European Journal of Applied Mathematics (2024), 1–28
doi:10.1017/S0956792524000196

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PAPER

Unified framework for the separation property in binary phase-segregation processes with singular entropy densities

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Received: 08 November 2023; **Revised:** 12 March 2024; **Accepted:** 24 March 2024

Keywords: Cahn-Hilliard equations; singular potential; entropy; separation property; De Giorgi's iterations

2020 Mathematics Subject Classification: 35B36, 35K55, 35Q82, 35R09 (Primary); 82C24 (Secondary)

Abstract
This paper investigates the separation property in binary phase-segregation processes modelled by Cahn-Hilliard type equations with constant mobility, singular entropy densities and different particle interactions. Under general assumptions on the entropy potential, we prove the strict separation property in both two and three-space dimensions. Namely, in 2D, we notably extend the minimal assumptions on the potential adopted so far in the literature, by only requiring a mild growth condition of its first derivative near the singular points ± 1, without any pointwise additional assumption on its second derivative. For all cases, we provide a compact proof using De Giorgi's iterations. In 3D, we also extend the validity of the asymptotic strict separation property to the case of fractional Cahn-Hilliard equation, as well as show the validity of the separation when the initial datum is close to an 'energy minimizer'. Our framework offers insights into statistical factors like particle interactions, entropy choices and correlations governing separation, with broad applicability.

1. Introduction

During various scientific investigations, we frequently come across phase segregation or separation phenomena in a range of materials, spanning from simple binary mixtures to complex systems [12, 31]. To comprehensively understand these phenomena, one must adopt a versatile approach based on a diverse array of experiments, each underpinned by a unique set of statistical assumptions. By introducing variations in particle interactions and mixing entropy functions, we scrutinise how these assumptions impact phase segregation. Essential to this study is the concept of free energy, as the driving force for the separation of two or more phases. The Helmhotz-free energy[1] of a system is the sum of its internal energy E_{int}, related to a Hamiltonian operation H_{int}, the product of its mixing entropy[2] $e(\phi) := -F(\phi)$ and temperature[3] $\theta > 0$, and a quadratic function related to demixing effects. Order parameter phase segregation is a type of phase segregation that is characterised by the emergence of an order parameter. For binary mixtures of two components, A and B, the order parameter $\phi := c_A - c_B \in [-1, 1]$ is the

[1]In the absence of any external fields, such as electric and magnetic fields.
[2]The entropy is by definition a strictly concave function.
[3]We scale θ to be the unit in what follows.

difference in concentrations $c_A, c_B \in [0, 1]$ of the two components in the two phases. The difference in the free energy

$$E(\phi) = E_{int}(\phi) - \theta e(\phi) - \frac{\alpha_0}{2}\phi^2, \qquad \alpha_0 > \min_{s\in(-1,1)} F''(s) > 0, \tag{1.1}$$

between the mixed and segregated states is what drives the process of phase-segregation (or separation). We recall that the energy involved in phase segregation of a mixture of two metals can be significantly large,[4] when the number of atoms in a typical sample is considered. As one possible approach to design and control these processes, we employ the classical formulation of the transport equation[5] that ensures mass conservation within the system, i.e.,

$$\partial_t\phi + \nabla\cdot\mathcal{J} = 0, \quad \text{in } \Omega\times(0,T), \quad \phi(0,x) = \phi_0, \quad \text{in } \Omega, \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^d$, $d \in \{1, 2, 3\}$, and $T > 0$ determines duration of the experiment. The mass conservation means

$$\int_\Omega \phi(t)\,dx = \int_\Omega \phi_0 dx,$$

for all $t \in (0, T)$. The flux $\mathcal{J}$ is directly proportional[6] to the gradient of the chemical potential μ, which flows from regions of high chemical potential to regions of low chemical potential, provided that

$$\mathcal{J}\cdot n = 0, \text{ on } \partial\Omega\times(0,T). \tag{1.3}$$

In particular, $\mu = \delta E(\phi)/\delta\phi$, is a measure of the change in total energy of the system when the order parameter is changed by a small amount. When designing physically justified models of segregation-separation processes, it is important to take into account the factors that affect the strict separation of the binary components. Overall, both the chemical potential and the free energy (1.1) provide versatile tools that can be used to study binary phase transitions or segregation processes from a variety of perspectives. In particular, the chemical potential is in balance between three competing factors in (1.1):

$$\mu = H_{int}(\phi) + F'(\phi) - \alpha_0\phi, \text{ in } \Omega\times(0,T). \tag{1.4}$$

The Hamiltonian $H_{int} = \delta E_{int}(\phi)/\delta\phi$ represents the total particle interactions in the system. Generally, the type of particle interactions determines the strength and range of the forces that drive segregation and separation. The entropy density function $(-F')$ measures the rate of mixing of particles in a system and favours state where the particles are well-mixed. The linear function $(-\alpha_0\phi)$ describes the tendency of particles to separate into different phases and favours state where the particles are separated. The parameter α_0 is a measure of the strength of the demixing effect. The larger the value of α_0, the stronger the demixing effect and the greater the tendency of the particles to separate into different phases. On the other hand, the geometry of the system Ω can also affect the flow of the binary components and the rate of segregation. Lattice gas models are simple models that can be used to model phase segregation in binary mixtures [12]. In a lattice gas model, the particles are located on a lattice of sites and the particle interactions are modelled by the operation H_{int} through a potential between them. In this work, we directly link the Hamiltonian H_{int} to several critical formulations:

(a) **The classical (Neumann) Laplace operator**. This essentially promotes states in which particles undergo rapid changes communicating with near site neighbours at short range. In simpler terms, the model favours state where particles move in a seemingly random and well-mixed way within the lattice, occurring at temperatures exceeding a certain critical threshold. The Laplace operator plays a pivotal role in steering and promoting this particle motion behaviour, while the internal energy E_{int} encapsulates the dynamics of interfacial regions that change gradually by penalising gradients.

[4] It has been estimated that the energy required for phase separation is of the order of 1 eV per atom [31].

[5] This assumption is specific to our current work. It's worth noting that other transport equations with nonlocal characteristics, which describe anomalous transport processes, are also applicable, as highlighted by [19].

[6] $\mathcal{J} = -m\nabla\mu$, for some positive constant mobility factor m.

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This foundational assumption draws inspiration from the pioneering work of Cahn and Hilliard [6] in the late 1950s. Indeed, given that

$$\Psi(s) = F(s) - \frac{\alpha_0}{2}s^2, \quad \forall s \in [-1, 1],$$

with $\alpha_0 > \alpha$, the free energy reads in this case

$$E_L(\phi) = \int_\Omega \frac{1}{2}|\nabla\phi|^2 + \Psi(\phi)\,\mathrm{d}x$$

(b) **A spectrally defined fractional Laplace operator** $(-\Delta_N)^s$, for $s \in (0, 1)$. This operator is intimately tied to Levy processes (or distributions) and is used in modelling systems with long-range particle jumps, allowing for stronger correlations between particle motions (see [20]). In a lattice gas model incorporating a fractional Laplace operator, particles tend to favour movement to more distant sites, considering the long-range correlations between them. This results in several alterations in particle motion behaviour. Notably, particles are prone to diffuse over extended distances and exhibit a propensity to assemble into clusters of similar particles, and this mechanism can give rise to intricate and complex phase structures. Additionally, it can lead to a deceleration in the phase-segregation process. In this case, the free energy also reads

$$E_{SF}(\phi) = \int_\Omega \frac{1}{2}|(-\Delta_N)^{s/2}\phi|^2 + \Psi(\phi)\mathrm{d}x.$$

(c) **Convolutions with a symmetric, integrable kernel** J, representing a microscopic potential that characterises how particle forces act between different particles. This Hamiltonian exerts a substantial influence on particle motion by accounting for both short-range and long-range effects in particle interactions. One of its crucial implications is the emergence of long-range correlations between particles. This occurs because the convolution operation considers interactions among all particles, not solely those immediately adjacent. Essentially, this model facilitates the application of sampling distributions[7] by convolving multiple probability distributions of particle motions. This concept is rooted in the research of Giacomin and Lebowitz [13], focusing on a nonlocal variant of the Cahn-Hilliard equation proposed in the late 1990s. In this case, the free energy reads as follows:

$$\begin{aligned} E_{NL}(\phi) := &\frac{1}{4}\int_\Omega\int_\Omega J(x-y)\,(\phi(y)-\phi(x))^2\,\mathrm{d}x\mathrm{d}y \\ &+ \int_\Omega F(\phi(x)) - \frac{(J*1)\,(x)}{2}\phi^2(x)\mathrm{d}x. \end{aligned}$$

In our exploration of a wide array of phase-segregation processes, it is crucial to consider an equally broad spectrum of entropy functions that fall under the umbrella of nonextensive statistical mechanics (NSM) [32], a framework that extends the well-established Boltzmann-Gibbs (BG) statistical mechanics, especially when BG is inadequate. While BG is commonly used for weakly interacting systems, like those in case (a), it lacks a robust foundation, particularly for strongly interacting systems with strong particle correlations and non-exponential behaviour as seen often in cases (b)–(c). NSM, on the other hand, has proven to be a potent tool for studying phase segregation in binary materials. It has demonstrated remarkable accuracy in reproducing experimental results, even when the BG framework falls short [32]. An example of an NSM candidate is Tsallis' entropy, denoted as $(-F_q)$, which deviates[8] from BG for any $q > 0, q \neq 1$, but converges to the standard BG entropy when $q = 1$ (see Figure 1):

$$F_q(x) := -g_q(1+x) - g_q(1-x), \qquad g_q(x) := x\ln_{\{q\}}(1/x).$$

[7] Employed in Monte Carlo simulations.

[8] Here, $\ln_{\{q\}}(\cdot)$ denotes the q-logarithm so that $\ln_{\{1\}}(\cdot) = \ln(\cdot)$, see Section 6.

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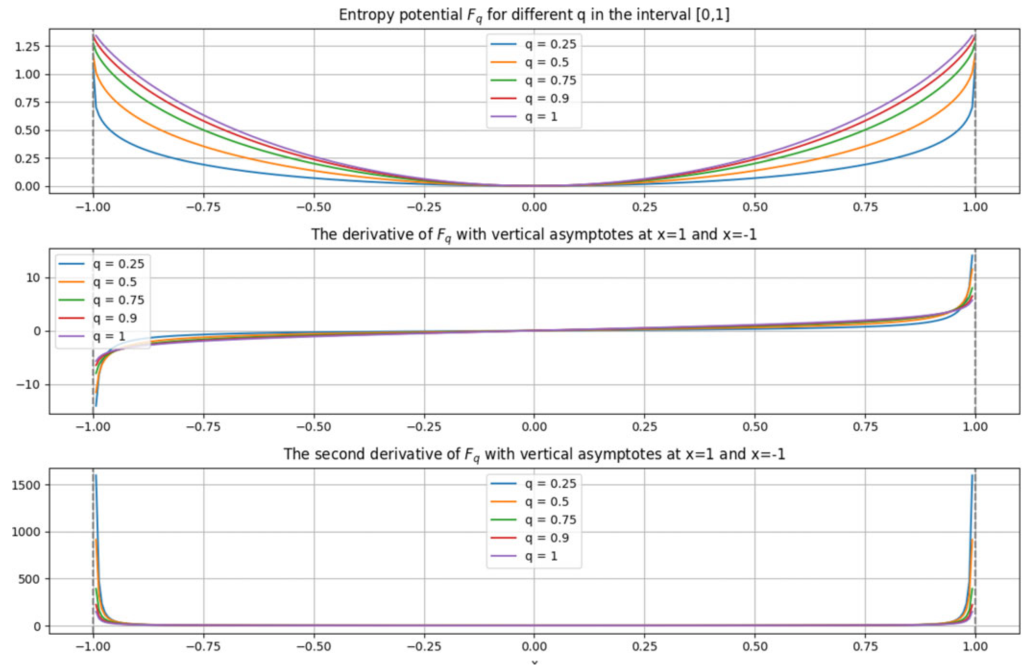

Figure 1. *Plots of Tsallis' and Boltzmann-Gibbs entropy potentials, and their singular behaviour of derivatives.*

The choice of the index $q \in (0, 1]$ may significantly depend on the system's correlations. Additionally, our approach allows for asymmetric entropy potentials across the phase domain $[-1, 1]$, enabling the modelling of systems favouring segregation or differing densities between phases (such as oil and water, by favouring the state where the oil is on top of the water). Indeed, our framework and the corresponding assumptions allow also for double-well potentials $\Psi(\phi) = F(\phi) - (\alpha_0/2)\phi^2$, with varying well depths that control system behaviour. In particular, deeper wells make it more likely for the system to reside in the deeper well, enhancing stability.

All these considerations are valuable for modelling phase behaviour and designing phase-separation processes. Our proposed framework helps us fathom the intricate interplay of statistical factors and parameters, offering insights valuable to a broad audience of applied scientists. It not only deepens our understanding of phase segregation but also guides research and applications across multiple disciplines, from materials science to biotechnology.

In this study, we aim at strongly emphasising the vital importance of preserving the strict separation of binary chemical components during segregation processes. This property,

$$|\phi(x, t)| \leq 1 - \delta_0, \; (x, t) \in \Omega \times (0, T), \; \text{ for some } \delta_0 \in (0, 1), \tag{1.5}$$

is crucial for ensuring product purity, preventing undesired by-products and maintaining control over the final product's properties (in particular, with respect to elaborating on quantitative estimates for δ_0; see also below). Industries such as pharmaceuticals, food production and electronics rely on this property to guarantee product quality and safety. Efficient phase-separation techniques and enhanced product quality stem from the strict adherence to (1.5). For instance, in pharmaceutical production, it is imperative to strictly separate the active ingredient from other components to maintain the drug's purity and effectiveness. Similarly, in food production, separating different ingredients prevents cross-contamination and ensures product safety. In electronic device manufacturing (of semiconductors), distinct material separation is critical to achieving the desired electrical properties (see, e.g., [17, 31]). Moreover, granular understanding of the degree of separation δ_0's dependence on various control factors during phase

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segregation is key to controlling the final product's properties. It's crucial in designing physically justified models, considering factors like particle interactions, mixing entropy, temperature and system geometry. On the other hand, the strict separation property holds a crucial role within the segregation model, serving as a fundamental mathematical property. It enables the development of Gevrey regularity in solutions, which, in turn, is an indispensable component in formulating and designing well-defined optimisation problems related to segregation processes. These encompass not only optimal control but also a range of related issues (e.g., [1, 6–9, 13, 14–25, 27–29] and references therein).

