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NONEXISTENCE OF AN EXTREMAL GRAPH OF A CERTAIN TYPE

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Abstract

Cubic Moore graphs of diameter k on 3.2^k-2 vertices do not exist for $k>2$. This paper exhibits the first known case of nonexistence for generalized cubic Moore graphs when the number of vertices is just less than the critical number for a Moore graph : the generalized Moore graph on 44 vertices does not exist.

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1. Introduction

Generalized Moore graphs of valence 3 are graphs which have the property that, for *any* vertex in the graph, there are three vertices at distance 1, six at distance 2, twelve at distance 3, etc. In short, each line of the distance matrix contains one zero, three ones, six twos, twelve threes, and 3.2^{j-1} entries of j for $1 \leqslant j$, so long as

$$N_1 = 1+3+6+12+\ldots+3.2^{j-1} \leqslant N,$$

where N is the number of vertices in the graph; the remaining entries per line are k, where $k-1$ is the maximal value of j, and where there are $N-N_1$ entries of k. Clearly k is the diameter of the graph.

Graphs of this type have been discussed by Cerf *et al.* (1973, 1974A, B, C, 1975, 1976); an up-to-date summary with a number of further results is given in McKay and Stanton (1978). We take this opportunity to make one minor correction to the latter paper; the number of graphs M(14, 3) should be 7 (graph 26*G* in Cerf *et al.* (1974a) should have been rejected because of the distance condition).

In the present discussion, we investigate the graphs M(44, 3), and show that they do not exist. Since M(46, 3) and M(48, 3) do not exist (see McKay and Stanton (1978) and Cerf *et al.* (1973)), and since a computer search by B. D. McKay has ruled out

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M(50, 3) and M(52, 3), evidence is strong for the existence of a range about $M(3.2^k-2, 3)$ for which these graphs do not exist.

2. Basic results

As usual, we represent the skeleton of M(44, 3) in Figure 1, recalling that *any* vertex can serve as root node R. Since the total number of lines is 66, there are 45 further lines to be inserted in the skeleton.

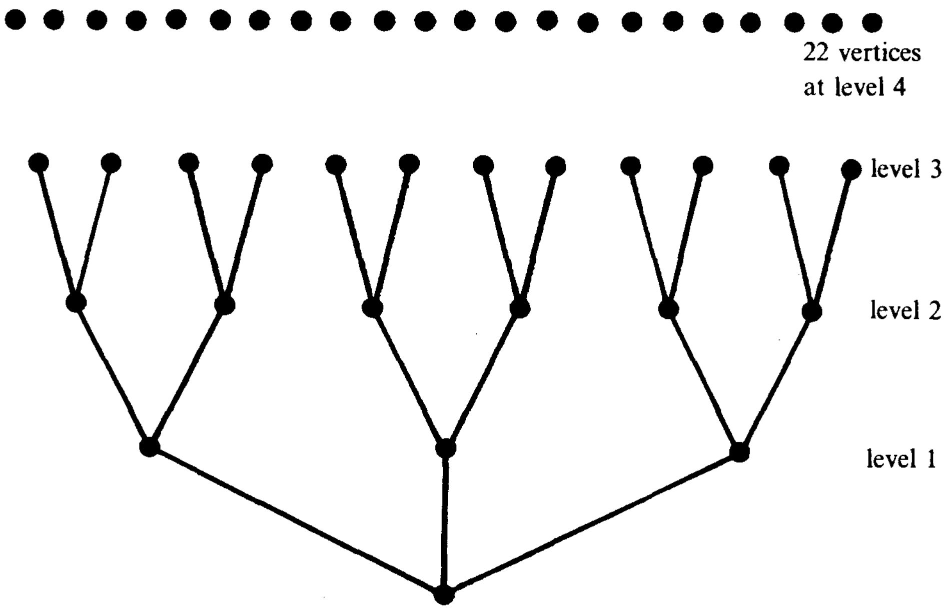

FIGURE 1. Skeleton of M(44, 3).

