CHAPTER 1

Separability

Efficiency simplifies an option's evaluation by trimming considerations that affect all options equally. A preliminary step separates considerations into those to be processed and those to be put aside. An accurate evaluation processes considerations that are separable or independent from those dropped.

Imagine composites divided into parts of specific types, for example, baskets of groceries divided into cereal, dairy, meat, and produce parts. Given a part's separability from other parts, a ranking of composites according to the part's contents agrees with a ranking of composites in which the other parts have constant contents. For example, if the cereal part is separable from the other parts, then ranking baskets according to contents of the cereal part agrees with ranking baskets that have constant contents in the other parts. Focusing on the cereal part is a shortcut to ranking baskets.

This chapter treats separability of considerations generally, exploring its philosophical foundations rather than its technical applications. It treats separability generally so that its applications to probability and various types of utility have behind them the explanatory power of general principles. Separability belongs to a cluster of similar phenomena, and exploring its relation to the cluster's other elements clarifies it. After defining a type of separability, the chapter treats separability's relation to complementarity and compositionality. It presents the conceptual geography of separability to create a context for arguments supporting separability and to identify promising argumentative strategies. The chapter finishes with a review of methods of establishing separability. It shows how to establish separability as a norm for ideal agents. Later chapters construct arguments for specialized norms of separability, grounding, for example, expected-utility analyses of comprehensive utilities. They introduce various types of utility and argue for the principles of separability that apply to them.

In deliberations, separating considerations is the first step toward efficiency; thus, this chapter introduces the technical concept of separability at

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length. Later chapters use principles of separability to support proposals about efficient evaluation of the options in a decision problem.

1.1 Separability's definition

Assuming complete information, an option's comprehensive utility evaluates the option's world. A narrower evaluation targets the option's future instead of the option's world. It guides decisions well if the order of options according to futures is the same as the order of options according to worlds. In general, simplified deliberations may evaluate an option using part of the option's world instead of the option's entire world if evaluations using the part replicate comprehensive utility's order of options using worlds.

This method of simplifying choices assumes that a part's evaluation affects the world's utility independently of the other parts. The type of independence that justifies putting aside considerations is called *separability*. Works in economics on separability and in measurement theory on decomposability, another name for separability, generally define utility using preferences. As Chapter 2 explains, a realist interpretation of utility that defines utility using degrees of desire rather than preferences strengthens norms of utility. This section modifies the literature's definition of separability to suit a definition of utility using degrees of desire. Also, the literature adjusts separability's definition according to separability's role in a theory. This section's definition tailors separability for its role in simplifying deliberations.[1]

The chapter's main target is separability of utility. The definition of separability, being general, applies to all types of utility, including utility attaching to goods as well as utility attaching to propositions, although later chapters use exclusively utility attaching to propositions, taking propositions to represent worlds, outcomes, and possession of goods.

The definition of utility's separability derives from the separability of parts of composites in a preference ranking. For uniformity among the objects of preference and utility, preferences hold between realizations of propositions when utility attaches to a proposition's realization. A preference for the truth of one proposition rather than another shows in decisions to realize the first proposition rather than the second and typically arises from beliefs about the consequences of each proposition's truth. A norm requires preferring one proposition's realization to another's realization if

[1] Topology uses another kind of separability, namely, a property of metric spaces limiting their size, to define separable spaces.

the desire for the first's realization is stronger than the desire for the second's realization. Chapter 2 elaborates these points.

Although this section's objective is to provide an account of utility's separability, it starts with an account of separable preferences among composites. To obtain this account, it formulates a general definition of separability for an order of composites.

1.1.1 Separable orders

Reviewing cases of separability and the general features of separability reveals two candidates for separability's definition. One defines separability as an order's independence from conditions. The other defines separability as a relation between an order and a suborder.

Suppose that wholes and their parts have various instantiations, as a two-part commodity bundle of apples and bananas has various instantiations depending on its number of apples and number of bananas. Also suppose that the instantiations of wholes have an order, that for each part its instantiations have an order, and that for each subset of parts its instantiations have an order. Preferences may supply the order. Separability of parts and subsets of parts holds with respect to an order of instantiations of the wholes, their parts, and subsets of their parts. For example, it may apply to parts of commodity bundles as preferences order their instantiations. Agreement of the order of a part's instantiations with the order of compatible instantiations of wholes, for all instantiations of the other parts, signals the separability of the part from the other parts. In two-part commodity bundles of apples and bananas, apples are separable from bananas if the order of the bundles goes by the number of apples, for any constant number of bananas. A set of parts, not just a single part, is separable from a whole's other parts just in case the order of instantiations of the set's parts agrees with the order of compatible instantiations of wholes, for any fixed instantiation of the other parts.[2]

A set of n-tuples of values of variables, or n-placed vectors with occurrences at locations in the vector, may represent a set of composites. The

[2] McClennen (1990) defines separability for preferences at different moments of time. His page 12 says that given separability, preferences at a future time depend only on consequences realizable at the future time and so are independent of earlier preferences. His pages 120–21 say that separability holds if and only if choice does not depend on earlier choices. His page 122 characterizes separability using decision trees: past choices reduce a tree without changing the tree's remainder. Hammond (1988) and Seidenfeld (1988) define similar types of temporal separability for preferences. Hammond calls his type of temporal separability *dynamic consistency*, and Seidenfeld calls his type of temporal separability *dynamic feasibility*.

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Table 1.1 *Illustration of terminology*

(x, y, z)	vector of variables, a composite's structure
(x_1, y_1, z_1)	vector of variables' values, a composite
(x, y)	subvector of variables, a subset of parts' structure
(x_1, y_1)	subvector of variables' values, a subset of parts' instantiation

sequence of variables may represent the sequence of a composite's parts, if they come in a sequence. For example, a vector giving the amount of each good in a set of n goods represents a shopping basket of the goods. A variable represents each good, and a value of the variable stands for the amount of the good in the basket. The vector's sequence of goods represents their sequence, if any, in the basket. A consumer's preferences order possible shopping baskets and so order vectors that represent the baskets.

Table 1.1 illustrates this terminology. The variables x, y, and z have as values, respectively, x_1, y_1, and z_1. A composite's structure has parts with variable values. Variables represent the parts, and vectors of variables represent the structure of composites. Assigning values to the variables produces a vector of values that represents a composite. A subvector of variables represents a subset of parts' structure, and a subset of values of variables represents an instantiation of the parts.