The main goal of this work is then to establish minimal assumptions on the entropy density function, which ensure the validity of the strict separation property. Indeed, while this property is already well-established under stronger assumptions on the entropy $(-F)$ (see, for instance, [14, 15, 28, 29]), here we propose a different proof method (completely new for the cases (a)-(b), a refinement for the case (c)). Namely, we solely rely on the balance equation for the chemical potential in (1.4), capitalising on its inherent Sobolev regularity. This is coupled with the application of a 'microscopic' De Giorgi-type argument. Thanks to our new proofs, our analysis in two-dimensional bounded domains now accommodates a broader range of entropy functions than previously available in the literature. Specifically, for cases (a) and (b) as described earlier, the least restrictive assumption on the entropy potential F involves the following growth condition (as seen in [15]): there exists $C_\sharp > 0$ such that

$$F''(s) \leq C_\sharp \mathrm{e}^{C_\sharp |F'(s)|^{\upsilon}}, \quad \forall s \in (-1,1)\,, \quad \text{for some } \upsilon \in [1,2), \tag{1.6}$$

In the case (c), a less demanding assumption, even in three dimensions, was first proposed in [29], introducing a requirement on the second derivative of the function F as well. However, for the first time in the two-dimensional framework, we propose a more general set of assumptions (refer to equation (2.3) below) that does not involve any further assumption on F'', as in (1.6), apart from its strict positivity (as shown in (1.1)). It solely requires a mild growth condition of F' near the singular points ± 1. For a more detailed discussion on this topic, please see Remark 2.1 below. In conclusion, we also offer insights into all previously discussed cases (a)–(c) within the context of three-dimensional space, highlighting certain mathematical limitations encountered when applying De Giorgi's scheme. Furthermore, by means of a new proof based again upon De Giorgi's iterations, we extend prior findings (see, e.g., [1]) concerning the 3D asymptotic separation property of the local Cahn-Hilliard equation (a) to the fractional Cahn-Hilliard equation (b), specifically when the fractional range s falls within 1/4 and 1. Additionally, we provide insights into the separation property in the vicinity of 'energy minimizers' for this problem within the same fractional range.

The paper's structure is as follows. In Section 2, we introduce the fractional operator A_N^s and state general assumptions on potential F. Section 3 provides a concise proof of the separation property in the local case (associated with case (a) in two space dimensions). In Section 4, we introduce a new method to derive the separation property for the nonlocal problem associated with case (c) in two space dimensions. Section 5 presents a comprehensive analysis of the spectral-fractional Cahn-Hilliard equation in dimension two, while the essential mathematical tools used to prove the strict separation property are detailed in the Appendix. Section 6 provides a discussion of cases (a)–(c) within the framework of three-dimensional bounded domains, also accounting for the validity of the asymptotic strict separation property. In the final Section 7, we solidify the high impact of our extended framework and underscore the minimal set of assumptions needed to establish the essential separation property in a variety of physical applications.

2. Mathematical setting

2.1. Functional framework

Let Ω be a smooth bounded domain in $\mathbb{R}^d$, where $d = 2$ or $d = 3$. We denote Sobolev spaces as $W^{k,p}(\Omega)$, where $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, equipped with the norm $\| \cdot \|_{W^{k,p}(\Omega)}$. The Hilbert space $W^{k,2}(\Omega)$ is represented

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as $H^k(\Omega)$, and its norm is $\|\cdot\|_{H^k(\Omega)}$. Additionally, we use $H^s(\Omega)$, where $s>0$ and $s\notin\mathbb{N}$, to denote standard fractional Sobolev spaces with the norm $\|\cdot\|_{H^s(\Omega)}$. For convenience, we define the following spaces:

$$H=L^2(\Omega),\quad V=H^1(\Omega),\quad V_2=\{v\in H^2(\Omega)\colon \partial_{\mathbf{n}}v=0 \text{ on } \partial\Omega\}.$$

Furthermore, when dealing with vector spaces, $\mathbf{X}$ denotes a space of vectors with d components belonging to X. We use $(\cdot,\cdot)$ to represent the inner product in H and $\|\cdot\|$ for the corresponding induced norm. In V, we use $(\cdot,\cdot)_V$ and $\|\cdot\|_V$ for the inner product and its induced norm. The integral mean of a function f is denoted as:

$$\bar{f}:=\frac{\int_\Omega f(x)dx}{|\Omega|},$$

where $|\Omega|$ represents the d-dimensional Lebesgue measure of the set Ω. We also define the following spaces:

$$H_0=\{v\in H{:}\bar{f}=0\},\quad V_0=\{v\in V{:}\bar{f}=0\},\quad V_0'=\{v\in V'\colon |\Omega|^{-1}\langle f,1\rangle=0\},$$

equipped with the norms of H, V and V'. The Laplace operator $A_N^0{:}V_0\to V_0'$, defined by $<A_N^0u,v>=(\nabla u,\nabla v)$, forms an isomorphism. We denote by $\mathcal{N}$ its inverse map, and we set $\|f\|_*:=\|\nabla\mathcal{N}f\|$, which serves as a norm on V_0' equivalent to the standard one. Furthermore, we have:

$$\|f-\bar{f}\|_*^2+|\bar{f}|^2$$

as a norm in V', which is equivalent to the canonical one. We consider A_N^0 as an unbounded operator in H_0, with domain $\mathfrak{D}(A_N^0):=H_0\cap V_2$, corresponding to the Laplace operator with homogeneous Neumann boundary conditions. It is standard to notice that the operator A_N^0 is selfadjoint on H_0 with compact inverse. Therefore, by spectral theory, there exists a sequence of real and positive eigenvalues of A_N^0, β_j, $j\in\mathbb{N}$, such that $\beta_j\nearrow\infty$ as $j\to\infty$. The corresponding eigenvectors $w_j\in\mathfrak{D}(A_N^0)$ solve $A_N^0w_j=\beta_jw_j$, and form an orthonormal basis of H_0. We thus have the following spectral decomposition:

$$u=\sum_{j=1}^\infty(u,w_j)w_j,\quad \forall u\in H_0,\quad \text{and}\quad A_N^0v=\sum_{j=1}^\infty\beta_j(u,w_j)w_j,\quad \forall v\in\mathfrak{D}(A_N^0).$$

By considering the constant function $w_0\equiv1$ and the related eigenvalue $\beta_0=0$, we can construct an orthonormal basis of H, such that, for any $u\in H$, $u=\sum_{j=0}^\infty(u,w_j)w_j$, and define the extended operator $A_N{:}\mathfrak{D}(A_N)\subset H\to H$, where $\mathfrak{D}(A_N)=V_2$, and

$$A_Nv=\sum_{j=0}^\infty\beta_j(u,v_j)w_j,\quad \forall v\in\mathfrak{D}(A_N).$$

The operator A_N is no longer a one-to-one operator. We can now introduce the positive fractional powers of order $s\in(0,1)$ of A_N as

$$A_N^su:=\sum_{j=0}^N\beta_j^s(u,w_j)w_j,\quad \forall u\in\mathfrak{D}(A_N^s),$$

where

$$\mathfrak{D}(A_N^s):=\{u\in H: A_N^su\in H_0\}.$$

For any $s>0$, the domain $\mathfrak{D}(A_N^s)$ is a Hilbert space associated with the inner product and norm

$$(u,v)_s:=(u,v)+(A_N^su,A_N^sv),\quad \|u\|_s^2:=\|u\|^2+\|A_N^su\|^2,$$

respectively. We recall the following 2D Sobolev-Gagliardo-Nirenberg-type inequalities (see, e.g., [2, Ch. 9], [3]):

$$\|u\|_{L^p(\Omega)}\le K_1\sqrt{p}\|u\|^{\frac{2}{p}}\|u\|_V^{1-\frac{2}{p}},\qquad \forall\,u\in V,\qquad \forall p\in[2,\infty),\tag{2.1}$$

$$\|u\|_{L^p(\Omega)} \leq K_2(p,s)\|u\|^{\frac{2}{ps}-\frac{1}{s}+1}\|u\|_{s/2}^{\frac{1}{s}-\frac{2}{ps}}, \qquad \forall\, u \in \mathfrak{D}(A_N^{\frac{s}{2}}), \quad s \in (0,1), \quad \forall p \in \left[2, \frac{2}{1-s}\right). \tag{2.2}$$

Here, the constant K_1 is independent of p, while the constant $K_2(p, s)$ may depend on s and p.

2.2. Assumptions on the entropy function

When $\Omega \subset \mathbb{R}^2$, the only assumptions we require on the singular entropy function $(-F)$ are the following:

1. $F \in C([-1, 1]) \cap C^2(-1, 1)$ satisfies

$$\lim_{s\to-1} F'(s) = -\infty, \quad \lim_{s\to 1} F'(s) = +\infty, \quad F''(s) \geq \alpha > 0, \quad \forall\, s \in (-1, 1).$$

We extend $F(s) = +\infty$ for any $s \notin [-1, 1]$. Without loss of generality, $F(0) = 0$ and $F'(0) = 0$. In particular, this means that $F(s) \geq 0$ for any $s \in [-1, 1]$.

2. As $\delta \to 0^+$, we assume, for some $\beta > 1/2$,

$$\frac{1}{F'(1-2\delta)} = O\left(\frac{1}{|\ln(\delta)|^\beta}\right), \qquad \frac{1}{|F'(-1+2\delta)|} = O\left(\frac{1}{|\ln(\delta)|^\beta}\right). \tag{2.3}$$

As anticipated in the Introduction, we then define:

$$\Psi(s) = F(s) - \frac{\alpha_0}{2}s^2, \quad \forall s \in [-1, 1], \tag{2.4}$$

with $\alpha_0 > \alpha$.

Remark 2.1. As observed in the Introduction, note that we do not make any assumption[9] about the pointwise relations between F' and F'', as in [15]. In fact, even the extra assumption (2) does not include F'', as the results seem to depend primarily on the first derivative of the entropy $(-F)$. This somewhat contradicts the conjecture proposed in the Introduction of [14], which claims that the crucial role in the validity of the separation property belongs to the second derivative of the entropy and not to the first. Here we show that in the local, nonlocal and fractional Cahn-Hilliard equations, only an assumption on F' near the pure phases ± 1 is required in 2D. This assumption naturally pertains only to F' near the endpoints and avoids any pointwise hypothesis. This is not surprising, since a similar assumption, not relying on F'' but rather only on F', also works in the proof of the separation property in the case of the conserved Allen-Cahn equation (see [16, 25]). Clearly, assumptions (1)–(2) are satisfied by the logarithmic entropy and by all other continuous entropies (including the Tsallis' entropy F_q, $q \in (0, 1)$) that exhibit stronger singularities in their derivatives at the endpoints (see [15, Appendices], for other examples). Given that $\beta > 1/2$, these assumptions also accommodate entropy densities with milder singularities than the logarithmic potential density.

Remark 2.2. We want to emphasise that, in greater generality, the potential F does not necessarily need to be symmetric with respect to $s = 0$, as long as condition (2) is satisfied.

3. The separation property for the local Cahn-Hilliard equation

In this section, we consider the local Cahn-Hilliard equation:

$$\begin{cases} \partial_t\phi - \Delta\mu = 0, & \text{in } \Omega \times (0,\infty), \\ \mu = \Psi'(\phi) - \Delta\phi, & \text{in } \Omega \times (0,\infty), \\ \partial_{\mathbf{n}}\phi = \partial_{\mathbf{n}}\mu = 0, & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{3.1}$$

[9] In particular, the growth condition $F''(s) \lesssim e^{C_\sharp |F'(s)|^\upsilon}$, for all $s \in (-1, 1)$, for some $\upsilon \in [1, 2)$ is now removed.

