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Moore's conjecture for connected sums

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Abstract. We show that under mild conditions, the connected sum $M\#N$ of simply connected, closed, orientable n-dimensional Poincaré Duality complexes M and N is hyperbolic and has no homotopy exponent at all but finitely many primes, verifying a weak version of Moore's conjecture. This is derived from an elementary framework involving CW-complexes satisfying certain conditions.

1 Introduction

Let X be a pointed, simply connected finite CW-complex. It is said to be *elliptic* if the rank of $\pi_*(X)\otimes\mathbb{Q}$ is finite and is otherwise *hyperbolic*. The rational dichotomy of Félix, Halperin, and Thomas [FHT, Chapter 33] says that, remarkably, if X is hyperbolic, then the rank of $\oplus_{k=2}^{m}(\pi_k(X)\otimes\mathbb{Q})$ grows exponentially with m. In particular, there is no hyperbolic space whose rational homotopy groups have polynomial growth. Classifying those spaces that are elliptic or hyperbolic is a major problem in rational homotopy theory.

Turning to torsion homotopy groups, for a fixed prime p, the *homotopy exponent* of X is the least power of p that annihilates the p-torsion in $\pi_*(X)$. If this least power is p^r, write $\exp_p(X)=p^r$. If no such power exists, that is, if $\pi_*(X)$ has p-torsion of arbitrarily high order, then write $\exp_p(X)=\infty$. Determining precise exponents, or at least good exponent bounds, is a major problem in unstable homotopy theory.

Moore's conjecture posits a deep relationship between the rational and torsion homotopy groups.

Conjecture 1.1 (Moore) *Let X be a simply connected finite dimensional CW-complex. Then the following are equivalent:*

(a) *X is elliptic.*
(b) $\exp_p(X)<\infty$ *for some prime p.*
(c) $\exp_p(X)<\infty$ *for all primes p.*

As prototypes, a sphere is rationally elliptic and has finite exponent at 2 by James [J2] and at odd primes by Toda [To]. A wedge of two or more spheres is hyperbolic by the Hilton–Milnor theorem and has no exponent at any prime p by [NS]. Moore's conjecture is known to hold for several families of spaces, including torsion-free

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suspensions [Se2], odd primary Moore spaces [N], finite H-spaces [L], H-spaces with finitely generated homology [CPSS], most spaces with two or three cells [NS], moment-angle complexes and generalized moment-angle complexes [HST], real moment-angle complexes [K], and certain families of Poincaré Duality complexes [BB, BT, Th] such as those that are $(n-1)$-connected and $2n$-dimensional. There are also some partial results: a hyperbolic loop space with p-torsion-free homology has no exponent at p [St], an elliptic space has a finite exponent at all but finitely many primes [MW], and a 2-cone satisfies Moore's conjecture at all but finitely many primes [A].

In this paper, we consider Moore's conjecture in the context of connected sums of Poincaré Duality complexes. Our main result is of the "all but finitely many primes" form, although there is an integral result in a special case. The statement of the result depends on Hurewicz images and localization. Let $h_*(X;\mathbb{Q})$ be the submodule of $H_*(X;\mathbb{Q})$ that consists of Hurewicz images.

Theorem 1.2 *Let M and N be simply connected, closed, orientable n-dimensional Poincaré Duality complexes. Suppose that M and N are not rationally homotopy equivalent to S^n and one of $h_*(M;\mathbb{Q})$ or $h_*(N;\mathbb{Q})$ has either a generator in odd degree or rank ≥ 2. Then $M\#N$ is hyperbolic and has no exponent at all but finitely many primes.*

The hypotheses on M and N are rational. To tease out what they are saying, observe that the condition that M and N are not rationally homotopy equivalent to S^n implies that the $(n-1)$-skeletons of M and N are not rationally contractible. Rationally, there is a Hurewicz image in the $(n-1)$-skeleton in each case coming from the inclusion of the bottom cell. So the ranks of $h_*(M;\mathbb{Q})$ and $h_*(N;\mathbb{Q})$ are both ≥ 1. If one of these Hurewicz images is in odd degree, then the hypotheses of Theorem 1.2 are fulfilled. Otherwise, all Hurewicz images are concentrated in even degrees and the additional hypothesis that the rank one of $h_*(M;\mathbb{Q})$ or $h_*(N;\mathbb{Q})$ is ≥ 2 is invoked. In general, this additional hypothesis is needed: an example follows the statement of Theorem 1.3.

The exact set of primes that are inverted is identified in the proof. There are two types. First, a rational Hurewicz image corresponds to a map whose integral Hurewicz image may be divisible by a finite number. We localize away from the primes dividing that number, and do so for the two or three Hurewicz images needed depending on the rank conditions on $h_*(M;\mathbb{Q})$ and $h_*(N;\mathbb{Q})$. Second, these Hurewicz images have cohomological duals that correspond to maps to Eilenberg–MacLane spaces. We invert sufficient primes to approximate these Eilenberg–MacLane spaces through dimension n by spaces Y with the property that $\Sigma\Omega Y$ is homotopy equivalent to a wedge of spheres.

The strategy of proof is to show that after localizing at a prime p not in the finite set of excluded primes, there is a wedge W of at least two simply connected spheres with the property that ΩW retracts off $\Omega(M\#N)$. Then, as W is hyperbolic and has no exponent at p, it follows that $M\#N$ is hyperbolic and has no exponent at p. In fact, the proof works in the more general context of a certain family of CW-complexes, as described in Section 3.

There is a special case for which no localization is necessary. If M and N are simply connected and both $H_2(M;\mathbb{Z})$ and $H_2(N;\mathbb{Z})$ have integral summands, these are Hurewicz images, so we may dispense with the first type of prime to invert. We may

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dispense with the second type of prime as, in this case, $\Omega K(\mathbb{Z}, 2) \simeq S^1$ is already a sphere.

Theorem 1.3 *Let M and N be simply connected, closed, orientable n-dimensional Poincaré Duality complexes where $n \geq 3$. Suppose that $H_2(M;\mathbb{Z})$ has a $\mathbb{Z}$-module summand and $H_2(N;\mathbb{Z})$ has a $\mathbb{Z} \oplus \mathbb{Z}$-module summand. Then $M\#N$ is hyperbolic and has no exponent at any prime p.*

For example, let N be a Poincaré Duality complex of dimension $2n$ with rank $H_2(N;\mathbb{Z}) \geq 2$. Then $\mathbb{C}P^n\#N$ is hyperbolic and has no exponent at any prime p.

