned}
&\int_{C^r} g(Y_{u,s}(x),Y_{s,t}(x))\,dw_\varphi^r(x)\\
&= \Big[\frac{1}{(2\pi)^3u(s-u)(t-s)}\Big]^{r/2}\int_{\mathbb{R}^{3r}}\chi_B\Big(\vec v_1-\frac{u}{s}\vec v_2,\vec v_2-\frac{s}{t}\vec v_3\Big)\exp\Big\{-\frac{1}{2u}\|\vec v_1\|_2^2\\
&\quad -\frac{1}{2(s-u)}\|\vec v_1-\vec v_2\|_2^2-\frac{1}{2(t-s)}\|\vec v_2-\vec v_3\|_2^2\Big\}\,d(\vec v_1,\vec v_2,\vec v_3).
\end{aligned}
$$

Let

$$\vec u_1=\vec v_1-\frac{u}{s}\vec v_2 \quad\text{and}\quad \vec u_2=\vec v_2-\frac{s}{t}\vec v_3.$$

Then

$$\vec v_2=\frac{s}{u}(\vec v_1-\vec u_1) \quad\text{and}\quad \vec v_3=\frac{t}{u}(\vec v_1-\vec u_1)-\frac{t}{s}\vec u_2,$$

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and hence, by Fubini's theorem and the change of variable theorem,

$$\begin{aligned}
&\int_{C^r} g(Y_{u,s}(x), Y_{s,t}(x))\, dw_\varphi^r(x) \\
&= \left[\frac{1}{(2\pi)^3 u(s-u)(t-s)}\right]^{r/2} \left(\frac{t}{u}\right)^r \int_{\mathbb{R}^{3r}} \chi_B(\vec{u}_1, \vec{u}_2) \exp\left\{-\frac{1}{2u}\|\vec{v}_1\|_2^2 - \frac{1}{2(s-u)}\right. \\
&\quad \times \left.\left\|\frac{u-s}{u}\vec{v}_1 + \frac{s}{u}\vec{u}_1\right\|_2^2 - \frac{1}{2(t-s)}\left\|\frac{t-s}{u}\vec{v}_1 - \frac{t-s}{u}\vec{u}_1 - \frac{t}{s}\vec{u}_2\right\|_2^2\right\} d(\vec{v}_1, \vec{u}_1, \vec{u}_2) \\
&= \left[\frac{1}{(2\pi)^3 u(s-u)(t-s)}\right]^{r/2} \left(\frac{t}{u}\right)^r \int_{\mathbb{R}^{2r}} \int_{\mathbb{R}^r} \chi_B(\vec{u}_1, \vec{u}_2) \exp\left\{-\frac{t}{2u^2}\left\|\vec{v}_1 - \vec{u}_1\right.\right. \\
&\quad \left.\left. - \frac{u}{s}\vec{u}_2\right\|_2^2 - \frac{s}{2u(s-u)}\|\vec{u}_1\|_2^2 - \frac{t}{2s(t-s)}\|\vec{u}_2\|_2^2\right\} d\vec{v}_1\, d(\vec{u}_1, \vec{u}_2) \\
&= \left[\frac{s}{2\pi u(s-u)}\right]^{r/2} \left[\frac{t}{2\pi s(t-s)}\right]^{r/2} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \chi_B(\vec{u}_1, \vec{u}_2) \exp\left\{-\frac{s}{2u(s-u)}\|\vec{u}_1\|_2^2\right. \\
&\quad \left. - \frac{t}{2s(t-s)}\|\vec{u}_2\|_2^2\right\} d\vec{u}_1\, d\vec{u}_2 \\
&\overset{(*)}{=} \int_{C^r} \int_{C^r} g(Y_{u,s}(x), Y_{s,t}(y))\, dw_\varphi^r(x)\, dw_\varphi^r(y),
\end{aligned}$$

where $(*)$ follows from the first part of the theorem. The other equality can be easily obtained from Fubini's theorem. □

DEFINITION 2.4. Let $F : C^r[0, t] \to \mathbb{C}$ be integrable and let X be a random vector on $C^r[0, t]$ assuming that the value space of X is a normed space with the Borel σ-algebra. Then, we have the conditional expectation $E[F|X]$ of F given X from a well-known probability theory. Further, there exists a P_X-integrable complex-valued function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for w_φ^r-almost everywhere $x \in C^r[0, t]$, where P_X is the probability distribution of X. The function ψ is called the conditional w_φ^r-integral of F given X and it is also denoted by $E[F|X]$.

Let $0 = t_0 < t_1 < \cdots < t_n = t$ be a partition of $[0, t]$. For any x in $C[0, t]$, define the polygonal function $[x]$ on $[0, t]$ by

$$[x](s) = x(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})), \quad t_{j-1} \le s \le t_j,\ j = 1, \ldots, n.$$

Similarly, for $\vec{\xi} = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, we define the polygonal function $[\vec{\xi}]$, replacing $x(t_j)$ by ξ_j. Then both $[x]$ and $[\vec{\xi}]$ are continuous on $[0, t]$, their graphs are line segments on each subinterval $[t_{j-1}, t_j]$, and $[x](t_j) = x(t_j)$ and $[\vec{\xi}](t_j) = \xi_j$ at each t_j.

In the following theorem, we introduce a simple formula for conditional w_φ-integrals on $C[0, t]$. The proof is given in [4].

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THEOREM 2.5. *Let $F : C[0, t] \to \mathbb{C}$ be integrable and $X : C[0, t] \to \mathbb{R}^{n+1}$ be given by $X(x) = (x(t_0), x(t_1), \ldots, x(t_n))$. Then, for P_X-almost everywhere $\vec{\xi} \in \mathbb{R}^{n+1}$,*

$$E[F|X](\vec{\xi}) = E[F(x - [x] + [\vec{\xi}])],$$

where P_X is the probability distribution of X on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$.

Let $\tau_j : 0 = t_{j0} < t_{j1} < \cdots < t_{jn_j} = t$ be a partition of $[0, t]$ and define $X_j : C[0, t] \to \mathbb{R}^{n_j+1}$ by $X_j(x) = (x(t_{j0}), x(t_{j1}), \ldots, x(t_{jn_j}))$ for $j = 1, \ldots, r$. We obtain the following theorem from Theorem 2.5.

THEOREM 2.6. *Let $F : C^r[0, t] \to \mathbb{C}$ be integrable and $X_t : C^r[0, t] \to \prod_{j=1}^r \mathbb{R}^{n_j+1}$ be given by*

$$X_t(x) = (X_1(x_1), \ldots, X_r(x_r)) \tag{2.1}$$

for $x = (x_1, \ldots, x_r) \in C^r[0, t]$. Then, for P_{X_t}-almost everywhere $\vec{\xi} = (\vec{\xi}_1, \ldots, \vec{\xi}_r) \in \prod_{j=1}^r \mathbb{R}^{n_j+1}$,

$$E[F|X_t](\vec{\xi}) = E[F(x_1 - [x_1] + [\vec{\xi}_1], \ldots, x_r - [x_r] + [\vec{\xi}_r])], \tag{2.2}$$

where P_{X_t} is the probability distribution of X_t on $(\prod_{j=1}^r \mathbb{R}^{n_j+1}, \mathcal{B}(\prod_{j=1}^r \mathbb{R}^{n_j+1}))$.

For a function $F : C^r[0, t] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-1/2}x)$ and $X_t^\lambda(x) = X_t(\lambda^{-1/2}x)$, where X_t is given by (2.1). Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. By the definition of the conditional w_φ^r-integral (Definition 2.4) and (2.2),

$$E[F^\lambda|X_t^\lambda](\vec{\xi}) = E[F(\lambda^{-1/2}(x_1 - [x_1]) + [\vec{\xi}_1], \ldots, \lambda^{-1/2}(x_r - [x_r]) + [\vec{\xi}_r])]$$

for $P_{X_t^\lambda}$-almost everywhere $\vec{\xi} = (\vec{\xi}_1, \ldots, \vec{\xi}_r) \in \prod_{j=1}^r \mathbb{R}^{n_j+1}$, where $P_{X_t^\lambda}$ is the probability distribution of X_t^λ on the Borel class of $\prod_{j=1}^r \mathbb{R}^{n_j+1}$. Throughout this paper, let

$$I_F^\lambda(\vec{\xi}) = E[F(\lambda^{-1/2}(x_1 - [x_1]) + [\vec{\xi}_1], \ldots, \lambda^{-1/2}(x_r - [x_r]) + [\vec{\xi}_r])] \tag{2.3}$$

unless otherwise specified. If $I_F^\lambda(\vec{\xi})$ has the analytic extension $J_\lambda^*(F)(\vec{\xi})$ on $\mathbb{C}_+$ as a function of λ, then it is called the conditional analytic Wiener w_φ^r-integral of F given X_t with the parameter λ and is denoted by

$$E^{anw_\lambda}[F|X_t](\vec{\xi}) = J_\lambda^*(F)(\vec{\xi})$$

for $\vec{\xi} \in \prod_{j=1}^r \mathbb{R}^{n_j+1}$. Moreover, if for a nonzero real q, $E^{anw_\lambda}[F|X_t](\vec{\xi})$ has the limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ^r-integral of F given X_t with the parameter q and denoted by

$$E^{anf_q}[F|X_t](\vec{\xi}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_t](\vec{\xi}).$$