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First, we review a well-established result regarding the well-posedness of the local Cahn-Hilliard equation for a sufficiently smooth bounded domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$. The proof of this result can be found in various references, such as [1, 8, 15, 23, 28]. In particular, the crucial estimate (3.6), which is central to our new proof, is presented in [15, (25)]. We state the following theorem:

Theorem 3.1. *Assuming that (1) holds and that $\phi_0 \in V \cap L^\infty(\Omega)$ such that $\|\phi_0\|_{L^\infty} \le 1$ and $|\overline{\phi}_0| = m < 1$, then there exists a unique weak solution to (3.1). This solution satisfies the following properties for any $T > 0$:*

$$\phi \in L^\infty(\Omega \times (0,T)): \text{for all } t > 0, \text{ almost everywhere in } \Omega,$$

$$\phi \in L^\infty(0,T;V) \cap L^2(0,T;V_2) \cap H^1(0,T;V'),$$

$$\mu \in L^2(0,T;V), \quad F'(\phi) \in L^2(0,T;L^p(\Omega)), \text{ for all } p \in [2,\infty) \text{ if } d = 2, \text{ or } p = 6 \text{ if } d = 3,$$

satisfying the following equations:

$$\langle \partial_t \phi, v \rangle_{V',V} + (\nabla \mu, \nabla v) = 0, \quad \text{for all } v \in V, \text{ almost everywhere in } (0,T), \tag{3.2}$$

$$\mu = \Psi'(\phi) - \Delta\phi, \quad \text{almost everywhere in } \Omega \times (0,T), \tag{3.3}$$

and $\phi(\cdot, 0) = \phi_0(\,\cdot\,)$ in Ω. Moreover, for any $\tau > 0$:

$$\sup_{t \ge \tau} \|\partial_t \phi(t)\|_{V'} + \sup_{t \ge \tau} \|\partial_t \phi\|_{L^2(t,t+1,V)} \le \frac{C_0}{\sqrt{\tau}}, \tag{3.4}$$

$$\sup_{t \ge \tau} \|\mu(t)\|_V + \sup_{t \ge \tau} \|\phi(t)\|_{W^{2,r}(\Omega)} \le \frac{C_0}{\sqrt{\tau}}, \quad \text{for all } r \in [2,\infty) \text{ if } d = 2, \text{ or } r = 6 \text{ if } d = 3, \tag{3.5}$$

$$\|F'(\phi)\|_{L^\infty(\tau,t;L^p(\Omega))} + \|\mu\|_{L^\infty(\tau,t;L^p(\Omega))} \le C_1(\tau)\sqrt{p}, \quad \text{for all } t \ge \tau, \text{ for all } p \in [2,\infty) \text{ if } d = 2, \tag{3.6}$$

$$\|F'(\phi)\|_{L^\infty(\tau,t;L^6(\Omega))} + \|\mu\|_{L^\infty(\tau,t;L^6(\Omega))} \le C_2(\tau), \quad \text{for all } t \ge \tau, \text{ if } d = 3, \tag{3.7}$$

$$\|\mu\|_{L^2(t,t+1,V_2)} \le C_3(\tau), \quad \text{for all } t \ge \tau. \tag{3.8}$$

Here, the positive constant C_0 depends only on the energy

$$\mathcal{E}_L(\phi_0) := \int_\Omega \frac{1}{2}|\nabla \phi_0|^2 + \Psi(\phi_0(x))\,\mathrm{d}x,$$

as well as the domain Ω, $\overline{\phi}_0$ and the other parameters of the system. The constants $C_1 = C_1(\tau)$, $C_2 = C_2(\tau)$ and $C_3 = C_3(\tau)$ also depend on τ.

Remark 3.2. It is worth noting that for $d = 2$, ϕ satisfies the continuity property:

$$\phi \in C([0,\infty); H^{\frac{3}{2}}(\Omega)) \hookrightarrow C(\overline{\Omega} \times [\tau,\infty)),$$

as mentioned in [24, Remark 3.11].

We now introduce a new approach to establish the instantaneous strict separation property. While this property is already well-established under stronger assumptions on the entropy potential F (as mentioned in the Introduction, e.g., in [28]), we propose a different proof method based on De Giorgi's iterations applied to the elliptic equation $(3.1)_2$. This approach allows us to relax the assumptions on the entropy function. Our main result is stated in the following theorem, which we prove immediately after.

Theorem 3.3. *Consider a sufficiently smooth bounded domain $\Omega \subset \mathbb{R}^2$. Assume that assumptions (1)–(2) hold for the entropy potential F. Also, assume that $\phi_0 \in V \cap L^\infty(\Omega)$ with $\|\phi_0\|_{L^\infty} \le 1$ and $|\overline{\phi}_0| = m < 1$. Then, for any $\tau > 0$, there exists $\delta \in (0,1)$, depending on τ, Ω, m, the initial energy $\mathcal{E}_L(\phi_0)$, and the parameters of the system, such that the unique weak solution to problem (3.1) satisfies:*

$$|\phi(x,t)| \le 1 - \delta, \quad \text{for all } (x,t) \in \overline{\Omega} \times [\tau, +\infty), \tag{3.9}$$

which means that the instantaneous strict separation property from the pure phases ± 1 holds.

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Remark 3.4. The proof of Theorem 3.3 provides us with a direct and explicit way to estimate the separation scale δ in (3.9) in relation to the physical scales inherent to the problem.

Proof of Theorem 3.3. We start by observing that all the assumptions of Theorem 3.1 are satisfied, and therefore, this theorem applies to the solution (ϕ, μ) under consideration. Our proof is based on applying De Giorgi's iteration scheme to the equation for the chemical potential μ, which is given by (3.3). Let us begin by fixing $\tau > 0$ and $\delta \in (0, 1)$ (we will choose its value later). We introduce the sequence:

$$k_n = 1 - \delta - \frac{\delta}{2^n}, \quad \forall n \geq 0, \tag{3.10}$$

where:

$$1 - 2\delta < k_n < k_{n+1} < 1 - \delta, \quad \forall n \geq 1, \quad k_n \to 1 - \delta \text{ as } n \to \infty. \tag{3.11}$$

Next, we define:

$$\phi_n(x, t) = (\phi - k_n)^+, \tag{3.12}$$

and, for any $n \geq 0$, we introduce the set:

$$A_n(t) = \{x \in \Omega : \phi(x, t) - k_n \geq 0\}, \quad \forall t \in [\tau, \infty).$$

Clearly, we have:

$$A_{n+1}(t) \subseteq A_n(t), \quad \forall n \geq 0, \quad \forall t \in [\tau, \infty).$$

In conclusion, we define:

$$z_n(t) = \int_{A_n(t)} 1 dx, \quad \forall n \geq 0.$$

Now, let us fix $t \in [\tau, \infty)$ (from now on, we will not repeat the dependence on t). For any $n \geq 0$, we consider the test function $v = \phi_n$, multiply equation (3.3) by v and integrate over Ω. After an integration by parts, taking into account the boundary conditions, we obtain:

$$\|\nabla \phi_n\|^2 + \int_\Omega F'(\phi)\phi_n dx = \alpha_0 \int_\Omega \phi\phi_n dx + \int_\Omega \mu\phi_n dx,$$

for any $t \in [\tau, \infty)$. Here, we used the identity:

$$\int_{A_n} \nabla\phi \cdot \nabla\phi_n dx = \|\nabla\phi_n\|^2. \tag{3.13}$$

For any $x \in A_n(t)$, it holds:

$$F'(\phi(x, t)) = F'(k_n) + F''(c(x, t))(\phi(x, t) - k_n), \tag{3.14}$$

with $c(x, t) \in [k_n, \phi(x, t)]$. Therefore, considering that $k_n > 1 - 2\delta$ and, by (1), $F''(c(x, t)) \geq \alpha$ for any $x \in A_n(t)$ and $t \geq \tau$, we can write:

$$\begin{aligned} \int_\Omega F'(\phi)\phi_n dx &= \int_{A_n(t)} F'(\phi)\phi_n dx \\ &\geq F'(k_n) \int_\Omega \phi_n dx + \alpha \int_\Omega \phi_n^2 dx \\ &\geq F'(1 - 2\delta) \int_\Omega \phi_n dx + \alpha \int_\Omega \phi_n^2 dx. \end{aligned} \tag{3.15}$$

Furthermore, we have:

$$\alpha_0 \int_\Omega \phi\phi_n dx \leq \alpha_0 \int_\Omega \phi_n dx, \tag{3.16}$$

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and by (3.6) and Hölder's inequality, recalling that $0 \leq \phi_n \leq 2\delta$:

$$\begin{aligned}\int_\Omega \mu\phi_n dx &= \int_{A_n} \mu\phi_n dx \\ &\leq \|\phi_n\|_{L^\infty(\Omega)}\|\mu\|_{L^p(\Omega)}\left(\int_{A_n} 1dx\right)^{1-\frac{1}{p}} \\ &\leq 2\delta\|\mu\|_{L^p(\Omega)} z_n^{1-\frac{1}{p}}, \quad \text{for } p \geq 2. \end{aligned} \tag{3.17}$$

To summarise, we have:

$$\|\nabla\phi_n\|^2 + (F'(1-2\delta)-\alpha_0)\int_\Omega \phi_n dx + \alpha\int_\Omega \phi_n^2 dx \leq C(\tau)\delta\sqrt{p}z_n^{1-\frac{1}{p}}, \quad \text{for } p \geq 2. \tag{3.18}$$

Clearly, by choosing δ sufficiently small, we ensure that, as per assumption (1), $F'(1-2\delta) > \alpha_0$. Moreover, for any $t \in [\tau, \infty)$ and for any $x \in A_{n+1}(t)$, we can observe:

$$\begin{aligned}\phi_n(x,t) &= \phi(x,t) - \left[1 - \delta - \frac{\delta}{2^n}\right] \\ &= \underbrace{\phi(x,t) - \left[1 - \delta - \frac{\delta}{2^{n+1}}\right]}_{\phi_{n+1}(x,t)\geq 0} + \delta\left[\frac{1}{2^n} - \frac{1}{2^{n+1}}\right] \geq \frac{\delta}{2^{n+1}}, \end{aligned} \tag{3.19}$$

which implies:

$$\int_\Omega |\phi_n|^3 dx \geq \int_{A_{n+1}(t)} |\phi_n|^3 dx \geq \left(\frac{\delta}{2^{n+1}}\right)^3 \int_{A_{n+1}(t)} 1dx = \left(\frac{\delta}{2^{n+1}}\right)^3 z_{n+1}.$$

Applying Hölder's inequality, we get:

$$\begin{aligned}\left(\frac{\delta}{2^{n+1}}\right)^3 z_{n+1} &\leq \int_\Omega |\phi_n|^3 dx \\ &= \int_{A_n(t)} |\phi_n|^3 dx \leq \left(\int_\Omega |\phi_n|^4 dx\right)^{\frac{3}{4}} \left(\int_{A_n(t)} 1dx\right)^{\frac{1}{4}}. \end{aligned} \tag{3.20}$$

Utilising Sobolev-Gagliardo-Nirenberg's inequality (2.1) with $p = 4$, and taking into account (3.18), we can write:

$$\begin{aligned}\int_\Omega |\phi_n|^4 dx &\leq 4(K_1)^2\|\phi_n\|_V^2\|\phi_n\|^2 \\ &\leq 4(K_1)^2\left(\|\phi_n\|^2 + \|\nabla\phi_n\|^2\right)\|\phi_n\|^2 \\ &\leq 4(K_1)^2(C(\tau)\delta\sqrt{p}z_n^{1-\frac{1}{p}} + C(\tau)\delta\sqrt{p}z_n^{1-\frac{1}{p}})C(\tau)\delta\sqrt{p}z_n^{1-\frac{1}{p}} \\ &\leq 4(K_1)^2 C(\tau)\delta^2 p z_n^{2-\frac{2}{p}} \leq K(\tau)\delta^2 p z_n^{2-\frac{2}{p}}, \end{aligned}$$

where we have selected an equivalent norm on V. Here, the constants $C(\tau)$ and $K(\tau)$ denote generic constants that may vary from line to line. Returning to (3.20), we immediately obtain:

$$\begin{aligned}\left(\frac{\delta}{2^{n+1}}\right)^3 z_{n+1} &\leq \left(\int_\Omega |\phi_n|^4 dx\right)^{\frac{3}{4}} z_n^{\frac{1}{4}} \\ &\leq K(\tau)^{\frac{3}{4}}\delta^{\frac{3}{2}}p^{\frac{3}{4}}z_n^{\frac{7}{4}-\frac{3}{2p}}, \end{aligned} \tag{3.21}$$

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where we have chosen and fixed a generic value of $p>2$. In conclusion, we obtain:

$$z_{n+1} \le 2^{3n+3}\delta^{-\frac{3}{2}}K(\tau)^{\frac{3}{4}}p^{\frac{3}{4}}z_n^{\frac{7}{4}-\frac{3}{2p}}. \tag{3.22}$$

Thus, we can apply Lemma A.1. In particular, we have $b=2^3>1$, $C=2^3\delta^{-\frac{3}{2}}K(\tau)^{\frac{3}{4}}p^{\frac{3}{4}}>0$ and $\varepsilon=\frac{3}{4}-\frac{3}{2p}=\frac{3}{4}\frac{p-2}{p}$, which allows us to conclude that $z_n\to 0$, as long as:

$$z_0 \le C^{-\frac{4p}{3(p-2)}}b^{-\frac{16p^2}{9(p-2)^2}},$$

or, by combining some constants:

$$z_0 \le \frac{\delta^{\frac{2p}{p-2}}}{2^{\frac{4p}{p-2}+\frac{16p^2}{3(p-2)^2}}K(\tau)^{\frac{p}{p-2}}p^{\frac{p}{p-2}}} := C(\tau)\frac{\delta^{\frac{2p}{p-2}}}{p^{\frac{p}{p-2}}}, \tag{3.23}$$

where the constant C can be made independent of $p>2$ if we fix p sufficiently large. We are left with one final estimate: thanks to (2.3), recalling that F' is monotone in a neighbourhood of $+1$, we infer that for any $q\ge 2$ and $\delta>0$ sufficiently small:

$$\begin{aligned} z_0 &= \int_{A_0(t)} 1dx \le \int_{\{x\in\Omega:\ \phi(x,t)\ge 1-2\delta\}} 1dx \\ &\le \int_{A_0(t)} \frac{|F'(\phi)|^q}{F'(1-2\delta)^q}dx \le \frac{\int_\Omega |F'(\phi)|^q dx}{F'(1-2\delta)^q} \\ &\le \frac{C_1(\tau)^q(\sqrt{q})^q}{F'(1-2\delta)^q}. \end{aligned}$$

If we ensure that:

$$\frac{C_1(\tau)^q(\sqrt{q})^q}{F'(1-2\delta)^q} \le C(\tau)\frac{\delta^{\frac{2p}{p-2}}}{p^{\frac{p}{p-2}}},$$

then (3.23) holds. This can be obtained as follows: let us first recall that by assumption (2), there exists $C_F>0$ such that, for δ sufficiently small:

$$\frac{1}{F'(1-2\delta)} \le \frac{C_F}{|\ln(\delta)|^\beta}.$$

We now make the crucial step: we fix $\delta=e^{-q}$ for $q\ge 2$ sufficiently large. Then:

$$\frac{C_1(\tau)^q(\sqrt{q})^q}{F'(1-2\delta)^q} \le \frac{C_1(\tau)^q C_F^q(\sqrt{q})^q}{|\ln(\delta)|^{\beta q}} = \frac{C_1(\tau)^q C_F^q(\sqrt{q})^q}{q^{\beta q}} = \frac{C_1(\tau)^q C_F^q}{q^{q(\beta-\frac{1}{2})}}. \tag{3.24}$$