The condition on the rank of $H_2(N;\mathbb{Z})$ is best possible in the sense that $\mathbb{C}P^n\#\mathbb{C}P^n$ is known to be elliptic. Further, by [HT], if $n = 2m$, then there is a homotopy equivalence $\Omega(\mathbb{C}P^{2m}\#\mathbb{C}P^{2m}) \simeq S^1 \times S^1 \times \Omega S^3 \times \Omega S^{4m-1}$ and if $n = 2m+1$, then, after localizing away from 2, there is a homotopy equivalence $\Omega(\mathbb{C}P^{2m+1}\#\mathbb{C}P^{2m+1}) \simeq S^1 \times S^1 \times \Omega S^3 \times \Omega S^{4m+1}$. Thus, if n is even, then $\mathbb{C}P^n\#\mathbb{C}P^n$ has a finite homotopy exponent at every prime p, and if n is odd, then $\mathbb{C}P^n\#\mathbb{C}P^n$ has a finite homotopy exponent at every odd prime.

In the last section of the paper, a generalization is made to bundles over connected sums.

2 Some properties of the James construction

Let X be a pointed, path-connected space. For $n \geq 1$, let $X^{\times n}$ and $X^{\wedge n}$, respectively, be the n-fold product and smash product of X with itself. Define $J_n(X)$ as the quotient space

$$J_n(X) = X^{\times n}/\sim,$$

where $(x_1, \ldots, x_{i-1}, *, x_{i+1}, x_{i+2}, \ldots, x_n) \sim (x_1, \ldots, x_{i-1}, x_{i+1}, *, x_{i+2}, \ldots, x_n)$. Observe that there is an inclusion

$$J_n(X) \longrightarrow J_{n+1}(X)$$

given by sending $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_n, *)$. Let

$$J(X) = \operatorname{colim} J_n(X).$$

The space $J(X)$ is called the *James construction* on X.

Concatenation of sequences gives $J(X)$ the structure of an associative monoid. James [J] showed that there is a homotopy equivalence of H-spaces

$$J(X) \simeq \Omega\Sigma X.$$

He used this to give a homotopy decomposition of $\Sigma\Omega\Sigma X$.

Lemma 2.1 *Let X be a pointed, path-connected space. Then there is a homotopy equivalence*

$$\Sigma\Omega\Sigma X \simeq \bigvee_{n=1}^{\infty} \Sigma X^{\wedge n}.$$

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Now, specialize to the case where $X = S^{2m}$ and localize at a prime p. There is a homotopy fibration (the EHP fibration)

$$J_{p-1}(S^{2m}) \xrightarrow{E} \Omega S^{2m+1} \xrightarrow{H} \Omega S^{2mp+1},$$

where E is the inclusion of $J_{p-1}(S^{2m})$ into $J(S^{2m}) \simeq \Omega S^{2m+1}$ and H is the p^{th}-James-Hopf invariant. The space $J_{p-1}(S^{2m})$ will play an important role in what follows. Three properties are needed.

First, if $p = 2$, then $J_{p-1}(S^{2m}) = S^{2m}$ and $\Sigma\Omega S^{2m}$ decomposes as a wedge of spheres by Lemma 2.1. Moore (a proof appears in [Se1]) proved an analogous result for odd primes. Let

$$\varepsilon_{2m}: S^{2m-1} \longrightarrow \Omega J_{p-1}(S^{2m})$$

be the inclusion of the bottom cell.

Lemma 2.2 *If p is odd and $m \geq 1$, the space $\Sigma\Omega J_{p-1}(S^{2m})$ is homotopy equivalent to a wedge of spheres. In particular, $\Sigma\varepsilon_{2m}$ has a left homotopy inverse.*

Corollary 2.3 *If p is odd and $m, n \geq 1$, then the map*

$$\Sigma S^{2m-1} \wedge S^{2n-1} \xrightarrow{\Sigma\varepsilon_{2m} \wedge \varepsilon_{2n}} \Sigma\Omega J_{p-1}(S^{2m}) \wedge \Omega J_{p-1}(S^{2n})$$

has a left homotopy inverse.

Proof By Lemma 2.2, $\Sigma\varepsilon_{2m}$ has a right homotopy inverse $\delta_{2m}: \Sigma\Omega J_{p-1}(S^{2m}) \longrightarrow \Sigma S^{2m-1}$. The composite

$$\begin{aligned}\Sigma\Omega J_{p-1}(S^{2m}) \wedge \Omega J_{p-1}(S^{2n}) &\xrightarrow{\Sigma\delta_{2m} \wedge 1} \Sigma S^{2m-1} \wedge \Omega J_{p-1}(S^{2m}) \xrightarrow{\ \simeq\ } S^{2m-1} \\ \wedge\, \Sigma\Omega J_{p-1}(S^{2n}) &\xrightarrow{1 \wedge \delta_{2n}} S^{2m-1} \wedge \Sigma S^{2n-1} \xrightarrow{\ \simeq\ } \Sigma S^{2m-1} \wedge S^{2n-1}\end{aligned}$$

is therefore a left homotopy inverse for $\Sigma\varepsilon_{2m} \wedge \varepsilon_{2n}$. ∎

The second property needed is a certain factorization. Let

$$\varepsilon_{2m}: S^{2m} \longrightarrow J_{p-1}(S^{2m})$$

be the inclusion of the bottom cell, and let

$$E: S^n \longrightarrow \Omega S^{n+1}$$

be the suspension, which is adjoint to the identity map on S^{n+1}. (This is the same as the map E in the EHP sequence if $n = 2m$ and $p = 2$; the duplication of notation is not ideal, but the context will make clear which is meant.) Note that localized at an odd prime p, the sphere S^{2m-1} is an H-space, implying that the suspension $S^{2m-1} \xrightarrow{E} \Omega S^{2m}$ has a left homotopy inverse.

Lemma 2.4 *If p is odd and $m \geq 1$, then there is a homotopy commutative diagram*

$$\begin{tikzcd} \Omega S^{2m} \arrow[r, "\Omega\varepsilon_{2m}"] \arrow[d, "r"] & \Omega J_{p-1}(S^{2m}) \arrow[d, equal] \\ S^{2m-1} \arrow[r, "\varepsilon_{2m}"] & \Omega J_{p-1}(S^{2m}) \end{tikzcd}$$

where r is a left homotopy inverse for E.