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If $E[F^\lambda]$ itself has the analytic extension $J^*_\lambda(F)$ on $\mathbb{C}_+$, then we call $J^*_\lambda(F)$ the analytic Wiener w^r_φ-integral of F over $C^r[0, t]$ with the parameter λ and it is denoted by

$$E^{anw_\lambda}[F] = J^*_\lambda(F).$$

Furthermore, if, for a nonzero real q, $E^{anw_\lambda}[F]$ has the limit as λ approaches $-iq$ through $\mathbb{C}_+$, then it is called the analytic Feynman w^r_φ-integral of F over $C^r[0, t]$ with the parameter q and is denoted by

$$E^{anf_q}[F] = \lim_{\lambda\to -iq} E^{anw_\lambda}[F].$$

REMARK 2.7. For each $j \in \{1, \ldots, r\}$, the paths $[x_j]$ and $[\vec{\xi}_j]$ in Theorem 2.6 must be understood as the polygonal functions obtained by using the partition τ_j. Furthermore, if $\tau_1 = \tau_2 = \cdots = \tau_r$, then we denote $[x]$ and $[\vec{\xi}]$ as

$$[x] = ([x_1], \ldots, [x_r]) \quad \text{and} \quad [\vec{\xi}] = ([\vec{\xi}_1], \ldots, [\vec{\xi}_r])$$

for $x = (x_1, \ldots, x_r) \in C^r[0, t]$ and $\vec{\xi} = (\vec{\xi}_1, \ldots, \vec{\xi}_r) \in \prod_{j=1}^r \mathbb{R}^{n_j+1}$.

3. The conditional analytic Feynman w^r_φ-integrals

We begin with this section by introducing the Banach algebra $\mathcal{S}^r_{w_\varphi}$ corresponding to Cameron and Storvick's Banach algebra $\mathcal{S}$ [1]. Let $\mathcal{M}(L^r_2[0, t])$ be the space of the complex Borel measures on $L^r_2[0, t]$ and let $\mathcal{S}^r_{w_\varphi}$ be the space of the functions of the form, for $\sigma \in \mathcal{M}(L^r_2[0, t])$,

$$F(x) = \int_{L^r_2[0,t]} \exp\left\{ i \sum_{j=1}^r (v_j, x_j) \right\} d\sigma(v_1, \ldots, v_r) \tag{3.1}$$

for $x = (x_1, \ldots, x_r) \in C^r[0, t]$. For each $j = 1, \ldots, r$ and $k = 1, \ldots, n_j$, let

$$\alpha_{jk}(s) = \frac{1}{\sqrt{t_{jk} - t_{jk-1}}} \chi_{(t_{jk-1}, t_{jk}]}(s).$$

For $j = 1, \ldots, r$, let V_j be the subspace of $L_2[0, t]$ generated by $\{\alpha_{j1}, \ldots, \alpha_{jn_j}\}$, let $V^\perp_j$ denote the orthogonal complement of V_j, and let $\mathcal{P}_j$ and $\mathcal{P}^\perp_j$ be the orthogonal projections from $L_2[0, t]$ to V_j and $V^\perp_j$, respectively.

The following two theorems give the evaluation formulas for the analytic and conditional analytic Feynman w^r_φ-integrals of $F \in \mathcal{S}^r_{w_\varphi}$. The first one follows immediately from Lemma 2.2.

THEOREM 3.1. *Let $F \in \mathcal{S}^r_{w_\varphi}$ be given by (3.1). Then, for $\lambda \in \mathbb{C}_+$,*

$$E^{anw_\lambda}[F] = \int_{L^r_2[0,t]} \exp\left\{ -\frac{1}{2\lambda} \sum_{j=1}^r \|v_j\|_2^2 \right\} d\sigma(\vec{v}), \tag{3.2}$$

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where $\vec{v} = (v_1, \ldots, v_r)$. Moreover, for a nonzero real q, $E^{anf_q}[F]$ is given by the right-hand side of (3.2), replacing λ by $-iq$.

THEOREM 3.2. *Let X_t and $F \in S^r_{w_\varphi}$ be given by (2.1) and (3.1), respectively. Then, for $\lambda \in \mathbb{C}_+$,*

$$E^{anw_\lambda}[F|X_t](\vec{\xi}) = \int_{L_2^r[0,t]} \exp\left\{-\frac{1}{2\lambda}\sum_{j=1}^r \|\mathcal{P}_j^\perp v_j\|_2^2 + i\sum_{j=1}^r (v_j, [\vec{\xi}_j])\right\} d\sigma(\vec{v}) \quad (3.3)$$

for $\vec{\xi} \in \prod_{j=1}^r \mathbb{R}^{n_j+1}$, where $\vec{\xi} = (\vec{\xi}_1, \ldots, \vec{\xi}_r)$ and $\vec{v} = (v_1, \ldots, v_r)$. Moreover, for a nonzero real q, $E^{anf_q}[F|X_t](\vec{\xi})$ is given by the right-hand side of (3.3), replacing λ by $-iq$.

PROOF. For $\lambda > 0$ and $\vec{\xi} \in \prod_{j=1}^r \mathbb{R}^{n_j+1}$, let $I_F^\lambda(\vec{\xi})$ be given by (2.3). Then, by Fubini's theorem and Lemma 2.2,

$$\begin{aligned} I_F^\lambda(\vec{\xi}) &= \int_{L_2^r[0,t]}\int_{C^r} \exp\left\{i\lambda^{-1/2}\sum_{j=1}^r (v_j, x_j - [x_j]) + i\sum_{j=1}^r (v_j, [\vec{\xi}_j])\right\} dw_\varphi^r(x)\, d\sigma(\vec{v}) \\ &= \int_{L_2^r[0,t]} \exp\left\{-\frac{1}{2\lambda}\sum_{j=1}^r \|\mathcal{P}_j^\perp v_j\|_2^2 + i\sum_{j=1}^r (v_j, [\vec{\xi}_j])\right\} d\sigma(\vec{v}) \end{aligned}$$

because $(v_j, x_j - [x_j]) = (v_j - \mathcal{P}_j v_j, x_j) = (\mathcal{P}_j^\perp v_j, x_j)$ for each j. The theorem follows from Morera's theorem and the dominated convergence theorem. □

COROLLARY 3.3. *Let $n_j = 1$, that is, $t_{j1} = t_{jn_j} = t$ for $j = 1, \ldots, r$. Then under the assumptions and notation of Theorem 3.2, for $\lambda \in \mathbb{C}_+$,*

$$\begin{aligned} &E^{anw_\lambda}[F|X_t](\vec{\xi}) \\ &\quad = \int_{L_2^r[0,t]} \exp\left\{-\frac{1}{2\lambda t}[t\|\vec{v}\|_2^2 - \|\vec{V}_t\|_2^2] + \frac{i}{t}\langle \vec{\xi}_1 - \vec{\xi}_0, \vec{V}_t\rangle\right\} d\sigma(\vec{v}) \end{aligned} \quad (3.4)$$

for $\vec{\xi} = ((\xi_{10}, \xi_{11}), \ldots, (\xi_{r0}, \xi_{r1})) \in \mathbb{R}^{2r}$, where

$$(\vec{\xi}_0, \vec{\xi}_1) = ((\xi_{10}, \ldots, \xi_{r0}), (\xi_{11}, \ldots, \xi_{r1}))$$

and

$$\vec{V}_t = \left(\int_0^t v_1(s)\, ds, \ldots, \int_0^t v_r(s)\, ds\right).$$

Moreover, for a nonzero real q, $E^{anf_q}E[F|X_t](\vec{\xi})$ is given by the right-hand side of (3.4), replacing λ by $-iq$.