If we suppose there to be a lines at level 4, b lines joining levels 4 and 3, c lines at level 3, we immediately deduce that

$$a+b+c = 45,$$

$$2a+b = 66,$$

$$b+2c = 24.$$

Since $b \geqslant 22$, we find that the only solutions are:

$$\text{I}: \; a = b = 22, \qquad c = 1;$$

$$\text{II}: \; a = 21, b = 24, \; c = 0.$$

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Let there be x vertices of Type I, y vertices of Type II; then clearly $x+y = 44$. Furthermore, there is exactly one heptagon through a vertex R of Type I, none through a vertex R of Type II. Hence the number of distinct heptagons in the graph is $x/7$, and so $x = 0, 7, 14, 21, 28, 35$ or 42. Our first task will be to show that $x = 0$.

3. Possibilities for Type I vertices

For a Type I vertex, $c = 1$, that is, there is a line at level 3. As usual, we denote the structures which arise from S, T, U as flowers F_1, F_2, F_3. Then the line at level 3 must join vertices in two distinct flowers (otherwise, we would obtain a circuit of length 5 or smaller); so there is no loss in generality in letting this line join a vertex f in F_2 to a vertex g in F_3.

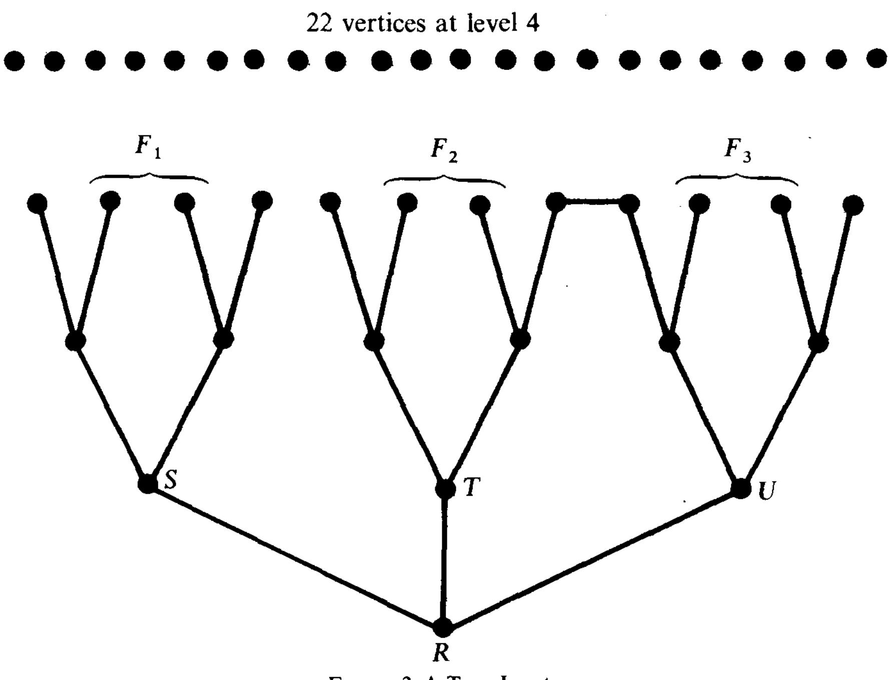

FIGURE 2. A Type I vertex.

Since there are 22 vertices and 22 lines at level 4, the lines at level 4 must be contained in the union of distinct polygons. Also, these polygons must contain at least 7 sides, since there are no shorter circuits in the graph. We tabulate the possibilities as Cases (a) 7, 7, 8; (b) 7, 15; (c) 8, 14; (d) 9, 13; (e) 10, 12; (f) 11, 11; (g) 22.