Given an order of vectors of values of variables and an order of subvectors of values of subsets of the variables, a subset of variables is separable from the other variables if and only if, for all values of the other variables, the order of subvectors of values of the subset of variables agrees with the order of vectors of values of all variables. Hence, for a vector of locations, a subvector of the locations, and corresponding vectors and subvectors of occupants of the locations, the subvector of locations is separable from its complement if and only if the order of subvectors of occupants agrees with the order of vectors of occupants. For example, suppose that the variables x and y represent two goods. The constants x_1 and x_2 represent values of x, and the constants y_1 and y_2 represent values of y. The vectors of values of the variables are (x_1, y_1), (x_2, y_1), (x_1, y_2), (x_2, y_2). Suppose that the list gives their order from lowest to highest so that by using the symbol "$<$" to represent their order, it would be $(x_1, y_1) < (x_2, y_1) < (x_1, y_2) < (x_2, y_2)$. For any value of y, the order of x's values given by their subscripts agrees with the order of vectors. Hence, x is separable from y. Taking the subscripts of y's values to show their order, y is also separable from x because given each value of x, the ranking of y's values agrees with the ranking of pairs. Separability exists within a set of variables generating the components of composites, given a way of ordering the composites and subsets of their components.[3]

The agreement of orders that separability entails may hold by definition or as a consequence of separability's definition and features of the orders. Rather than define separability as an agreement of orders, some theorists define it as an independence of conditional orders that they define using a nonconditional order.

The order of vectors of values of all variables defines an order of subvectors of values of a subset of variables given a way of fixing the subvectors' complement. That is, it defines a *conditional order* of the subvectors. In the example, $\{x\}$ is a subset of variables, and (x_1) and (x_2) are subvectors of its values. The definition declares that (x_2) ranks above (x_1) when y's value is y_1 because vectors of values of x and y containing x_2 rank above vectors of values of x and y containing x_1 when y's value is y_1. In the example, for all ways of fixing the value of y, the vectors of values of x and y containing x_1 have the same rank with respect to vectors of values of x and y containing x_2. If the conditional order of subvectors defined using the order of vectors is the same for all values of y, then the common conditional order of subvectors may define the subvectors' nonconditional order.[4] The subvectors' order so defined agrees with the order of vectors; the variable x is separable from the variable y. The variable x's separability from the variable y amounts to the vectors' order generating the same order of subvectors of x's values given any value of y. Starting with the order $(x_1, y_1) < (x_2, y_1) < (x_1, y_2) < (x_2, y_2)$, the conditional order $(x_1) < (x_2)$ given y_1 derives from the order of pairs. Also, the conditional order $(x_1) < (x_2)$ given y_2 derives from the order of pairs. The variable x is separable from the variable y because these conditional orders of (x_1) and (x_2) agree. The variable y is similarly separable from the variable x. In general, the order of subvectors of values of a subset of variables is the same given all ways of fixing the values of the other variables – that is, the conditional orders of the subvectors are the same given all ways of fixing the values of the other variables – just in case the variables generating the subvectors are separable from the other variables.

To illustrate, imagine a diner who prefers a ham sandwich to a beef sandwich, whether or not the sandwiches have cheese, but prefers each type

[3] Broome (1991: 22–25, 65–80) states that in a set of n-placed vectors, a set of subvectors is separable if and only if comparison of the subvectors, conditional on a constant complement of the subvectors, is the same as comparison of the whole vectors. Imagine an order of a set of vectors. A subset of locations in these vectors is separable if and only if the order of subvectors of occurrences at these locations, given all ways of fixing occurrences at other locations, agrees with the order of vectors.

[4] Broome (1991: 67) defines the order of subvectors using the order of vectors.

of sandwich with cheese. This order of vectors represents the diner's preferences: (no cheese, beef) < (cheese, beef) < (no cheese, ham) < (cheese, ham). Applying the definition of conditional preference, using these nonconditional preferences, the diner prefers cheese to no cheese given beef and prefers cheese to no cheese given ham. Because the conditional preference for cheese holds regardless of the meat, the variable indicating whether the sandwich includes cheese is separable from the variable indicating the type of meat.

The order of vectors need not define the order of subvectors, however. For example, the diner may prefer cheese to no cheese even if the cheese is not in a sandwich. The preference for cheese need not derive from the preference ranking of sandwiches.[5]

If the order of vectors does not define the order of subvectors, then conditional independence of the order of subvectors does not entail its agreement with the order of vectors. Conditional independence is not sufficient for agreement. The order of subvectors may be constant for any way of fixing the subvectors' complement but contrary to the order of vectors. In the example, the conditional independence of the order of x_1 and x_2 with respect to the value of y authorizes using the order of pairs to define the order $x_1 < x_2$. However, the order derived from the order of pairs may disagree with the actual order $x_1 > x_2$. For example, taken alone, less pepper may taste better than more pepper, although taken with either a carrot salad or a preferred beet salad, two dashes of pepper may taste better than one dash of pepper. A set of vectors creates a context for their elements' realizations, and the context may influence preferences among their elements' realizations.[6]

A definition of separability that takes it as conditional independence of an order of subvectors does not require agreement of orders of subvectors and vectors unless the order of vectors defines the order of subvectors. Because agreement of orders, rather than conditional independence, is crucial for simplifying deliberations, to obtain it from separability without using the order of vectors to define the order of subvectors, this section defines

[5] Preference may also order subvectors that a subset of variables generates given a set of values for the other variables. Preference's order may differ from the order of subvectors that the order of vectors imposes given the values of the other variables, although for a rational ideal agent the two orders of subvectors agree.

[6] Binmore (2009: 47–49) takes separability for an order of pairs to obtain if preferences concerning one factor are independent of the value of the other factor. The pairs may involve goods or gambles concerning goods. For a set of pairs, let the relation < represent preference for the pair on the right, and the relation ≤ represent preference for the pair on the right or indifference between the pair on the left and the pair on the right. If two variables with L and L' and M and M' as values, respectively, are separable from each other, then $(L, M) < (L, M')$ implies $(L', M) \leq (L', M')$, and $(L, M) < (L', M)$ implies $(L, M') \leq (L', M')$.

separability as agreement of orders and not conditional independence. It is then a relation between an order of subvectors and the order of vectors, rather than a feature of the order of vectors. Accordingly, separability has this definition:

> In a set of variables, a subset is *separable* from the others if and only if for all ways of fixing the values of the other variables, the order of subvectors of values of variables in the subset agrees with the order of vectors of values of all variables.

In the example about the variables x and y, the order of vectors of their values is $(x_1, y_1) < (x_2, y_1) < (x_1, y_2) < (x_2, y_2)$. Suppose that each variable's values are amounts of an economic good and that preferences order its amounts taking the good in isolation. A consumer prefers more to less of good x and also of good y, and the subscripts of a variable's values indicate amounts of the good the variable represents, so that the values of the variables have the order of their subscripts. Then, x is separable from y because the order of subvectors of x's values agrees with the order of vectors of values of all variables for any fixed value of y.[7]

Separability taken as a relation between an order of vectors and an order of subvectors is equivalent to separability taken as a property of the order of vectors, if the order of vectors defines a conditional order of subvectors that in turn defines the nonconditional order of subvectors. According to separability taken as a property, in a list of variables a subset of variables is separable from the others if and only if no matter how the others are fixed, the subvectors of values of the variables in the subset have the same order. The subvectors' constant conditional order defines their nonconditional order. Their nonconditional order agrees with the order of the vectors because the order of vectors defines the order of the subvectors of values of variables in the subset conditional on fixed values of the other variables. Hence, given the definitions, the subvectors of values of the variables in the subset have the same order no matter how the other variables are fixed if and only if they have a nonconditional order that agrees with the order of the vectors.

