Canad. Math. Bull. Vol. **56** (4), 2013 pp. 745–758
http://dx.doi.org/10.4153/CMB-2012-040-4

Dimension Functions of Self-Affine Scaling Sets

Xiaoye Fu and Jean-Pierre Gabardo

Abstract. In this paper, the dimension function of a self-affine generalized scaling set associated with an $n \times n$ integral expansive dilation A is studied. More specifically, we consider the dimension function of an A-dilation generalized scaling set K assuming that K is a self-affine tile satisfying $BK = (K+d_1) \cup (K+d_2)$, where $B = A^t$, A is an $n \times n$ integral expansive matrix with $|\det A| = 2$, and $d_1, d_2 \in \mathbb{R}^n$. We show that the dimension function of K must be constant if either $n = 1$ or 2 or one of the digits is 0, and that it is bounded by $2|K|$ for any n.

1 Introduction

Suppose A is an $n \times n$ integral expansive matrix, *i.e.*, a matrix with integer entries whose eigenvalues are all of modulus greater than one. A finite set $\Psi = \{\psi^1, \dots, \psi^M\} \subset L^2(\mathbb{R}^n)$ is called an *A-dilation orthonormal multiwavelet* if the system

$$\{|\det A|^{\frac{j}{2}} \psi^\ell(A^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \dots, M\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^n)$. If an A-dilation orthonormal multiwavelet Ψ consists of a single element $\psi \in L^2(\mathbb{R}^n)$, then we call ψ an A-dilation orthonormal wavelet. For the general theory and characterization of orthonormal multiwavelets, we refer the readers to [3, 12, 13]. The dimension function of an A-dilation orthonormal multiwavelet $\Psi = \{\psi^1, \dots, \psi^M\} \subset L^2(\mathbb{R}^n)$ is defined by

$$D_\Psi(\xi) = \sum_{\ell=1}^{M} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| \hat{\psi}^\ell \big(B^j(\xi + k) \big) \right|^2, \quad \xi \in \mathbb{R}^n, \tag{1.1}$$

where $B = A^t$ and $\hat{g}$ denotes the Fourier transform of g.

The (multi)wavelet dimension function, which was first introduced and investigated by Auscher in [1], is an important tool in the theory of wavelets and it has been extensively studied by many researchers for the class of integer dilations [3, 7, 19, 27] and for the class of real dilations [5, 6]. Its importance is due to the fact that it can be used to prove that certain wavelets are associated with a multiresolution analysis (MRA [9, 26]). Initially, Lemarié-Rieusset [24, 25] used the wavelet dimension function to show that all compactly supported wavelets are associated with an MRA in the

Received by the editors November 8, 2011; revised September 25, 2012.
Published electronically January 10, 2013.
AMS subject classification: **42C40**.
Keywords: scaling set, self-affine tile, orthonormal multiwavelet, dimension function.

one-dimensional dyadic case. After that, Gripenberg [13] and Wang [29] independently used it to characterize all wavelets that arise from an MRA and they proved that a wavelet ψ is an MRA wavelet if and only if its dimension function $D_\psi(\xi)$ is equal to 1 for almost every $\xi \in \mathbb{R}$. Baggett, Medina and Merrill [3, 4] made a systematic study of the properties of the multiwavelet dimension function associated with an $n \times n$ integral expansive matrix A, including the case where the wavelets are not associated with an MRA. They showed that the dimension function D_Ψ of the multiwavelet Ψ satisfies the consistency equation

$$\sum_{d \in \mathcal{D}} D_\Psi(\xi + B^{-1}d) = D_\Psi(B\xi) + L, \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

where $B = A^t$ and $\mathcal{D}$ is a complete set of coset representatives for the group $\mathbb{Z}^n / B\mathbb{Z}^n$. Furthermore, they exploited this consistency equation to provide a constructive procedure for producing all A-dilation wavelet sets in $\mathbb{R}^n$. A measurable set $Q \subset \mathbb{R}^n$ is called an A-dilation wavelet set if χ_Q is the Fourier transform of some single A-dilation wavelet ψ. Dai, Larson and Speegle [8] proved the existence of wavelet sets associated with any $n \times n$ real expansive matrix. Bownik, Rzeszotnik and Speegle [7] introduced the notion of (generalized) scaling set which can be associated with any (multi)wavelet set and they showed that (generalized) scaling sets and (multi)wavelet sets are determined by one another. Wavelet sets are also characterized as the support sets of minimally supported frequency (MSF) wavelets (see [11, 18]).

The connection between the theory of compactly supported wavelet bases and the theory of self-affine tiles was first provided by Gröchenig and Madych [16]. After that, many authors studied the connection between self-affine tiles and wavelets, more particularly, in relation with the construction of multi-dimensional wavelet bases having compact support, or Haar-type wavelets (*e.g.*, [10, 14, 20, 28]). Moreover, Lagarias and Wang [21–23] gave an in-depth study of the structure and tiling properties of self-affine tiles, since these problems concern the construction of orthonormal wavelet bases in $\mathbb{R}^n$. With a view toward constructing wavelets which are compactly supported in the frequency domain, Gabardo and Yu [17] considered the problem of constructing wavelet sets using integral self-affine tiles and they gave some necessary and sufficient conditions for an integral self-affine tile to be an A-dilation scaling set.

This paper is devoted to characterizing the properties of the dimension function of all A-dilation generalized scaling sets in $\mathbb{R}^n$ that satisfy a self-affine equation, where A is an $n \times n$ integral expansive matrix with $|\det A| = 2$ which is both necessary and sufficient for the existence of an A-dilation MRA wavelet (*e.g.*, see [2, 15]). The problem can also be considered when $|\det A| > 2$ for self-affine generalized scaling sets associated with the matrix A. However, using (iv) of Theorem 2.9, we have $\sum_{d \in \mathcal{D}} e^{-2\pi i k \cdot d} = 1$ for $k \in \mathbb{Z}^n \setminus \{0\}$ such that $\hat{\chi}_K(k) \neq 0$, where $\mathcal{D}$ is the digit set. In the general case, an explicit relation between digits in $\mathcal{D}$ just as (2.11) cannot be obtained and the techniques used in the case $|\det A| = 2$ cannot be generalized. In view of this fact, we restrict our discussion to dilations A with $|\det A| = 2$ in our manuscript.