The condition (3.24), with this choice of δ, is thus ensured if:

$$\frac{C_1(\tau)^q C_F^q}{q^{q(\beta-\frac{1}{2})}} \le C(\tau)\frac{e^{-\frac{2qp}{p-2}}}{p^{\frac{p}{p-2}}},$$

or:

$$\frac{p^{\frac{p}{p-2}}}{C(\tau)} \le C_1(\tau)^{-q}C_F^{-q}e^{-\frac{2qp}{p-2}}q^{q(\beta-\frac{1}{2})}. \tag{3.25}$$

However, we have:

$$C_1(\tau)^{-q}C_F^{-q}e^{-\frac{2qp}{p-2}}q^{q(\beta-\frac{1}{2})} \to +\infty \qquad \text{as } q\to\infty.$$

Indeed, since $\beta>\frac{1}{2}$, it holds, for any $a_1>0$ fixed, as $q\to\infty$:

$$e^{-a_1 q}(C_1(\tau)C_F)^{-q}q^{q(\beta-\frac{1}{2})} = e^{-a_1 q - q\ln(C_1(\tau)C_F)+(\beta-\frac{1}{2})q\ln(q)} \to \infty.$$

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Therefore, by choosing q sufficiently large, corresponding to a sufficiently small $\delta = e^{-q}$, we can ensure (3.25), and thus (3.24), obtaining that $z_n \to 0$ as $n \to \infty$. Passing to the limit in z_n as $n \to \infty$, we have shown that, for any $t \geq \tau$:

$$\|(\phi(t) - (1-\delta))^+\|_{L^\infty(\Omega)} = 0,$$

since, as $n \to \infty$:

$$z_n(t) \to |\{x \in \Omega : \phi(x,t) \geq 1-\delta\}|,$$

and $z_n(t) \to 0$ as $n \to \infty$, for any fixed $t \geq \tau$. It is clear from (3.25) that δ depends on τ but *not* on the specific $t \geq \tau$, so that the uniform strict separation holds. We now repeat exactly the same argument, thanks to the essential assumption (2.3), for the case $(\phi - (-1+\delta))^-$ (using $\phi_n(t) = (\phi(t) + k_n)^-$ and testing (3.3) by $v = -\phi_n$). We can then choose the minimum between the δ obtained in the two cases. In the end, recalling the space-time continuity of ϕ (see Remark 3.2), we conclude that there exists $\delta(\tau) > 0$ (whose dependencies are seen in (3.25)) such that:

$$-1+\delta \leq \phi(x,t) \leq 1-\delta, \qquad \text{in } \overline{\Omega} \times [\tau, \infty).$$

The proof is thus concluded. □

4. The separation property for the nonlocal Cahn-Hilliard equation

In this section, we investigate the nonlocal Cahn-Hilliard equation:

$$\begin{cases} \partial_t \phi - \Delta \mu = 0, & \text{in } \Omega \times (0,\infty), \\ \mu = F'(\phi) - J * \phi, & \text{in } \Omega \times (0,\infty), \\ \partial_{\mathbf{n}} \mu = 0, & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{4.1}$$

where J is a suitably regular kernel. Specifically, we assume that $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$, is a sufficiently smooth bounded domain, and we require that:

(J) $J \in W^{1,1}_{loc}(\mathbb{R}^d)$, $d = 2, 3$, with $J(x) = J(-x)$.

We now present a well-known theorem, the proof of which can be found in references such as [14] and [9] (for additional references, see [29, Remark 3.4]).

Theorem 4.1. *Under the assumptions that F satisfies (1), J satisfies (**J**) and that $\phi_0 \in L^\infty(\Omega)$ with $\|\phi_0\|_{L^\infty} \leq 1$ and $|\overline{\phi}_0| = m < 1$, there exists a unique weak solution to (4.1). This solution satisfies the following properties for any $T > 0$:*

$$\phi \in L^\infty(\Omega \times (0,T)) \text{: for all } t > 0,\ |\phi(t)| < 1 \text{ almost everywhere in } \Omega,$$

$$\phi \in L^2(0,T;V) \cap H^1(0,T;H),$$

$$\mu \in L^2(0,T;V),\ F'(\phi) \in L^2(0,T;V),$$

subject to the following equations:

$$\langle \partial_t \phi, v \rangle_{V',V} + (\nabla \mu, \nabla v) = 0 \text{ for all } v \in V, \text{ almost everywhere in } (0,T), \tag{4.2}$$

$$\mu = F'(\phi) - J * \phi \text{ almost everywhere in } \Omega \times (0,T), \tag{4.3}$$

and with the initial condition $\phi(\cdot, 0) = \phi_0(\cdot)$ in Ω. Moreover, for any $\tau > 0$, the following estimates hold:

$$\sup_{t \geq \tau} \|\partial_t \phi(t)\|_{V'} + \sup_{t \geq \tau} \|\partial_t \phi\|_{L^2(t,t+1,H)} \leq \frac{C_4}{\sqrt{\tau}}, \tag{4.4}$$

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$$\sup_{t\geq\tau}\|\mu(t)\|_V+\sup_{t\geq\tau}\|\phi(t)\|_V\leq\frac{C_4}{\sqrt{\tau}},\tag{4.5}$$

$$\|F'(\phi)\|_{L^\infty(\tau,t;V)}+\|\mu\|_{L^2(t,t+1,V_2)}\leq C_5(\tau),\ \text{for all } t\geq\tau,\tag{4.6}$$

where the positive constant C_4 depends solely on the initial energy

$$\begin{aligned}\mathcal{E}_{NL}(\phi_0):=\frac{1}{4}\int_\Omega\int_\Omega J(x-y)\,(\phi_0(y)-\phi_0(x))^2\,\mathrm{d}x\mathrm{d}y\\ +\int_\Omega F(\phi_0(x))-\frac{(J*1)\,(x)}{2}\phi_0^2(x)\,\mathrm{d}x,\end{aligned}$$

$\overline{\phi}_0$, Ω *and the system's parameters. The constant $C_5(\tau)$ also depends on τ.*

Remark 4.2. For two-dimensional domains, according to (2.1) and (4.6), it can be readily observed that, for any $\tau>0$, the following inequality holds:

$$\|F'(\phi)\|_{L^\infty(\tau,\infty;L^p(\Omega))}\leq C_6(\tau)\sqrt{p},\tag{4.7}$$

where $C_6(\tau)$ depends on K_1 and $C_5(\tau)$ (also see [15, (34)]). Note that in this context the double-well potential is

$$\Psi\,(\phi)=F\,(\phi)-\frac{\alpha_0\,(x)}{2}\phi^2,\ \alpha_0\,(x)\ :=\ (J*1)\,(x)$$

while the internal energy reads

$$E_{int}\,(\phi_0)=\frac{1}{4}\int_\Omega\int_\Omega J(x-y)\,(\phi_0(y)-\phi_0(x))^2\,\mathrm{d}x\mathrm{d}y.$$

Our objective now is to establish the instantaneous strict separation property with minimal assumptions. As outlined in the Introduction, the validity of the instantaneous strict separation property for the nonlocal Cahn-Hilliard equation was initially proven in 2D in [14] (and later in [15] with a more relaxed set of assumptions), and in 3D (with proof applicable to 2D as well) in [29] (also see the subsequent work [21]). As previously mentioned, we present a simplified set of assumptions for the two-dimensional case. Specifically, we only require the additional assumption (2), which is significantly weaker than the assumptions in the aforementioned works. In particular, we have the following theorem.

Theorem 4.3. *Let $\Omega\subset\mathbb{R}^2$ be a smooth bounded domain, and assume that conditions (1)–(2) hold for the entropy potential F, along with assumption (**J**) for the kernel. Suppose $\phi_0\in L^\infty(\Omega)$ with $\|\phi_0\|_{L^\infty}\leq 1$ and $|\overline{\phi}_0|=m<1$. Then, for any $\tau>0$, there exists $\delta\in(0,1)$, depending on τ, Ω, m, the initial energy $E_{NL}(\phi_0)$ and the system's parameters, such that the unique weak solution to problem (4.1) satisfies*

$$|\phi(x,t)|\leq 1-\delta,\qquad \text{for almost every } (x,t)\in\Omega\times(\tau,+\infty).$$

We point out that Remark 3.4 holds also in this case.

Proof of Theorem 4.3. The proof of this theorem closely follows the approach taken in [15, Theorem 4.1] and [29, Theorem 4.3]. Therefore, we will primarily emphasise the key distinctions while omitting some of the finer details. Our proof relies on a De Giorgi's iteration scheme. Let us begin by selecting $\tau>0$ and choosing an arbitrary $\widetilde{\tau}>0$ such that it satisfies the condition:

$$2\widetilde{\tau}+\frac{\tau}{2}\leq\tau,\tag{4.8}$$

For example, we can set $\widetilde{\tau}:=\frac{\tau}{4}$, making it a function solely of τ. Next, we fix $T>0$ in a way that ensures $T-3\widetilde{\tau}\geq\frac{\tau}{2}$. One possible choice is to start with $T=3\widetilde{\tau}+\frac{\tau}{2}$. Now, we introduce a parameter $\delta\in(0,1)$ (to be determined later). We define the sequence k_n as in (3.10), recalling that it satisfies:

$$1-2\delta<k_n<k_{n+1}<1-\delta,\qquad\forall n\geq 1,\qquad k_n\to 1-\delta\qquad\text{as } n\to\infty,\tag{4.9}$$

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Additionally, we define a sequence of times as follows:

$$\begin{cases} t_{-1} = T - 3\widetilde{\tau}, \\ t_n = t_{n-1} + \frac{\widetilde{\tau}}{2^n}, \qquad n \geq 0, \end{cases}$$

These times satisfy:

$$t_{-1} < t_n < t_{n+1} < T - \widetilde{\tau}, \qquad \forall n \geq 0.$$

To aid in our analysis, we introduce a cut-off function $\eta_n \in C^1(\mathbb{R})$ defined as:

$$\eta_n(t) := \begin{cases} 0, & t \leq t_{n-1}, \\ 1, & t \geq t_n, \end{cases} \quad \text{and} \quad |\eta_n'(t)| \leq \frac{2^{n+1}}{\widetilde{\tau}}. \tag{4.10}$$

With these preparations in place, we define ϕ_n as in (3.12). For any $n \geq 0$, we also introduce the interval $I_n = [t_{n-1}, T]$ and the set:

$$A_n(t) := \{x \in \Omega : \phi(x,t) - k_n \geq 0\}, \quad \forall t \in I_n.$$

It is evident that:

$$I_{n+1} \subseteq I_n, \qquad \forall n \geq 0,$$

$$A_{n+1}(t) \subseteq A_n(t), \qquad \forall n \geq 0, \qquad \forall t \in I_{n+1}.$$

In conclusion, we define the sequence y_n as follows:

$$y_n = \int_{I_n} \int_{A_n(s)} 1 dxds, \qquad \forall n \geq 0.$$

For each $n \geq 0$, we consider the test function $v = \phi_n \eta_n^2$ and integrate it over the interval $[t_{n-1}, t]$, where $t_n \leq t \leq T$. This yields (as shown in [15, 21, 29]):

$$\int_{t_{n-1}}^t < \partial_t \phi, \phi_n \eta_n^2 >_{V',V} ds + \int_{t_{n-1}}^t \int_{A_n(s)} F''(\phi) \nabla\phi \cdot \nabla\phi_n \eta_n^2 dxds$$
$$= \int_{t_{n-1}}^t \int_{A_n(s)} \eta_n^2 (\nabla J * \phi) \cdot \nabla \phi_n dxds. \tag{4.11}$$

We can then use assumption (1) to derive the following inequality:

$$\int_{t_{n-1}}^t \eta_n^2 \int_{A_n(s)} F''(\phi)\nabla\phi \cdot \nabla\phi_n dxds \geq \alpha \int_{t_{n-1}}^t \eta_n^2 \|\nabla\phi_n\|^2 ds, \tag{4.12}$$

And for the right-hand side of Equation (4.11), given that $|\phi| < 1$ almost everywhere in $\Omega \times (0, +\infty)$, we find:

$$\begin{aligned} &\int_{t_{n-1}}^t \int_{A_n(s)} (\nabla J * \phi) \cdot \nabla \phi_n \eta_n^2 dxds \\ &\leq \frac{\alpha}{2} \int_{t_{n-1}}^t \eta_n^2 \|\nabla\phi_n\|^2 ds + \frac{1}{2\alpha} \int_{t_{n-1}}^t \int_{A_n(s)} \eta_n^2 |\nabla J * \phi|^2 dxds \\ &\leq \frac{\alpha}{2} \int_{t_{n-1}}^t \eta_n^2 \|\nabla\phi_n\|^2 ds + \frac{1}{2\alpha} \int_{t_{n-1}}^t \|\nabla J * \phi\|^2_{L^\infty(\Omega)} \int_{A_n(s)} 1 dxds \\ &\leq \frac{\alpha}{2} \int_{t_{n-1}}^t \eta_n^2 \|\nabla\phi_n\|^2 ds + \frac{\|\nabla J\|^2_{L^1(B_r)}}{2\alpha} y_n, \end{aligned} \tag{4.13}$$

where we have utilised the inequality (see, e.g., [2, Thm. 4.33]):

$$\|\nabla J * \phi\|_{L^\infty(\Omega)} \leq \|\nabla J\|_{L^1(B_r)} \|\phi\|_{L^\infty(\Omega)} \leq \|\nabla J\|_{L^1(B_r)}, \tag{4.14}$$