[7] For a statement of the definition in technical notation, consider a vector $\mathbf{X}$ of variables $X_1, X_2, \ldots X_n$, a subvector $\mathbf{X_i}$ of these variables, and the subvector's complement $\mathbf{X_j}$ with respect to $\mathbf{X}$. x_i and $x_{i'}$ are values of a variable X_i belonging to $\mathbf{X_i}$, and x_j is a value of a variable X_j belonging to $\mathbf{X_j}$. $\mathbf{x_i}$ and $\mathbf{x_{i'}}$ are respectively vectors of values x_i and $x_{i'}$ of variables in $\mathbf{X_i}$ according to the variables' order in $\mathbf{X_i}$. $\mathbf{x_j}$ is a vector of values of variables in $\mathbf{X_j}$ according to the variables' order in $\mathbf{X_j}$. $\mathbf{x_{ij}}$ is the vector of values of the variables in $\mathbf{X}$ according to the variables' order in $\mathbf{X}$ formed from the subvectors $\mathbf{x_i}$ and $\mathbf{x_j}$, and $\mathbf{x_{i'j}}$ is a similar vector formed from the subvectors $\mathbf{x_{i'}}$ and $\mathbf{x_j}$. Let $\geq$ represent the order of vectors and subvectors of values. The subvector of variables $\mathbf{X_i}$ is *separable* from its complement $\mathbf{X_j}$ if and only if for any subvector $\mathbf{x_j}$, for all subvectors $\mathbf{x_i}$ and $\mathbf{x_{i'}}$, $\mathbf{x_i} \geq \mathbf{x_{i'}}$ if and only if $\mathbf{x_{ij}} \geq \mathbf{x_{i'j}}$.

To illustrate the equivalence, consider preferences among worlds taken as conjunctions of a past and a future (that includes the present). Values of a pair of variables – p for the past and f for the future – form a vector that represents a world. Suppose, using < between two propositions to represent preference for the second's truth to the first's truth, that the conditional preference $f_1 < f_2$ given p_1 by definition derives from the preference between conjunctions $(p_1 \,\&\, f_1) < (p_1 \,\&\, f_2)$, and similarly for other conditional preferences. By definition, the order of worlds settles the order of futures given the past. Also, suppose that the future is separable from the past in the property sense. Given this separability, $f_1 < f_2$ given p_1 if and only if $f_1 < f_2$ given p_2 or, equivalently, $(p_1 \,\&\, f_1) < (p_1 \,\&\, f_2)$ if and only if $(p_2 \,\&\, f_1) < (p_2 \,\&\, f_2)$. The order of futures conditional on the past, as derived from the order of worlds, is the same for all ways of fixing the past. Take the nonconditional order of futures as that constant conditional order. Then, the nonconditional ranking of futures agrees with the ranking of worlds whatever the past. So, the future is separable from the past in the relational sense. Similar inferences move from separability in the relational sense to separability in the property sense.

In a decision problem, when using evaluation of options' futures to simplify evaluation of options, it is best not to use the order of options' worlds to define the order of options' futures. Applying the definition to obtain the order of futures requires obtaining the order of worlds, and so it does not simplify evaluations. To simplify evaluations, it is best to infer the order of worlds from a definitionally independent order of futures, using the future's separability from the past in a relational sense. In general, a world part's separability from its complement in the relational sense justifies evaluations that attend only to the part.

Separability is a type of independence for components of composites. Elegance suggests using the order of composites to define conditional orders of composites, independence of these conditional orders, and then separability of components. However, the project of simplifying choices needs a nonderivative type of independence of components. Because simplification derives the order of composites from the order of components, it requires that the order of components be definitionally independent of the order of composites. Hence, it requires the relational definition of separability.

1.1.2 Utility

An order is separable just in case it orders composites that have parts that are separable with respect to the composites' order and the parts' order. Separability of a part entails independence of the order of its instantiations from other parts' instantiations. This independence entails an analogous independence of a function representing the order of the part's instantiations, in particular, a utility function representing preferences among the part's instantiations. Separability of a set of parts also entails an analogous independence of functional representations of the order of the set's instantiations.

Suppose that a commodity bundle's utility is a sum of its parts' utilities. Then, each set of parts is separable from its complement, using the preferences that a utility function represents to order instantiations of wholes and sets of parts. Ordering a part's instantiations according to preferences agrees with ordering particular bundles containing its instantiations according to preferences, given that the other parts have a constant instantiation. For example, if x_2 has greater utility than x_1, then, no matter what the utility of the variable y's value, the pair (x_2, y) has greater utility than the pair (x_1, y) because by supposition the utility of a pair equals the sum of its constituents' utilities.

Given separability of a world's part from the world's other parts, the order of the part's instantiations agrees with the order of worlds for all instantiations of the other parts. Given the future's separability from the past, the order of futures agrees with the order of worlds for every account of the past. A world's parts are separable if evaluations of a world's parts sum to the world's utility. The world's utility increases as a part's evaluation increases, given any way of fixing other parts. So, the order of the part's instantiations, holding other parts fixed, agrees with the order of worlds containing the part's instantiations.

Separability of an order of vectors constrains utility functions representing the order of vectors. So, separability may extend from orders of vectors to utility functions representing the orders. This section introduces separability of an ordered subset of the argument variables of a utility function representing an order of vectors. It assumes that a utility function exists over subvectors of values of the subset of argument variables that represents the order of the subvectors, but it does not assume that the order of the vectors and their utilities defines the order and the utilities of the subvectors. A proposition may express the state any vector or subvector represents, so utility defined using strength of desire applies to the vectors and subvectors through the propositions that represent them.

Separability of variables giving the arguments of a utility function follows from separability of variables generating the vectors to which the utility function attaches and whose order the utility function represents. Take the function $U(x, y)$. The variable x is separable from the variable y just in case x

is separable from y with respect to the vectors (x, y) whose order U represents. For a utility function, separability of a subset of argument variables from its complement holds if and only if the order of subvectors the variables generate agrees with the order of vectors no matter how the other variables are fixed. Given separability, moving up the order of subvectors, holding their complements constant, entails moving up the order of vectors, so an equivalent characterization of the subset's separability uses *strictly increasing monotonicity*: increasing the utility of the subset of argument variables, holding fixed argument variables in its complement, increases the function's value. If the function is strictly monotonically increasing at an argument place, increasing the argument's value increases the function's value, holding fixed the values of arguments at other places. The monotonicity condition entails the separability of the argument place, and because its separability entails the monotonicity condition, the monotonicity condition is equivalent to the argument place's separability. A similar equivalence holds for a set of argument places.

Suppose that the utility of composites is a *separable function* of the utilities of components. Then, the function is strictly monotonically increasing at each argument place. This holds if the function is addition. Utility's additivity, if it obtains, implies the utility function's separability.