The paper is organized as follows. In Section 2, we introduce some notations and review some known results about generalized scaling sets and multiwavelet sets. We also define the dimension function of a generalized scaling set associated with an $n \times n$ integral expansive matrix. In Sections 3 and 4, we prove that the dimension function of any A-dilation self-affine generalized scaling set K is a constant and is equal to the Lebesgue measure of the set K in $\mathbb{R}$ and in $\mathbb{R}^2$ respectively, where A is an $n \times n$ integral expansive matrix with $|\det A| = 2$. This result shows that all A-dilation self-affine scaling sets must be A-dilation MRA scaling sets in dimensions one and two. In Section 5, we consider our problem in arbitrary dimension and prove that the dimension function of a self-affine generalized scaling set is bounded by twice its Lebesgue measure.

2 Notations, Definitions and Preliminary Results

Let $M_n^{(2)}(\mathbb{Z})$ be the set of all $n \times n$ integral expansive matrices with determinant equal to 2 or -2. In this section, we will always assume that A is an $n \times n$ integral expansive matrix and we let $B = A^t$. We write $E \cong F$ for two measurable sets in $\mathbb{R}^n$ if their symmetric difference, $(F \setminus E) \cup (E \setminus F)$, has zero Lebesgue measure. The Lebesgue measure of a measurable set $K \subset \mathbb{R}^n$ is denoted by $|K|$. The Fourier transform of any function $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is defined by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x)\, dx$$

where $x \cdot \xi$ is the standard inner product of the vectors $x, \xi \in \mathbb{R}^n$. The inverse Fourier transform will be denoted by $\mathcal{F}^{-1}$.

Definition 2.1 A measurable set $Q \subset \mathbb{R}^n$ is called an *A-dilation multiwavelet set* of order M if $Q = \bigcup_{i=1}^{M} Q_i$ for some almost disjoint sets $Q_i \subset \mathbb{R}^n$, $1 \leq i \leq M$, with the property that the finite collection $\Psi := \{\mathcal{F}^{-1}(\chi_{Q_1}), \dots, \mathcal{F}^{-1}(\chi_{Q_M})\}$ is an A-dilation orthonormal multiwavelet.

A multiwavelet set of order 1 is called an A-dilation wavelet set, and it is called an A-dilation MRA wavelet set if $\mathcal{F}^{-1}(\chi_Q)$ is associated with an MRA. An A-dilation multiwavelet set of order M can be characterized by its tiling properties.

Theorem 2.2 ([7]) *A measurable set $Q \subset \mathbb{R}^n$ is an A-dilation multiwavelet set of order M if and only if*

(i) $\sum_{k \in \mathbb{Z}^n} \chi_Q(\xi + k) = M$ *for a.e.* $\xi \in \mathbb{R}^n$,
(ii) $\sum_{j \in \mathbb{Z}} \chi_Q(B^j \xi) = 1$ *for a.e.* $\xi \in \mathbb{R}^n$.

If $M = 1$, then Theorem 2.2 is reduced to the criterion given by Dai, Larson and Speegle [8] for a measurable set $Q \subset \mathbb{R}^n$ to be an A-dilation wavelet set. Under this case, the condition (i) in Theorem 2.2 is equivalent to saying that

$$\bigcup_{k \in \mathbb{Z}^n} (Q + k) \cong \mathbb{R}^n \quad \text{and} \quad Q \cap (Q + \ell) \cong \varnothing \quad \text{for } \ell \neq 0. \tag{2.1}$$

We call $Q \subset \mathbb{R}^n$ a $\mathbb{Z}^n$-tiling set if Q satisfies (2.1).

https://doi.org/10.4153/CMB-2012-040-4 Published online by Cambridge University Press

Definition 2.3 A measurable set $K \subset \mathbb{R}^n$ is called an *A-dilation scaling set* if $BK \setminus K$ is an A-dilation wavelet set.

Definition 2.4 For fixed $M \in \mathbb{N}$, a measurable set $K \subset \mathbb{R}^n$ is called a *generalized scaling set of order M* associated with an $n \times n$ integral expansive matrix A if $BK \setminus K$ is an A-dilation multiwavelet set of order M.

Definition 2.5 A *self-affine tile* in $\mathbb{R}^n$ is a measurable set K with positive Lebesgue measure satisfying the set valued equation $BK = \bigcup_{d \in \mathcal{D}}(K + d)$ for some $n \times n$ real expansive matrix B with $|\det B| = m \in \mathbb{Z}$ and $\mathcal{D} = \{d_1, d_2, \dots, d_m\} \subseteq \mathbb{R}^n$ is a set of m digits.

In this paper, we will restrict our discussion to the self-affine A-dilation generalized scaling sets K, where $A \in M_n^{(2)}(\mathbb{Z})$ is expansive, *i.e.*, the set K is an A-dilation generalized scaling set and it satisfies the set equation $BK = (K + d_1) \cup (K + d_2)$, for some vectors $d_1, d_2 \in \mathbb{R}^n$.

The notion of an A-dilation (generalized) scaling set, introduced by Bownik, Rzeszotnik and Speegle [7], can be defined in several equivalent ways. The following result (see Proposition 3.2 of [7]) can also serve as a definition.

Lemma 2.6 *A measurable set $K \subseteq \mathbb{R}^n$ is an A-dilation generalized scaling set of order M if and only if $K = \bigcup_{j=1}^{\infty} B^{-j}Q$ for some A-dilation multiwavelet set Q of order M.*

Since $Q = BK \setminus K$, if K and Q are as in the previous lemma, it follows that an A-dilation generalized scaling set can be associated with a unique A-dilation multiwavelet set and vice-versa. In the following, we will deduce, for the reader's convenience, the definition of dimension function of an A-dilation generalized scaling set K as provided by Bownik, Rzeszotnik and Speegle in Section 3 of [7]. Given an A-dilation generalized scaling set K of order M, we can define an A-dilation multiwavelet set $Q := BK \setminus K$ of order M. By the definition of multiwavelet set, we can find essentially disjoint sets Q_i, $i = 1, \dots, M$ with $Q = \bigcup_{i=1}^{M} Q_i$ such that the set $\Psi := \{\mathcal{F}^{-1}(\chi_{Q_1}), \dots, \mathcal{F}^{-1}(\chi_{Q_M})\}$ is an A-dilation orthonormal multiwavelet. Thus the dimension function of Ψ is well defined and, using (1.1), we have

$$\begin{aligned} D_\Psi(\xi) &= \sum_{\ell=1}^{M} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} |\chi_{Q_\ell}\big(B^j(\xi + k)\big)|^2 \\ &= \sum_{\ell=1}^{M} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \chi_{Q_\ell}\big(B^j(\xi + k)\big) \quad \text{for a.e. } \xi \in \mathbb{R}^n. \end{aligned}$$

Using the representation given in Lemma 2.6 of the generalized scaling set K associated with the multiwavelet set Q, we have

$$D_\Psi(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \chi_Q\big(B^j(\xi + k)\big) = \sum_{k \in \mathbb{Z}^n} \chi_K(\xi + k) \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Motivated by this last identity, we have the following definition.

https://doi.org/10.4153/CMB-2012-040-4 Published online by Cambridge University Press

Definition 2.7 The *dimension function* of an A-dilation generalized scaling set K is the function $D\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$D(\xi) = \sum_{k\in\mathbb{Z}^n} \chi_K(\xi + k), \quad \xi \in \mathbb{R}^n.$$

As we mentioned before, the condition $|\det A| = 2$ is necessary and sufficient for the existence of an A-dilation MRA wavelet. Gu and Han [15] proved that there exists an A-dilation wavelet set associated with an MRA for any expanding matrix $A \in M_n^{(2)}(\mathbb{Z})$. We will call any such wavelet set an A-dilation MRA wavelet set and its corresponding scaling set is called an A-dilation MRA scaling set. An A-dilation MRA scaling set can also be characterized by its dimension function.