PROOF. Since $\mathcal{P}_j^2 = \mathcal{P}_j$ and $\mathcal{P}_j$ is self-adjoint,

$$\|P_j^\perp v_j\|_2 = \|v_j\|_2^2 - \|\mathcal{P}_j v_j\|_2^2 = \|v_j\|_2^2 - \frac{1}{t}\left(\int_0^t v_j(s)\, ds\right)^2$$

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and

$$(v_j, [\vec{\xi}_j]) = \frac{\xi_{j1} - \xi_{j0}}{t} \int_0^t v_j(s)\, ds.$$

Now the corollary follows. □

For notational convenience, if $\vec{\xi}_0 = (\xi_{10}, \ldots, \xi_{r0}) \in \mathbb{R}^r$ and $\vec{\xi}_1 = ((\xi_{11}, \ldots, \xi_{1n_1}), (\xi_{21}, \ldots, \xi_{2n_2}), \ldots, (\xi_{r1}, \ldots, \xi_{rn_r})) \in \prod_{j=1}^r \mathbb{R}^{n_j}$, we write $\vec{\xi} = (\vec{\xi}_0, \vec{\xi}_1) = ((\xi_{10}, \xi_{11}, \ldots, \xi_{1n_1}), (\xi_{20}, \xi_{21}, \ldots, \xi_{2n_2}), \ldots, (\xi_{r0}, \xi_{r1}, \ldots, \xi_{rn_r}))$, which is a vector in $\prod_{j=1}^r \mathbb{R}^{n_j+1}$. Furthermore, the product measure of φ on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$ is denoted by φ^r. Let f be defined on $\prod_{j=1}^r \mathbb{R}^{n_j}$. We adopt the following notation which coincides with [9, (6.1)] when $n_j = 1$:

$$\overline{\int}_{\prod_{j=1}^r \mathbb{R}^{n_j}} f(\vec{\xi}_1)\, d\vec{\xi}_1 = \lim_{A \to \infty} \int_{\prod_{j=1}^r \mathbb{R}^{n_j}} f(\vec{\xi}_1) \exp\bigg\{-\frac{1}{2A} \sum_{j=1}^r \sum_{k=1}^{n_j} (\xi_{jk} - \xi_{jk-1})^2\bigg\}\, d\vec{\xi}_1$$

if the limit exists, where $(\xi_{10}, \ldots, \xi_{r0}) = (0, \ldots, 0)$.

With this notation, the following theorem provides an inversion formula for $E^{anf_q}[F|X_t]$.

THEOREM 3.4. *With the assumptions and notation of Theorem 3.2, for $\lambda \in \mathbb{C}_+$,*

$$\begin{aligned} E^{anw_\lambda}[F] = \int_{\prod_{j=1}^r \mathbb{R}^{n_j}} \int_{\mathbb{R}^r} \prod_{j=1}^r \prod_{k=1}^{n_j} \bigg[\frac{\lambda}{2\pi(t_{jk} - t_{jk-1})}\bigg]^{1/2} \\ \times \exp\bigg\{-\frac{\lambda}{2} \sum_{j=1}^r \sum_{k=1}^{n_j} \frac{(\xi_{jk} - \xi_{jk-1})^2}{t_{jk} - t_{jk-1}}\bigg\} E^{anw_\lambda}[F|X_t](\vec{\xi}_0, \vec{\xi}_1)\, d\varphi^r(\vec{\xi}_0)\, d\vec{\xi}_1 \end{aligned}$$

and, for a nonzero real q,

$$\begin{aligned} E^{anf_q}[F] = \overline{\int}_{\prod_{j=1}^r \mathbb{R}^{n_j}} \int_{\mathbb{R}^r} \prod_{j=1}^r \prod_{k=1}^{n_j} \bigg[\frac{q}{2\pi i(t_{jk} - t_{jk-1})}\bigg]^{1/2} \\ \times \exp\bigg\{\frac{iq}{2} \sum_{j=1}^r \sum_{k=1}^{n_j} \frac{(\xi_{jk} - \xi_{jk-1})^2}{t_{jk} - t_{jk-1}}\bigg\} E^{anf_q}[F|X_t](\vec{\xi}_0, \vec{\xi}_1)\, d\varphi^r(\vec{\xi}_0)\, d\vec{\xi}_1. \end{aligned}$$

PROOF. For $\lambda \in \mathbb{C}_+$, let

$$\begin{aligned} I = \int_{\prod_{j=1}^r \mathbb{R}^{n_j}} \int_{\mathbb{R}^r} \prod_{j=1}^r \prod_{k=1}^{n_j} \bigg[\frac{\lambda}{2\pi(t_{jk} - t_{jk-1})}\bigg]^{1/2} \exp\bigg\{-\frac{\lambda}{2} \sum_{j=1}^r \sum_{k=1}^{n_j} \frac{(\xi_{jk} - \xi_{jk-1})^2}{t_{jk} - t_{jk-1}}\bigg\} \\ \times E^{anw_\lambda}[F|X_t](\vec{\xi}_0, \vec{\xi}_1)\, d\varphi^r(\vec{\xi}_0)\, d\vec{\xi}_1. \end{aligned}$$

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By Theorem 3.2 and Fubini's theorem,

$$I=\prod_{j=1}^{r}\prod_{k=1}^{n_j}\bigg[\frac{\lambda}{2\pi(t_{jk}-t_{jk-1})}\bigg]^{1/2}\int_{L_2^r[0,t]}\exp\bigg\{-\frac{1}{2\lambda}\sum_{j=1}^{r}[\|v_j\|_2^2-\|\mathcal{P}_jv_j\|_2^2]\bigg\}$$
$$\times\int_{\mathbb{R}^r}\int_{\prod_{j=1}^{r}\mathbb{R}^{n_j}}\exp\bigg\{-\frac{\lambda}{2}\sum_{j=1}^{r}\sum_{k=1}^{n_j}\frac{(\xi_{jk}-\xi_{jk-1})^2}{t_{jk}-t_{jk-1}}+i\sum_{j=1}^{r}\sum_{k=1}^{n_j}(\mathcal{P}_jv_j)(t_{jk})$$
$$\times(\xi_{jk}-\xi_{jk-1})\bigg\}\,d\vec{\xi}_1d\varphi^r(\vec{\xi}_0)\,d\sigma(\vec{v}).$$

Using the well-known integral formula

$$\int_{\mathbb{R}}\exp\{-au^2+ibu\}\,du=\Big(\frac{\pi}{a}\Big)^{1/2}\exp\bigg\{-\frac{b^2}{4a}\bigg\}\tag{3.5}$$

for $a\in\mathbb{C}_+$ and any real b,

$$I=\int_{L_2^r[0,t]}\exp\bigg\{-\frac{1}{2\lambda}\sum_{j=1}^{r}\|v_j\|_2^2+\frac{1}{2\lambda}\sum_{j=1}^{r}\sum_{k=1}^{n_j}\langle\alpha_{jk},v_j\rangle^2$$
$$-\frac{1}{2\lambda}\sum_{j=1}^{r}\sum_{k=1}^{n_j}(t_{jk}-t_{jk-1})[\langle\alpha_{jk},v_j\rangle\alpha_{jk}(t_{jk})]^2\bigg\}\,d\sigma(\vec{v})$$
$$=\int_{L_2^r[0,t]}\exp\bigg\{-\frac{1}{2\lambda}\sum_{j=1}^{r}\|v_j\|_2^2\bigg\}\,d\sigma(\vec{v})=E^{anw_\lambda}[F],$$

where the last equality immediately follows Theorem 3.1.

To prove the second equality, for $A>0$ and nonzero real q, let

$$J(A)=\int_{\prod_{j=1}^{r}\mathbb{R}^{n_j}}\int_{\mathbb{R}^r}\prod_{j=1}^{r}\prod_{k=1}^{n_j}\bigg[\frac{q}{2\pi i(t_{jk}-t_{jk-1})}\bigg]^{1/2}\exp\bigg\{\frac{iq}{2}\sum_{j=1}^{r}\sum_{k=1}^{n_j}\frac{(\xi_{jk}-\xi_{jk-1})^2}{t_{jk}-t_{jk-1}}$$
$$-\frac{1}{2A}\sum_{j=1}^{r}\bigg[\sum_{k=2}^{n_j}(\xi_{jk}-\xi_{jk-1})^2+\xi_{j1}^2\bigg]\bigg\}E^{anf_q}[F|X_t](\vec{\xi}_0,\vec{\xi}_1)\,d\varphi^r(\vec{\xi}_0)\,d\vec{\xi}_1.$$

By Theorem 3.2 and Fubini's theorem,

$$J(A)=\prod_{j=1}^{r}\prod_{k=1}^{n_j}\bigg[\frac{q}{2\pi i(t_{jk}-t_{jk-1})}\bigg]^{1/2}\int_{L_2^r[0,t]}\exp\bigg\{\frac{1}{2qi}\sum_{j=1}^{r}\|\mathcal{P}_j^{\perp}v_j\|_2^2\bigg\}$$
$$\times\int_{\mathbb{R}^r}\int_{\prod_{j=1}^{r}\mathbb{R}^{n_j}}\exp\bigg\{\frac{iq}{2}\sum_{j=1}^{r}\sum_{k=1}^{n_j}\frac{(\xi_{jk}-\xi_{jk-1})^2}{t_{jk}-t_{jk-1}}-\frac{1}{2A}\sum_{j=1}^{r}\bigg[\sum_{k=2}^{n_j}(\xi_{jk}$$
$$-\xi_{jk-1})^2+\xi_{j1}^2\bigg]+i\sum_{j=1}^{r}\sum_{k=1}^{n_j}(\mathcal{P}_jv_j)(t_{jk})(\xi_{jk}-\xi_{jk-1})\bigg\}\,d\vec{\xi}_1d\varphi^r(\vec{\xi}_0)\,d\sigma(\vec{v}).$$