Proof Let $w\colon S^{4m-1} \longrightarrow S^{2m}$ be the Whitehead product of the identity map on S^{2m} with itself. At odd primes, the homotopy fiber of w is S^{2m-1}, resulting in a homotopy fibration

$$\Omega S^{2m-1} \xrightarrow{\Omega w} \Omega S^{2m} \xrightarrow{\partial} S^{2m-1}.$$

Notice that $\partial \circ E$ is degree one in homology and so is homotopic to the identity map. Therefore, the composite

$$e\colon S^{2m-1} \times \Omega S^{4m-1} \xrightarrow{E\times\Omega w} \Omega S^{2m} \times \Omega S^{2m} \xrightarrow{\mu} \Omega S^{2m}$$

is a homotopy equivalence, where μ is the standard loop multiplication. Consider the diagram

$$\begin{tikzcd} S^{2m-1} \times \Omega S^{4m-1} \arrow[r, "E\times\Omega w"] \arrow[d, "\pi_1"] & \Omega S^{2m} \times \Omega S^{2m} \arrow[r, "\mu"] \arrow[d, "\Omega\varepsilon_{2m}\times\Omega\varepsilon_{2m}"] & \Omega S^{2m} \arrow[d, "\Omega\varepsilon_{2m}"] \\ S^{2m-1} \arrow[r, "j_1\times\varepsilon_{2m}"] & \Omega J_{p-1}(S^{2m}) \times J_{p-1}(S^{2m}) \arrow[r, "\mu"] & \Omega J_{p-1}(S^{2m}) \end{tikzcd}$$

where π_1 is the projection onto the first factor and j_1 is the inclusion of the first factor. The left square homotopy commutes since $\varepsilon_{2m} \circ w$ is null homotopic (as $J_2(S^{2m})$ is the homotopy cofiber of w and p odd implies $p-1 \geq 2$). The right square homotopy commutes since $\Omega\varepsilon_{2m}$ is an H-map. The upper row is the homotopy equivalence e, and the bottom row is homotopic to ε_{2m}. The homotopy commutativity of the diagram therefore implies that $\Omega\varepsilon_{2m} \circ e \simeq \varepsilon_{2m} \circ \pi_1$. Now, precompose this diagram with the map $\Omega S^{2m} \xrightarrow{e^{-1}} S^{2m-1} \times \Omega S^{4m-1}$ to obtain $\Omega\varepsilon_{2m} \simeq \varepsilon_{2m} \circ \pi_1 \circ e^{-1}$. Define $r\colon \Omega S^{2m} \longrightarrow S^{2m-1}$ by $r = \pi_1 \circ e^{-1}$. Then $\Omega\varepsilon \simeq \varepsilon_{2m} \circ r$, giving the homotopy commutative square in the statement of the lemma. Also, observe that r is degree one in $H_{2m-1}(\ )$, so $r \circ E$ is the identity map in homology and therefore is homotopic to the identity map. ∎

Let $\mathbb{Z}_{(p)}$ be the integers localized at p. The third property needed is an approximation of the Eilenberg–MacLane space $K(\mathbb{Z}_{(p)}, 2m)$ by $J_{p-1}(S^{2m})$. Let $S^{2m+1} \longrightarrow K(\mathbb{Z}_{(p)}, 2m+1)$ be the inclusion of the bottom cell. Loop to obtain a map $\Omega S^{2m+1} \longrightarrow K(\mathbb{Z}_{(p)}, 2m)$. Let φ be the composite

(2.1) $$\varphi\colon J_{p-1}(S^{2m}) \xrightarrow{E} \Omega S^{2m+1} \longrightarrow K(\mathbb{Z}_{(p)}, 2m).$$

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As the homotopy fiber of E is $\Omega^2 S^{2mp+1}$, the map E induces an isomorphism on π_n for $n \leq 2mp - 2$. On the other hand, the first nontrivial torsion homotopy group of ΩS^{2m+1} occurs in dimension $2p - 3 + 2m$, so the right map in (2.1) induces an isomorphism on π_n for $n < 2p - 3 + 2m$. It is straightforward to see that $2mp - 2 \geq 2p - 3 + 2m$ for all $m \geq 2$. Therefore, φ induces an isomorphism on π_n for all $n < 2p - 3 + 2m$. Consequently, we obtain the following.

Lemma 2.5 *Let X be a CW-complex of dimension n, and suppose that there is a map $X \xrightarrow{r} K(\mathbb{Z}_{(p)}, 2m)$. If $n < 2p - 3 + 2m$, then there is a lift*

$$\begin{tikzcd} & J_{p-1}(S^{2m}) \arrow[d, "\varphi"] \\ X \arrow[ur, "\bar{r}"] \arrow[r, "r"] & K(\mathbb{Z}_{(p)}, 2m) \end{tikzcd}$$

for some map $\bar{r}$.

This completes the preliminaries needed for the James construction, but it is useful to record here the companion approximation of the Eilenberg–MacLane space $K(\mathbb{Z}_{(p)}, 2m+1)$ by S^{2m+1}. Rationally, the inclusion $S^{2m+1} \longrightarrow K(\mathbb{Q}, 2m+1)$ of the bottom cell is a homotopy equivalence. Localized at a prime p, the least nonvanishing torsion homotopy group of S^{2m+1} occurs in dimension $2p - 2 + 2m$. Therefore, if $2p - 2 + 2m > n$, then the inclusion $S^{2m+1} \longrightarrow K(\mathbb{Z}_{(p)}, 2m+1)$ of the bottom cell induces an isomorphism on π_m for $m \leq n$. Consequently, we obtain the following.

Lemma 2.6 *Let X be a CW-complex of dimension n, and suppose that there is a map $X \xrightarrow{r} K(\mathbb{Z}_{(p)}, 2m+1)$. If $n < 2p - 2 + 2m$, then there is a lift*

$$\begin{tikzcd} & S^{2m+1} \arrow[d] \\ X \arrow[ur, "\bar{r}"] \arrow[r, "r"] & K(\mathbb{Z}_{(p)}, 2m+1) \end{tikzcd}$$

for some map $\bar{r}$.

3 Conditions for the non-existence of an exponent at p

In this section, an elementary approach involving Hurewicz images and localization is described that leads to conditions implying that a space has no exponent at a given prime p. Specific application to connected sums of Poincaré Duality complexes will be in the next section.

We begin with a preliminary result proved by Ganea [G].

Lemma 3.1 *Let X and Y be path-connected spaces. Including the wedge into the product, there is a homotopy fibration*

$$\Sigma\Omega X \wedge \Omega Y \longrightarrow X \vee Y \longrightarrow X \times Y$$

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that splits after looping to give a homotopy equivalence

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega(\Sigma\Omega X \wedge \Omega Y).$$

Further, the fibration and the homotopy equivalence are natural for maps $X \longrightarrow X'$ and $Y \longrightarrow Y'$.