Theorem 2.8 ([7, 29]) *Let A be an expansive matrix in $M_n^{(2)}(\mathbb{Z})$ and K be an A-dilation generalized scaling set. Then K is associated with an MRA if and only if its dimension function $D(\xi) = \sum_{k\in\mathbb{Z}^n} \chi_K(\xi + k) = 1$ for a.e. $\xi \in \mathbb{R}^n$.*

We now state the following criterion of Bownik, Rzeszotnik and Speegle [7] for a measurable set $K \subseteq \mathbb{R}^n$ to be an A-dilation generalized scaling set of order M.

Theorem 2.9 *Let A be an $n \times n$ integral expanding matrix with $|\det A| = q$. A measurable set $K \subset \mathbb{R}^n$ is an A-dilation generalized scaling set of order M if and only if*

(i) $|K| = \frac{M}{q-1}$,
(ii) $K \subset BK$,
(iii) $\lim_{m\to\infty} \chi_K(B^{-m}\xi) = 1$ *for a.e.* $\xi \in \mathbb{R}^n$,
(iv) $\sum_{d\in\mathcal{D}} D(\xi + B^{-1}d) = D(B\xi) + M$ *a.e., where* $D(\xi) = \sum_{k\in\mathbb{Z}^n} \chi_K(\xi + k)$ *and* $\mathcal{D}$ *is a complete set of coset representatives for the group* $\mathbb{Z}^n/B\mathbb{Z}^n$.

Lemma 2.10 below, which is well known, characterizes the condition (i) of Theorem 2.2 in the Fourier domain and will be used in later sections.

Lemma 2.10 *Let $K \subseteq \mathbb{R}^n$ be a measurable set with finite Lebesgue measure. Then the identity $\sum_{k\in\mathbb{Z}^n} \chi_K(x + k) = L$ holds for a.e. $x \in \mathbb{R}^n$ if and only if $\hat{\chi}_K(j) = L\delta_{j,0}$ for all $j \in \mathbb{Z}^n$.*

It follows from Lemma 2.10 that if the dimension function of an A-dilation generalized scaling set of order M is a constant, then it is equal to the measure of the set K. That is, we have $D(\xi) = \sum_{j\in\mathbb{Z}^n} \chi_K(\xi + j) = \frac{M}{q-1}$ for a.e. $\xi \in \mathbb{R}^n$ using (i) of Theorem 2.9. In this case, the condition (iv) in Theorem 2.9 holds automatically.

From Theorem 2.8 and Theorem 2.9, we obtain an equivalent criterion for a measurable set $K \subseteq \mathbb{R}^n$ to be an A-dilation MRA scaling set, where $A \in M_n^{(2)}(\mathbb{Z})$ is expansive.

Corollary 2.11 *Let $A \in M_n^{(2)}(\mathbb{Z})$ be expansive. A measurable set $K \subset \mathbb{R}^n$ is an A-dilation MRA scaling set if and only if*

(i) *K is a $\mathbb{Z}^n$-tiling set,*
(ii) $K \subset BK$,
(iii) $\lim_{m\to\infty} \chi_K(B^{-m}\xi) = 1$ *for a.e.* $\xi \in \mathbb{R}^n$.

In the following, our goal will be to consider the dimension function of a generalized scaling set associated with an expansive dilation $A \in M_n^{(2)}(\mathbb{Z})$, which is a self-affine tile. For short, we call such dimension function the dimension function of an A-dilation generalized self-affine scaling set.

Assume that K is an A-dilation generalized self-affine scaling set of order L. Then K satisfies the set equation $BK = (K + d_1) \cup (K + d_2)$, for some $d_1, d_2 \in \mathbb{R}^n$ and K satisfies (i), (ii), (iii) and (iv) of Theorem 2.9. Moreover, (iv) of Theorem 2.9 can also be written as

$$\sum_{d \in \mathcal{D}} D\big(B^{-1}(\xi + d)\big) = D(\xi) + L \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{2.2}$$

Then, since $\mathcal{D}$ is a complete set of coset representatives for $\mathbb{Z}^n / B\mathbb{Z}^n$, we have

$$\begin{aligned} \sum_{d \in \mathcal{D}} D\big(B^{-1}(\xi + d)\big) &= \sum_{d \in \mathcal{D}} \sum_{k \in \mathbb{Z}^n} \chi_K\big(B^{-1}(\xi + d) + k\big) \\ &= \sum_{d \in \mathcal{D}} \sum_{k \in \mathbb{Z}^n} \chi_{BK}(\xi + d + Bk) \\ &= \sum_{k \in \mathbb{Z}^n} \chi_{BK}(\xi + k). \end{aligned} \tag{2.3}$$

Since K satisfies that $BK = (K + d_1) \cup (K + d_2)$, for some $d_1, d_2 \in \mathbb{R}^n$, (2.3) implies that

$$\sum_{d \in \mathcal{D}} D\big(B^{-1}(\xi + d)\big) = \sum_{k \in \mathbb{Z}^n} \chi_{K+d_1}(\xi + k) + \sum_{k \in \mathbb{Z}^n} \chi_{K+d_2}(\xi + k). \tag{2.4}$$