Case (a). We illustrate the level-4 configuration in Figure 3. The 22 lines joining level 4 to level 3 must be such that vertex S in Figure 2 can reach every level 4 vertex

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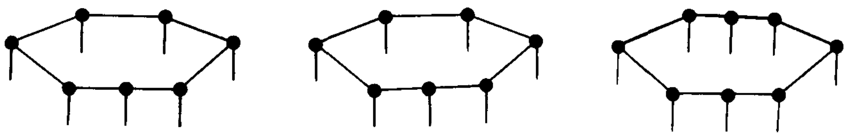

FIGURE 3.

in at most 4 steps. This requires that there be 3 lines from F_1 going to each of the 3 polygons; but this uses up 9 lines from F_1. This is impossible; so Case (a) cannot occur.

Case (b). In this case, the level-4 configuration consists of a heptagon and a 15-gon. We suppose that the lines going from f and g in Figure 2 up to level 4 are named a and b respectively.

By the same argument as in Case (a), there must be 3 lines from F_1 to the heptagon. In order that T and U reach the points of the heptagon in at most 4 steps, and in order that only the 4 available lines are used, we see that a and b must both join up to points on the heptagon (Figure 4). But then, no matter how we place a and b, we find that an illicit circuit (of length 6 or smaller) is created. (In the diagram, we place a and b joining with 1 and 4; this makes 1 f g 4 5 6 7 a licit circuit; but then 1 f g 4 3 2 is not.) Thus, Case (b) is ruled out.

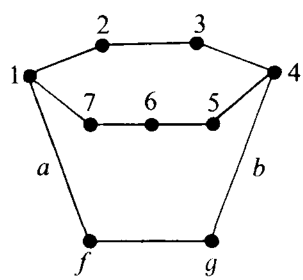

FIGURE 4.

Case (c). In this case, the level-4 configuration is an octagon and a 14-gon. We refer again to Figure 2.

In order for S, T, U to reach all the points of the octagon in 4 steps or fewer, we require three lines to the octagon from F_1 and three lines each from F_2 and F_3 (whether we use a and b or not). But this requires 9 lines from level-3 vertices up to the octagon. This is impossible; so we reject Case (c).

Case (f). The discussion here is as in Case (c). There are two 11-gons at level 4. In order for S, T, U to reach an 11-gon in at most 4 steps, we need 4 lines to the 11-gon from each of F_1, F_2, F_3. This is impossible; so Case (f) is ruled out.

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Case (e). In Case (e), there is a decagon and a 12-gon at level 4. We shall require two lemmas.

LEMMA 1. *A vertex of Type I has exactly one heptagon through it* (*see Figure* 2), *and a vertex of Type II has no heptagons through it.*

LEMMA 2. *A vertex of Type I has no octagons passing through it* (*for the* 22 *vertices at level* 4 *lie in the union of disjoint polygons*).

We again refer to Figure 2; it follows, as in Case (b), that F_1 is joined to the decagon by 4 lines; F_2 and F_3 are joined to the decagon by 3 lines each, and these lines must include a and b. In Figure 5, we examine how a and b can join up to the decagon. We start by joining f to a vertex 1; to avoid a circuit of length 6 or smaller, the line b cannot join g to 2, 3, 4, 8, 9, 10. If b joins g to 5 or 7, then f is a Type I vertex

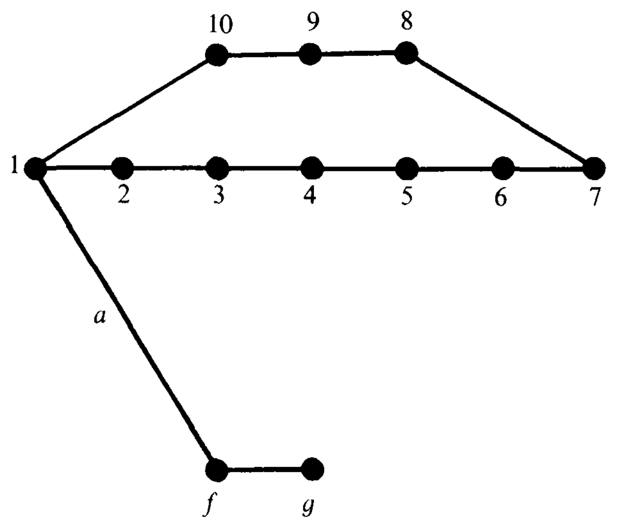

FIGURE 5.

lying on 2 heptagons (impossible by Lemma 1). If b joins g to 6, then f is a Type I vertex lying on an octagon (impossible by Lemma 2). Thus we have shown that Case (e) is not possible.