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Here, B_r represents a ball centred at $\mathbf{0}$ with radius $r > 0$, chosen sufficiently large such that $x - \Omega \subset B_r$ for any $x \in \Omega$. Additionally, we have the expression:

$$\int_{t_{n-1}}^{t} \left\langle \partial_t \phi, \phi_n \eta_n^2 \right\rangle_{V',V} ds = \frac{1}{2}\|\phi_n(t)\|^2 - \int_{t_{n-1}}^{t} \|\phi_n(s)\|^2 \eta_n \partial_t \eta_n ds. \tag{4.15}$$

Since $|\phi| < 1$ a.e. in Ω, for any $t \geq \frac{\tau}{2}$, we have $0 \leq \phi_n \leq 2\delta$ a.e. in Ω, $\forall t \geq \frac{\tau}{2}$ (as shown in [29]). Therefore, applying the above inequality, we obtain, as in [29, (4.27)]:

$$\int_{t_{n-1}}^{t} \|\phi_n(s)\|^2 \eta_n \partial_t \eta_n ds \leq \frac{2^{n+3}\delta^2}{\widetilde{\tau}} y_n. \tag{4.16}$$

By inserting Equations (4.12), (4.13), (4.15) and (4.16) into equation (4.11), we obtain:

$$\begin{aligned}
&\frac{1}{2}\|\phi_n(t)\|^2 + \frac{\alpha}{2}\int_{t_{n-1}}^{t} \eta_n^2 \|\nabla \phi_n(s)\|^2 ds \\
&\leq 2^{n+1} \max\left\{ \frac{\|\nabla J\|^2_{L^1(B_r)}}{2\alpha}, \frac{8\delta^2}{\widetilde{\tau}} \right\} y_n \leq 2^{n+1} \frac{\|\nabla J\|^2_{L^1(B_r)}}{2\alpha} y_n,
\end{aligned}$$

This holds for any $t \in [t_n, T]$, provided that δ is chosen sufficiently small. Specifically, we need δ to satisfy:

$$\frac{8\delta^2}{\widetilde{\tau}} \leq \frac{\|\nabla J\|^2_{L^1(B_r)}}{2\alpha}. \tag{4.17}$$

As a result, we obtain the following inequalities:

$$\max_{t \in I_{n+1}} \|\phi_n(t)\|^2 \leq X_n, \qquad \alpha \int_{I_{n+1}} \|\nabla \phi_n\|^2 ds \leq X_n, \tag{4.18}$$

where

$$X_n := 2^{n+1} \frac{\|\nabla J\|^2_{L^1(B_r)}}{\alpha} y_n.$$

On the other hand, for any $t \in I_{n+1}$ and for almost any $x \in A_{n+1}(t)$, we have, as shown in (3.19),

$$\phi_n(x,t) \geq \frac{\delta}{2^{n+1}},$$

implying

$$\int_{I_{n+1}} \int_{\Omega} |\phi_n|^3 dx ds \geq \int_{I_{n+1}} \int_{A_{n+1}(s)} |\phi_n|^3 dx ds \geq \left(\frac{\delta}{2^{n+1}}\right)^3 \int_{I_{n+1}} \int_{A_{n+1}(s)} 1 dx ds = \left(\frac{\delta}{2^{n+1}}\right)^3 y_{n+1}.$$

Thus, by applying Hölder's inequality, we get:

$$\begin{aligned}
\left(\frac{\delta}{2^{n+1}}\right)^3 y_{n+1} &\leq \left(\int_{I_{n+1}} \int_{\Omega} |\phi_n|^4 dx ds\right)^{\frac{3}{4}} \left(\int_{I_{n+1}} \int_{A_n(s)} 1 dx ds\right)^{\frac{1}{4}} \\
&\leq (K_1)^{\frac{3}{2}} \left(1 + \frac{1}{\alpha}\right)^{\frac{3}{4}} \frac{2^{\frac{3n}{2}+\frac{3}{2}} \|\nabla J\|^3_{L^1(B_r)}}{\alpha^{\frac{3}{2}}} y_n^{\frac{7}{4}}.
\end{aligned} \tag{4.19}$$

In conclusion, we can write:

$$y_{n+1} \leq \delta^{-3} (K_1)^{\frac{3}{2}} \left(1 + \frac{1}{\alpha}\right)^{\frac{3}{4}} \frac{2^{\frac{9n}{2}+\frac{9}{2}} \|\nabla J\|^3_{L^1(B_r)}}{\alpha^{\frac{3}{2}}} y_n^{\frac{7}{4}}, \qquad \forall n \geq 0, \tag{4.20}$$

where we introduce the constant:

$$C_\alpha := (K_1)^{\frac{3}{2}} \left(1 + \frac{1}{\alpha}\right)^{\frac{3}{4}} \frac{2^{\frac{9}{2}} \|\nabla J\|^3_{L^1(B_r)}}{\alpha^{\frac{3}{2}}}.$$

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Therefore, we can apply Lemma A.1. In particular, we have $z_n = y_n$, $b = 2^{\frac{9}{2}} > 1$, $C = \delta^{-3} C_\alpha > 0$, $\varepsilon = \frac{3}{4}$, and we conclude that $y_n \to 0$, as long as

$$y_0 \leq C^{-\frac{4}{3}} b^{-\frac{16}{9}},$$

i.e.,

$$y_0 \leq \frac{2^{-8}\delta^4}{C_\alpha^{\frac{4}{3}}}. \tag{4.21}$$

We are now faced with a final estimate, distinct from those found in [15, 21, 29]. We proceed in a manner akin to the proof of Theorem 3.3 presented earlier. Specifically, as F' is monotonically increasing, we have:

$$\begin{aligned}
y_0 &= \int_{I_0} \int_{A_0(t)} 1 dxds \\
&\leq \int_{I_0} \int_{\{x\in\Omega:\ \phi(x,t)\geq 1-2\delta\}} 1 dxds \\
&\leq \int_{I_0} \int_{A_0(t)} \frac{|F'(\phi)|^q}{F'(1-2\delta)^q} dxds \\
&\leq \frac{3\widetilde{\tau}\|F'(\phi)\|^q_{L^\infty(\tau,\infty;L^q(\Omega))}}{F'(1-2\delta)^q} \\
&\leq \frac{3\widetilde{\tau} C_6(\tau)^q (\sqrt{q})^q}{F'(1-2\delta)^q},
\end{aligned}$$

where we have used (4.7). Importantly, under assumption (2), there exists $C_F > 0$ such that, for sufficiently small δ:

$$\frac{1}{F'(1-2\delta)} \leq \frac{C_F}{|\ln(\delta)|^\beta}.$$

Now, let us set $\delta = e^{-q}$, with $q \geq 2$ sufficiently large. The estimate for y_0 becomes:

$$y_0 \leq \frac{3\widetilde{\tau} C_6(\tau)^q (\sqrt{q})^q}{F'(1-2\delta)^q} \leq \frac{3\widetilde{\tau} C_F^q C_6(\tau)^q (\sqrt{q})^q}{|\ln(\delta)|^{\beta q}} = \frac{3\widetilde{\tau} C_F^q C_6(\tau)^q}{q^{q(\beta-\frac{1}{2})}}.$$

To ensure (4.21) with $\delta = e^{-q}$, we must assume:

$$\frac{3\widetilde{\tau} C_F^q C_6(\tau)^q}{q^{q(\beta-\frac{1}{2})}} \leq \frac{2^{-8} e^{-4q}}{C_\alpha^{\frac{4}{3}}}, \tag{4.22}$$

which leads to:

$$3\widetilde{\tau}\left(2^8 C_\alpha^{\frac{4}{3}}\right) \leq e^{-4q} C_F^{-q} C_6(\tau)^{-q} q^{q(\beta-\frac{1}{2})}. \tag{4.23}$$

It's important to note that:

$$e^{-4q} C_F^{-q} C_6(\tau)^{-q} q^{q(\beta-\frac{1}{2})} \to \infty \quad \text{as } q \to \infty,$$

since $\beta > \frac{1}{2}$. Therefore, to satisfy (4.23), and thus (4.21), it is sufficient to choose a sufficiently large value for q. In summary, with this choice of q, corresponding to $\delta = e^{-q}$, and by taking the limit as $n \to \infty$, we conclude that:

$$\|(\phi - (1-\delta))^+\|_{L^\infty(\Omega\times(T-\widetilde{\tau},T))} = 0,$$

since, as $n \to \infty$,

$$y_n \to |\{(x,t) \in \Omega \times [T-\widetilde{\tau}, T] : \phi(x,t) \geq 1-\delta\}|,$$

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and $y_n \to 0$ as $n \to \infty$. We can then repeat the same argument, with the same T and $\widetilde{\tau}$ fixed, for the case $(\phi - (-1+\delta))^-$ (using $\phi_n(t) = (\phi(t) + k_n)^-$). The argument remains exactly the same due to assumption (2.3). We can choose the minimum of the δ obtained in the two cases, ensuring that:

$$-1 + \delta \le \phi(x,t) \le 1 - \delta, \quad \text{a.e. in } \Omega \times (T - \widetilde{\tau}, T).$$

In conclusion, since $T - \widetilde{\tau} = 2\widetilde{\tau} + \frac{\tau}{2} \le \tau$, we can repeat the same procedure on the interval $(T, T + \widetilde{\tau})$ (with a new starting time at $t_{-1} = T - 2\widetilde{\tau} \ge \frac{\tau}{2}$) and so on, eventually covering the entire interval $[\tau, +\infty)$. Notably, δ remains constant throughout the interval $[\tau, +\infty)$, and the time horizon T does not affect any of the estimates. The dependencies of δ on Ω, τ, m, $\mathcal{E}_{NL}(\phi_0)$ and the parameters of the problem can be deduced from the smallness assumptions (4.17) and (4.23). This concludes the proof. □

5. The separation property for the fractional Cahn-Hilliard equation

In this section, we consider the fractional Cahn-Hilliard equation, often referred to as (5.1):

$$\begin{cases} \partial_t \phi - \Delta \mu = 0, & \text{in } \Omega \times (0,\infty), \\ \mu = A_N^s \phi + \Psi'(\phi), & \text{in } \Omega \times (0,\infty), \\ \partial_{\mathbf{n}} \mu = 0, & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{5.1}$$

Here, $s \in (0,1)$ and A_N is the homogeneous Neumann Laplacian operator defined in Section 2. The primary result regarding the well-posedness and instantaneous regularisation of weak solutions to (5.1) is established in [7] for weak solutions and [7, Theorems 5.1–5.4] for strong solutions and regularisation. Notably, the essential estimate (5.5) below aligns with [15, (85)], and its proof relies on (2.2). Specifically, we have:

Theorem 5.1. *For $s \in (0,1)$, assuming F satisfies (1), and given that $\phi_0 \in \mathfrak{D}(A_N^{\frac{s}{2}}) \cap L^\infty(\Omega)$ such that $\|\phi_0\|_{L^\infty} \le 1$ and $|\overline{\phi}_0| = m < 1$, there exists a unique weak solution to (4.1). This solution satisfies, for any $T > 0$:*

$$\begin{aligned} &\phi \in L^\infty(\Omega \times (0,T)): \quad \forall t > 0, \quad |\phi(t)| < 1, \quad \text{a.e. in } \Omega, \\ &\phi \in L^\infty(0,T;\mathfrak{D}(A_N^{\frac{s}{2}})) \cap L^4(0,T;D(A_N^s)) \cap L^{2(1+s)}(0,T;\mathfrak{D}(A_N^{\frac{1+s}{2}})) \cap H^1(0,T;V'), \\ &\mu \in L^2(0,T;V), \end{aligned}$$

such that:

$$\langle \partial_t \phi, v \rangle_{V',V} + (\nabla \mu, \nabla v) = 0, \quad \forall v \in V, \quad \text{a.e. in } (0,T), \tag{5.2}$$

$$\mu = \Psi'(\phi) + A_N^s \phi, \quad \text{a.e. in } \Omega \times (0,T), \tag{5.3}$$

and $\phi(\cdot,0) = \phi_0(\,\cdot\,)$ in Ω. Furthermore, for any $\tau > 0$:

$$\sup_{t \ge \tau} \|\partial_t \phi\|_{L^2(t,t+1,V)} \le \frac{C_7}{\sqrt{\tau}}, \quad \forall t \ge \tau, \tag{5.4}$$

$$\|F'(\phi)\|_{L^\infty(\tau,t;L^p(\Omega))} + \|\mu\|_{L^\infty(\tau,\infty,L^p(\Omega))} \le C_8(\tau)\sqrt{p}, \quad \forall p \in [2,\infty), \tag{5.5}$$

Here, the positive constant C_7 depends only on the initial datum energy

$$\mathcal{E}_{SF}(\phi_0) = \int_\Omega \frac{1}{2} |A_N^{\frac{s}{2}} \phi_0(x)|^2 + \Psi(\phi_0(x)) \mathrm{d}x,$$

the domain Ω, $\overline{\phi}_0$ and the parameters of the system, while $C_8(\tau)$ also depends on τ, but not on p.

We present a novel proof of the instantaneous strict separation property in 2D for problem (5.1). The validity of the strict separation property in two-dimensional bounded domains has been established in

[15, Theorem 5.4] under the assumption of a pointwise (exponential) relation between F'' and F'. As observed in Remark 2.1, here we only require a condition on F' near its endpoints ± 1.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain, $s \in (0,1)$, and F satisfying (1)–(2). Assume that $\phi_0 \in \mathfrak{D}(A_N^{\frac{s}{2}}) \cap L^\infty(\Omega)$ with $\|\phi_0\|_{L^\infty} \leq 1$ and $|\overline{\phi}_0| = m < 1$. Then, for any $\tau > 0$, there exists $\delta \in (0,1)$, depending on s, τ, Ω, m, the initial energy $\mathcal{E}_{SF}(\phi_0)$, and the system parameters, such that the unique weak solution to problem (5.1) satisfies:*

$$|\phi(x,t)| \leq 1 - \delta, \quad \textit{for almost every } (x,t) \in \Omega \times (\tau, +\infty).$$

Remark 3.4 is also applicable in this case.