It follows from (2.2) and (2.4) that

$$\sum_{k \in \mathbb{Z}^n} \chi_K(\xi - d_1 + k) + \sum_{k \in \mathbb{Z}^n} \chi_K(\xi - d_2 + k) = \sum_{k \in \mathbb{Z}^n} \chi_K(\xi + k) + L \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Since $D(\xi)$ is $\mathbb{Z}^n$-periodic, it can be expanded as the Fourier series

$$D(\xi) = \sum_{k \in \mathbb{Z}^n} a_k e^{2\pi i k \cdot \xi},$$

where $a_k = \hat{\chi}_K(k)$, $k \in \mathbb{Z}^n$. Thus, we obtain

$$\sum_{k \in \mathbb{Z}^n} \hat{\chi}_K(k) e^{2\pi i k \cdot \xi} (e^{-2\pi i k \cdot d_1} + e^{-2\pi i k \cdot d_2}) = \sum_{k \in \mathbb{Z}^n} \hat{\chi}_K(k) e^{2\pi i k \cdot \xi} + L \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

or equivalently,

$$(e^{-2\pi i k \cdot d_1} + e^{-2\pi i k \cdot d_2}) \hat{\chi}_K(k) = \hat{\chi}_K(k) + L\delta_{k,0}, \quad k \in \mathbb{Z}^n. \tag{2.5}$$

https://doi.org/10.4153/CMB-2012-040-4 Published online by Cambridge University Press

We deduce from (2.5) that $\hat{\chi}_K(0) = L$ and

$$e^{-2\pi i k\cdot d_1} + e^{-2\pi i k\cdot d_2} = 1 \quad \text{for } k \in \mathbb{Z}^n \setminus \{0\} \quad \text{such that } \hat{\chi}_K(k) \neq 0. \tag{2.6}$$

Note that if z_1, z_2 are two complex numbers with $|z_1| = |z_2| = 1$ and $z_1 + z_2 = 1$, then $|z_1 + z_2|^2 = 1$ implies that $\mathrm{Re}(z_2\overline{z_1}) = -\frac{1}{2}$ or $z_2 = e^{\pm 2\pi i/3} z_1$. If $z_2 = e^{2\pi i/3} z_1$, we have $z_1(1 + e^{2\pi i/3}) = 1$ and thus $z_1 = e^{-\pi i/3}$, $z_2 = e^{\pi i/3}$. Similarly, if $z_2 = e^{-2\pi i/3} z_1$, we have $z_1 = e^{\pi i/3}$, $z_2 = e^{-\pi i/3}$. It follows thus from (2.6) that, if $k \in T := \{ j \in \mathbb{Z}^n \setminus \{0\} : \hat{\chi}_K(j) \neq 0 \}$, then

$$\begin{cases} k\cdot d_1 \in \frac{1}{6} + \mathbb{Z}, \\ k\cdot d_2 \in -\frac{1}{6} + \mathbb{Z}, \end{cases} \qquad \text{or} \qquad \begin{cases} k\cdot d_1 \in -\frac{1}{6} + \mathbb{Z}, \\ k\cdot d_2 \in \frac{1}{6} + \mathbb{Z}. \end{cases}$$

On the other hand, (2.3) implies that $\sum_{d\in\mathcal{D}} D\big(B^{-1}(\xi + d)\big) = \sum_{k\in\mathbb{Z}^n} \chi_{BK}(\xi + j)$ is also $\mathbb{Z}^n$-periodic. Denoting its Fourier series by $\sum_{k\in\mathbb{Z}^n} b_k\, e^{2\pi i k\cdot\xi}$, then as before, we get $b_k = \hat{\chi}_{BK}(k) = 2\hat{\chi}_K(B^t k)$. Then (2.2) is also equivalent to the following equation

$$\sum_{k\in\mathbb{Z}^n} 2\hat{\chi}_K(B^t k) e^{2\pi i k\cdot\xi} = \sum_{k\in\mathbb{Z}^n} \hat{\chi}_K(k) e^{2\pi i k\cdot\xi} + L \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{2.7}$$

From (2.7), we have $\hat{\chi}_K(0) = L$ and

$$2\hat{\chi}_K(B^t k) = \hat{\chi}_K(k) \quad \text{for } k \in \mathbb{Z}^n \setminus \{0\}. \tag{2.8}$$

(2.8) implies that $\hat{\chi}_K(B^t k) \neq 0$ for $k \in T$, and iteratively that, $\hat{\chi}_K\big((B^t)^m k\big) \neq 0$ for any $m \geq 0$ and $k \in T$. Thus, if $k \in T$, we have

$$\begin{cases} (B^t)^m k\cdot d_1 \in \frac{1}{6} + \mathbb{Z}, \\ (B^t)^m k\cdot d_2 \in -\frac{1}{6} + \mathbb{Z}, \end{cases} \qquad \text{or} \qquad \begin{cases} (B^t)^m k\cdot d_1 \in -\frac{1}{6} + \mathbb{Z}, \\ (B^t)^m k\cdot d_2 \in \frac{1}{6} + \mathbb{Z}, \end{cases} \tag{2.9}$$

for any $m \geq 0$.

Note that if 0 is one of the digits, then the equation (2.9) cannot hold for any $k \neq 0$ and thus the dimension function must be equal to the constant L. This is true in any dimension. We have the following theorem.

Theorem 2.12 *Let A be an $n\times n$ integral expansive matrix with $|\det A| = 2$ and let K be a self-affine tile satisfying $BK = K \cup (K + d)$ for some $d \in \mathbb{R}^n$. If K is a generalized scaling set, then its dimension function $D(\xi)$ is constant and equal to $|K|$.*

Then, in the following sections, we will concentrate on dealing with the much more difficult case where both digits are not zero.

3 The Dimension Function of Self-affine Generalized Scaling Sets in $\mathbb{R}$

We first consider the one-dimensional problem. In dimension one, since $|\det A| = 2$, we have two possibilities: $A = 2$ or $A = -2$.

Theorem 3.1 *Let $A \in M_1^{(2)}(\mathbb{Z})$ and K be a self-affine tile satisfying the set-valued equation $BK = (K + d_1) \cup (K + d_2)$ for some $d_1, d_2 \in \mathbb{R}$. If K is a generalized scaling set, then its dimension function $D(\xi)$ is a constant and equal to $|K|$.*

Proof $A \in M_1^{(2)}(\mathbb{Z})$ implies that $A = 2$ or $A = -2$. Here we only consider the case $A = 2$. The proof of the case $A = -2$ is similar. Assume that $|K| = L$. To prove that $D(\xi) = |K| = L$ for a.e. $\xi \in \mathbb{R}$, we only need to prove that $\hat{\chi}_K(j) = 0$ for $j \in \mathbb{Z} \setminus \{0\}$ since $\hat{\chi}_K(0) = |K| = L$ by Lemma 2.10. Assume that there exists $j_0 \in \mathbb{Z} \setminus \{0\}$ such that $\hat{\chi}_K(j_0) \neq 0$. Since K is a generalized scaling set and K is a self-affine tile satisfying $BK = (K + d_1) \cup (K + d_2)$ for some $d_1, d_2 \in \mathbb{R}$, we have, using (2.9), that

$$\begin{cases} 2^m j_0 d_1 \in \frac{1}{6} + \mathbb{Z}, \\ 2^m j_0 d_2 \in -\frac{1}{6} + \mathbb{Z}, \end{cases} \qquad \text{or} \qquad \begin{cases} 2^m j_0 d_1 \in -\frac{1}{6} + \mathbb{Z}, \\ 2^m j_0 d_2 \in \frac{1}{6} + \mathbb{Z}, \end{cases}$$

for any $m \geq 0$. In particular, $j_0 \cdot d_1 \in \frac{1}{6} + \mathbb{Z}$ or $j_0 \cdot d_1 \in -\frac{1}{6} + \mathbb{Z}$, but in that case, if $m = 1$, we have $2 j_0 \cdot d_1 \notin \frac{1}{6} + \mathbb{Z}$ and $2 j_0 \cdot d_1 \notin -\frac{1}{6} + \mathbb{Z}$, which is a contradiction. This proves our statement. ∎