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For $j = 1, \ldots, r$ and $k = 1, \ldots, n_j$, let $u_{jk} = \xi_{jk} - \xi_{jk-1}$. Then, by the change of variable theorem,

$$\begin{aligned} J(A) &= \prod_{j=1}^{r}\prod_{k=1}^{n_j}\bigg[\frac{q}{2\pi i(t_{jk}-t_{jk-1})}\bigg]^{1/2}\int_{L_2^r[0,t]}\exp\bigg\{\frac{1}{2qi}\sum_{j=1}^{r}\|\mathcal{P}_j^{\perp}v_j\|_2^2\bigg\} \\ &\quad\times\int_{\mathbb{R}^r}\int_{\prod_{j=1}^r\mathbb{R}^{n_j}}\exp\bigg\{\frac{iq}{2}\sum_{j=1}^{r}\sum_{k=1}^{n_j}\frac{u_{jk}^2}{t_{jk}-t_{jk-1}}-\frac{1}{2A}\sum_{j=1}^{r}\bigg[\sum_{k=2}^{n_j}u_{jk}^2 \\ &\quad+(u_{j1}+\xi_{j0})^2\bigg]+i\sum_{j=1}^{r}\sum_{k=1}^{n_j}(\mathcal{P}_jv_j)(t_{jk})u_{jk}\bigg\}\,d\vec{u}\,d\varphi^r(\vec{\xi}_0)\,d\sigma(\vec{v}) \\ &= \prod_{j=1}^{r}\prod_{k=1}^{n_j}\bigg[\frac{q}{2\pi i(t_{jk}-t_{jk-1})}\bigg]^{1/2}\int_{L_2^r[0,t]}\exp\bigg\{\frac{1}{2qi}\sum_{j=1}^{r}\|\mathcal{P}_j^{\perp}v_j\|_2^2\bigg\} \\ &\quad\times\int_{\mathbb{R}^r}\int_{\prod_{j=1}^r\mathbb{R}^{n_j}}\exp\bigg\{-\frac{1}{2}\sum_{j=1}^{r}\sum_{k=1}^{n_j}\frac{t_{jk}-t_{jk-1}-iqA}{A(t_{jk}-t_{jk-1})}u_{jk}^2 \\ &\quad+i\sum_{j=1}^{r}\sum_{k=2}^{n_j}(\mathcal{P}_jv_j)(t_{jk})u_{jk}+i\sum_{j=1}^{r}\bigg[(\mathcal{P}_jv_j)(t_{j1})+\frac{i}{A}\xi_{j0}\bigg]u_{j1} \\ &\quad-\frac{1}{2A}\sum_{j=1}^{r}\xi_{j0}^2\bigg\}\,d\vec{u}\,d\varphi^r(\vec{\xi}_0)\,d\sigma(\vec{v}), \end{aligned}$$

where $\vec{u} = ((u_{11}, \ldots, u_{1n_1}), \ldots, (u_{r1}, \ldots, u_{rn_r}))$. Now, by (3.5),

$$\begin{aligned} J(A) &= \prod_{j=1}^{r}\prod_{k=1}^{n_j}\bigg[\frac{-iqA}{t_{jk}-t_{jk-1}-iqA}\bigg]^{1/2}\int_{L_2^r[0,t]}\int_{\mathbb{R}^r}\exp\bigg\{\frac{1}{2qi}\sum_{j=1}^{r}\|\mathcal{P}_j^{\perp}v_j\|_2^2 \\ &\quad-\frac{A}{2}\sum_{j=1}^{r}\bigg[\sum_{k=2}^{n_j}\frac{(t_{jk}-t_{jk-1})[(\mathcal{P}_jv_j)(t_{jk})]^2}{t_{jk}-t_{jk-1}-iqA}+\frac{t_{j1}}{t_{j1}-iqA}\bigg[(\mathcal{P}_jv_j)(t_{j1}) \\ &\quad+\frac{i}{A}\xi_{j0}\bigg]^2\bigg]-\frac{1}{2A}\sum_{j=1}^{r}\xi_{j0}^2\bigg\}\,d\varphi^r(\vec{\xi}_0)\,d\sigma(\vec{v}). \end{aligned}$$

Let K be the exponential function of the last integral. Then

$$\begin{aligned} |K| &\le \bigg|\exp\bigg\{-\frac{A}{2}\sum_{j=1}^{r}\frac{t_{j1}^2+it_{j1}qA}{t_{j1}^2+(qA)^2} \\ &\quad\times\bigg[[(\mathcal{P}_jv_j)(t_{j1})]^2-\frac{\xi_{j0}^2}{A^2}+\frac{2i}{A}\xi_{j0}(\mathcal{P}_jv_j)(t_{j1})\bigg]-\frac{1}{2A}\sum_{j=1}^{r}\xi_{j0}^2\bigg\}\bigg| \end{aligned}$$

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$$= \exp\biggl\{-\frac{1}{2A}\sum_{j=1}^{r}\biggl[\biggl[1-\frac{t_{j1}^2}{t_{j1}^2+(qA)^2}\biggr]\xi_{j0}^2-\frac{2t_{j1}qA^2\xi_{j0}}{t_{j1}^2+(qA)^2}(\mathcal{P}_jv_j)(t_{j1}) + \frac{(At_{j1})^2}{t_{j1}^2+(qA)^2}[(\mathcal{P}_jv_j)(t_{j1})]^2\biggr]\biggr\}$$

$$= \exp\biggl\{-\frac{A}{2}\sum_{j=1}^{r}\frac{[q\xi_{j0}-t_{j1}(\mathcal{P}_jv_j)(t_{j1})]^2}{t_{j1}^2+(qA)^2}\biggr\} \le 1$$

so that also, by the dominated convergence theorem,

$$\lim_{A\to\infty} J(A) = \int_{L_2^r[0,t]}\exp\biggl\{\frac{1}{2qi}\sum_{j=1}^{r}\|v_j\|_2^2-\frac{1}{2qi}\sum_{j=1}^{r}\sum_{k=1}^{n_j}\langle\alpha_{jk},v_j\rangle^2 + \frac{1}{2qi}\sum_{j=1}^{r}\sum_{k=1}^{n_j}(t_{jk}-t_{jk-1})[\langle\alpha_{jk},v_j\rangle\alpha_{jk}(t_{jk})]^2\biggr\}\,d\sigma(\vec{v})$$

$$= \int_{L_2^r[0,t]}\exp\biggl\{\frac{1}{2qi}\sum_{j=1}^{r}\|v_j\|_2^2\biggr\}\,d\sigma(\vec{v}) = E^{anf_q}[F],$$

where the last equality immediately follows from Theorem 3.1. □

The following example shows that there exists an unbounded function G such that $E^{anf_q}[G|X_t]$ exists.

Example 3.5. Let $\psi \in L_p(\mathbb{R}^r)$ be Borel measurable for $1 \le p \le \infty$. Moreover, let X_t and $F \in \mathcal{S}_{w_\varphi}^r$ be given by (2.1) and (3.1), respectively, and let $G(x) = F(x)\psi(x(t))$ for $x = (x_1, \ldots, x_r) \in C^r[0, t]$. Then for $\lambda > 0$, by Lemma 2.1 and Hölder's inequality,

$$\int_{C^r}|G^\lambda(x)|\,dw_\varphi^r(x) \le \|\sigma\|\biggl(\frac{1}{2\pi t}\biggr)^{r/2} \times \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}|\psi(\lambda^{-1/2}\vec{u}_1)|\exp\biggl\{-\frac{\|\vec{u}_1-\vec{u}_0\|_2^2}{2t}\biggr\}\,d\vec{u}_1\,d\varphi^r(\vec{u}_0)$$

$$\le \|\sigma\|p_1^{r/2p_1}\|\psi\|_p\biggl(\frac{\lambda}{2\pi t}\biggr)^{r/2p}$$

with $1/p + 1/p_1 = 1$, because φ^r is a finite measure. For $\vec{\xi} = (\vec{\xi}_1, \ldots, \vec{\xi}_r) = ((\xi_{10}, \ldots, \xi_{1n_1}), \ldots, (\xi_{r0}, \ldots, \xi_{rn_r})) \in \prod_{j=1}^r \mathbb{R}^{n_j+1}$, let $I_G^\lambda(\vec{\xi})$ be given by (2.3) with F replaced by G. Since

$$\psi(\lambda^{-1/2}(x_1-[x_1])(t)+[\vec{\xi}_1](t), \ldots, \lambda^{-1/2}(x_r-[x_r])(t)+[\vec{\xi}_r](t)) = \psi(\xi_{1n_1}, \ldots, \xi_{rn_r}),$$

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then $I_G^\lambda(\vec{\xi}) = \psi(\xi_{1n_1}, \ldots, \xi_{rn_r}) I_F^\lambda(\vec{\xi})$. Now, by Theorem 3.2, for any nonzero real q,

$$E^{anf_q}[G|X_t](\vec{\xi}) = \psi(\xi_{1n_1}, \ldots, \xi_{rn_r}) E^{anf_q}[F|X_t](\vec{\xi}).$$

4. The integral equation

In this section, we present a solution of an integral equation including the integral equation which is formally equivalent to the Schrödinger equation.