Let X be a simply connected CW-complex of dimension $m \geq 2$. If X has dimension 2, then it is homotopy equivalent to $\bigvee_{i=1}^{d} S^2$ for some $d \geq 1$ (assuming that X is not trivial). If $d = 1$, then $X = S^2$ is elliptic and has an exponent at every prime p. If $d > 1$, then X is a wedge of at least two simply connected spheres, implying that it is hyperbolic and has no exponent at any prime p. So, from here on, assume that $n \geq 3$.

Localize at a prime p. Suppose that there are maps

$$i\colon S^k \longrightarrow X \qquad j\colon S^\ell \longrightarrow X$$

whose Hurewicz images a and b, respectively, generate distinct $\mathbb{Z}_{(p)}$ summands in $H_k(X;\mathbb{Z}_{(p)})$ and $H_\ell(X;\mathbb{Z}_{(p)})$. The universal coefficient theorem implies that a and b have dual classes $\bar{a} \in H^k(X;\mathbb{Z}_{(p)})$ and $\bar{b} \in H^\ell(X;\mathbb{Z}_{(p)})$. The cohomology classes $\bar{a}$ and $\bar{b}$ are represented by maps

$$r\colon X \longrightarrow K(\mathbb{Z}_{(p)}, k) \qquad s\colon X \longrightarrow K(\mathbb{Z}_{(p)}, \ell).$$

Thus, the composites

$$\alpha\colon S^k \xrightarrow{i} X \xrightarrow{r} K(\mathbb{Z}_{(p)}, k) \qquad \beta\colon S^\ell \xrightarrow{j} X \xrightarrow{s} K(\mathbb{Z}_{(p)}, \ell) \tag{3.1}$$

are homotopic to the inclusions of the bottom cells.

Now, an adjustment is made. Let $L(k) = S^k$ if k is odd, and let $L(k) = J_{p-1}(S^k)$ if k is even; let $L(\ell) = S^\ell$ if ℓ is odd, and let $L(\ell) = J_{p-1}(S^\ell)$ if ℓ is even. Suppose that $p > \max\{\frac{n-k+3}{2}, \frac{n-\ell+3}{2}\}$. Then, as X has dimension n, Lemmas 2.5 and 2.6 imply that there are lifts

$$\begin{tikzcd} & L(k) \arrow[d] \\ X \arrow[ur, "\bar{r}"] \arrow[r, "r"] & K(\mathbb{Z}_{(p)}, k) \end{tikzcd} \qquad \begin{tikzcd} & L(\ell) \arrow[d] \\ X \arrow[ur, "\bar{s}"] \arrow[r, "s"] & K(\mathbb{Z}_{(p)}, \ell) \end{tikzcd}$$

for some maps $\bar{r}$ and $\bar{s}$. Observe that the composites

$$\alpha\colon S^k \xrightarrow{i} X \xrightarrow{\bar{r}} L(k) \qquad \beta\colon S^\ell \xrightarrow{j} X \xrightarrow{\bar{s}} L(\ell)$$

are homotopic to the identity map when k or ℓ is odd and homotopic to the inclusion of the bottom cell when k or ℓ is even.

Since r and s represent cohomology classes dual to the Hurewicz images generated by i and j, we have $r \circ j$ and $s \circ i$ null homotopic. Therefore, $\bar{r} \circ j$ and $\bar{s} \circ i$ are also null homotopic since the approximations $L(k)$ and $L(\ell)$ to the Eilenberg–MacLane spaces $K(\mathbb{Z}_{(p)}, k)$ and $K(\mathbb{Z}_{(p)}, \ell)$ induce isomorphisms on π_m for $m \leq n$. Suppose that there is a lift

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$$
\begin{tikzcd}
 & L(k)\vee L(\ell) \arrow[d] \\
X \arrow[ur] \arrow[r, "\bar{r}\times\bar{s}"] & L(k)\times L(\ell).
\end{tikzcd}
\tag{3.2}
$$

In general, given maps $f\colon A \longrightarrow Z$ and $g\colon B \longrightarrow Z$, let $f \perp g\colon A \vee B \longrightarrow Z$ be the map uniquely determined by having its restrictions to A and B being f and g, respectively. Then, from (3.2), we obtain a composite

$$
S^k \vee S^\ell \xrightarrow{i\perp j} X \longrightarrow L(k)\vee L(\ell)
\tag{3.3}
$$

that is homotopic to $\alpha \vee \beta$.

Composing each map in (3.3) with the inclusion $L(k)\vee L(\ell) \longrightarrow L(k)\times L(\ell)$ and taking homotopy fibers gives a homotopy fibration diagram

$$
\begin{tikzcd}
Z \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \Sigma\Omega L(k)\wedge\Omega L(\ell) \arrow[d] \\
S^k\vee S^\ell \arrow[r, "i\perp j"] \arrow[d] & X \arrow[r] \arrow[d] & L(k)\vee L(\ell) \arrow[d] \\
L(k)\times L(\ell) \arrow[r, equal] & L(k)\times L(\ell) \arrow[r, equal] & L(k)\times L(\ell),
\end{tikzcd}
\tag{3.4}
$$

where the homotopy fibration in the right column is from Lemma 3.1 and the fibration diagram defines the spaces Y and Z.

Lemma 3.2 *There is a homotopy commutative diagram*

$$
\begin{tikzcd}
\Omega(\Sigma\Omega S^k\wedge\Omega S^\ell) \arrow[d] \arrow[dr, "\Omega(\Sigma\Omega\alpha\wedge\Omega\beta)"] & \\
\Omega X \arrow[r] & \Omega(\Sigma\Omega L(k)\wedge\Omega L(\ell)).
\end{tikzcd}
$$

Proof Observe that the middle row of (3.4) is homotopic to $\alpha\vee\beta$. Thus, Z is the homotopy pullback of $\alpha\vee\beta$ and $\Sigma\Omega L(k)\wedge\Omega L(\ell) \longrightarrow L(k)\vee L(\ell)$. On the other hand, the naturality of Lemma 3.1 implies that there is a homotopy fibration diagram

$$
\begin{tikzcd}
\Sigma\Omega S^k\wedge\Omega S^\ell \arrow[r, "\Sigma\Omega\alpha\wedge\Omega\beta"] \arrow[d] & \Sigma\Omega L(k)\wedge\Omega L(\ell) \arrow[d] \\
S^k\vee S^\ell \arrow[r, "\alpha\vee\beta"] \arrow[d] & L(k)\vee L(\ell) \arrow[d] \\
S^k\times S^\ell \arrow[r, "\alpha\times\beta"] & L(k)\times L(\ell).
\end{tikzcd}
$$

The homotopy commutativity of the upper square therefore implies that there is a pullback map $\Sigma\Omega S^k \wedge \Omega S^\ell \longrightarrow Z$ such that the composite $\Sigma\Omega S^k \wedge \Omega S^\ell \longrightarrow Z \longrightarrow Y \longrightarrow \Sigma\Omega L(k) \wedge \Omega L(\ell)$ is homotopic to $\Sigma\Omega\alpha \wedge \Omega\beta$.