4 The Dimension Function of Self-affine Generalized Scaling Sets in $\mathbb{R}^2$

In this section, we deal with the two dimensional case. Consider the following matrices:

$$C_1 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}.$$

Two $n \times n$ integral matrices A and $\widetilde{A}$ are called *integrally similar* if there exists an $n \times n$ integral matrix P with $|\det P| = 1$ such that $P^{-1}AP = \widetilde{A}$. Lagarias and Wang [20] completely classified all integral expansive matrices $A \in M_2^{(2)}(\mathbb{Z})$ and they showed that there are exactly six integrally similar classes of such integral matrices. Representatives from each of these classes are given by $C_1, C_2, C_3, -C_3, C_4, -C_4$, respectively.

Proposition 4.1 ([20]) *Let $A \in M_2^{(2)}(\mathbb{Z})$ be expansive. If $\det A = -2$, then A is integrally similar to C_1. If $\det A = 2$, then A is integrally similar to one of the matrices $C_2, \pm C_3, \pm C_4$.*

Theorem 4.2 *Let $A \in M_2^{(2)}(\mathbb{Z})$ be expanding and K be a self-affine tile satisfying the set equation $BK = (K + d_1) \cup (K + d_2)$ for some $d_1, d_2 \in \mathbb{R}^2$. If K is a generalized scaling set, then its dimension function $D(\xi)$ must be a constant and equal to $|K|$.*

Proof Since $A \in M_2^{(2)}(\mathbb{Z})$, the characteristic polynomial of matrix B, where $B = A^t$, has a general form $\lambda^2 + b\lambda + c = 0$, where $b = -\operatorname{tr} B \in \mathbb{Z}$ and $c = \det B = 2$ or -2. Assume that $|K| = L$. To prove that $D(\xi) = |K| = L$ for a.e. $\xi \in \mathbb{R}^2$, we only need

https://doi.org/10.4153/CMB-2012-040-4 Published online by Cambridge University Press

to prove that $\hat{\chi}_K(j) = 0$ for $j \in \mathbb{Z} \setminus \{0\}$ since $\hat{\chi}_K(0) = |K| = L$. Assume that there exists $j_0 \in \mathbb{Z}^2 \setminus \{0\}$ such that $\hat{\chi}_K(j_0) \neq 0$. By assumption, (2.9) holds, *i.e.*,

$$(4.1) \qquad \begin{cases} (B^t)^m j_0 \cdot d_1 \in \frac{1}{6} + \mathbb{Z}, \\ (B^t)^m j_0 \cdot d_2 \in -\frac{1}{6} + \mathbb{Z}, \end{cases} \quad \text{or} \quad \begin{cases} (B^t)^m j_0 \cdot d_1 \in -\frac{1}{6} + \mathbb{Z}, \\ (B^t)^m j_0 \cdot d_2 \in \frac{1}{6} + \mathbb{Z}, \end{cases}$$

holds for any $m \geq 0$. Since two similar matrices have the same eigenvalues, Proposition 4.1 implies that for any expansive matrix $A \in M_2^{(2)}(\mathbb{Z})$, the matrix B has two distinct eigenvalues. Let $\lambda_0, \lambda_1, \lambda_0 \neq \lambda_1$, be the eigenvalues of matrix B and v_0, v_1 be the corresponding eigenvectors respectively, *i.e.*, $Bv_0 = \lambda_0 v_0$ and $Bv_1 = \lambda_1 v_1$. Then $\mathbb{C}^2 = \operatorname{span}\{v_0, v_1\}$ and thus $d_1 \in \mathbb{R}^2$ can be represented as a linear combination of v_0 and v_1, *i.e.*, there exist some $c_0, c_1 \in \mathbb{C}$ such that $d_1 = c_0 v_0 + c_1 v_1$, and

$$B^m d_1 = c_0 B^m v_0 + c_1 B^m v_1 = c_0 \lambda_0^m v_0 + c_1 \lambda_1^m v_1 = \lambda_0^m c_0 v_0 + \lambda_1^m c_1 v_1.$$

From the above equality, we obtain

$$(4.2) \qquad (B^t)^m j_0 \cdot d_1 = j_0 \cdot B^m d_1 = \overline{\lambda_0^m} \overline{c_0} j_0 \cdot v_0 + \overline{\lambda_1^m} \overline{c_1} j_0 \cdot v_1.$$

Let $a_0 = \overline{c_0} j_0 \cdot v_0$, $a_1 = \overline{c_1} j_0 \cdot v_1$ and $r_m = (B^t)^m j_0 \cdot d_1$. From (4.1) and (4.2), we have

$$(4.3) \qquad r_m = a_0 \overline{\lambda_0^m} + a_1 \overline{\lambda_1^m} \in \frac{1}{6} + \mathbb{Z} \quad \text{or} \quad -\frac{1}{6} + \mathbb{Z} \quad \text{for any } m \geq 0.$$

Thus for each $m \geq 0$, there exists some $\delta_m \in \{-1, 1\}$ and some $k_m \in \mathbb{Z}$ such that

$$(4.4) \qquad r_m = \frac{\delta_m + 6k_m}{6}.$$

Using the representation of the characteristic polynomial of the matrix B, for any $m \geq 0$ and $i = 0, 1$, we have

$$(4.5) \qquad \begin{aligned} \lambda_i^2 + b\lambda_i + c = 0 &\Longleftrightarrow \lambda_i^2 = -b\lambda_i - c, \\ &\Longleftrightarrow \lambda_i^{2+m} = -b\lambda_i^{m+1} - c\lambda_i^m. \end{aligned}$$

For any $m \geq 0$, it follows from (4.3), (4.4) and (4.5) that

$$(4.6) \qquad r_{m+2} = -br_{m+1} - cr_m = \frac{-b\delta_{m+1} - c\delta_m}{6} - bk_{m+1} - ck_m \in \pm\frac{1}{6} + \mathbb{Z}.$$

Since $b, c, k_m, k_{m+1} \in \mathbb{Z}$, (4.6) implies that for any $m \geq 0$,

$$(4.7) \qquad -b\delta_{m+1} = \pm 1 + c\delta_m \bmod 6, \quad \text{where } \delta_m, \delta_{m+1} \in \{-1, 1\}.$$

Since two similar matrices have the same characteristic polynomial, it follows from Proposition 4.1 that the characteristic polynomial of the matrix $B \in M_2^{(2)}(\mathbb{Z})$ can be expressed as $\lambda^2 - 2 = 0$ if $\det B = -2$ and $\lambda^2 + b\lambda + 2 = 0$, where $b \in \{0, -1, 1, -2, 2\}$ if $\det B = 2$.