Case (d). Here we have an enneagon and a 13-gon at level 4. In order for S, T, U to reach the 13-gon in 4 or fewer steps, we need both a and b to join up to the 13-gon.

Now there are exactly four lines from F_3 (including b) that go up to the 13-gon. First, let line a join to point 1 (Figure 6). In order for U to reach all vertices in the 13-gon in 4 or fewer steps, we need these lines to meet the 13-gon at points 3, 6, 9, 12.

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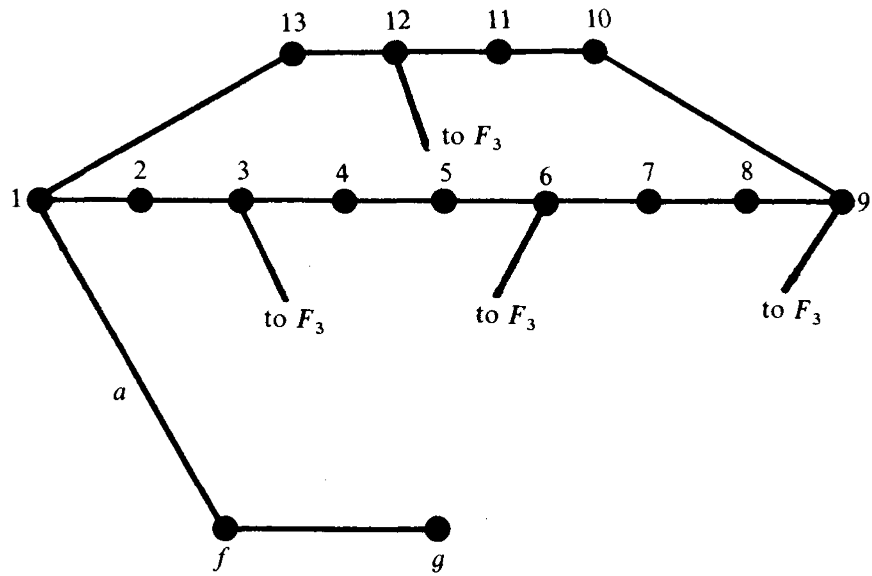

FIGURE 6.

Now one of these lines is b (joining to g). If b meets the 13-gon at 3 or 12, a pentagon will be created. However, if b meets the 13-gon at 6 or 9, then an octagon will be created. This octagon passes through the vertex f of Type I (a contradiction, by Lemma 2). So we have ruled out Case (d).

The result of these discussions is

THEOREM 1. *If there is a vertex of Type I, then the corresponding level-four configuration is a single 22-circuit.*

4. The case of a single level-four circuit

We refer again to Figure 2, and let a join to point 1. It follows that the vertices in F_3 must join to points 3, 6, 9, 12, 15, 18, 21; one of these joins must be b.

The join b cannot be to 3 or 21, since that would produce a pentagon. Nor can it be to 6 or 18 (that would produce an octagon through f, a Type I vertex, in contradiction with Lemma 2). So we have two cases : b joins to 9 (15) or b joins to 12.

If b joins to 9, then F_2 must be joined to points 4, 7, 10, 11, 14, 17, 20 and 1 (line a). This situation is shown in Figure 7. We ask to where point e in Figure 7 is joined.

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Points 7 and 11 are ruled out (this would create a second heptagon through f); similarly, points 4 and 20 are ruled out. Hence e must be joined to points 17 and 14; this creates a pentagon (e 17 16 15 14). So we are left with our only possibility as in Figure 8 : b joins to 12.