Next, consider the diagram

$$\begin{tikzcd}
\Omega(\Sigma\Omega S^k \wedge \Omega S^\ell) \arrow[d] \arrow[dr, "\Omega(\Sigma\Omega\alpha\wedge\Omega\beta)"] & & \\
\Omega Y \arrow[r] \arrow[d] & \Omega(\Sigma\Omega L(k) \wedge \Omega L(\ell)) \arrow[d] \arrow[dr, equal] & \\
\Omega X \arrow[r] & \Omega(L(k) \vee L(\ell)) \arrow[r] & \Omega(\Sigma\Omega L(k) \wedge \Omega L(\ell)).
\end{tikzcd}$$

The upper-left triangle homotopy commutes by the preceding paragraph. The lower-left square homotopy commutes by (3.4). By Lemma 3.1, the map $\Sigma\Omega L(k) \wedge \Omega L(\ell) \longrightarrow L(k) \vee L(\ell)$ has a left homotopy inverse after looping. Using this left homotopy inverse, the lower-right triangle homotopy commutes. The outer perimeter of this diagram then gives the homotopy commutative diagram asserted by the lemma. ∎

Next, a full or partial left homotopy inverse of $\Sigma\Omega\alpha \wedge \Omega\beta$ is considered.

Lemma 3.3 *The following hold:*

(a) *If k and ℓ are both odd, then $\Sigma\Omega\alpha \wedge \Omega\beta$ has a left homotopy inverse.*

(b) *If k is odd and ℓ is even, then the composite*

$$\Sigma\Omega S^k \wedge S^{\ell-1} \xrightarrow{\Sigma 1\wedge E} \Sigma\Omega S^k \wedge \Omega S^\ell \xrightarrow{\Sigma\Omega\alpha\wedge\Omega\beta} \Sigma\Omega L(k) \wedge \Omega L(\ell)$$

has a left homotopy inverse.

(c) *If k is even and ℓ is odd, then the composite*

$$\Sigma S^{k-1} \wedge \Omega S^{\ell-1} \xrightarrow{\Sigma E\wedge 1} \Sigma\Omega S^k \wedge \Omega S^\ell \xrightarrow{\Sigma\Omega\alpha\wedge\Omega\beta} \Sigma\Omega L(k) \wedge \Omega L(\ell)$$

has a left homotopy inverse.

Proof If k and ℓ are both odd, then, by definition, $L(k) = S^k$ and $L(\ell) = S^\ell$ and both α and β are homotopic to the identity maps. This proves part (a).

If k is odd and ℓ is even, then, by definition, $L(k) = S^k$, $L(\ell) = J(S^\ell)$, α is homotopic to the identity map, and β is homotopic to the inclusion ε_ℓ of the bottom cell. Thus, the composite

$$\Sigma\Omega S^k \wedge S^{\ell-1} \xrightarrow{\Sigma 1\wedge E} \Sigma\Omega S^k \wedge \Omega S^\ell \xrightarrow{\Sigma\Omega\alpha\wedge\Omega\beta} \Sigma\Omega L(k) \wedge \Omega L(\ell)$$

is homotopic to

$$\Sigma\Omega S^k \wedge S^{\ell-1} \xrightarrow{\Sigma 1\wedge\varepsilon_\ell} \Sigma\Omega S^k \wedge \Omega J(S^\ell).$$

By Lemma 2.2, $\Sigma\varepsilon_\ell$ has a left homotopy inverse. Therefore, so does $\Sigma 1 \wedge \varepsilon_\ell$. This proves part (b).

The argument for part (c) is the same as for part (b) but with the roles of k and ℓ exchanged. ∎

Collecting what has been done so far gives the following.

Proposition 3.4 *Let X be a finite simply connected CW-complex of dimension $n \geq 3$. Localize at a prime p. Suppose that there are maps $S^k \xrightarrow{i} X$ and $S^\ell \xrightarrow{j} X$ whose Hurewicz images generate distinct $\mathbb{Z}_{(p)}$ summands of $H_*(X;\mathbb{Z}_{(p)})$ and there is a lift as in (3.2). If $p > \max\{\frac{n-k+3}{2}, \frac{n-\ell+3}{2}\}$ and one of k or ℓ is odd, then X is hyperbolic and has no homotopy exponent at p.*

Proof Since one of k or ℓ is odd, each of the three cases in Lemma 3.3 implies that there is a countable wedge W of simply connected spheres with the property that the composite $W \longrightarrow \Sigma\Omega S^k \wedge \Omega S^\ell \xrightarrow{\Sigma\Omega\alpha\wedge\Omega\beta} \Sigma\Omega L(k) \wedge \Omega L(\ell)$ has a left homotopy inverse. Thus, the homotopy commutativity of the diagram in the statement of Lemma 3.3 implies that ΩW retracts off ΩX. Therefore, as W is hyperbolic and has no exponent at p, the same is true of X. ∎

The case when both k and ℓ are even is different. In Lemma 3.3, the map $\Sigma\Omega\alpha \wedge \Omega\beta$ now takes the form $\Sigma\Omega S^k \wedge \Omega S^\ell \xrightarrow{\Sigma\Omega\varepsilon_k\wedge\Omega\varepsilon_\ell} \Sigma\Omega J_{p-1}(S^k) \wedge \Omega J_{p-1}(S^\ell)$. By Lemma 2.4 applied to both $\Omega\varepsilon_k$ and $\Omega\varepsilon_\ell$, the map $\Sigma\Omega\varepsilon_k \wedge \Omega\varepsilon_\ell$ factors through $\Sigma S^{k-1} \wedge S^{\ell-1}$. Thus, only one of the spheres in $\Sigma\Omega S^k \wedge \Omega S^\ell$ retracts off $\Sigma\Omega J_{p-1}(S^k) \wedge \Omega J_{p-1}(S^\ell)$, whereas at least two are needed for hyperbolicity and no exponent. To go forward in this case, an extra initial hypothesis is necessary.