Case 1 $\det A = -2$. In this case, $b = 0$, $c = -2$. (4.7) cannot hold for any $m \geq 0$ since $0 \neq \pm 1 - 2\delta_m \bmod 6$ for any $m \geq 0$ and $\delta_m \in \{-1, 1\}$.

Case 2 $\det A = 2$. In this case, $b \in \{0, -1, 1, -2, 2\}$ and $c = 2$. It follows from (4.7) that the possible values for b are -1 and 1. Suppose that

$$r_0 = \frac{\delta_0 + 6k_0}{6}, \quad r_1 = \frac{\delta_1 + 6k_1}{6},$$

where $\delta_0, \delta_1 \in \{-1, 1\}$ and $k_0, k_1 \in \mathbb{Z}$. We obtain, using (4.7), that

$$r_2 \in \pm\frac{1}{6} + \mathbb{Z} \Longleftrightarrow -b\delta_1 - 2\delta_0 = \pm 1 \bmod 6, \tag{4.8}$$

and

$$r_3 \in \pm\frac{1}{6} + \mathbb{Z} \Longleftrightarrow (b^2 - 2)\delta_1 + 2b\delta_0 = \pm 1 \bmod 6, \tag{4.9}$$

for some $\delta_0, \delta_1 \in \{-1, 1\}$.

If $b = 1$, (4.8) implies that $\delta_0 = -1, \delta_1 = 1$ or $\delta_0 = 1$, $\delta_1 = -1$. Substituting these values into the right hand side of (4.9), we get $(b^2 - 2)\delta_1 + 2b\delta_0 = 3$ or -3. Hence, $r_3 \notin \pm\frac{1}{6} + \mathbb{Z}$, which contradicts (4.3). If $b = -1$, (4.8) implies that $\delta_0 = \delta_1 = 1$ or $\delta_0 = \delta_1 = -1$. Substituting these values into the right hand side of (4.9), we get $(b^2 - 2)\delta_1 + 2b\delta_0 = -3$ or 3. Hence, $r_3 \notin \pm\frac{1}{6} + \mathbb{Z}$, which again yields a contradiction. This proves our claim. ■

From Theorem 3.1 and Theorem 4.2, we have the following corollary.

Corollary 4.3 *Let $A \in M_n^{(2)}(\mathbb{Z})$ be expansive and K be a self-affine tile satisfying the set equation $BK = (K + d_1) \cup (K + d_2)$ for some $d_1, d_2 \in \mathbb{R}^n$, where $n = 1, 2$. If K is an A-dilation scaling set, then K must be an A-dilation MRA scaling set.*

5 The Dimension Function of Self-affine Generalized Scaling Sets in $\mathbb{R}^n$, for $n \geq 3$

Unlike the one or two dimensional case, in dimension $n \geq 3$ we do not know if the dimension function of a self-affine generalized scaling set must be a constant. The method used in dimension one and dimension two could, in theory, be applied in any dimension if there were a characterization such as in Proposition 4.1 for each dimension n. Even if this were the case, the method could become impractical as each distinct equivalence class would have to be handled separately and their number would increase with the dimension. However, we will show that, when n is arbitrary, the dimension function is always bounded above by $2|K|$.

Theorem 5.1 *Let $A \in M_n^{(2)}(\mathbb{Z})$ be expansive and K be a self-affine tile satisfying the set equation $BK = (K + d_1) \cup (K + d_2)$ for some $d_1, d_2 \in \mathbb{R}^n$. If K is a generalized scaling set, then its dimension function satisfies $D(\xi) \leq 2|K|$.*

Proof Assume that $|K| = L$. Let $T = \{ j \in \mathbb{Z}^n \setminus \{0\} : \hat{\chi}_K(j) \neq 0 \}$. Let

$$\Lambda = \{ j \in \mathbb{Z}^n : (d_1 \cdot j, d_2 \cdot j) \in \mathbb{Z}^2 \},$$

$$\Gamma_k = \Big\{ j \in \mathbb{Z}^n : (d_1 \cdot j, d_2 \cdot j) \in \Big(\frac{k}{6}, -\frac{k}{6} \Big) + \mathbb{Z}^2 \Big\}, \quad k = 0, 1, 2, 3, 4, 5.$$

Then $\Gamma_0 = \Lambda$, $\Gamma_1 = j_1 + \Lambda$, $\Gamma_5 = -j_1 + \Lambda$, where $j_1 \in \Gamma_1$, and $T \subseteq \Gamma_1 \cup \Gamma_5$. Since Λ is a subgroup of $\mathbb{Z}^n$, there exists an integral matrix C such that $\Lambda = C\mathbb{Z}^n$. From the discussion above and that in Section 2, we have, using (2.9) with $m = 0$, that

$$\begin{aligned} \text{(5.1)} \quad D(\xi) &= \sum_{j \in \mathbb{Z}^n} \hat{\chi}_K(j) e^{2\pi i j \cdot \xi} \\ &= L + \sum_{j \in \mathbb{Z}^n} \hat{\chi}_K(j_1 + Cj) e^{2\pi i (j_1 + Cj) \cdot \xi} + \sum_{j \in \mathbb{Z}^n} \hat{\chi}_K(-j_1 - Cj) e^{-2\pi i (j_1 + Cj) \cdot \xi} \\ &= L + \mathrm{Re}\{ 2 e^{2\pi i j_1 \cdot \xi} \sum_{j \in \mathbb{Z}^n} \hat{\chi}_K(j_1 + Cj) e^{2\pi i Cj \cdot \xi} \}. \end{aligned}$$