Let $\mathcal{M}(\mathbb{R}^r)$ be the class of all complex Borel measures on $\mathbb{R}^r$ and $\mathcal{G}$ be the set of all $\mathbb{C}$-valued functions θ on $[0, \infty) \times \mathbb{R}^r$ which have the form

$$\theta(s, \vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{v}_0\rangle\}\, d\sigma_s(\vec{v}_0), \tag{4.1}$$

where $\{\sigma_s \mid s \in [0, \infty)\}$ is the family from $\mathcal{M}(\mathbb{R}^r)$ satisfying the following conditions:

(1) for each Borel subset E of $\mathbb{R}^r$, $\sigma_s(E)$ is a Borel measurable function of s on $[0, t]$;

(2) $\|\sigma_s\| \in L_1[0, t]$.

We now adopt the following notational conventions: for $\vec{\xi}_0 = (\xi_{10}, \xi_{20}, \ldots, \xi_{r0}) \in \mathbb{R}^r$ and $\vec{\xi}_1 = (\xi_{11}, \xi_{21}, \ldots, \xi_{r1}) \in \mathbb{R}^r$, let $\vec{\xi} = (\vec{\xi}_0, \vec{\xi}_1) = ((\xi_{10}, \xi_{11}), \ldots, (\xi_{r0}, \xi_{r1})) \in \mathbb{R}^{2r}$. Moreover, $I_{F_s}^\lambda(\vec{\xi}_0, \vec{\xi}_1)$ means $I_{F_s}^\lambda(\vec{\xi}_0, \vec{\xi}_1) = I_{F_s}^\lambda(\vec{\xi})$ for such $\vec{\xi} = (\vec{\xi}_0, \vec{\xi}_1)$, where $I_{F_s}^\lambda$ is given by (2.3). Further, in the expression for $I_{F_s}^\lambda(\vec{\xi})$, each polygonal function is given by using the partition $0 < s$.

THEOREM 4.1. *Let X_s be given as in Corollary 3.3, replacing t by the varying parameter $s (0 < s \le t)$, and let*

$$F_t(x) = \exp\left\{\int_0^t \theta(u, x(u))\, du\right\} \tag{4.2}$$

for $x \in C^r[0, t]$, where $\theta \in \mathcal{G}$ is given by (4.1). Furthermore, for $(t, \xi_1, \lambda) \in (0, \infty) \times \mathbb{R}^r \times (0, \infty)$, let

$$H(t, \vec{\xi}_1, \lambda) = \left(\frac{\lambda}{2\pi t}\right)^{r/2} \int_{\mathbb{R}^r} \exp\left\{-\frac{\lambda}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} I_{F_t}^\lambda(\vec{\xi}_0, \vec{\xi}_1)\, d\varphi^r(\vec{\xi}_0),$$

where $I_{F_t}^\lambda$ is given by (2.3) in the sense of the conditioning function X_t given as in Corollary 3.3. Then $F_t \in \mathcal{S}_{w_\varphi^r}$ and H satisfies the integral equation

$$\begin{aligned} H(t, \vec{\xi}_1, \lambda) = {}& \left(\frac{\lambda}{2\pi t}\right)^{r/2} \int_{\mathbb{R}^r} \exp\left\{-\frac{\lambda}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} d\varphi^r(\vec{\xi}_0) + \int_0^t \left[\frac{\lambda}{2\pi(t-s)}\right]^{r/2} \\ & \times \int_{\mathbb{R}^r} \theta(s, \vec{u}) H(s, \vec{u}, \lambda) \exp\left\{-\frac{\lambda}{2(t-s)}\|\vec{\xi}_1 - \vec{u}\|_2^2\right\} d\vec{u}\, ds \end{aligned} \tag{4.3}$$

for $(t, \xi_1, \lambda) \in (0, \infty) \times \mathbb{R}^r \times (0, \infty)$.

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PROOF. It is not difficult to show that $F_t \in \mathcal{S}_{w_\varphi^r}$ using the same process used in [1]. For $s \in [0, t]$, let F_s be given by (4.2), replacing t by s. Differentiating F_s with respect to s and then integrating the derivative on $[0, t]$,

$$F_t(x) = 1 + \int_0^t \theta(s, x(s)) F_s(x)\, ds, \quad \text{for } x \in C^r[0, t].$$

Since $|\theta(s, x(s))| \le \|\sigma_s\|$ for $s \in [0, t]$,

$$\int_0^t |\theta(s, x(s)) F_s(x)|\, ds \le \exp\left\{\int_0^t \|\sigma_s\|\, ds\right\} \int_0^t \|\sigma_s\|\, ds < \infty$$

so that by Fubini's theorem, for $\lambda > 0$ and $\vec{\xi} = (\vec{\xi}_0, \vec{\xi}_1) = ((\xi_{10}, \xi_{11}), \ldots, (\xi_{r0}, \xi_{r1})) \in \mathbb{R}^{2r}$,

$$\begin{aligned}
I^\lambda_{F_t}(\vec{\xi}_0, \vec{\xi}_1) &= 1 + \int_0^t \int_{C^r} \theta(s, \lambda^{-1/2}(x(s) - [x](s)) + [\vec{\xi}](s)) F_s \\
&\quad \times (\lambda^{-1/2}(x - [x]) + [\vec{\xi}])\, dw^r_\varphi(x)\, ds \\
&= 1 + \int_0^t \int_{C^r} \theta(s, \lambda^{-1/2} Y_{s,t}(x) + [\vec{\xi}](s)) \exp\left\{\int_0^s \theta\Big(u, \lambda^{-1/2}\right. \\
&\quad \left. \times \Big(x(u) - x(0) - \frac{u}{t}(x(t) - x(0))\Big) + \frac{u}{t}(\vec{\xi}_1 - \vec{\xi}_0) + \vec{\xi}_0\Big)\, du\right\} dw^r_\varphi(x)\, ds \\
&= 1 + \int_0^t \int_{C^r} \theta(s, \lambda^{-1/2} Y_{s,t}(x) + [\vec{\xi}](s)) \exp\left\{\int_0^s \theta\Big(u, \lambda^{-1/2} Y_{u,s}(x)\right. \\
&\quad \left. + \frac{u}{s}(\lambda^{-1/2} Y_{s,t}(x) + [\vec{\xi}](s)) + \Big(1 - \frac{u}{s}\Big)\vec{\xi}_0\Big)\, du\right\} dw^r_\varphi(x)\, ds,
\end{aligned}$$

where $Y_{u,s}$ and $Y_{s,t}$ are given as in Lemma 2.3. Now, by Lemma 2.3 and Fubini's theorem,

$$\begin{aligned}
I^\lambda_{F_t}(\vec{\xi}_0, \vec{\xi}_1) &= 1 + \int_0^t \left[\frac{\lambda t}{2\pi s(t-s)}\right]^{r/2} \int_{\mathbb{R}^r} \theta(s, \vec{u}) \int_{C^r} \exp\left\{-\frac{\lambda t}{2s(t-s)}\|\vec{u} - [\vec{\xi}](s)\|_2^2\right. \\
&\quad \left. + \int_0^s \theta\Big(u, \lambda^{-1/2} Y_{u,s}(x) + \frac{u}{s}(\vec{u} - \vec{\xi}_0) + \vec{\xi}_0\Big)\, du\right\} dw^r_\varphi(x)\, d\vec{u}\, ds \\
&= 1 + \int_0^t \left[\frac{\lambda t}{2\pi s(t-s)}\right]^{r/2} \int_{\mathbb{R}^r} \theta(s, \vec{u}) I^\lambda_{F_s}(\vec{\xi}_0, \vec{u}) \exp\left\{-\frac{\lambda t}{2s(t-s)}\left\|\vec{u}\right.\right. \\
&\quad \left.\left. - \frac{s}{t}(\vec{\xi}_1 - \vec{\xi}_0) - \vec{\xi}_0\right\|_2^2\right\} d\vec{u}\, ds.
\end{aligned}$$