Assume that there is another map

$$j'\colon S^{\ell'} \longrightarrow X$$

whose Hurewicz image b' generates a $\mathbb{Z}_{(p)}$ summand in $H_{\ell'}(X;\mathbb{Z}_{(p)})$ that is distinct from those generated by a and b. The dual class $\overline{b'} \in H^{\ell'}(X;\mathbb{Z}_{(p)})$ is represented by a map

$$s'\colon X \longrightarrow K(\mathbb{Z}_{(p)}, \ell').$$

If ℓ' is odd, then we may replace ℓ by ℓ' in Proposition 3.4 and we are done. So assume that ℓ' is even. If $p > \frac{n-\ell'+3}{2}$, then Lemma 2.5 implies that s' factors as a composite $X \xrightarrow{\overline{s'}} J_{p-1}(S^{\ell'}) \longrightarrow K(\mathbb{Z}_{(p)}, \ell')$ for some map $\overline{s'}$. Observe that the composite $S^{\ell'} \longrightarrow X \xrightarrow{\overline{s'}} J_{p-1}(S^{\ell'})$ is homotopic to the inclusion of the bottom cell. Define γ by the composite

$$\gamma\colon S^\ell \vee S^{\ell'} \xrightarrow{i\perp i'} X \xrightarrow{\bar{s}\times\overline{s'}} J_{p-1}(S^\ell) \times J_{p-1}(S^{\ell'}).$$

Observe that γ is homotopic to $(j_1 \circ \varepsilon_\ell) \perp (j_2 \circ \varepsilon_{\ell'})$, where j_1 and j_2 are the inclusions of the first and second factors into $J_{p-1}(S^\ell) \times J_{p-1}(S^{\ell'})$.

In place of (3.2), suppose that there is a lift

$$\begin{tikzcd} & L(k)\vee(L(\ell)\times L(\ell')) \arrow[d] \\ X \arrow[ur] \arrow[r, "\overline{r}\times(\overline{s}\times\overline{s'})"] & L(k)\times(L(\ell)\times L(\ell')). \end{tikzcd} \tag{3.5}$$

Then the composite

$$S^k\vee(S^\ell\vee S^{\ell'})\xrightarrow{i\perp(j\perp j')}X\longrightarrow L(k)\vee(L(\ell)\times L(\ell'))$$

is homotopic to $\alpha\vee\gamma$. Now, replace j and $L(\ell)$ in (3.4) with $j\perp j'$ and $L(\ell)\times L(\ell')$ and argue as in Lemma 3.2 to obtain the following.

Lemma 3.5 *There is a homotopy commutative diagram*

$$\begin{tikzcd} \Omega(\Sigma\Omega S^k\wedge\Omega(S^\ell\vee S^{\ell'})) \arrow[d] \arrow[dr, "\Omega(\Sigma\Omega\alpha\wedge\Omega\gamma)"] & \\ \Omega X \arrow[r] & \Omega(\Sigma\Omega L(k)\wedge(\Omega L(\ell)\times\Omega L(\ell')). \end{tikzcd}$$

Lemma 3.6 *The composite*

$$\Sigma S^{k-1}\wedge(S^{\ell-1}\vee S^{\ell'-1})\xrightarrow{\Sigma E\wedge E}\Sigma\Omega S^k\wedge\Omega(S^\ell\vee S^{\ell'})$$
$$\xrightarrow{\Sigma\Omega\alpha\wedge\Omega\gamma}\Sigma\Omega J_{p-1}(S^k)\wedge(\Omega J_{p-1}(S^\ell)\times\Omega J_{p-1}(S^{\ell'}))$$

has a left homotopy inverse.

Proof By their definitions, $\overline{\alpha}$ is homotopic to ε_k and $\overline{\gamma}$ is homotopic to $(j_1\circ\varepsilon_\ell)\perp(j_2\circ\varepsilon_{\ell'})$. Thus, $\Omega\alpha\circ E\simeq\varepsilon_k$ and $\Omega\gamma\circ E\simeq(\Omega j_1\circ\varepsilon_\ell)\perp(\Omega j_1\circ\varepsilon_{\ell'})$. By Lemma 2.2, each of $\Sigma\varepsilon_k$, $\Sigma\varepsilon_\ell$, and $\Sigma\varepsilon_{\ell'}$ has a left homotopy inverse, denoted by δ_k, δ_ℓ, and $\delta_{\ell'}$ respectively. A left homotopy inverse of $(\Sigma\Omega\alpha\wedge\Omega\beta)\circ(\Sigma E\wedge E)$ is then given by the composite

$$\begin{aligned}\Sigma\Omega J_{p-1}(S^k)\wedge(\Omega J_{p-1}(S^\ell)\times\Omega J_{p-1}(S^{\ell'}))&\xrightarrow{\delta_k\times 1}\Sigma S^{k-1}\wedge(\Omega J_{p-1}(S^\ell)\times\Omega J_{p-1}(S^{\ell'}))\\&\xrightarrow{\simeq}S^{k-1}\wedge\Sigma(\Omega J_{p-1}(S^\ell)\times\Omega J_{p-1}(S^{\ell'}))\\&\xrightarrow{1\wedge t}S^{k-1}\wedge(\Sigma\Omega J_{p-1}(S^\ell)\vee\Sigma\Omega J_{p-1}(S^{\ell'}))\\&\xrightarrow{1\wedge(\delta_\ell\vee\delta_{\ell'})}S^{k-1}\wedge(\Sigma S^{\ell-1}\vee\Sigma S^{\ell'-1}),\end{aligned}$$

where t comes from the splitting of $\Sigma(A\times B)$ as $\Sigma A\vee\Sigma B\vee(\Sigma A\wedge B)$. ■

The analogue of Proposition 3.4 in this case is the following.

Proposition 3.7 *Let X be a finite simply connected CW-complex of dimension $n\geq 3$. Localize at a prime p. Suppose that there are maps $S^k\xrightarrow{i}X$, $S^\ell\xrightarrow{j}X$ and $S^{\ell'}\xrightarrow{j'}X$ whose Hurewicz images generate distinct $\mathbb{Z}_{(p)}$ summands of $H_*(X;\mathbb{Z}_{(p)})$ and there is* *a lift as in (3.5). If* $p > \max\{\frac{n-k+3}{2}, \frac{n-\ell+3}{2}, \frac{n-\ell'+3}{2}\}$ *and each of* k, ℓ, *and* ℓ' *is even, then* X *is hyperbolic and has no homotopy exponent at* p.