The last equality is obtained from the fact that $\hat{\chi}_K(-\xi) = \overline{\hat{\chi}_K(\xi)}$, $\xi \in \mathbb{R}^n$. Let

$$\Lambda^* = \{ x \in \mathbb{R}^n : Cj \cdot x \in \mathbb{Z}, \text{ for all } j \in \mathbb{Z}^n \},$$

and

$$S(\xi) = \sum_{j \in \mathbb{Z}^n} \hat{\chi}_K(j_1 + Cj) e^{2\pi i Cj \cdot \xi}.$$

Then, for any $x \in \Lambda^*$, $S(\xi + x) = S(\xi)$. The fact that $j_1 \in \Gamma_1$ implies that $6j_1 \in \Lambda$ and thus $e^{2\pi i 6 j_1 \cdot x} = 1$, if $x \in \Lambda^*$, which implies that

$$e^{2\pi i j_1 \cdot x} \in \{ e^{\frac{k\pi i}{3}}, k = 0, 1, 2, 3, 4, 5 \}.$$

By the definition of Λ, $d_1, d_2 \in \Lambda^*$. Hence, $e^{\frac{\pi i}{3}}, e^{-\frac{\pi i}{3}} \in \{ e^{2\pi i j_1 \cdot x}, x \in \Lambda^* \}$ using the definition of Γ_1. Therefore, we have

$$\begin{aligned} \text{(5.2)} \qquad \{ e^{2\pi i j_1 \cdot \ell x} : \ell \in \mathbb{Z}, x \in \Lambda^* \} &= \{ e^{2\pi i j_1 \cdot \ell x} : 0 \le \ell \le 5, \ell \in \mathbb{Z}, x \in \Lambda^* \} \\ &= \{ e^{\frac{k\pi i}{3}}, 0 \le k \le 5, k \in \mathbb{Z} \}. \end{aligned}$$

Let $z = 2e^{2\pi i j_1 \cdot \xi} S(\xi)$. Consider the functions $D(\xi + \ell x)$, where $x \in \Lambda^*$ and $\ell \in \{0, 1, 2, 3, 4, 5\}$. By (5.1), we have

$$\begin{aligned} D(\xi + \ell x) &= L + \mathrm{Re}\{ 2 e^{2\pi i j_1 \cdot (\xi + \ell x)} S(\xi + \ell x) \} = L + \mathrm{Re}\{ 2 e^{2\pi i j_1 \cdot \xi} S(\xi) e^{2\pi i j_1 \cdot \ell x} \} \\ &= L + \mathrm{Re}\{ z e^{2\pi i j_1 \cdot \ell x} \}. \end{aligned}$$

From the definition of $D(\xi)$, $D(\xi) \ge 0$ for a.e. $\xi \in \mathbb{R}^n$. Hence, we deduce from (5.2) that

$$\text{(5.3)} \qquad L + \mathrm{Re}\{ z e^{\frac{\pi i \ell}{3}} \} \ge 0 \quad \text{for each } \ell \in \{0, 1, 2, 3, 4, 5\}.$$

(5.3) implies that $|\mathrm{Re}(z)| \le L$ and thus $D(\xi) \le L + |\mathrm{Re}(z)| \le 2L = 2|K|$ for $\xi \in \mathbb{R}^n$. ∎

From Theorem 5.1, we can get the following result.

Corollary 5.2 *Let $A \in M_n^{(2)}(\mathbb{Z})$ be expansive and K be a self-affine A-dilation scaling set satisfying the set equation $BK = (K+d_1) \cup (K+d_2)$ for some $d_1, d_2 \in \mathbb{R}^n$. If K is not an A-dilation MRA scaling set, then its dimension function $D(\xi)$ will take values* 0, 1, *and* 2.

There exist many self-affine A-dilation scaling sets in dimension bigger than 2 whose dimension function is a constant. We give such an example next.

Example 5.1 In dimension 3, consider the self-affine set K which satisfies

$$BK = (K+d_1)\cup(K+d_2), \text{ where } B = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, d_1 = \begin{pmatrix} 0 \\ -1/2 \\ 0 \end{pmatrix} \text{ and } d_2 = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}.$$

In this example, B is an expansive matrix and $\det B = 2$. It is easy to see that K is the unit cube $[-\frac{1}{2}, \frac{1}{2}]^3$ and that $K \subset BK$. Then, obviously, K is a $\mathbb{Z}^3$-tiling set and K contains a neighborhood of 0. Thus K is an A-dilation MRA scaling set and its dimension function $D_K(\xi) = 1$ a.e. $\xi \in \mathbb{R}^3$, where $A = B^t$.

We know that the equality (2.9) is necessary for the dimension function of a self-affine generalized scaling set to be a non-constant. In higher dimension, it is not easy to find an example of a self-affine generalized A-dilation scaling set, where $A \in M_n^{(2)}(\mathbb{Z})$, satisfying (2.9). Below, we show an example, found with the help of Matlab, of a self-affine tile K associated with an expansive matrix $B \in M_n^{(2)}(\mathbb{Z})$ in dimension 4 which satisfies (2.9). However, we are unable at this point to prove whether or not this self-affine tile is an A-dilation generalized scaling set, where $A = B^t$. In particular, we do not know if the inclusion $K \subset BK$ holds or not.

Example 5.2 Let K be a self-affine set satisfying $BK = (K + d_1) \cup (K + d_2)$, where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad d_1 = \frac{1}{6}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, d_2 = \frac{1}{6}\begin{pmatrix} -1 \\ -1 \\ -2 \\ -2 \end{pmatrix}.$$

For this example, it is easy to prove that the matrix $B \in M_n^{(2)}(\mathbb{Z})$ is expansive and the digit set $\mathcal{D} := \{6d_1, 6d_2\}$ is a complete set of coset representatives for the group $\mathbb{Z}^4/B\mathbb{Z}^4$. Let $K(B,\mathcal{D})$ be the set satisfying $BK = \bigcup_{d\in\mathcal{D}} K + d$. Thus, $|K(B,\mathcal{D})|$ is a positive integer by Theorem 1.1 in [21]. Therefore, $|K| = \frac{1}{6}|K(B,\mathcal{D})| > 0$ and K is a self-affine tile. If we take $j = \left(\begin{smallmatrix} 4 \\ 3 \\ 2 \\ 1 \end{smallmatrix}\right)$, we can also check that for any $m \geq 0$,

$$(B^t)^m j \cdot d_1 \in \frac{1}{6} + \mathbb{Z} \quad \text{and} \quad (B^t)^m j \cdot d_2 \in -\frac{1}{6} + \mathbb{Z}.$$

To summarize, if an A-dilation generalized scaling set is a self-affine tile, then its corresponding dimension function must be bounded in $\mathbb{R}^n$ and particularly, it is a constant in $\mathbb{R}$ and in $\mathbb{R}^2$. On the other hand, if we remove the self-affine condition, the corresponding dimension function can be arbitrary large, as some examples in Section 5 of [7] illustrate.