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Multiplying by $(\lambda/2\pi t)^{r/2}\exp\{-(\lambda/(2t))\|\vec{\xi}_1-\vec{\xi}_0\|_2^2\}$ on the both sides of the equality and then integrating over $\mathbb{R}^r$ with respect to φ^r, Fubini's theorem leads to

$$\begin{aligned}
H(t,\vec{\xi}_1,\lambda) &= \left(\frac{\lambda}{2\pi t}\right)^{r/2}\int_{\mathbb{R}^r}\exp\left\{-\frac{\lambda}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|_2^2\right\}d\varphi^r(\vec{\xi}_0)+\int_0^t\left[\frac{\lambda t}{2\pi s(t-s)}\right]^{r/2}\\
&\quad\times\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\left(\frac{\lambda}{2\pi t}\right)^{r/2}\theta(s,\vec{u})I_{F_s}^{\lambda}(\vec{\xi}_0,\vec{u})\exp\left\{-\frac{\lambda}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|_2^2-\frac{\lambda t}{2s(t-s)}\right.\\
&\quad\times\left.\left[\|\vec{u}-\vec{\xi}_0\|_2^2+\frac{s^2}{t^2}\|\vec{\xi}_1-\vec{\xi}_0\|_2^2-\frac{2s}{t}\langle\vec{u}-\vec{\xi}_0,\vec{\xi}_1-\vec{\xi}_0\rangle\right]\right\}d\varphi^r(\vec{\xi}_0)\,d\vec{u}\,ds\\
&= \left(\frac{\lambda}{2\pi t}\right)^{r/2}\int_{\mathbb{R}^r}\exp\left\{-\frac{\lambda}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|_2^2\right\}d\varphi^r(\vec{\xi}_0)+\int_0^t\left[\frac{\lambda}{2\pi(t-s)}\right]^{r/2}\\
&\quad\times\int_{\mathbb{R}^r}\theta(s,\vec{u})\exp\left\{-\frac{\lambda}{2(t-s)}\|\vec{\xi}_1-\vec{u}\|_2^2\right\}\int_{\mathbb{R}^r}\left(\frac{\lambda}{2\pi s}\right)^{r/2}I_{F_s}^{\lambda}(\vec{\xi}_0,\vec{u})\\
&\quad\times\exp\left\{-\frac{\lambda}{2s}\|\vec{u}-\vec{\xi}_0\|_2^2\right\}d\varphi^r(\vec{\xi}_0)\,d\vec{u}\,ds\\
&= \left(\frac{\lambda}{2\pi t}\right)^{r/2}\int_{\mathbb{R}^r}\exp\left\{-\frac{\lambda}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|_2^2\right\}d\varphi^r(\vec{\xi}_0)+\int_0^t\left[\frac{\lambda}{2\pi(t-s)}\right]^{r/2}\\
&\quad\times\int_{\mathbb{R}^r}\theta(s,\vec{u})H(s,\vec{u},\lambda)\exp\left\{-\frac{\lambda}{2(t-s)}\|\vec{\xi}_1-\vec{u}\|_2^2\right\}d\vec{u}\,ds
\end{aligned}$$

which completes the proof. Note that the justification for using Fubini's theorem will be contained in the proof of the next theorem. □

THEOREM 4.2. *With the same assumptions and notation as in Theorem 4.1, let for* $(t,\vec{\xi}_1,\lambda)\in(0,\infty)\times\mathbb{R}^r\times\mathbb{C}_+$,

$$H(t,\vec{\xi}_1,\lambda)=\left(\frac{\lambda}{2\pi t}\right)^{r/2}\int_{\mathbb{R}^r}\exp\left\{-\frac{\lambda}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|_2^2\right\}E^{anw_\lambda}[F_t|X_t](\vec{\xi}_0,\vec{\xi}_1)\,d\varphi^r(\vec{\xi}_0).$$

Then $H(t,\vec{\xi}_1,\lambda)$ *satisfies the integral equation (4.3) given in Theorem 4.1.*

PROOF. Take $\sigma_{F_t}\in\mathcal{M}(L_2^r[0,t])$ such that F_t and σ_{F_t} are related by (3.1) since $F_t\in\mathcal{S}_{w_\varphi}^r$. Because $|E^{anw_\lambda}[F_t|X_t](\vec{\xi}_0,\vec{\xi}_1)|\le\|\sigma_{F_t}\|$ by Theorem 3.2 and φ^r is a probability measure, $H(t,\vec{\xi}_1,\lambda)$ is well defined and analytic on $\mathbb{C}_+$ by Morera's theorem. If the right-hand side of (4.3) is analytic on $\mathbb{C}_+$, then the theorem follows by the uniqueness of the analytic continuation and Theorem 4.1. Since $|\exp\{-(\lambda/2t)\|\vec{\xi}_1-\vec{\xi}_0\|_2^2\}|\le 1$ for $\lambda\in\mathbb{C}_+$ and φ^r is a probability measure,

$$\left(\frac{\lambda}{2\pi t}\right)^{r/2}\int_{\mathbb{R}^r}\exp\left\{-\frac{\lambda}{2t}\|\vec{\xi}_1-\vec{\xi}_0\|_2^2\right\}d\varphi^r(\vec{\xi}_0)$$

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is analytic on $\mathbb{C}_+$ by Morera's theorem. Now, for $\lambda \in \mathbb{C}_+$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} |\theta(s, \vec{u})||E^{anw_\lambda}[F_s|X_s](\vec{\xi}_0, \vec{u})|\left[\frac{|\lambda|^2}{(2\pi)^2 s(t-s)}\right]^{r/2} \exp\left\{-\frac{\operatorname{Re}\lambda}{2(t-s)}\right. \\
&\qquad \times \left.\|\vec{\xi}_1 - \vec{u}\|_2^2 - \frac{\operatorname{Re}\lambda}{2s}\|\vec{u} - \vec{\xi}_0\|_2^2\right\} d\vec{u}\, d\varphi^r(\vec{\xi}_0)\, ds \\
&\quad \le \exp\left\{\int_0^t \|\sigma_u\|\, du\right\} \int_0^t \|\sigma_s\| \left[\frac{|\lambda|^2}{(2\pi)^2 s(t-s)}\right]^{r/2} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \exp\left\{-\frac{t \operatorname{Re}\lambda}{2s(t-s)}\right. \\
&\qquad \times \left\|\vec{u} - \frac{s}{t}\vec{\xi}_1 - \frac{t-s}{t}\vec{\xi}_0\right\|_2^2 + \frac{t \operatorname{Re}\lambda}{2s(t-s)}\left\|\frac{s}{t}\vec{\xi}_1 + \frac{t-s}{t}\vec{\xi}_0\right\|_2^2 - \frac{\operatorname{Re}\lambda}{2(t-s)}\|\vec{\xi}_1\|_2^2 \\
&\qquad \left. - \frac{\operatorname{Re}\lambda}{2s}\|\vec{\xi}_0\|_2^2\right\} d\vec{u}\, d\varphi^r(\vec{\xi}_0)\, ds \\
&\quad = \left(\frac{|\lambda|^2}{2\pi t \operatorname{Re}\lambda}\right)^{r/2} \exp\left\{\int_0^t \|\sigma_u\|\, du\right\} \int_0^t \|\sigma_s\| \int_{\mathbb{R}^r} \exp\left\{-\frac{\operatorname{Re}\lambda}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} \\
&\qquad \times d\varphi^r(\vec{\xi}_0)\, ds \\
&\quad \le \left(\frac{|\lambda|^2}{2\pi t \operatorname{Re}\lambda}\right)^{r/2} \exp\left\{\int_0^t \|\sigma_u\|\, du\right\} \int_0^t \|\sigma_s\|\, ds < \infty.
\end{aligned}$$

By Morera's theorem,

$$\int_0^t \left[\frac{\lambda}{2\pi(t-s)}\right]^{r/2} \int_{\mathbb{R}^r} \theta(s, \vec{u}) H(s, \vec{u}, \lambda) \exp\left\{-\frac{\lambda}{2(t-s)}\|\vec{\xi}_1 - \vec{u}\|_2^2\right\} d\vec{u}\, ds$$

is analytic on $\mathbb{C}_+$ as a function of λ, which is the desired result. □

THEOREM 4.3. *With the assumptions and notation of Theorem 4.2, let, for* $(t, \vec{\xi}_1, q) \in (0, \infty) \times \mathbb{R}^r \times (\mathbb{R} - \{0\})$,

$$H(t, \vec{\xi}_1, -iq) = \left(\frac{q}{2\pi i t}\right)^{r/2} \int_{\mathbb{R}^r} \exp\left\{\frac{iq}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} E^{anf_q}[F_t|X_t](\vec{\xi}_0, \vec{\xi}_1)\, d\varphi^r(\vec{\xi}_0).$$

Then $H(t, \vec{\xi}_1, -iq)$ *satisfies the integral equation*

$$\begin{aligned}
H(t, \vec{\xi}_1, -iq) = {} & \left(\frac{q}{2\pi i t}\right)^{r/2} \int_{\mathbb{R}^r} \exp\left\{\frac{iq}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} d\varphi^r(\vec{\xi}_0) + \int_0^t \left[\frac{q}{2\pi i(t-s)}\right]^{r/2} \\
& \times \overline{\int}_{\mathbb{R}^r} \theta(s, \vec{u}) H(s, \vec{u}, -iq) \exp\left\{\frac{iq}{2(t-s)}\|\vec{\xi}_1 - \vec{u}\|_2^2\right\} d\vec{u}\, ds.
\end{aligned}$$

PROOF. Let

$$G(s, \vec{u}, \lambda) = \left[\frac{\lambda}{2\pi(t-s)}\right]^{r/2} \theta(s, \vec{u}) H(s, \vec{u}, \lambda) \exp\left\{-\frac{\lambda}{2(t-s)}\|\vec{\xi}_1 - \vec{u}\|_2^2\right\}$$