When a subset of variables giving arguments of a utility function is separable from the other variables, some function of the utilities of the subvectors and of their complements yields the utilities of the vectors. For example, $U(x, y) = F(U(x), y)$ for some F given x's separability from y. Given that separability, a utility subfunction $U(x)$ represents the order of x's values, that is, the subvectors (x), given any value for y. The utility subfunction replaces the variable x in the move from $U(x, y)$ to $F(U(x), y)$. The function $F(U(x), y)$ represents the order of vectors, and, with respect to that function, $U(x)$ is separable from y. Given a fixed value of y, if $U(x)$ increases, then so do $F(U(x), y)$ and $U(x, y)$. A subset of a utility function's argument variables is separable from the others if and only if the corresponding function of the utilities of the subsets' values and of their complements is strictly monotonically increasing in the utilities of the subsets' values given any way of fixing its complement's values. Separability of utility functions of variables and of utility functions in which a subutility function replaces a set of variables arises from the separability of the variables.[8]

[8] Varian (1984: sec. 3.14) defines utility's *functional separability* as the utility of a composite's being increasing given increases in a component's utility. The utilities of components are functionally separable if and only if holding constant all components but one, while increasing the utility of that

Given that utility settles order, in the order of vectors (x, y) the variables x and y are mutually separable if and only if in the utility function $U(x, y)$ the variables are also mutually separable. Also, the arguments of the utility function $U(x, y)$ are mutually separable if and only if the arguments of the function $F(U(x), U(y))$ are mutually separable. Hence, $U(x)$ and $U(y)$ are mutually separable in $F(U(x), U(y))$ if and only if x and y are mutually separable in (x, y). This equivalence holds for n-tuples as well as pairs.

For a preference order of vectors of two variables, this section distinguishes one variable's effect on preferences concerning the other variable from its effect on utilities concerning the other variable because it takes utilities to represent not just preferences but also strengths of desire. According to this section's accounts of separability and utility, for mutually separable variables x and y the equation $U(x, y) = F(U(x), U(y))$ does not state a feature of the order of vectors of values of x and y, but instead it states a relation of three utility functions not defined by the order of vectors. An argument for the equation appeals not just to features of the order of vectors but also to relations between utility functions that represent the order of vectors and the orders of subvectors.

The literature on separability often treats utility attaching to combinations of goods, but points about separability carry over to utility attaching to conjunctions of propositions. Many structural points about utility apply whether utility attaches to goods and combinations of goods or to propositions and conjunctions of propositions. Consider instead of $U(x, y)$, with utility applying to pairs of goods, $U(x \,\&\, y)$, with utility applying to conjunctions of propositions.

If a type of utility U applies to binary conjunctions, the value of the variable x yields the first conjunct, the value of the variable y yields the second conjunct, and the variables are mutually separable, then $U(x_1 \,\&\, y_1) = F(U(x_1), U(y_1))$ and $U(x_1 \,\&\, y_2) = F(U(x_1), U(y_2))$ for some strictly monotonically increasing function F. To compare $U(x_1 \,\&\, y_1)$ with $U(x_1 \,\&\, y_2)$, letting F be the reduction of F to a one-place function given $U(x_1)$ as fixed first argument of F, one may compare $F^*(U(y_1))$ with $F^*(U(y_2))$ or equivalently $U(y_1)$ with $U(y_2)$ because F is strictly monotonically increasing. By separability, the order of conjunctions according to F agrees with the order of second conjuncts according to $U(y)$ after fixing the value of x.

A simplified evaluation of composites using separability excises the common element from the composites and compares them using the

component, a composite's utility increases. According to utilitarianism, individuals' utilities are functionally separable. The utility function for one individual settles the ranking of worlds given fixed utilities for other individuals.

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remainders' utilities, according to an appropriate type of utility. Chapter 4 compares worlds by excising the past from the worlds and evaluating their futures. Suppose that p stands for the past and f stands for the future. Given the mutual separability of f and p, because $U(p, f) = F(U(p), U(f))$, the order of (p_1, f_1) and (p_1, f_2) follows from the order of f_1 and f_2. Moreover, because of separability, if $F(U(p_1), U(f_1)) \leq F(U(p_1), U(f_2))$, then $F(U(p_2), U(f_1)) \leq F(U(p_2), U(f_2))$. The utilities of futures settles the utilities of worlds no matter what the past is like. Hence, in a decision problem, deliberators may ignore the past given the utilities of futures.

Separability in this chapter's sense imposes constraints on utility functions for vectors and subvectors; however, it does not entail that the utility curve for a subvector has the same shape for all ways of fixing its complement. For vectors (x, y), the shape of $U(x)$ may depend on the value of y. Then, the utility of a value of x depends on the context, even though the values of x have a constant order. Consider baskets of apples and bananas. A basket's utility depends on the basket's number of apples and number of bananas. Fix the number of bananas and so the utility of bananas. Then, a basket's utility increases as the number of apples increases, but the increase in its utility depends on the number of bananas in the basket. The greater the number of bananas in the basket, the less utility more apples add to the basket. The number of bananas affects the utility of apples. So, the shape of the utility curve for apples changes as the number of bananas in the basket changes. An apple's marginal utility is not independent of the number of bananas. It declines as the number of bananas increases. The utility curve for apples flattens as the number of bananas increases. However, for each quantity of bananas, the order of baskets agrees with the order of quantities of apples. Increasing the quantity of bananas, the utility of apples goes down, but the preference order is the same – the more apples the better.

The effect carries over to cases in which probabilities govern possible outcomes. In such cases, a *gamble* in the economic sense represents a prospect of gaining apples by specifying the probability of each number of apples that may issue from the prospect. The quantity of bananas may affect preferences among gambles concerning apples, even though the agent still prefers more apples to fewer so that the type of separability this chapter introduces obtains.[9]

[9] Some definitions of separability attend to marginal utility. Black (2002) states that if x and y are variables having as values amounts of a commodity, a utility function for a pair (x, y) is separable if and only if the marginal utility of x is independent of y, and the marginal utility of y is independent of x. Gollier (2001: 202) states that apples and bananas are not separable if the marginal utility of apples

Let us call the type of separability that preserves order *ordinal separability* and a more demanding type that preserves the shape of the utility function *metric separability*. Metric separability assumes that a utility function represents more than order. It may represent intensities of preference as well as preferences. Ordinal, but not metric, separability holds for the baskets of apples and bananas. Because ordinal separability suffices for simplification of choices, I treat only it.

1.1.3 Complementarity

Some types of *complementarity* oppose separability. Suppose that a first variable has either a left glove or a right glove as value, and similarly a second variable has either a left glove or a right glove as value. The order of pairs formed using a value of each variable does not generate a single order of the first variable's values for all ways of fixing the second variable's value. The first glove is not separable from the second glove because the order of values for the first glove, right and left, depends on whether the value of the second glove is right or left. Suppose that the order of pairs is (l, l), (r, r), (l, r), (r, l) from lowest to highest, except with indifference between the last two pairs. With a left second glove, the order puts a right first glove higher than a left first glove. In contrast, with a right second glove, it puts a left first glove higher than a right first glove. No way of defining the order of first elements and the order of second elements makes the order of either set of elements agree with the order of pairs given each element of the other set. Neither variable is separable from the other because the values of the two variables are complementary; the good pairs have one right glove and one left glove.[10]

Not all types of complementarity oppose separability. Habit creates complementarity between past and future events. Because of habit, past

varies with the number of bananas already in a consumption bundle so that the value of an extra apple depends on how many bananas are in the bundle. Blackorby, Primont, and Russell (2008) define separability in terms of marginal rates of substitution of commodities. Two variables are separable from a third if and only if the marginal rate of substitution between the two variables is independent of the third. This type of separability implies an aggregation function (a subutility function) over the two variables that is independent of the third. For a single variable, its separability from the others is its having the same marginal utility given fixed values of the other variables. The definition assumes that the values of variables are continuous so that marginal rates of substitution are defined. Separability defined in terms of marginal utilities puts subutility functions for subvectors on the same scale and grounds a representation of an order of vectors that adds the subutilities of subvectors partitioning a vector to obtain the vector's utility.