Proof Argue as for Proposition 3.4, replacing Lemmas 3.2 and 3.3 with Lemmas 3.5 and 3.6. Note that the wedge W of spheres in this case has two summands. ∎

4 Moore's conjecture for connected sums

Let M and N be simply connected, closed, orientable n-dimensional Poincaré duality spaces. If $\overline{M}$ and $\overline{N}$ are the $(n-1)$-skeletons of M and N, respectively, there are homotopy cofibrations

$$S^{n-1} \xrightarrow{f} \overline{M} \longrightarrow M \qquad S^{n-1} \xrightarrow{g} \overline{N} \longrightarrow N,$$

where f and g attach the n-cell to M and N. Geometrically, the connected sum $M\#N$ is obtained by cutting a small n-disk from each of M and N and then gluing together along the boundary of the disk. Topologically, this implies that the $(n-1)$-skeleton of $M\#N$ is $\overline{M} \vee \overline{N}$ and there is a homotopy cofibration

$$S^{n-1} \xrightarrow{f+g} \overline{M} \vee \overline{N} \longrightarrow M\#N,$$

where $f+g$ is the composite $S^{n-1} \xrightarrow{\sigma} S^{n-1} \vee S^{n-1} \xrightarrow{f\vee g} \overline{M} \vee \overline{N}$, with σ being the standard comultiplication. Collapsing the "collar" of the connected sum – the boundary of the n-disk along which the two were glued – gives a quotient map

$$q\colon M\#N \longrightarrow M \vee N.$$

Proof of Theorem 1.2 By hypothesis, M and N are simply connected and not rationally homotopy equivalent to S^n. This implies that if the rational connectivities of M and N are $k-1$ and $\ell-1$, respectively, then $2 \le k, \ell < n$. Therefore, both $H_k(M;\mathbb{Q})$ and $H_\ell(N;\mathbb{Q})$ have a $\mathbb{Q}$-summand in the image of the rational Hurewicz homomorphism. This implies that there are maps

$$i\colon S^k \longrightarrow M \qquad j\colon S^\ell \longrightarrow N$$

with Hurewicz images $m_1 \cdot a \in H_k(M;\mathbb{Z})$ and $m_2 \cdot b \in H_\ell(M;\mathbb{Z})$ where a and b generate $\mathbb{Z}$-summands and $m_1, m_2 \in \mathbb{Z}$. Let $\mathcal{P}_1$ be the set of all primes that divide either m_1 or m_2, let $\mathcal{P}_2$ be the set of all primes $p \le \max\{\frac{n-k+3}{2}, \frac{n-\ell+3}{2}\}$, and let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Note that $\mathcal{P}$ is finite, and possibly empty.

Case 1: one of k *or* ℓ *is odd.* Localize at a prime $p \notin \mathcal{P}$. As $p \notin \mathcal{P}_1$, the Hurewicz images of both i and j generate $\mathbb{Z}_{(p)}$-summands of $H_k(M;\mathbb{Z}_{(p)})$ and $H_\ell(N;\mathbb{Z}_{(p)})$. If the dual classes in $\mathbb{Z}_{(p)}$ cohomology are represented by maps $r\colon M \longrightarrow K(\mathbb{Z}_{(p)}, k)$ and $s\colon N \longrightarrow K(\mathbb{Z}_{(p)}, \ell)$, respectively, then as we are localized at $p \notin \mathcal{P}_2$, Lemmas 2.6 and 2.5 imply the maps r and s lift to maps $\overline{r}\colon M \longrightarrow L(k)$ and $\overline{s}\colon N \longrightarrow L(\ell)$. The composite

$$M\#N \xrightarrow{q} M \vee N \xrightarrow{\overline{r}\vee\overline{s}} L(k) \vee L(\ell)$$

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is then a lift of $\overline{r} \times \overline{s}$ as in (3.2). Proposition 3.4 now implies that $M\#N$ is hyperbolic and has no exponent at p.

Case 2: both k and ℓ are even. By hypothesis, there is a second rational Hurewicz image in $H_{\ell'}(N;\mathbb{Q})$ that generates a $\mathbb{Q}$-summand independent from the Hurewicz image of j. This implies that there is a map $j'\colon S^{\ell'} \longrightarrow N$ with Hurewicz image $m_2' b' \in H_{\ell'}(N;\mathbb{Z})$, where b' generates a $\mathbb{Z}$-summand and $m_2' \in \mathbb{Z}$. Let $\mathcal{P}_1'$ be the set of all primes that divide any one of m_1, m_2, or m_2'. Note that $\mathcal{P}_1 \subseteq \mathcal{P}_1'$. Also, as N is rationally $(\ell-1)$-connected, we have $\ell' \geq \ell$, implying that $\frac{n-\ell'+3}{2} \leq \frac{n-\ell+3}{2}$, so no adjustment is needed to $\mathcal{P}_2$. Let $\mathcal{P}' = \mathcal{P}_1' \cup \mathcal{P}_2$. Localize at $p \notin \mathcal{P}'$. As $p \notin \mathcal{P}_1'$, the Hurewicz images of i, j, and j' generate $\mathbb{Z}_{(p)}$-summands in $H_k(M;\mathbb{Z}_{(p)})$, $H_\ell(N;\mathbb{Z}_{(p)})$, and $H_{\ell'}(N;\mathbb{Z}_{(p)})$, respectively. If the dual classes in $\mathbb{Z}_{(p)}$-cohomology are represented by maps $r\colon M \longrightarrow K(\mathbb{Z}_{(p)}, k)$, $s\colon N \longrightarrow K(\mathbb{Z}_{(p)}, \ell)$, and $s'\colon N \longrightarrow K(\mathbb{Z}_{(p)}, \ell')$, respectively, then as we are localized at $p \notin \mathcal{P}_2$ and each of k, ℓ, and ℓ' is even, Lemma 2.5 implies that r, s, and s' lift to maps $\overline{r}\colon M \longrightarrow J_{p-1}(S^k)$, $\overline{s}\colon N \longrightarrow J_{p-1}(S^\ell)$, and $\overline{s'}\colon N \longrightarrow J_{p-1}(S^{\ell'})$. The composite

$$M\#N \xrightarrow{\quad q \quad} M \vee N \xrightarrow{\overline{r} \vee (\overline{s} \times \overline{s'})} J_{p-1}(S^k) \vee (J_{p-1}(S^\ell) \times J_{p-1}(S^{\ell'}))$$

is then a lift of $\overline{r} \times (\overline{s} \times \overline{s'})$ as in (3.5). Proposition 3.7 now implies that $M\#N$ is hyperbolic and has no exponent at p. ∎

5 An integral case

An integral statement holds in the context of Proposition 3.7 when $k = \ell = \ell' = 2$.