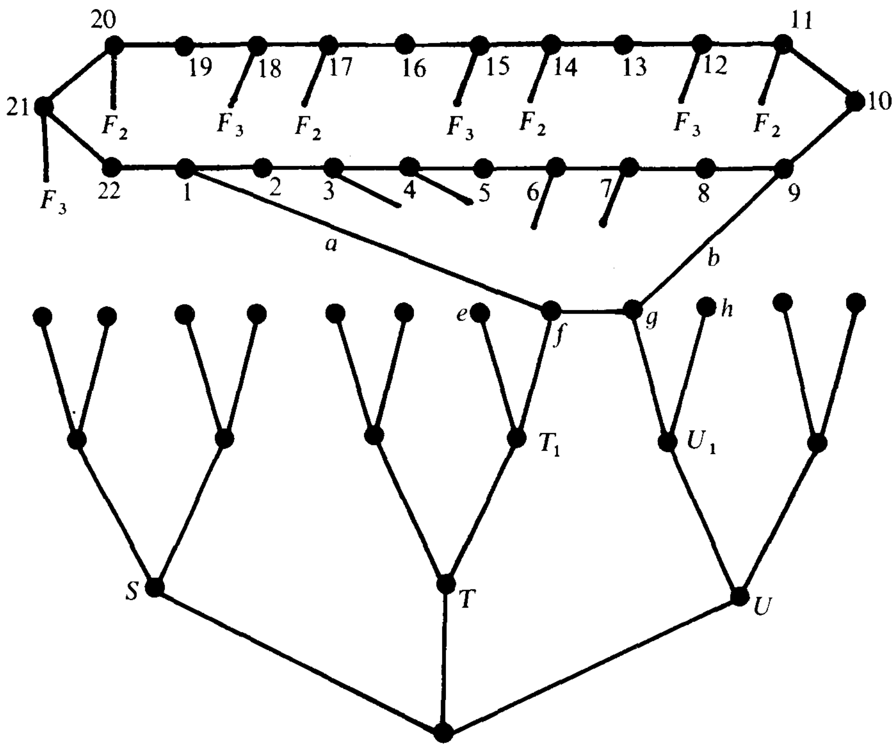

FIGURE 7.

The same kind of arguments as used before show that F_3 joins up with 3, 6, 9, 15, 18, 21; F_2 joins up with 4, 7, 10, 14, 17, 20. Now e cannot join 4 or 20 (a heptagon would be created); nor can it join 10 or 14 (for the same reason). Hence e must join 17 and 7.

By the same arguments, h is forced to join with 6 and 18. Then (17 28 h 6 7 e) is a hexagon, and we have a contradiction.

Thus, our last possibility is ruled out, and we have

THEOREM 2. *If the graph* M(44, 3) *exists, then all vertices are of Type II (thus, the graph contains no heptagons).*

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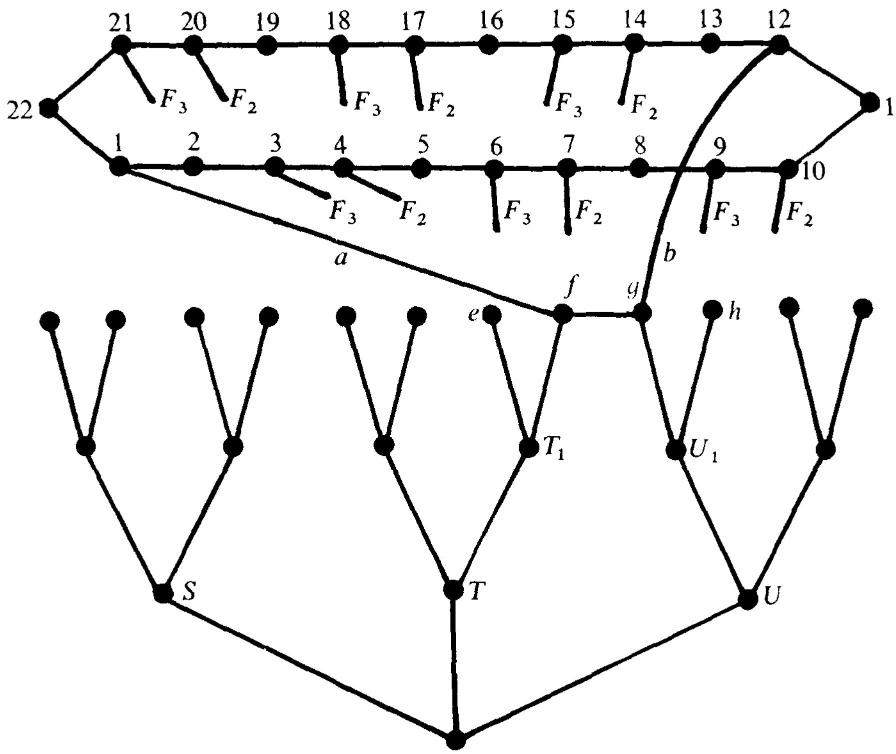