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for Re $\lambda \geq 0$ and $\lambda \neq 0$. By Theorem 4.2, for nonzero real q,

$$\lim_{\lambda \to -iq} H(t, \vec{\xi}_1, \lambda) = H(t, \vec{\xi}_1, -iq)$$

and

$$\lim_{\lambda \to -iq} \left(\frac{\lambda}{2\pi t}\right)^{r/2} \int_{\mathbb{R}^r} \exp\left\{-\frac{\lambda}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} d\varphi^r(\vec{\xi}_0)$$
$$= \left(\frac{q}{2\pi i t}\right)^{r/2} \int_{\mathbb{R}^r} \exp\left\{\frac{iq}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} d\varphi^r(\vec{\xi}_0)$$

by the dominated convergence theorem so that $\lim_{\lambda \to -iq} \int_0^t \int_{\mathbb{R}^r} G(s, \vec{u}, \lambda)\, d\vec{u}\, ds$ exists. To complete the proof, we must show that

$$\lim_{\lambda \to -iq} \int_0^t \int_{\mathbb{R}^r} G(s, \vec{u}, \lambda)\, d\vec{u}\, ds = \int_0^t \overline{\int}_{\mathbb{R}^r} G(s, \vec{u}, -iq)\, d\vec{u}\, ds.$$

For $A > 0$, let

$$K(s) = \int_{\mathbb{R}^r} G(s, \vec{u}, \lambda) \exp\left\{-\frac{\|\vec{u}\|_2^2}{2A}\right\} d\vec{u}.$$

Then, by Fubini's theorem,

$$K(s) = \left[\frac{\lambda^2}{(2\pi)^2 s(t-s)}\right]^{r/2} \int_{L_2^r[0,s]} \exp\left\{-\frac{1}{2\lambda s}[s\|\vec{v}\|_2^2 - \|\vec{V}_s\|_2^2]\right\}$$
$$\times \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \exp\left\{i\langle \vec{u}, \vec{v}_0\rangle - \frac{\lambda}{2s}\|\vec{u} - \vec{\xi}_0\|_2^2 + \frac{i}{s}\langle \vec{u} - \vec{\xi}_0, \vec{V}_s\rangle\right.$$
$$\left. - \frac{\lambda}{2(t-s)}\|\vec{\xi}_1 - \vec{u}\|_2^2 - \frac{\|\vec{u}\|_2^2}{2A}\right\} d\vec{u}\, d\varphi^r(\vec{\xi}_0)\, d\sigma_s(\vec{v}_0)\, d\sigma_{F_s}(\vec{v}),$$

where $\vec{V}_s$ is given as in Corollary 3.3, replacing t by s, and F_s and σ_{F_s} are related by (3.1). Thus, by (3.5),

$$K(s) = \left[\frac{\lambda^2}{(2\pi)^2 s(t-s)}\right]^{r/2} \int_{L_2^r[0,s]} \exp\left\{-\frac{1}{2\lambda s}[s\|\vec{v}\|_2^2 - \|\vec{V}_s\|_2^2]\right\}$$
$$\times \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \exp\left\{-\frac{\lambda t A + s(t-s)}{2s(t-s)A}\|\vec{u}\|_2^2\right.$$
$$+ i\left\langle \vec{v}_0 - \frac{\lambda i}{s}\vec{\xi}_0 + \frac{1}{s}\vec{V}_s - \frac{\lambda i}{t-s}\vec{\xi}_1, \vec{u}\right\rangle$$

$$
- \frac{\lambda}{2s}\|\vec{\xi}_0\|_2^2 - \frac{i}{s}\langle\vec{\xi}_0, \vec{V}_s\rangle - \frac{\lambda}{2(t-s)}\|\vec{\xi}_1\|_2^2\bigg\} \, d\vec{u}\, d\varphi^r(\vec{\xi}_0)\, d\sigma_s(\vec{v}_0)\, d\sigma_{F_s}(\vec{v})
$$

$$
= \left(\frac{\lambda}{2\pi t}\right)^{r/2}\left[\frac{\lambda t A}{\lambda t A + s(t-s)}\right]^{r/2}\int_{L_2^r[0,s]} \exp\bigg\{-\frac{1}{2\lambda s}[s\|\vec{v}\|_2^2 - \|\vec{V}_s\|_2^2]\bigg\}
$$

$$
\times \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \exp\bigg\{-\frac{s(t-s)A}{2[\lambda t A + s(t-s)]}\bigg\|\vec{v}_0 - \frac{\lambda i}{s}\vec{\xi}_0 + \frac{1}{s}\vec{V}_s - \frac{\lambda i}{t-s}\vec{\xi}_1\bigg\|_2^2
$$

$$
- \frac{\lambda}{2s}\|\vec{\xi}_0\|_2^2 - \frac{i}{s}\langle\vec{\xi}_0, \vec{V}_s\rangle - \frac{\lambda}{2(t-s)}\|\vec{\xi}_1\|_2^2\bigg\} \, d\varphi^r(\vec{\xi}_0)\, d\sigma_s(\vec{v}_0)\, d\sigma_{F_s}(\vec{v}).
$$

Let $a = \operatorname{Re}\lambda$, $b = \operatorname{Im}\lambda$, $\alpha = atA + s(t-s)$ and

$$
K(\lambda) = -\frac{s(t-s)A}{2[\lambda t A + s(t-s)]}\bigg\|\vec{v}_0 - \frac{\lambda i}{s}\vec{\xi}_0 + \frac{1}{s}\vec{V}_s - \frac{\lambda i}{t-s}\vec{\xi}_1\bigg\|_2^2 - \frac{\lambda}{2s}\|\vec{\xi}_0\|_2^2
$$

$$
-\frac{i}{s}\langle\vec{\xi}_0, \vec{V}_s\rangle - \frac{\lambda}{2(t-s)}\|\vec{\xi}_1\|_2^2.
$$

Then

$$
K(\lambda) = -\frac{s(t-s)A(\alpha - btAi)}{2[\alpha^2 + (btA)^2]}\bigg[\bigg\|\vec{v}_0 + \frac{1}{s}\vec{V}_s + \frac{b}{s}\vec{\xi}_0 + \frac{b}{t-s}\vec{\xi}_1\bigg\|_2^2
$$

$$
- a^2\bigg\|\frac{1}{s}\vec{\xi}_0 + \frac{1}{t-s}\vec{\xi}_1\bigg\|_2^2 - 2ai\bigg\langle\vec{v}_0 + \frac{1}{s}\vec{V}_s + \frac{b}{s}\vec{\xi}_0 + \frac{b}{t-s}\vec{\xi}_1,
$$

$$
\frac{1}{s}\vec{\xi}_0 + \frac{1}{t-s}\vec{\xi}_1\bigg\rangle\bigg] - \frac{a+bi}{2s}\|\vec{\xi}_0\|_2^2 - \frac{i}{s}\langle\vec{\xi}_0, \vec{V}_s\rangle - \frac{a+bi}{2(t-s)}\|\vec{\xi}_1\|_2^2,
$$

so that

$$
\operatorname{Re} K(\lambda) = -\frac{s(t-s)A}{2[\alpha^2 + (btA)^2]}\bigg[\alpha\bigg[\bigg\|\vec{v}_0 + \frac{1}{s}\vec{V}_s + \frac{b}{s}\vec{\xi}_0 + \frac{b}{t-s}\vec{\xi}_1\bigg\|_2^2
$$

$$
- a^2\bigg\|\frac{1}{s}\vec{\xi}_0 + \frac{1}{t-s}\vec{\xi}_1\bigg\|_2^2\bigg] - 2abtA\bigg\langle\vec{v}_0 + \frac{1}{s}\vec{V}_s + \frac{b}{s}\vec{\xi}_0 + \frac{b}{t-s}\vec{\xi}_1,
$$

$$
\frac{1}{s}\vec{\xi}_0 + \frac{1}{t-s}\vec{\xi}_1\bigg\rangle\bigg] - \frac{a}{2s}\|\vec{\xi}_0\|_2^2 - \frac{a}{2(t-s)}\|\vec{\xi}_1\|_2^2
$$

$$
= -\frac{s(t-s)A\alpha}{2[\alpha^2 + (btA)^2]}\bigg\|\vec{v}_0 + \frac{1}{s}\vec{V}_s + \frac{b}{s}\vec{\xi}_0 + \frac{b}{t-s}\vec{\xi}_1 - \frac{abtA}{\alpha}\bigg[\frac{1}{s}\vec{\xi}_0
$$

$$
+ \frac{1}{t-s}\vec{\xi}_1\bigg]\bigg\|_2^2 - \frac{a(t-s)}{2\alpha}\|\vec{\xi}_0\|_2^2 - \frac{as}{2\alpha}\|\vec{\xi}_1\|_2^2 - \frac{a^2A}{2\alpha}\|\vec{\xi}_0 - \vec{\xi}_1\|_2^2
$$

$$
\le 0.
$$

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Consequently, $\lim_{A\to\infty} K(s)$ exists and, in particular, $\overline{\int}_{\mathbb{R}^r} G(s, \vec{u}, -iq)\, d\vec{u}$ exists by the dominated convergence theorem. Since

$$|G(s, \vec{u}, \lambda)| \exp\left\{-\frac{\|\vec{u}\|_2^2}{2A}\right\} \le \left[\frac{|\lambda|^2}{(2\pi)^2 s(t-s)}\right]^{r/2} \|\sigma_s\| \|\sigma_{F_s}\| \exp\left\{-\frac{\|\vec{u}\|_2^2}{2A}\right\},$$

by the dominated convergence theorem again

$$\int_{\mathbb{R}^r} G(s, \vec{u}, -iq) \exp\left\{-\frac{\|\vec{u}\|_2^2}{2A}\right\} d\vec{u} = \lim_{\lambda\to -iq} \int_{\mathbb{R}^r} G(s, \vec{u}, \lambda) \exp\left\{-\frac{\|\vec{u}\|_2^2}{2A}\right\} d\vec{u}.$$