[10] The order of pairs of gloves need not define the order of gloves in isolation. Preferences may order gloves taken in isolation and not just pairs of gloves. In a rational ideal agent, the order of single gloves may derive from each glove's evaluation according to a probability-weighted average of evaluations of pairs it may form.

activities affect the value of future activities. A typical person enjoys exercising an acquired skill. For example, a tennis pro enjoys playing tennis. Past training affects the value of future tennis. Despite this complementarity between past and future events, the order of worlds agrees with the order of futures. When future tennis increases the value of a future, it also increases the value of a world with the future given any compatible settling of the past. This correlation of values makes the future separable from the past. Chapter 4 elaborates the point.[11]

1.1.4 Compositionality

The utility of a composite often depends on the utilities of its components. For example, the utility of a basket of goods may depend on the utilities of the goods. This section introduces the compositionality of a property's application to composites and explains its relation to the separability of the property's application to the composites. It treats utility as a quantitative property that a proposition's realization may have to various degrees.

A property's application to a composite is *compositional*; that is, it decomposes into its applications to the composite's parts if and only if the property's application to the composite is a function of the property's application to its parts, taken in their order if the composite orders them. Because compositionality may hold for one division but not for another division of a whole into parts, a complete claim of compositionality specifies not only a property but also a division of a whole into parts. A complete claim explicitly states the division into parts that a property's compositionality assumes if context does not settle the division. Compositionality for a property and a division of composites, for some range of composites and a context, asserts the existence of a function that obtains the property's application to a composite from the property's application to the composite's parts. For utility's application to a proposition, a principle of compositionality may specify a method of dividing the proposition into parts. One principle of compositionality, for a type of utility, may divide a conjunction into its conjuncts.

[11] The mutual separability of two variables does not entail that the utility of the first's value and the utility of the second's value contribute independently to their combination's utility. Mutual separability holds trivially for a single vector (x_1, y_1). It fails only for an order of multiple vectors and subvectors. So the argument variables of $U(x, y)$ are mutually separable given the unique vector of variable values (x_1, y_1). However, independent contribution may fail for a single vector. The utilities of the values of two variables may not contribute independently to the vector's utility. The values of the variables may be complementary, as in the case of a right glove and a left glove, despite the variables' mutual separability given their unique vector of values.

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For a kind of utility U applied to an ordered composite $(x_1, x_2, \ldots, x_n)$, compositionality claims that $U(x_1, x_2, \ldots, x_n) = F(U(x_1), U(x_2), \ldots, U(x_n))$. This equation does not assert that the utilities of a whole's parts settle the whole's utility. It recognizes that the order of the parts may contribute to the whole's utility. The composite may present events in their temporal order. The events' temporal order may affect the composite's utility. Taking an introductory logic course before an advanced logic course is better than taking the courses in the reverse order. The order of courses in a sequence settles the order of arguments in the function that obtains the sequence's utility from the courses' utilities. If composites order parts, compositionality uses a function that incorporates their order. Utility's compositionality for pairs (x, y) entails that $U(x)$ and $U(y)$ in this order settle $U(x, y)$ so that no variation in conditions that does not affect $U(x)$ and $U(y)$ affects $U(x, y)$.

According to the principle of *interchange of equivalents*, for a property applying to objects and the composites they form, replacing a composite's part with an object of the same value yields a new composite with the same value as the original, provided that the new part does not interact with remaining parts to generate an additional part. Take a composite with several parts, and assume that a utility function applies to each part as well as to the whole composite. What happens to the composite's utility after replacing some part with an object of the same utility? If the principle of interchange of equivalents holds, then the composite's utility stays the same, assuming that the exchange is not productive. For example, suppose that a shopper's basket of goods includes yogurt, and the shopper exchanges the yogurt for another brand of the same utility. If the principle applies, the shopping basket's utility before the exchange is the same as after the exchange.

Compositionality and interchangeability both treat relations between a property's application to wholes and its application to parts. In fact, interchangeability is equivalent to compositionality; that is, it is necessary and sufficient for compositionality, given a common range of composites.[12]

Compositionality and separability live in the same neighborhood. Compositionality governs a property's application to composites and their components. Separability governs orders of composites and their components, and a property such as utility may generate the orders. Both compositionality and separability apply to variables, for their ranges of values. Both are trivial if each variable has just one value. Types of complementarity oppose both compositionality and separability.

[12] Westerståhl and Pagin (2011) define compositionality and note its equivalence with interchange of equivalents, given observance of domain restrictions.

A difference in quantification distinguishes compositionality and separability. A utility function $U(x, y)$'s compositionality and its mutual separability for x and y entail, respectively, generalizations with quantifiers of different scope. According to compositionality, for all x, y, $U(x, y) = F(U(x), U(y))$ for some F. According to mutual separability, for all y, for all x and x', $U(x) \geq U(x')$ if and only if $U(x, y) \geq U(x', y)$, and similarly for all x, for all y and y'. Compositionality and separability differ over arrangements of factors to fix and to let vary.

In the utility function $U(x, y)$, the first variable's separability from the second does not suffice for compositionality. If x is separable from y, then $U(x, y) = F'(U(x), y)$ for some function F'. If two values of y have the same utility, the utility of a value of x and a value of y may depend on y's value and not on the utility of y's value despite the separability of x from y. Then, it is not the case that $U(x, y) = F(U(x), U(y))$ for some function F. Compositionality follows from the variables' separability from each other, however. If x and y are separable from each other, then $U(x, y) = F(U(x), U(y))$ for some function F. More generally, in a utility function for vectors, separability of each argument variable from the others implies compositionality. If the order of each argument's values, holding the other arguments fixed, agrees with the vectors' order, then the utility of a vector is a function of the utilities of the arguments' values.[13]

In the example, the implication is trivial if no values of x have equal utility, and no values of y have equal utility. To make the implication nontrivial, suppose that y_1 and y_2 have the same rank and so the same utility, let (x_1, y_1) and (x_1, y_2) have the same rank, and let (x_2, y_1) and (x_2, y_2) have the same rank. Furthermore, let x_1 come below x_2, and let (x_1, y_2) come below (x_2, y_1). Then, x and y are mutually separable. Substituting equivalents, y_2 for y_1, to move from (x_1, y_1) to (x_1, y_2) does not change the composite's utility. The principle of interchange, which is equivalent to compositionality, holds.