Proof of Theorem 5.2. The proof closely follows the one for Theorem 3.3, with slight differences to account for the fractional Laplacian case. Using the same notation as in the previous proof, we test equation (5.3) by $v = \phi_n$, where ϕ_n is defined in (3.12). Utilising Lemma A.2 and estimates (3.15)–(3.17), which still apply, we arrive at:

$$\left\| A_N^{\frac{s}{2}} \phi_n \right\|^2 + (F'(1-2\delta) - \alpha_0) \int_\Omega \phi_n dx + \alpha \int_\Omega \phi_n^2 dx \leq C(\tau)\delta \sqrt{p} z_n^{1-\frac{1}{p}}, \quad \text{for } p \geq 2. \tag{5.6}$$

This holds for any $t \geq \tau$. Furthermore, it is recalled that (see (3.19)) for any $t \in [\tau, \infty)$ and almost $x \in A_{n+1}(t)$, it holds:

$$\phi_n(x,t) \geq \frac{\delta}{2^{n+1}},$$

which implies:

$$\int_\Omega |\phi_n|^2 dx \geq \int_{A_{n+1}(t)} |\phi_n|^2 dx \geq \left(\frac{\delta}{2^{n+1}}\right)^2 \int_{A_{n+1}(t)} 1 dx = \left(\frac{\delta}{2^{n+1}}\right)^2 z_{n+1}.$$

Using Hölder's inequality and selecting $2 < \gamma < \frac{2}{1-s}$, we obtain:

$$\begin{aligned}\left(\frac{\delta}{2^{n+1}}\right)^2 z_{n+1} &\leq \int_\Omega |\phi_n|^2 dx \\ &= \int_{A_n(t)} |\phi_n|^2 dx \leq \left(\int_\Omega |\phi_n|^\gamma dx\right)^{\frac{2}{\gamma}} \left(\int_{A_n(t)} 1 dx\right)^{1-\frac{2}{\gamma}}.\end{aligned} \tag{5.7}$$

By the Sobolev-Gagliardo-Nirenberg-type inequality (2.2) (with $p = \gamma$), we then get, setting $\vartheta := \frac{2}{\gamma s} - \frac{1}{s} + 1 \in (0,1)$, by (5.6):

$$\begin{aligned}\int_\Omega |\phi_n|^\gamma dx &\leq (K_2)^\gamma \left(\|A_N^{\frac{s}{2}}\phi_n\|^2 + \|\phi_n\|^2\right)^{\frac{\gamma(1-\vartheta)}{2}} \|\phi_n\|^{\gamma\vartheta} \\ &\leq (K_2)^\gamma \left(C(\tau)\delta \sqrt{p} z_n^{1-\frac{1}{p}} + C(\tau)\delta \sqrt{p} z_n^{1-\frac{1}{p}}\right)^{\frac{\gamma(1-\vartheta)}{2}} C(\tau)^{\frac{\gamma\vartheta}{2}} \delta^{\frac{\gamma\vartheta}{2}} (\sqrt{p})^{\frac{\gamma\vartheta}{2}} z_n^{\frac{\gamma\vartheta}{2}(1-\frac{1}{p})} \\ &\leq K(\tau)\delta^{\frac{\gamma}{2}} p^{\frac{\gamma}{4}} z_n^{\frac{\gamma}{2}(1-\frac{1}{p})}.\end{aligned}$$

Here again, the constants $C(\tau), K(\tau)$ represent generic constants that may vary from line to line. Returning to (5.7), we immediately deduce:

$$\begin{aligned}\left(\frac{\delta}{2^{n+1}}\right)^2 z_{n+1} &\leq \left(\int_\Omega |\phi_n|^\gamma dx\right)^{\frac{2}{\gamma}} z_n^{1-\frac{2}{\gamma}} \\ &\leq K(\tau)^{\frac{2}{\gamma}} \delta \sqrt{p} z_n^{2-\frac{1}{p}-\frac{2}{\gamma}},\end{aligned} \tag{5.8}$$

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Here, we choose and fix a generic $p > 2$ such that:

$$2 - \frac{1}{p} - \frac{2}{\gamma} > 1,$$

which implies $p > \frac{\gamma}{\gamma - 2}$. In conclusion, we arrive at:

$$z_{n+1} \leq 2^{2n+2} K(\tau)^{\frac{2}{\gamma}} \delta^{-1} \sqrt{p} z_n^{2 - \frac{1}{p} - \frac{2}{\gamma}}.$$

Therefore, we can apply Lemma A.1. Specifically, with $b = 2^2 > 1$, $C = 2^2 K(\tau)^{\frac{2}{\gamma}} \delta^{-1} \sqrt{p} > 0$ and $\varepsilon = 1 - \frac{1}{p} - \frac{2}{\gamma}$, we obtain:

$$z_n \to 0,$$

as long as:

$$z_0 \leq C^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}.$$

Incorporating various constants and recalling that p and γ are fixed and depend only on s, we arrive at:

$$z_0 \leq C(\tau, s) \delta^{\frac{1}{1 - \frac{1}{p} - \frac{2}{\gamma}}}.$$

The last estimate is again similar to the proof of Theorem 3.3. Considering that F' is monotonically increasing, we deduce for any $q \geq 2$:

$$z_0 = \int_{A_0(t)} 1 dx \leq \int_{A_0(t)} \frac{|F'(\phi)|^q}{F'(1 - 2\delta)^q} dx \leq \frac{C_8(\tau)^q (\sqrt{q})^q}{F'(1 - 2\delta)^q}, \tag{5.9}$$

where we have used the essential (5.5). Therefore, if we ensure that:

$$\frac{C_8(\tau)^q (\sqrt{q})^q}{F'(1 - 2\delta)^q} \leq C(\tau, s) \delta^{\frac{1}{1 - \frac{1}{p} - \frac{2}{\gamma}}},$$

then (5.9) holds. This can be obtained as follows: according to assumption (2), there exists $C_F > 0$ such that, for sufficiently small δ:

$$\frac{1}{F'(1 - 2\delta)} \leq \frac{C_F}{|\ln(\delta)|^\beta}.$$

Now, we fix $\delta = e^{-q}$ for $q \geq 2$ sufficiently large. We then have:

$$\frac{C_8(\tau)^q C_F^q}{q^{q(\beta - \frac{1}{2})}} \leq \frac{C_8(\tau)^q C_F^q}{q^{q(\beta - \frac{1}{2})}}. \tag{5.10}$$

The condition (5.10), with this choice of δ, is then ensured if:

$$\frac{C_8(\tau)^q C_F^q}{q^{q(\beta - \frac{1}{2})}} \leq C(\tau, s) e^{-\frac{q}{1 - \frac{1}{p} - \frac{2}{\gamma}}}, \tag{5.11}$$

or:

$$\frac{1}{C(\tau, s)} \leq C_8(\tau)^{-q} C_F^{-q} e^{-\frac{q}{1 - \frac{1}{p} - \frac{2}{\gamma}}} q^{q(\beta - \frac{1}{2})}.$$

It is clear that:

$$C_8(\tau)^{-q} C_F^{-q} e^{-\frac{q}{1 - \frac{1}{p} - \frac{2}{\gamma}}} q^{q(\beta - \frac{1}{2})} \to \infty \quad \text{as } q \to \infty,$$

so that, by choosing q sufficiently large (corresponding to a sufficiently small $\delta = e^{-q}$), we can ensure (5.11), and thus (5.10), thereby obtaining that $z_n \to 0$ as $n \to \infty$. Continuing with this process for the case $(\phi - (-1 + \delta))^-$ (by using $\phi_n(t) = (\phi(t) + k_n)^-$ and testing (5.3) by $v = -\phi_n$), we can choose the

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minimum δ obtained from the two cases. This ensures that there exists $\delta(\tau) > 0$ (with dependencies explained in (5.11)) such that:

$$-1 + \delta \le \phi(x, t) \le 1 - \delta, \quad \text{a.e. in } \Omega \times [\tau, \infty).$$

The proof is thus concluded. □

6. Remarks about the 3D cases

6.1. Instantaneous strict separation

In this section, we present some insights on the possibility of applying similar techniques in the case of three-dimensional bounded domains, to show the validity of the strict separation property under general assumptions on the entropy function.

6.1.1. The local Cahn-Hilliard equation

Assume that $\phi_0 \in V \cap L^\infty(\Omega)$ with $\|\phi_0\|_{L^\infty} \le 1$ and $|\overline{\phi}_0| = m < 1$. In the case $d = 3$, i.e., for $\Omega \subset \mathbb{R}^3$, the proof presented in Section 3 is not directly applicable. However, with slight adaptations, the proof can be extended under more restrictive assumptions on the entropy potential F. To see this, let us first assume additionally that

(3) There exists $\xi \in (0, 1)$ such that F'' is nondecreasing in $[1 - \xi, 1)$ and non-increasing in $(-1, -1 + \xi]$.

We can now repeat almost word by word the proof of Theorem 3.3, with some slight changes. In particular, with the same notation, under the additional assumption (1) we also have, from (3.14),

$$\int_\Omega F'(\phi)\phi_n dx = \int_{A_n(t)} F'(\phi)\phi_n dx \ge F'(1 - 2\delta) \int_\Omega \phi_n dx + F''(1 - 2\delta) \int_\Omega \phi_n^2 dx. \tag{6.1}$$

Following the same aforementioned proof, we then end up with

$$\|\nabla\phi_n\|^2 + (F'(1 - 2\delta) - \alpha_0) \int_\Omega \phi_n dx + F''(1 - 2\delta) \int_\Omega \phi_n^2 dx \le C(\tau)\delta z_n^{\frac{5}{6}}, \tag{6.2}$$

for δ sufficiently small, so that, also, $F'(1 - 2\delta) - \alpha_0 \ge 0$. Clearly, in this case, we have chosen $p = 6$ as a maximum (see (3.7)). We now only need to adapt the Sobolev-Gagliardo-Nirenberg's inequality to three-dimensional bounded domains: we have

$$\int_\Omega |\phi_n|^{\frac{10}{3}} dx \le C\|\phi_n\|^{\frac{4}{3}} \|\nabla\phi_n\|^2.$$

Therefore, adapting (3.20),

$$\left(\frac{\delta}{2^{n+1}}\right)^3 z_{n+1} \le \int_{A_n(t)} |\phi_n|^3 dx \le \left(\int_\Omega |\phi_n|^{\frac{10}{3}} dx\right)^{\frac{9}{10}} \left(\int_{A_n(t)} 1 dx\right)^{\frac{1}{10}}. \tag{6.3}$$

Utilising Sobolev-Gagliardo-Nirenberg's inequality and taking into account (6.2), we can write:

$$\int_\Omega |\phi_n|^{\frac{10}{3}} dx \le C\left(\|\phi_n\|^2 + \|\nabla\phi_n\|^2\right) \|\phi_n\|^{\frac{4}{3}} \le \frac{K(\tau)\delta^{\frac{5}{3}}}{F''(1 - 2\delta)^{\frac{2}{3}}} z_n^{\frac{25}{18}},$$

and thus we conclude

$$z_{n+1} \le \frac{2^{3n+3}\delta^{-\frac{3}{2}} K(\tau)^{\frac{9}{10}}}{F''(1 - 2\delta)^{\frac{3}{5}}} z_n^{\frac{27}{20}}. \tag{6.4}$$

We then apply Lemma A.1 with $b = 2^3 > 1$, $C = \frac{2^3 \delta^{-\frac{3}{2}} K(\tau)^{\frac{9}{10}}}{F''(1-2\delta)^{\frac{3}{5}}} > 0$ and $\varepsilon = \frac{20}{7}$, which allows us to conclude that $z_n \to 0$, as long as

$$z_0 \leq \frac{\delta^{\frac{30}{7}}}{2^{3\frac{400}{49}+\frac{60}{7}} K(\tau)^{\frac{18}{7}}} := C(\tau)\delta^{\frac{30}{7}} F''(1-2\delta)^{\frac{12}{7}}. \tag{6.5}$$

Now, for $\delta > 0$ sufficiently small, recalling (3.7),

$$\begin{aligned} z_0 &= \int_{A_0(t)} 1dx \leq \int_{\{x\in\Omega:\ \phi(x,t)\geq 1-2\delta\}} 1dx \\ &\leq \int_{A_0(t)} \frac{|F'(\phi)|^6}{F'(1-2\delta)^6} dx \leq \frac{C}{F'(1-2\delta)^6}, \end{aligned} \tag{6.6}$$

so that, to ensure (6.5), we need that

$$\frac{C}{F'(1-2\delta)^6} \leq C(\tau)\delta^{\frac{30}{7}} F''(1-2\delta)^{\frac{12}{7}}. \tag{6.7}$$

In order to satisfy this condition, and thus to conclude that the strict separation holds, we refer to the Tsallis' entropy class presented in the Introduction, which stands out as a natural generalisation of the BG functional (logarithmic), in the sense that the latter can be obtained from the former in a suitable limit as $q \to 1$. Namely, for given $q \in \mathbb{R}_+$, we define the q-logarithm of a real number $r > 0$, as

$$\ln_{\{q\}} r := \begin{cases} \ln r, & \text{if } q = 1, \\ \frac{r^{1-q}-1}{1-q}, & \text{if } q > 0,\ q \neq 1. \end{cases}$$

By defining $g_q(x) := x \ln_{\{q\}} \frac{1}{x}$, the associated mixing potential is

$$F_q(x) := \begin{cases} (1+x)\ln(1+x) + (1-x)\ln(1-x), & q = 1, \\ -g_q(1+x) - g_q(1-x), & q \neq 1. \end{cases} \tag{6.8}$$

Notice that

$$F'_q(x) := \begin{cases} \ln(1+x) - \ln(1-x), & q = 1, \\ \frac{q}{q-1}(1+x)^{q-1} - \frac{q}{q-1}(1-x)^{q-1}, & q \neq 1, \end{cases} \tag{6.9}$$

and

$$F''_q(x) := \begin{cases} \frac{1}{1+x} + \frac{1}{1-x}, & q = 1, \\ q(1+x)^{q-2} + q(1-x)^{q-2}, & q \neq 1. \end{cases} \tag{6.10}$$

Therefore, it is clear that if we assume $q \in \left[0, \frac{2}{3}\right)$ then condition (6.7) is satisfied for F_q, with δ sufficiently small, and thus the instantaneous strict separation property, i.e., (3.9), holds for three-dimensional bounded domains. This result is very similar to the one obtained in [27] by means of a completely different argument. The challenge of establishing the instantaneous separation property for the natural logarithmic choice of F (i.e., F_q when $q = 1$) remains an open problem. However, De Giorgi's iterations could potentially contribute to the solution of this open problem in three dimensions.