FIGURE 8.

5. The case of Type II vertices

Vertices of Type II can be of Type IIA (Figure 9) or Type IIB (Figure 10). We suppose there are p and q of these respectively; then $p+q = 44$.

Consider Case IIA. Then the 3 lines from A must join to 3 different flowers (otherwise, we get circuits of length at most 6). So we have 3 octagons through the root node R.

Also, the other 21 vertices are such that the 2 adjacent vertices in any polygon go to different flowers (otherwise, we would have a circuit of length less than or equal to 7). So we have 21 enneagons through R, and state

LEMMA 3. *For Type IIA vertices, there are* 3 *octagons and* 21 *enneagons through the root node* R.

In Case IIB, we can limit r further. If $r = 0$, then one line from A and one from B go to the same flower to create a circuit of length 7 or smaller; so $r > 0$. Thus we have

LEMMA 4. *In Case IIB,* $r \geqslant 1$.

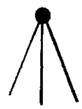

FIGURE 9. Level-4 configuration for Type IIA.

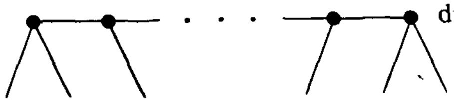

FIGURE 10. Level-4 configuration for Type IIB.

For the $20-r$ vertices in the union of disjoint polygons, two adjacent vertices must be joined to different flowers (just as in Case IIA). So these vertices produce $20-r$ enneagons through the root node R.

We now consider the $r+2$ vertices from A to B (inclusive). The two lines from A and the line from the vertex adjacent to A must go to 3 different flowers (or we get a circuit of length 7 or smaller); similarly, the two lines from B and the line from the vertex adjacent to B go to different flowers. For the r vertices between A and B, any two adjacent vertices must be joined to different flowers (to avoid a circuit of length 7 or smaller). Hence these $r+2$ vertices generate 2 octagons (one through A, one through B) and a total of $r-1+2(2) = r+3$ enneagons, all passing through the root node R. We thus have

LEMMA 5. *In Case IIB, there are* 2 *octagons and* $(20-r)+(r+3) = 23$ *enneagons through the root node* R.

We can now count the total number of distinct octagons as $(3p+2q)/8$ and the total number of distinct enneagons as $(21p+23q)/9$. Since these values are integers, we have

LEMMA 6. $3p+2q \equiv 0 \pmod 8$ *and* $3p+5q \equiv 0 \pmod 9$. *Now* $p+q = 44$; *thus* $p+q \equiv 4 \pmod 8$. *It follows that* $p \equiv 0 \pmod 8$.

We also have $p+q \equiv 8 \pmod 9$; thus $2p \equiv 4 \pmod 9$, $p \equiv 2 \pmod 9$. However, the solution to $p \equiv 0 \pmod 8$, $p \equiv 2 \pmod 9$, is $p \equiv 56 \pmod{72}$. This is impossible, since p is an integer in the range $0 \leqslant p \leqslant 44$. So we conclude with

THEOREM 3. *A graph* M(44, 3) *cannot be constructed.*

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