Although each variable's separability from the others implies compositionality, compositionality does not imply each variable's separability from the others (even if the order of vectors defines the order of subvectors). A

[13] Suppose that in $U(x, y)$, the variable x is separable from y, and the variable y is separable from x. That is, the order of (x, y)'s values agrees with the order of x's values however y's value is fixed, and the order of (x, y)'s values agrees with the order of y's values however x's value is fixed. Then, $U(x, y) = F'(U(x), y)$ for some function F' given x's separability from y, and $U(x, y) = F(U(x), U(y))$ for some function F given y's separability from x. The function F increases with increases either in $U(x)$ or in $U(y)$. In a utility function with more than two argument variables, despite separability of one argument variable from the others, aggregation of the other argument variables, and so the whole set of argument variables, may not be compositional. Separability of each argument variable from the others ensures compositionality, however. Broome (1991: 69) states this point.

property's compositionality does not entail the separability of components of composites with the property because separability concerns orders. The property may not impose an order on the composites and their components. Meaning is compositional, supposing that Frege is right about meaning; the meaning of a sentence is a function of the meaning of its parts. However, sentence meanings do not order sentences, and word meanings do not order words. No agreement between the order of subjects and the order of sentences obtains when a sentence's predicate is held constant but its subject is allowed to vary. The words of a sentence are not separable with respect to any function representing order.

Even if a compositional property imposes an order, compositionality does not entail separability. Consider an order of pairs (x_1, y_1), (x_2, y_2), (x_1, y_2), (x_2, y_1) from lowest to highest. The variable x is not separable from y (given any order of elements). With y_1 fixed, the agreeing order for the first element is $x_1 < x_2$, but with y_2 fixed, the agreeing order for the first element is $x_2 < x_1$. No single order of x_1 and x_2 holds however y's value is fixed. Although in $U(x, y)$ the variable x is not separable from the variable y because it is not separable from y in the order of vectors (x, y) that the function represents, $U(x, y)$ may be a function of $U(x)$ and $U(y)$, and so compositional. In the example, let the order of pairs give the pairs their utilities, let the order of elements by subscript give the elements their utilities, and replace the elements with their utilities to obtain a function from their utilities to the pairs' utilities. Imagine that $U(x_1) = 1$, $U(x_2) = 2$, $U(y_1) = 1$, $U(y_2) = 2$, $U(x_1, y_1) = 1$, $U(x_2, y_2) = 2$, $U(x_1, y_2) = 3$, and $U(x_2, y_1) = 4$. Then, construct F so that $F(1, 1) = 1$, $F(2, 2) = 2$, $F(1, 2) = 3$, and $F(2, 1) = 4$. As a result, $U(x, y) = F(U(x), U(y))$, which establishes compositionality.

Other cases in which compositionality and separability come apart begin with compositionality and show how separability fails. If a pair's utility equals a ratio of its elements' utilities, then the pair's utility is compositional, but the order of pairs inverts the order of the denominator's values given a fixed numerator, and so the denominator is not separable from the numerator. Next, suppose that x and y are variables with varying possible values. $U(x, y) = F(U(x), U(y))$, so compositionality holds. Suppose that $U(x_1, y_1) = F(U(x_1), U(y_1)) = 1$, $U(x_1, y_2) = F(U(x_1), U(y_2)) = 2$, $U(x_2, y_1) = F(U(x_2), U(y_1)) = 2$, and $U(x_2, y_2) = F(U(x_2), U(y_2)) = 1$. The order of y_1 and y_2 agreeing with the order of pairs changes as U's first argument shifts from x_1 to x_2. So, the second argument place is not separable from the first.

To illustrate concretely compositionality without separability, imagine that the utility of wealth and security together is a function of the utility of wealth and the utility of security. That is, no utility-divergent pairs of a level of wealth and a level of security have levels of wealth with the same utility and levels of security with the same utility. Suppose that if security is set low, then increasing the utility of wealth from a high level to a slightly higher level in a way that triggers publicity decreases overall utility by increasing the probability of kidnapping. In contrast, if security is set high, then increasing the utility of wealth increases overall utility. The utility of a composite may be a function of its components' utilities without being a separable function of their utilities, that is, a function in which each argument is separable from the others.

Separability entails a type of independence. Does a weaker type of independence combine with compositionality to yield separability? Independence may hold between variables, variables' values, or their utilities besides holding between orders of variables' values, as with separability. Some types of independence involve constancy as conditions change, and some involve the absence of interaction. Consider a set of variables, a subset of them, and vectors and subvectors that list values of the variables in the set and in the subset, respectively. Suppose that preferences order the vectors and subvectors and that for a single subvector, its utility is the same given any values of its complement. This is a type of independence involving constancy for utilities of variables' values. Suppose that it holds for all subvectors of values of variables in the subset. Then, the order of subvectors of the subset's values is the same for all values of the subset's complement. This is conditional independence of the order. Does compositionality and conditional independence imply separability? No, separability of the subset of variables from its complement requires that the order of subvectors agree with the order of vectors for all values of the subset's complement. Even though the order of subvectors is independent of its complement, it need not agree with the order of vectors.

To illustrate, suppose that one variable is for a world's past and another is for a world's future (including the present). Compositionality for utility entails that when the past is fixed, a world's utility is a function of its future's utility. It prohibits assigning different utilities to two composites whose elements have the same utility profiles. With the past fixed, futures of the same utility yield worlds of the same utility. A future's utility settles a world's utility. Nonetheless, the direction of a future's influence on a world's utility may vary from future to future. For one future, its utility may increase the world's utility, whereas for another future, its utility may decrease the world's utility. This happens if the utility of the past is two, the utility of the first future is one, the utility of the second future is two, and the future's utility adds to the utility of the past if the total of past and future utilities is three or less and subtracts from the utility of the past if the total of past and future utilities is more than three because of penalties for exceeding a limit on their sum. Conditional independence makes the direction of influence uniform for all ways of fixing the past. However, separability requires more. It requires that with the past fixed, no matter how, the order of futures agrees with the order of worlds. That a world's utility is a function of its parts' utilities follows from the separability of each part from the others but does not entail that in a utility function for the world, each argument is separable from the others.

Compositionality simplifies evaluation of options, as does separability. It makes a composite's utility calculable from the utilities of its parts. However, compositionality is not enough to simplify deliberations given incomplete information. Suppose that utility attaches to worlds, their pasts, and their futures, and the utility of a world is a function of the utility of its past and its future, so that holding the past fixed, the utility of a world's future settles the world's utility. That is, $U(p, f) = F(U(p), U(f)) = F'(p, U(f))$ for some F and F'. Letting p have the actual past as its value, the utilities of futures settle the utilities of worlds. Given conditional independence, a future's utility influences a world's utility in a uniform direction. However, the direction of influence may depend on the past. Separability rules out that dependence and so allows for ignorance of the past. Given that the future is separable from the past, $F(U(p), U(f))$ is strictly monotonically increasing with respect to $U(f)$. Strict monotonicity puts the order of futures in step with the order of worlds for any value of p. Given separability, an evaluator need not know the past to know that the order of futures agrees with the order of worlds. This simplifies options' evaluations.