6.1.2. The nonlocal Cahn-Hilliard equation and the fractional Cahn-Hilliard equation

As already observed, the validity of the strict separation property for three-dimensional bounded domains for the nonlocal Cahn-Hilliard equation has been first proven in [29] under the following assumptions on F, more restrictive than (2), but nevertheless accounting for the logarithmic potential (i.e., F_q with $q = 1$): as $\delta \to 0^+$,

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$$\frac{1}{F'(1-2\delta)} = O\left(\frac{1}{|\ln(\delta)|}\right), \quad \frac{1}{F''(1-2\delta)} = O(\delta) \tag{6.11}$$

and, similarly,

$$\frac{1}{|F'(-1+2\delta)|} = O\left(\frac{1}{|\ln(\delta)|}\right), \quad \frac{1}{F''(-1+2\delta)} = O(\delta). \tag{6.12}$$

Additionally, we need to assume that there exists $\xi \in (0,1)$ such that F'' is nondecreasing in $[1-\xi, 1)$ and non-increasing in $(-1, -1+\xi]$.

For the fractional Cahn-Hilliard equation, the strict separation for the logarithmic potential in three-dimensional bounded domains is still an open problem. In any case, we can make some remarks as for the local Cahn-Hilliard equation. In particular, by assuming the Tsallis's entropy, i.e., $F = F_q$, defined in (6.8), we can prove the instantaneous strict separation property when $s \in \left(\frac{1}{4}, 1\right)$ and $q \in \left[0, \frac{12-2\gamma(\vartheta+2)}{12-\gamma(\vartheta+5)}\right)$, for any $\gamma \in \left[2, \frac{6}{3-2s}\right)$ and $\vartheta := \frac{3}{\gamma s} - \frac{3}{2s} + 1$. Indeed, similarly to the local case, exploiting (6.1) and Lemma A.2, we end up with

$$\|A_N^{\frac{s}{2}}\phi_n\|^2 + (F'(1-2\delta) - \alpha_0)\int_\Omega \phi_n dx + F''(1-2\delta)\int_\Omega \phi_n^2 dx \le C(\tau)\delta z_n^{\frac{5}{6}}, \tag{6.13}$$

for δ sufficiently small, so that, also, $F'(1-2\delta) - \alpha_0 \ge 0$. We now only need to adapt the Sobolev-Gagliardo-Nirenberg's inequality to three dimensional bounded domains and then proceed as in the proof of Theorem 5.2: we have (see, e.g., [3])

$$\int_\Omega |\phi_n|^\gamma dx \le C\|\phi_n\|^\vartheta \|\nabla\phi_n\|^{1-\vartheta}, \quad \forall s \in (0,1), \quad \forall \gamma \in \left[2, \frac{6}{3-2s}\right), \tag{6.14}$$

with $\vartheta = \frac{3}{\gamma s} - \frac{3}{2s} + 1$. Therefore, from (6.13) we get, for a fixed $\gamma \in \left(2, \frac{6}{3-2s}\right)$,

$$\int_\Omega |\phi_n|^\gamma dx \le C\left(\|A_N^{\frac{s}{2}}\phi_n\|^2 + \|\phi_n\|^2\right)^{\frac{\gamma(1-\vartheta)}{2}} \|\phi_n\|^{\gamma\vartheta} \le \frac{K(\tau)\delta^{\frac{\gamma}{2}}}{F''(1-2\delta)^{\frac{\gamma\vartheta}{2}}} z_n^{\frac{5\gamma}{12}}.$$

Here again, the constants $C(\tau), K(\tau)$ are generic positive constants. Returning to (5.7), we deduce:

$$z_{n+1} \le \frac{2^{2n+2}K(\tau)^{\frac{2}{\gamma}}\delta^{-1}}{F''(1-2\delta)^\vartheta} z_n^{\frac{11}{6}-\frac{2}{\gamma}}, \tag{6.15}$$

where we need $\gamma > \frac{12}{5}$, entailing from (6.14) that we must restrict ourselves to the case $s \in \left(\frac{1}{4}, 1\right)$. We then apply Lemma A.1 with $b = 2^2 > 1$, $C = \frac{2^2K(\tau)^{\frac{2}{\gamma}}\delta^{-1}}{F''(1-2\delta)^\vartheta} > 0$ and $\varepsilon = \frac{5}{6} - \frac{2}{\gamma}$, which allows us to conclude that $z_n \to 0$, as long as

$$z_0 \le C(\tau)\delta^{\frac{6\gamma}{5\gamma-12}} F''(1-2\delta)^{\vartheta\frac{6\gamma}{5\gamma-12}}. \tag{6.16}$$

Now, for $\delta > 0$ sufficiently small, recalling (6.6), condition (6.16) is satisfied if

$$\frac{C}{F'(1-2\delta)^6} \le C(\tau)\delta^{\frac{6\gamma}{5\gamma-12}} F''(1-2\delta)^{\vartheta\frac{6\gamma}{5\gamma-12}},$$

since $z_0(t) \le \frac{C}{F'(1-2\delta)^6}$ for any $t \ge \tau$. If so, then the proof is concluded and the instantaneous strict separation holds. With the choice of $F = F_q$, this corresponds exactly to ask for $q \in \left[0, \frac{12-2\gamma(\vartheta+2)}{12-\gamma(\vartheta+5)}\right)$, for any $\gamma \in \left[2, \frac{6}{3-2s}\right)$ and $\vartheta := \frac{3}{\gamma s} - \frac{3}{2s} + 1$, as anticipated. Notice that, as $s \to 1$, we can choose $\gamma \to 6$ and thus $\vartheta \to 0$, entailing $q \to \frac{2}{3}$, which corresponds to the local Cahn-Hilliard case already analysed.

Remark 6.1. Due to the lower bound $s > 1/4$, we cannot let $s \to 0$. Nevertheless, we can observe that the technical reason for this issue is only related to the fact that $\mu \in L^\infty(\tau, \infty; L^6(\Omega))$. Indeed, if we could prove that $\mu \in L^\infty(\tau, \infty; L^p(\Omega))$ for any $p \ge 2$, then in (6.15) we could simplify δ^{-1} by assuming, for $F = F_q$, the value $q = 2 - \frac{1}{\vartheta}$. Letting then $s \to 0$, which would now be possible, we would have

$\gamma \to 2$ and then $\vartheta \to 1$, so that we could immediately deduce that $q \to 1$. Ideally this would mean that when $s = 0$, the strict separation property would hold also for the logarithmic potential (i.e., F_q with $q = 1$). This is in agreement with what already obtained in [29] for the nonlocal Cahn-Hilliard equation, corresponding in some sense to the fractional Cahn-Hilliard equation in the limit when $s \to 0$.

6.2. Asymptotic strict separation property

In three dimensions, the validity of the asymptotic strict separation property, meaning that the strict separation from pure phases holds only from some positive times (large enough), has been proven in [1] for the local Cahn-Hilliard equation for *any* singular entropy, just having assumption (1) on F in force. The proof is based upon showing that any solution to the stationary Cahn-Hilliard equation is strictly separated from pure phases and that any weak solution to the (local) Cahn-Hilliard equation converges to a single separated stationary state (solving the stationary Cahn-Hilliard equation) as $t \to \infty$ in some $H^r(\Omega)$-norm, with $r > \frac{3}{2}$. This entails, since $H^r(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega)$, that, from some time $T_s > 0$, depending on the initial datum φ_0, the strict separation holds (see [1, Section 6]). The same argument can be extended to the fractional Laplacian case, as long as $s \in \left(\frac{3}{4}, 1\right]$, so that the regularity of the solution ϕ (see Theorem 5.1) guarantees that $\phi \in L^\infty(\tau, \infty; H^{2s}(\Omega))$ for any $\tau > 0$. Since then $H^{2s}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega)$, the same argument of [1] works, entailing that the strict separation property holds at least asymptotically. Here we aim at giving a new proof of the same result, which also extends the fractional exponent range to $s \in \left(\frac{1}{4}, 1\right]$. This proof does not rely on any compactness property in $L^\infty(\Omega)$, but rather is based on the dissipative properties of the associated dynamical system. Namely, we have the following:

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^3$ be a smoothly bounded domain, $s \in \left(\frac{1}{4}, 1\right]$, and F satisfying (1). Assume that $\phi_0 \in \mathfrak{D}(A_N^{\frac{s}{2}}) \cap L^\infty(\Omega)$ with $\|\phi_0\|_{L^\infty} \le 1$ and $|\overline{\phi}_0| = m < 1$. Then, there exist $\delta \in (0, 1)$, depending on s, Ω, $\mathcal{E}_{SF}(\phi_0)$, m and the system parameters, and $T_f = T_f(\delta, \phi_0)$, depending additionally on δ and ϕ_0, such that the unique weak solution to problem (5.1) satisfies:*

$$|\phi(x,t)| \le 1 - \delta, \quad \textit{for almost every } (x,t) \in \Omega \times (T_f, +\infty). \tag{6.17}$$

Remark 6.3. Observe that, when $\Omega \subset \mathbb{R}^2$, the same result holds for the entire interval $s \in (0, 1]$, by slight modifications in the main argument (see the proof of Theorem 5.2). This means that the asymptotic separation property holds without any assumption on F additional to (1), thus in an even more general framework as presented in Sections 3 and 5.

Proof. Let us first recall that due to the energy inequality (see [15, Theorem 5.1]), it holds

$$\mathcal{E}_{SF}(\phi(t)) + \int_r^t \|\nabla \mu\|^2 d\tau \le \mathcal{E}_{SF}(\phi(r)), \tag{6.18}$$

for any $0 \le r \le t$. Moreover, having defined the difference quotient $\partial_t^h v(\cdot) := \frac{1}{h}(v(\cdot + h) - v(\cdot))$, $h > 0$, it holds by [15, (81)]

$$\frac{d}{dt}\|\partial_t^h \phi\|^2_{H^1(\Omega)'} + \|A_N^{\frac{s}{2}} \partial_t^h \phi\|^2 \le \|\partial_t^h \phi\|^2_{H^1(\Omega)'}, \tag{6.19}$$

for almost any $t \ge 0$. Note that by comparison and from (6.18), it holds, for $h > 0$ sufficiently small,

$$\int_t^{t+1} \|\partial_t^h \phi\|^2_{H^1(\Omega)'} \le C \int_t^{t+2} \|\partial_t \phi\|^2_{H^1(\Omega)'} \le C \int_t^{t+2} \|\nabla \mu\|^2 dr, \tag{6.20}$$

for any $t \ge 0$. Now, since $\mathcal{E}_{SF}(\phi(\cdot))$ is monotone non-increasing, there exists $\mathcal{E}_{SF,\infty}$ such that $\lim_{t\to\infty} \mathcal{E}_{SF}(\phi(t)) = \mathcal{E}_{SF,\infty}$. Therefore, from (6.18), for any $\epsilon > 0$, there exists $T_f = T_f(\epsilon, \phi_0)$, depending also on ϕ_0, such that

$$\int_t^{t+2} \|\nabla \mu\|^2 dr \le |\mathcal{E}_{SF}(\phi(t+2)) - \mathcal{E}_{SF}(\phi(t))| < \epsilon, \quad \forall t \ge T_f.$$

By the Uniform Gronwall Lemma (see, e.g., [30]), we thus infer that there exists $C>0$ such that

$$\sup_{t\geq T_f}\|\partial_t^h\phi(t)\|^2_{H^1(\Omega)'}\leq C\epsilon,$$

for any $h>0$ sufficiently small, and then, clearly,

$$\sup_{t\geq T_f}\|\partial_t\phi(t)\|^2_{H^1(\Omega)'}\leq C\epsilon.$$

Therefore, by comparison, there exists $C>0$ such that

$$\sup_{t\geq T_f}\|\nabla\mu(t)\|^2\leq C\epsilon. \tag{6.21}$$

We can now perform the De Giorgi iteration scheme for $t\geq T_f$ (assume w.l.o.g. $T_f>1$), similarly to Section 6.1.2. In particular, recalling (3.15), analogously to (6.13), we end up with

$$\|A_N^{\frac{s}{2}}\phi_n\|^2+(F'(1-2\delta)-\alpha_0)\int_\Omega\phi_n dx+\alpha\int_\Omega\phi_n^2dx\leq\int_\Omega\mu\phi_n dx=\int_\Omega(\mu-\overline{\mu})\phi_n dx+\overline{\mu}\int_\Omega\phi_n dx.$$