Proof of Theorem 1.3 Since M and N are simply connected, the hypotheses on $H_2(M;\mathbb{Z})$ and $H_2(N;\mathbb{Z})$ imply that there are maps

$$i\colon S^2 \longrightarrow M \qquad j, j'\colon S^2 \longrightarrow N$$

whose Hurewicz images generate a $\mathbb{Z}$ summand in $H_2(M;\mathbb{Z})$ and distinct $\mathbb{Z}$-summands $H_2(N;\mathbb{Z})$. The dual cohomology classes are represented by maps

$$r\colon M \longrightarrow \mathbb{C}P^\infty \qquad s, s'\colon N \longrightarrow \mathbb{C}P^\infty,$$

respectively. Observe that each of $r \circ i$, $s \circ j$, and $s' \circ j'$ is the inclusion of the bottom cell and the composite

$$M\#N \xrightarrow{\quad q \quad} M \vee N \xrightarrow{r \vee (s \times s')} \mathbb{C}P^\infty \vee (\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

is a lift of $r \times (s \times s')$ in the manner of (3.5).

Define α and γ by the composites

$$\alpha\colon S^2 \xrightarrow{i} M \xrightarrow{r} \mathbb{C}P^\infty \qquad \gamma\colon S^2 \vee S^2 \xrightarrow{j \perp j'} N \xrightarrow{s \times s'} \mathbb{C}P^\infty \times \mathbb{C}P^\infty.$$

Then the analogue of Lemma 3.5 is a factorization of

$$\Omega(\Sigma\Omega S^2 \wedge \Omega(S^2 \vee S^2)) \xrightarrow{\Omega(\Sigma\Omega\alpha \wedge \Omega\gamma)} \Omega(\Sigma\Omega\mathbb{C}P^\infty \vee (\Omega\mathbb{C}P^\infty \times \Omega\mathbb{C}P^\infty))$$

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through $\Omega(M\#N)$. As $\Omega\mathbb{C}P^\infty \simeq S^1$, the composite

$$\Sigma S^1 \wedge (S^1 \vee S^1) \xrightarrow{\Sigma E \wedge E} \Sigma\Omega S^2 \wedge \Omega(S^2 \vee S^2) \xrightarrow{\Sigma\Omega\alpha\wedge\Omega\gamma} S^1 \wedge (S^1 \times S^1)$$

has a left homotopy inverse. Therefore, there is a wedge W of two simply connected spheres retracting off $\Omega(M\#N)$, implying that it is hyperbolic and has no exponent at any prime p. ∎

6 A generalization to certain pullbacks

This section generalizes the result for connected sums. Let M and N be simply connected, closed, orientable n-dimensional manifolds. There is a map $\pi\colon M\#N \longrightarrow M$ that collapses N to a point. (Equivalently, π is the composite $M\#N \xrightarrow{q} M \vee N \xrightarrow{p_1} M$ where p_1 pinches onto the first wedge summand.) Suppose that there is a fibration $F \longrightarrow E \longrightarrow M$. Define the space E_N as the pullback of π and α, giving a homotopy fibration diagram

(6.1)
$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & E_N \arrow[r] \arrow[d] & M\#N \arrow[d, "\pi"] \\ F \arrow[r] & E \arrow[r] & M. \end{tikzcd}$$

The homotopy type of E_N has attracted attention recently. In [JS], examples were given to show that E_N is sometimes a connected sum and sometimes not; in [C], conditions were given for when ΩE_N has the homotopy type of a looped connected sum (even if E_N may not be homotopy equivalent to a connected sum itself); and in [HT], conditions were given for when E_N is a connected sum. In this paper, we study E_N from the point of view of Moore's conjecture.

Proposition 6.1 *Define the space E_N as in (6.1). If $M\#N$ satisfies the hypotheses of Theorem 1.2 and some multiple of the map $S^k \longrightarrow M$ realizing the rational Hurewicz image for M lifts to E, then E_N is rationally hyperbolic and has no exponent at all but finitely many primes.*

Proof By hypothesis, there is a map $i\colon S^k \longrightarrow M$ which, rationally, generates a $\mathbb{Q}$-summand in $H_k(M;\mathbb{Q})$. By hypothesis, $t \cdot i$ lifts to a map $\widehat{i}\colon S^k \longrightarrow E$. Note that as M is not rationally homotopy equivalent to S^n, its $(n-1)$-skeleton $\overline{M}$ is not contractible. Therefore, we may assume $k < n$, implying that i factors through $\overline{M}$. Also, denoting the factorization by i, consider the diagram

$$\begin{tikzcd} S^k \vee \overline{N} \arrow[rr, "t\cdot i\vee 1"] \arrow[dr, dashed, "\theta"] \arrow[dd, "p_1"'] & & \overline{M} \vee \overline{N} \arrow[d] \\ & E_N \arrow[r] \arrow[d] & M\#N \arrow[d, "\pi"] \\ S^k \arrow[r, "\widehat{i}"] & E \arrow[r] & M, \end{tikzcd}$$

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where p_1 is the pinch map onto the first wedge summand and θ will be defined momentarily. The inner square homotopy commutes by the definition of E_N as a pullback. Observe that the composite along the left column is homotopic to the composite $\overline{M} \vee \overline{N} \xrightarrow{p_1} \overline{M} \longrightarrow M$, where again p_1 is the pinch map onto the first wedge summand. Thus, the composite along the outer perimeter in the clockwise direction is homotopic to $S^{k-1} \vee \overline{N} \xrightarrow{p_1} S^{k-1} \xrightarrow{t \cdot i} M$. Since $\widehat{t}$ is a lift of $t \cdot i$, the outer perimeter of the diagram homotopy commutes. This implies that there is a homotopy pullback map θ that makes both the left and upper quadrilaterals homotopy commute.

Now, argue as in the proof of Theorem 1.2 with the composite $M\#N \xrightarrow{q} M \vee N \longrightarrow L(k) \vee L(\ell)$ replaced by $E_N \longrightarrow M\#N \xrightarrow{q} M \vee N \longrightarrow L(k) \vee L(\ell)$ (and similarly for the $L(k) \vee (L(\ell) \times L(\ell'))$ variant) to show that E_N is hyperbolic and has no exponent at all but finitely many primes. (In fact, the primes inverted are those for the $M\#N$ case and any additional primes dividing t.) ∎

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