1.2 Establishing separability

Various principles of separability govern the various types of utility. Chapter 3 argues for a fundamental form of separability, and Chapters 2 and 4–6 argue for derived forms of separability. This section assembles general strategies for arguments supporting separability.

1.2.1 Norms

Separability is a relation among orders of composites or utility functions representing the orders. Rationality requires some forms of separability, as later chapters argue. The separation of the future from the past is a

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normative requirement. For a rational ideal agent and type of utility, a world's utility separates into the past's utility and the future's utility. The separation encourages deliberations to be forward looking and to ignore sunk costs.

Utility is rational degree of desire in an ideal agent. For a type of desire and utility, showing that rationality requires an ideal agent's degrees of desire to be separable also shows that utilities, a rational ideal agent's degrees of desire, are separable. Utilities obey all principles of rationality including principles of separation. Utilities do not obey the principles by definition alone but also because of rationality's regulation of degrees of desire. A type of utility's separability is normative given that an ideal agent's rationality is responsible for it.

Methods of establishing a form of separability depend on whether the separability is normative or empirical. If it is empirical, then gathering evidence is a basic method of supporting it. For example, evidence supports the combined gas law, according to which the pressure of a gas is a separable function of the gas's temperature and the reciprocal of its volume. The scientific method directs justification of an empirical principle of separability. If separability is normative, then its support comes from normative principles and resolutions of cases. For example, the moral theory, utilitarianism, advances a normative principle claiming that utility for a group of people is a separable function of utilities for the group's members. Moral principles and resolutions of cases provide its support. This book treats normative separability, in particular, separability that rationality requires. Taking separability as a norm demands philosophical rather than empirical justification.

Suppose that rationality requires a type of separability for a utility assignment. Then, the norm is an a priori truth and every set of assumptions implies it. Giving assumptions that imply it does not establish the norm. An effective argument for it states assumptions that explain the norm, perhaps normative assumptions that logically or mathematically imply the norm.

Do representation theorems offer a method of establishing separability? A representation theorem for a preference order over composites demonstrates the possibility of representing the order with a utility function that is unique given a zero point and a unit. The appendix to this chapter reviews famous representation theorems that Gérald Debreu and William Gorman prove, stating that some types of ordinal separability of variables suffice for an additive utility-representation of the order of the variables' values. A sufficient type is called *additive separability* and obtains if each set of variables is separable from its complement.

Representation theorems use empirical and normative assumptions that regiment the preferences that utilities represent. A unique representation of

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an order of composites, given a utility scale, exists only if the order has suitable complexity and structure. An additive representation's existence and uniqueness require multiple composites and require putting on the same scale utility functions that represent the order of the composites and the order of the values of variables generating the composites. Some structural assumptions are empirical rather than normative. For two variables x and y specifying amounts of commodities, commutativity, a necessary condition of an additive representation, is a norm for utility: $U(x, y) = U(y, x)$. A rational ideal agent complies with the norm and so facilitates construction of an additive utility function that represents her preferences. However, the composites' generation from variables with multiple values, a requirement for the representation, is not a norm. Whether it holds is an empirical matter. The representation theorems that the appendix reviews take separability as an empirical fact about preferences that grounds a utility-representation of preferences with certain structural properties such as additivity. They do not offer a means of justifying separability as a norm.

Normative principles of separability typically use simplifying assumptions that create a normative model. Both philosophy and economics use models. Typically, economic models of choice serve empirical goals, whereas philosophical models of choice serve normative goals. Even when models in economics and in philosophy treat the same normative topic, they differ because the disciplines have different objectives. Philosophy does not exclude a factor from a model because it makes the model difficult to apply to practical problems, whereas economics does. Because the two disciplines' models serve different purposes, they are subject to different standards. This book justifies norms of separability within a philosophical model that includes the idealization that agents are rational and cognitively unlimited, and also the restrictive assumption that agents have basic intrinsic attitudes attaching to proper parts of worlds. The idealizations and restrictions limit the book's normative principles of separation.[14]

Utilitarianism creates a philosophical normative model. The model assumes interpersonal comparisons of utility for alternative courses of action affecting the people in a population. Being philosophical, the model accepts interpersonal comparisons of utility despite the practical difficulty of making such comparisons. In light of judgments about right acts, it advances the

[14] It is convenient to take a model as a possible world or set of possible worlds meeting a group of assumptions. Investigations of the model reveal the assumptions' implications. A typical normative model for decision theory is a possible world that meets a group of assumptions about agents, their decision problems, and their circumstances. Applications of principles of rationality within the model reveal the assumptions' normative implications.

principle that an act is right if and only if it maximizes collective utility. The normative principle assumes the additive separability of an act's collective utility; it is the sum of the act's utilities for the population's members. This separability is also normative unless part of collective utility's definition.

Norms of separability for personal preferences, desires, and utilities come in varying strengths. A normative principle for preferences requires the possibility of a representation that obtains the utility of a world from a separable function of the utility of the world's past and the utility of the world's future. A stronger normative principle requires, for a type of desire and utility taken as rational degree of desire, separation of a world's utility into its past's utility and its future's utility. Chapter 4 argues for the stronger normative principle. In general, the chapters presenting norms of separability target utilities taken as rational degrees of desire and not utilities defined using preferences.

What type of separability does simplification of choices require? At a minimum, it demands, for options' worlds and their preference ranking, the separability of a world part evaluated from a world part put aside, for example, assuming evaluation of options by their futures, the separability of the future from the past. However, the additivity of a world's utility, for a type of utility and division of the world, and the more demanding type of separability such additivity entails, is not superfluous.

Consider an analogy. Although utilitarianism's goal is a collective ranking of collective options, the introduction of collective utilities in addition to the collective ranking is not superfluous. The collective utilities generate the collective ranking. The options' rankings by individuals do not suffice for the collective ranking. The rankings by individuals yield, following the *Pareto principle*, according to which unanimous preference settles collective preference, only an incomplete collective ranking. Interpersonal utilities for individuals, by generating collective utilities, complete the collective ranking.

Similarly, although deliberations rank options, introducing utilities of option's worlds is not superfluous. The worlds' utilities generate a ranking of the worlds that in turn generates a ranking of the options. Ranking worlds' futures and ranking their pasts, the analog of two individuals ranking collective acts, is not enough to generate the ranking of worlds. The rankings of pasts and futures yield, following an analog of the Pareto principle, only an incomplete ranking of worlds. Utilities of pasts and futures, by generating utilities of worlds, complete the ranking of worlds. For example, take two worlds. In the first, the past is better than in the second, and in the second, the future is better than in the first. Neither world is Pareto superior to the other. However, given utilities for each past and for each future, and the additivity of worlds' utilities, a ranking of the worlds emerges. Utility's additivity makes a contribution beyond preference's separability.