Since $|\overline{\mu}|\in L^\infty(1,\infty)$ and $|\overline{\mu}|\leq C(\mathcal{E}_{SF}(\phi_0),\overline{\phi}_0)$, if we then assume $\delta=\delta(\mathcal{E}_{SF}(\phi_0),\overline{\phi}_0)$ sufficiently small (recall assumption (1) on F) so that

$$F'(1-2\delta)-\alpha_0-\overline{\mu}\geq 0,$$

we end up with

$$\begin{aligned}\|A_N^{\frac{s}{2}}\phi_n\|^2+\alpha\int_\Omega\phi_n^2dx&\leq\|A_N^{\frac{s}{2}}\phi_n\|^2+(F'(1-2\delta)-\alpha_0-\overline{\mu})\int_\Omega\phi_n dx+\alpha\int_\Omega\phi_n^2dx\\&\leq\int_\Omega(\mu-\overline{\mu})\phi_n dx\\&\leq C\delta\|\mu-\overline{\mu}\|_{L^\infty(T_f,\infty;L^6(\Omega))}z_n^{\frac{5}{6}} \qquad (6.22)\\&\leq C\delta\|\nabla\mu\|_{L^\infty(T_f,\infty;L^2(\Omega))}z_n^{\frac{5}{6}}\\&\leq C\epsilon^{\frac{1}{2}}\delta z_n^{\frac{5}{6}}, \qquad (6.23)\end{aligned}$$

by Sobolev embeddings and Poincaré's inequality, together with (6.21). By repeating word by word the same argument of Section 6.1.2, to obtain that $z_n(t)\to 0$ as $n\to\infty$, with $t\geq T_f$, $s\in(\frac{1}{4},1]$ and $\gamma\in(2,\frac{6}{3-2s})$, we need that

$$z_0(t)\leq\frac{C}{\epsilon^{\frac{3\gamma}{5\gamma-12}}}\delta^{\frac{6\gamma}{5\gamma-12}}. \tag{6.24}$$

Therefore, since $z_0(t)\leq|\Omega|$, it is enough to choose $\epsilon=\epsilon(\delta)$ small enough such that

$$|\Omega|\leq\frac{C}{\epsilon^{\frac{3\gamma}{5\gamma-12}}}\delta^{\frac{6\gamma}{5\gamma-12}}$$

to conclude that, for any $t\geq T_f(\epsilon,\phi_0)$ $(T_f>1)$, the strict separation property, i.e., (6.17), holds, concluding the proof of the theorem. □

6.3. *Strict separation property if the initial energy is small*

We conclude this three-dimensional overview with another observation about the validity of the (almost instantaneous) strict separation property under some smallness assumptions on the initial energy $\mathcal{E}_{SF}(\phi_0)$, and the sole assumption (1) for the singular function F. This means that somehow we need that the initial datum ϕ_0 is 'close' to a minimiser of the energy $\mathcal{E}_{SF}$. As far as we know, this result has been

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achieved only in [22, Theorem 6.1] for the local Cahn-Hilliard equation. In particular, the proof is based on the control of the $H^2(\Omega)$-norm of the solution ϕ (see [22, (6.31)]. If this norm is sufficiently small, then by the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, one can deduce the validity of the strict separation property. This argument can be extended to the fractional Cahn-Hilliard equation when $s \in \left(\frac{3}{4}, 1\right]$, thanks to the embedding $H^r(\Omega) \hookrightarrow L^\infty(\Omega)$, for any $r \in (\frac{3}{2}, 1]$. Here we propose a completely different approach to the problem, based on the dissipativity properties of the dynamical system. This allows, in 3D, to extend the validity of the strict separation to the range $s \in \left(\frac{1}{4}, 1\right]$. In particular, we have

Theorem 6.4. *Let $\Omega \subset \mathbb{R}^3$ be a smoothly bounded domain, $s \in \left(\frac{1}{4}, 1\right]$, and F satisfying (1). Assume that $\phi_0 \in \mathfrak{D}(A_N^{\frac{s}{2}}) \cap L^\infty(\Omega)$ with $\|\phi_0\|_{L^\infty} \leq 1$ and $|\overline{\phi}_0| = m < 1$. Then, for any $\tau > 0$, there exist $\delta \in (0, 1)$, depending on τ, m, s, Ω and the system parameters, and $\epsilon = \epsilon(\tau, \delta) > 0$, such that, if $|\mathcal{E}_{SF}(\phi_0)| \leq \epsilon$, then the unique weak solution to problem (5.1) satisfies:*

$$|\phi(x,t)| \leq 1 - \delta, \quad \textit{for almost every } (x,t) \in \Omega \times (\tau, +\infty). \tag{6.25}$$

Remark 6.5. We point out that, in the case $\Omega \subset \mathbb{R}^2$, Remark 6.3 still holds and the argument works for any $s \in (0, 1]$ without extra assumptions on F with respect to (1).

Proof. First, from (6.18), we deduce that

$$\int_t^{t+2} \|\nabla \mu\|^2 dr \leq \mathcal{E}_{SF}(\phi(t)) \leq \mathcal{E}_{SF}(\phi_0), \quad \forall t \geq 0,$$

since $\mathcal{E}_{SF}(\cdot)$ is non-increasing. Let us fix, w.l.o.g., $\epsilon \in (0, 1]$ to be chosen later on, and assume that $|\mathcal{E}_{SF}(\phi_0)| \leq \epsilon$. Recalling (6.19) and (6.20), we can apply the Uniform Gronwall Lemma to infer that there exists $C > 0$ such that

$$\sup_{t \geq \tau} \|\partial_t^h \phi(t)\|^2_{H^1(\Omega)'} \leq \frac{C}{\tau}\epsilon,$$

for any $h > 0$ sufficiently small, and then, by comparison, there exists also $C > 0$ so that

$$\sup_{t \geq \tau} \|\nabla \mu(t)\|^2 \leq \frac{C}{\tau}\epsilon. \tag{6.26}$$

By reasoning as in the proof of Theorem 6.17, we can now apply the De Giorgi iteration scheme. In particular, in this case, we obtain (see (6.23) for a reference), for any $t \geq \tau$ and $\delta = \delta(\tau, \overline{\phi}_0)$ sufficiently small (recall that $|\overline{\mu}|_{L^\infty(\tau,\infty)} \leq C(\tau, \overline{\phi}_0)$ by (5.5), and the constant is independent of ϵ, by assuming $\epsilon \leq 1$),

$$\begin{aligned} \|A_N^{\frac{s}{2}} \phi_n\|^2 + \alpha \int_\Omega \phi_n^2 dx &\leq \|A_N^{\frac{s}{2}} \phi_n\|^2 + (F'(1-2\delta) - \alpha_0 - \overline{\mu}) \int_\Omega \phi_n dx + \alpha \int_\Omega \phi_n^2 dx \\ &\leq C\delta \|\mu - \overline{\mu}\|_{L^\infty(\tau,\infty;L^6(\Omega))} z_n^{\frac{5}{6}} \leq C\delta \|\nabla \mu\|_{L^\infty(\tau,\infty;L^2(\Omega))} z_n^{\frac{5}{6}} \leq \frac{C\epsilon^{\frac{1}{2}}}{\tau^{\frac{1}{2}}} \delta z_n^{\frac{5}{6}}. \end{aligned} \tag{6.27}$$

By repeating the same argument of Section 6.1.2, we obtain that $z_n(t) \to 0$ as $n \to \infty$, with $t \geq \tau$, $s \in (\frac{1}{4}, 1]$ and $\gamma \in (2, \frac{6}{3-2s})$, as long as

$$z_0(t) \leq \frac{C\tau^{\frac{3\gamma}{5\gamma-12}}}{\epsilon^{\frac{3\gamma}{5\gamma-12}}} \delta^{\frac{6\gamma}{5\gamma-12}}. \tag{6.28}$$

Therefore, due to $z_0(t) \leq |\Omega|$, it is enough to choose $\epsilon = \epsilon(\tau, \delta) \in (0, 1]$ suitably small such that

$$|\Omega| \leq \frac{C\tau^{\frac{3\gamma}{5\gamma-12}}}{\epsilon^{\frac{3\gamma}{5\gamma-12}}} \delta^{\frac{6\gamma}{5\gamma-12}},$$

to conclude that, for any $t \geq \tau$, the strict separation property, i.e., (6.25), holds, concluding the proof. □

https://doi.org/10.1017/S0956792524000196 Published online by Cambridge University Press

7. Concluding remarks and open questions

As outlined in the Introduction, the separation property is a pivotal physical characteristic in phase-segregation processes. We have managed to minimise the mathematical assumptions, resulting not only in critical behaviour in such processes but also in significantly relaxed conditions concerning the entropy density, subject to various statistical characteristics of the solution flow. These advancements are poised to have a profound impact on a wide array of applications, encompassing complex fluid behaviour in two-phase flows, elastic surface phenomena in mathematical biology within intricate geometries, image inpainting, multi-phase material behaviour and more (refer to the previously mentioned references in the Introduction for details, cf. also [4, 5, 15, 18] and references therein).

Moreover, our approach holds the potential for further expansion into highly critical phase-segregation processes involving anomalous mass transport (cf. [19, 20]). This will involve harnessing the effects of fractional-order processes, an area of growing interest within the mathematical community. We are also considering extending our framework to encompass multi-phase fluid flows and other associated multi-component phase-segregation processes in the future.

Acknowledgements. The present research has been supported by MUR Grant Dipartimentodi Eccellenza 2023-2027.

Competing interests. The authors declare none.

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A. Appendix

One of the fundamental tools for applying De Giorgi's iteration argument is the following lemma concerning the geometric convergence of sequences. This lemma can be found in various sources, such as [10, Ch. I, Lemma 4.1], and can be proven by induction (see, for example, [29, Lemma 3.8]).

Lemma A.1. *Let* $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}\subset\mathbb{R}^+$ *satisfy the recursive inequalities:*

$$z_{n+1}\le Cb^n z_n^{1+\varepsilon},\qquad \forall n\ge 0, \tag{A.1}$$

for some constants $C>0$, $b>1$, *and* $\varepsilon>0$. *If* $z_0\le\theta:=C^{-\frac{1}{\varepsilon}}b^{-\frac{1}{\varepsilon^2}}$, *then*

$$z_n\le\theta b^{-\frac{n}{\varepsilon}},\qquad \forall n\ge 0, \tag{A.2}$$

and consequently, $z_n\to 0$ *as* $n\to\infty$.

The following lemma establishes a useful comparison between the energy forms related to the (spectral) fractional Laplacian $A_N^{s/2}$, where $s\in(0,1)$. This comparison is essential for carrying out De Giorgi's iterations. Specifically, we have the following result:

Lemma A.2. *Let* $\rho\in\mathbb{R}$ *and* $u\in\mathfrak{D}(A_N^{\frac{s}{2}})\hookrightarrow H^s(\Omega)$ *for* $s\in(0,1)$. *Then* $u_\rho:=(u-\rho)^+\in\mathfrak{D}(A_N^{\frac{s}{2}})$, *and for* $u\in\mathfrak{D}(A_N^s)$, *it holds that*

$$\|A_N^{\frac{s}{2}}u_\rho\|^2\le(A_N^s u,u_\rho).$$

Proof. *First, we note that the standard energy form* $d_{SF}:\mathfrak{D}(A_N^{\frac{s}{2}})\times\mathfrak{D}(A_N^{\frac{s}{2}})\to\mathbb{R}_+$, *defined as*

$$d_{SF}(u,v)=\int_\Omega A_N^{s/2}u(x)A_N^{s/2}v(x)dx,$$

is a nonnegative and symmetric bilinear form on $\mathfrak{D}(A_N^{\frac{s}{2}})\times\mathfrak{D}(A_N^{\frac{s}{2}})$, *provided that* $\mathfrak{D}(A_N^{\frac{s}{2}})$ *is equipped with the equivalent norm of* $H^s(\Omega)$. *Utilising the semigroup representation of* A_N^s *(as detailed in [15, Appendix A.1]), we can express* d_{SF} *as follows:*

$$d_{SF}(u,v)=C_s\int_0^\infty\int_\Omega\int_\Omega K_N(t,x,y)(u(x)-u(y))v(x)\,dydxt^{-1-s}dt, \tag{A.3}$$

where $K_N(t,x,y)$ *is the symmetric heat kernel associated with the Markovian semigroup:*

$$e^{-tA_N}=\int_\Omega K_N(t,x,y)f(y)dy,\qquad \int_\Omega K_N(t,x,y)dy=1, \tag{A.4}$$

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for the Neumann heat problem (see, e.g., [15, Appendix A.1]). From (A.3) and the properties of $K_N(t,x,y)$, we can infer that for any $u \in \mathfrak{D}(A_N^s)$ and $v \in \mathfrak{D}(A_N^{\frac{s}{2}})$, we also have:

$$d_{SF}(u,v) = (A_N^s u, v),$$

where $A_N^s u(x) = 2P.V. \int_\Omega \widetilde{K}_N(x,y)(u(x)-u(y))dy$. Here, $\widetilde{K}_N(x,y)$ is nonnegative and symmetric over $\Omega \times \Omega$ and satisfies:

$$\widetilde{K}_N(x,y) = C_s \int_0^\infty K_N(t,x,y) t^{-1-s} dt \asymp \widetilde{C}_s \, |x-y|^{-(d+2s)},$$

whenever $x \neq y$. In this expression, $a \asymp b$ indicates the existence of positive constants c_1 and c_2 such that $c_1 b \leq a \leq c_2 b$. Notably, these observations imply that d_{SF} is also a Dirichlet form on $\mathfrak{D}(A_N^{\frac{s}{2}})$ in the sense of [11, Chapter 1], as expressed below:

$$d_{SF}(u,v) = \frac{1}{2} \int_\Omega \int_\Omega \widetilde{K}_N(x,y)(u(x)-u(y))(v(x)-v(y))dydx.$$

This, in turn, implies that $u_\rho \in \mathfrak{D}(A_N^{\frac{s}{2}})$ and $d_{SF}(u,u_\rho) \geq d_{SF}(u_\rho,u_\rho)$, which concludes the proof of the lemma. □

Cite this article: Gal C.G and Poiatti A. Unified framework for the separation property in binary phase-segregation processes with singular entropy densities. *European Journal of Applied Mathematics*, https://doi.org/10.1017/S0956792524000196