Further, because

$$\begin{aligned} &|G(s, \vec{u}, \lambda)| \exp\left\{-\frac{\|\vec{u}\|_2^2}{2A}\right\} \\ &\quad \le \left[\frac{|\lambda|^2}{(2\pi)^2 s(t-s)}\right]^{r/2} \|\sigma_s\| \|\sigma_{F_s}\| \exp\left\{-\frac{\operatorname{Re}\lambda}{2(t-s)} \|\vec{\xi}_1 - \vec{u}\|_2^2\right\} \end{aligned}$$

for $\lambda \in \mathbb{C}_+$, it follows from the dominated convergence theorem that

$$\int_{\mathbb{R}^r} G(s, \vec{u}, \lambda)\, d\vec{u} = \lim_{A\to\infty} \int_{\mathbb{R}^r} G(s, \vec{u}, \lambda) \exp\left\{-\frac{\|\vec{u}\|_2^2}{2A}\right\} d\vec{u}$$

which implies that

$$\begin{aligned} \overline{\int}_{\mathbb{R}^r} G(s, \vec{u}, -iq)\, d\vec{u} &= \lim_{A\to\infty} \int_{\mathbb{R}^r} G(s, \vec{u}, -iq) \exp\left\{-\frac{\|\vec{u}\|_2^2}{2A}\right\} d\vec{u} \\ &= \lim_{A\to\infty} \lim_{\lambda\to -iq} \int_{\mathbb{R}^r} G(s, \vec{u}, \lambda) \exp\left\{-\frac{\|\vec{u}\|_2^2}{2A}\right\} d\vec{u} \\ &= \lim_{\lambda\to -iq} \lim_{A\to\infty} \int_{\mathbb{R}^r} G(s, \vec{u}, \lambda) \exp\left\{-\frac{\|\vec{u}\|_2^2}{2A}\right\} d\vec{u} \\ &= \lim_{\lambda\to -iq} \int_{\mathbb{R}^r} G(s, \vec{u}, \lambda)\, d\vec{u} \end{aligned}$$

if the double limit can be changed. Indeed, for any complex λ in the bounded set $\{\lambda \in \mathbb{C} : |\lambda + iq| \le |q|/2, \operatorname{Re}\lambda \ge 0\}$ and any $A > 0$,

$$\begin{aligned} |K(s)| &\le \left(\frac{|\lambda|}{2\pi t}\right)^{r/2} \left|\frac{\lambda t A}{\lambda t A + s(t-s)}\right|^{r/2} \|\sigma_s\| \|\sigma_{F_s}\| \\ &\le \left(\frac{3|q|}{4\pi t}\right)^{r/2} \|\sigma_s\| \exp\left\{\int_0^t \|\sigma_u\|\, du\right\} \end{aligned}$$

which is in the class $L_1[0, t]$ as a function of s and independent of λ and A. Now the change of the iterated limits is justified by the dominated convergence theorem, and

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hence

$$\int_0^t \overline{\int}_{\mathbb{R}^r} G(s, \vec{u}, -iq)\, d\vec{u}\, ds = \int_0^t \lim_{\lambda \to -iq} \int_{\mathbb{R}^r} G(s, \vec{u}, \lambda)\, d\vec{u}\, ds$$
$$= \lim_{\lambda \to -iq} \int_0^t \int_{\mathbb{R}^r} G(s, \vec{u}, \lambda)\, d\vec{u}\, ds$$

because $\lim_{\lambda \to -iq} \int_0^t \int_{\mathbb{R}^r} G(s, \vec{u}, \lambda)\, d\vec{u}\, ds$ exists and $\int_{\mathbb{R}^r} G(\cdot, \vec{u}, \lambda)\, d\vec{u} \in L_1[0, t]$ for $\lambda \in \mathbb{C}_+$. The proof is complete. □

COROLLARY 4.4. *With the assumptions and notation of Theorem 4.3, if φ^r is the Dirac measure concentrated at* 0*, then w_φ^r is exactly the r-dimensional Wiener measure on the Borel class of $C_0^r[0, t]$, and*

$$H(t, \vec{\xi}_1, -iq) = \left(\frac{q}{2\pi i t}\right)^{r/2} \exp\left\{\frac{iq}{2t}\|\vec{\xi}_1\|_2^2\right\} E^{anf_q}[F_t | X_t](\vec{0}, \vec{\xi}_1)$$

so that $H(t, \vec{\xi}_1, -iq)$ satisfies the following integral equation which is formally equivalent to the Schrödinger equation:

$$H(t, \vec{\xi}_1, -iq) = \left(\frac{q}{2\pi i t}\right)^{r/2} \exp\left\{\frac{iq}{2t}\|\vec{\xi}_1\|_2^2\right\} + \int_0^t \left[\frac{q}{2\pi i(t-s)}\right]^{r/2}$$
$$\times \overline{\int}_{\mathbb{R}^r} \theta(s, \vec{u}) H(s, \vec{u}, -iq) \exp\left\{\frac{iq}{2(t-s)}\|\vec{\xi}_1 - \vec{u}\|_2^2\right\} d\vec{u}\, ds$$

which is the integral equation given in [7, Theorem 6*].*

As an application of Theorem 4.3, the following theorem holds if $\varphi^r \ll m_L$, where m_L denotes the Lebesgue measure on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$.

THEOREM 4.5. *With the assumptions and notation of Theorem 4.3, suppose that $\varphi^r \ll m_L$, that is, φ^r has the probability density ψ on $\mathbb{R}^r$. Moreover, for $(t, \vec{\xi}_1, q) \in (0, \infty) \times \mathbb{R}^r \times (\mathbb{R} - \{0\})$, let*

$$H(t, \vec{\xi}_1, -iq) = \left(\frac{q}{2\pi i t}\right)^{r/2} \int_{\mathbb{R}^r} \exp\left\{\frac{iq}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} E^{anf_q}[F_t | X_t](\vec{\xi}_0, \vec{\xi}_1) \psi(\vec{\xi}_0)\, d\vec{\xi}_0.$$

Then $H(t, \vec{\xi}_1, -iq)$ satisfies the integral equation

$$H(t, \vec{\xi}_1, -iq) = \left(\frac{q}{2\pi i t}\right)^{r/2} \int_{\mathbb{R}^r} \psi(\vec{\xi}_0) \exp\left\{\frac{iq}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} d\vec{\xi}_0 + \int_0^t \left[\frac{q}{2\pi i(t-s)}\right]^{r/2}$$
$$\times \overline{\int}_{\mathbb{R}^r} \theta(s, \vec{u}) H(s, \vec{u}, -iq) \exp\left\{\frac{iq}{2(t-s)}\|\vec{\xi}_1 - \vec{u}\|_2^2\right\} d\vec{u}\, ds$$

which is formally equivalent to the Schrödinger equation.

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REMARK 4.6. If ψ is Lebesgue measurable, then we can take a Borel measurable function ψ_1 with $\psi(\vec{u}) = \psi_1(\vec{u})$ for m_L-almost everywhere $\vec{u} \in \mathbb{R}^r$, so that we can assume that ψ is Borel measurable. Furthermore, since $\psi \in L_1(\mathbb{R}^r)$, by the dominated convergence theorem

$$\overline{\int}_{\mathbb{R}^r} \psi(\vec{\xi}_0) \exp\left\{\frac{iq}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} d\vec{\xi}_0 = \int_{\mathbb{R}^r} \psi(\vec{\xi}_0) \exp\left\{\frac{iq}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} d\vec{\xi}_0,$$

so that $H(t, \vec{\xi}_1, -iq)$ satisfies the integral equation

$$H(t, \vec{\xi}_1, -iq) = \left(\frac{q}{2\pi it}\right)^{r/2} \overline{\int}_{\mathbb{R}^r} \psi(\vec{\xi}_0) \exp\left\{\frac{iq}{2t}\|\vec{\xi}_1 - \vec{\xi}_0\|_2^2\right\} d\vec{\xi}_0 + \int_0^t \left[\frac{q}{2\pi i(t-s)}\right]^{r/2} \times \overline{\int}_{\mathbb{R}^r} \theta(s, \vec{u}) H(s, \vec{u}, -iq) \exp\left\{\frac{iq}{2(t-s)}\|\vec{\xi}_1 - \vec{u}\|_2^2\right\} d\vec{u}\, ds,$$

which is formally equivalent to the Schrödinger equation [2, 3, 9].

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