Acknowledgements The authors would like to thank the reviewers for their valuable comments and suggestions which were very helpful for improving the paper.

References

[1] P. Auscher, *Solution of two problems on wavelets.* J. Geom. Anal. **5**(1995), 181–236. http://dx.doi.org/10.1007/BF02921675

[2] L. Baggett, A. Carey, W. Moran, and P. Ohring, *General existence theorems for orthonormal wavelets, an abstract approach.* Publ. Res. Inst. Math. Sci. **31**(1995), 95–111. http://dx.doi.org/10.2977/prims/1195164793

[3] L. W. Baggett, *An abstract interpretation of the wavelet dimension function using group representations.* J. Funct. Anal. **173**(2000), 1–20. http://dx.doi.org/10.1006/jfan.1999.3551

[4] L. W. Baggett, H. A. Medina, and K. D. Merrill, *Generalized multi-resolution analyses and a construction procedure for all wavelet sets in* $\mathbb{R}^n$. J. Fourier Anal. Appl. **5**(1999), 563–573. http://dx.doi.org/10.1007/BF01257191

[5] M. Bownik and D. Speegle, *The wavelet dimension function for real dilations and dilations admitting non-MSF wavelets.* In: Approximation Theory X: Wavelet splines, and applications, Vanderbilt University Press, Nashville, 2002, 63–85.

[6] M. Bownik and K. Hoover, *Dimension functions of rationally dilated GMRAs and wavelets.* J. Fourier Anal. Appl. **15**(2009), 585–615. http://dx.doi.org/10.1007/s00041-009-9058-0

[7] M. Bownik, Z. Rzeszotnik, and D. Speegle, *A characterization of dimension functions of wavelets.* Appl. Comput. Harmon. Anal. **10**(2001), 71–92. http://dx.doi.org/10.1006/acha.2000.0327

[8] X. Dai, D. Larson, and D. Speegle, *Wavelet sets in* $\mathbb{R}^N$. J. Fourier Anal. Appl. **3**(1997), 451–456. http://dx.doi.org/10.1007/BF02649106

[9] I. Daubechies, *Ten lectures on wavelets.* CBMS-NSF Regional Conf. Ser. in Appl. Math. **61**, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1992.

[10] I. Daubechies and J. C. Lagarias, *Two-scale difference equations I: existence and global regularity of solutions.* SIAM J. Math. Anal. **22**(1991), 1388–1410. http://dx.doi.org/10.1137/0522089

[11] X. Fang and X. Wang, *Construction of minimally-supported-frequencies wavelets.* J. Fourier Anal. Appl. **2**(1996), 315–327.

[12] M. Frazier, G. Garrigós, K. Wang, and G. Weiss, *A characterization of functions that generate wavelet and related expansion.* J. Fourier Anal. Appl. **3**(1997), 883–906. http://dx.doi.org/10.1007/BF02656493

[13] G. Gripenberg, *A necessary and sufficient condition for the existence of a father wavelet.* Stud. Math. **114**(1995), 207–226.

[14] K. Gröchenig and A. Haas, *Self-similar lattice tilings.* J. Fourier Anal. Appl. **1**(1994), 131–170. http://dx.doi.org/10.1007/s00041-001-4007-6

[15] Q. Gu and D. Han, *On multiresolution analysis (MRA) wavelets in* $\mathbb{R}^N$. J. Fourier Anal. Appl. **6**(2000), 437–447. http://dx.doi.org/10.1007/BF02510148

[16] K. Gröchenig and W. Madych, *Multiresolution analysis. haar bases, and self-similar tilings of* $\mathbb{R}^n$. IEEE Trans. Inform. Theory **38**(1992), 556–568. http://dx.doi.org/10.1109/18.119723

[17] J.-P. Gabardo and X. Yu, *Construction of wavelet sets with certain self-similarity properties.* J. Geom. Anal. **14**(2004), 629–651. http://dx.doi.org/10.1007/BF02922173

[18] E. Hernández, X. Wang, and G. Weiss, *Smoothing minimally supported frequency (MSF) wavelets: Part I.* J. Fourier Anal. Appl. **3**(1997), 329–340.

[19] E. Hernández and G. Weiss, *A first course on wavelets.* Stud. in Adv. Math., CRC Press, Boca Raton, FL, 1996.

[20] J. C. Lagarias and Y. Wang, *Haar-type orthonormal wavelet bases in* $\mathbb{R}^2$. J. Fourier Anal. Appl. **2**(1995), 1–14. http://dx.doi.org/10.1007/s00041-001-4019-2

[21] ———, *Integral self-affine tiles in* $\mathbb{R}^n$. *I. Standard and nonstandard digit sets.* J. London Math. Soc. **54**(1996), 161–179. http://dx.doi.org/10.1112/jlms/54.1.161

[22] ———, *Self-affine tiles in* $\mathbb{R}^n$. Adv. Math. **121**(1996), 21–49. http://dx.doi.org/10.1006/aima.1996.0045
[23] ———, *Integral self-affine tiles in* $\mathbb{R}^n$*, Part II: Lattice tilings.* J. Fourier Anal. Appl. **3**(1997), 83–102. http://dx.doi.org/10.1007/BF02647948
[24] P.-G. Lemarié-Rieusset, *Existence de "fonction-père" pour les ondelettes à support compact.* C. R. Acad. Sci. Paris Sér. I Math. **314**(1992), 17–19.
[25] P.-G. Lemarié-Rieusset, *Sur l'existence des analyses multi-résolutions en théorie des ondelettes.* Rev. Mat. Iberoam. **8**(1992), 457–474. http://dx.doi.org/10.4171/RMI/131
[26] S. G. Mallat, *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbb{R})$. Trans. Amer. Math. Soc. **315**(1989), 69–87.
[27] A. Ron and Z. Shen, *The wavelet dimension function is the trace function of a shift-invariant system.* Proc. Amer. Math. Soc. **131**(2003), 1385–1398. http://dx.doi.org/10.1090/S0002-9939-02-06677-7
[28] R. S. Strichartz, *Wavelets and self-affine tilings.* Constr. Approx. **9**(1993), 327–346. http://dx.doi.org/10.1007/BF01198010
[29] X. Wang, *The study of wavelets from the properties of their Fourier transform.* Ph.D. thesis, Washington University in St. Louis, 1995.

Department of Mathematics, the Chinese University of Hong Kong, Hong Kong
e-mail: xiaoyefu@gmail.com

Department of Mathematics and Statistics, McMaster University, Hamilton, ON, L8S 4K1
e-mail: gabardo@mcmaster.ca