Various types of separability apply to a set of parts. Additive separability of a world's components, with respect to preference, grounds addition of components' evaluations to obtain a world's utility. If a world's utility is additive (for a type of utility), then subtracting a part's evaluation is equivalent to a scale change for the world's utility. Taking world parts as variables, the order of a variable's values agrees with the order of their utilities' sum after fixing the values of the other variables. Although separation of a world's part from its complement is sufficient for separable evaluations that simplify choices, utility's additivity for some world parts is not superfluous. Its additivity for realizations of basic intrinsic attitudes grounds the definitions of temporal, spatiotemporal, and causal utility, and those types of utility yield an evaluation of a world's part that is separable from an evaluation of its complement. Utility's additivity for world parts grounds the separability of preferences that simplifies choices.

1.2.2 Fundamental separability

Some quantities are additive because of conventions of measurement. Weight, operationally defined, adopts as a *concatenation operation*, to represent with addition, combining two objects to obtain another object. Adopting a concatenation operation for a quantity's measurement establishes the additivity of the quantity measured with that operation. Weight is additive because the quantity's measure adopts combination as a concatenation operation. The weight of a combination of objects, according to the measure, is the sum of the weights of the objects it combines. The measure, the weight function, makes weight additive.

An alternative view of weight takes it as a theoretical quantity manifest in various phenomena such as pull on springs and tipping of balances. Assuming additivity in one manifestation may count against additivity in another manifestation. Weight according to balances, assumed to be additive, may not be additive according to springs. Failures of additivity in some manifestation may suggest that weight changes in combinations or that combinations produce new objects with weight. Representation theorems show how to use additivity to measure a quantity in some manifestation under assumptions, such as constancy in combinations, but do not define the theoretical quantity. Operationally defined using balances, an object's weight given a unit is relative to comparisons of a set of objects, including combinations, on a balance. However, taken as a theoretical quantity, its weight given a unit is not relative this way.[15]

Two conceptions of utility differ as the two conceptions of weight differ. According to one, utility's additivity (for a type of utility, range of composites, and their division) is a consequence of a conventional concatenation operation for composite objects. According to the other, utility is a theoretical quantity measurable in various ways. A concatenation operation, assuming its accuracy, offers a method of measuring but not defining the theoretical quantity. The first conception defines utility as a construct from preferences meeting certain constraints and makes a composite's utility given a unit relative to preferences over a set of composites. The second conception, taking utilities to be strengths of desire, makes utility given a unit conceptually independent of preferences and constraints on them, and nonrelative to preferences over a set of composites.

How may an argument establish additivity for a theoretical quantity such as degree of desire without establishing it as a convention of measurement, as does a typical operational definition of the quantity? A strategy uses representation theorems the appendix describes to show that if an order of composites meets certain conditions, including some separability conditions, then the order has an additive representation. However, this strategy, which is not general because of the conditions it imposes, uses the separability of an order of composites to establish the possibility of an additive representation of the order and does not establish the additivity of the theoretical quantity that generates the order.

Probability theory illustrates another method of deriving one type of separability from another. Support for additivity of probability, taken as rational degree of belief in ideal agents, may equate probabilities with additive quantities, such as ideal betting quotients. Degrees of belief yield betting quotients, and some groups of bets made with nonadditive betting quotients are vulnerable to sure losses. Ideal betting quotients are additive; the quotients for bets on disjunctions of exclusive propositions decompose into the quotients for bets on the disjuncts. Also, degrees of belief should match strengths of evidence, and the strength of evidence for a disjunction of exclusive propositions is a sum of the strengths of evidence for the disjuncts. So in an ideal agent, degrees of belief, if rational, inherit their additivity from strengths of evidence. Such arguments for probability's additivity take some type of

[15] Representation theorems explain measurement of probability and utility in ideal cases even if, for reasons that Meacham and Weisberg (2011) review, they do not offer good definitions of probability and utility.

additivity as given and derive probability's additivity from it. They show that probabilities are additive by showing that probabilities match other additive quantities, such as ideal betting quotients or strengths of evidence.

These arguments do not take probability's additivity as fundamental, so they push back the problem of establishing additivity. Suppose that probabilities are fundamentally additive. For a rational ideal agent, the degree of belief that a disjunction of incompatible propositions holds equals the sum of the degrees of belief that the propositions hold. Then, additivity receives support as an intuitive generalization of intuitions about examples. Support may also show that probability meets necessary conditions of additivity, such as separability. Given that the probability of a disjunction of exclusive propositions equals the sum of the probabilities of the disjuncts, the probability of the first disjunct is separable from the probability of the second disjunct. No matter how the probability of the second disjunct is fixed, increasing the probability of the first disjunct increases the probability of the disjunction. An order of disjunctions of mutually exclusive propositions is separable because the order of disjunctions with a common disjunct agrees with the order of the other disjuncts. Showing this separability, a necessary condition of probability's additivity, involves examining cases and generalizing from them. Showing that addition has no structural properties that the relations among degrees of belief lack, if the agent is ideal and the degrees of belief are rational, motivates measurement's taking disjunction of mutually exclusive propositions as a concatenation operation for probability. Probabilities are additive given this justified, as opposed to conventional, concatenation operation.

Arguments for a type of utility's additivity depend on whether it, and the separability it presumes, is fundamental or derived. An option's utility equals its expected utility, and an option's expected utility is a sum of the probability-utility products for the option's possible outcomes. Therefore, an option's utility is a sum of the utilities of the chances for its possible outcomes. The next chapter argues for this additivity of comprehensive utilities. It derives from probability's additivity.

Chapter 3 argues for intrinsic utility's separability and additivity. Its separability and additivity are fundamental, and so their support uses the independence of intrinsic utility's order of realizations of basic intrinsic attitudes. As Chapter 3 shows, if an agent has basic intrinsic desires for health and for wisdom, and realizing the first desire has greater weight, then it has greater weight no matter which other basic intrinsic attitudes are realized. This independence grounds intrinsic utility's separability and additivity. Independence of reasons for basic intrinsic attitudes supports the separability of the intrinsic utilities of combinations of their realizations. The argument uses a type of independence to support a type of independence but is not question begging because of the difference between independence of reasons and independence of utilities.

This chapter defined separability for a preference order and a utility function in a way that makes separability apt for simplifying choices. It also showed that complementarity opposes separability and that separability implies compositionality. Because later chapters argue for the separability of various types of utility, it identified suitable methods of arguing for separability.

1.3 Appendix: Theorems

Famous theorems treat an order's separability. They characterize relations between types of separability and show that extensive separability of variables in a set supports an additive representation of an order of vectors of the variables' values.

Suppose that a set of variables generates a rich array of vectors with an order (which may define orders of subvectors). In the set of variables, some subsets may be separable from other sets. In the set, *weak separability* holds if and only if each variable is separable from the others. As Section 1.1.4 notes, weak separability entails compositionality. *Strong separability* holds if and only if every set of variables is separable from its complement. Strong separability is necessary and sufficient for an additive representation (unique up to positive affine transformations) of the order of the vectors that the variables generate. When and only when an additive representation is possible, the variables are *additively separable*: using utilities to represent the order of occurrences at a location, for a vector the sum of occurrences' utilities represents the vector's place in the vectors' order. *Crosscutting or overlapping separability* (which has a complex definition) suffices for an additive representation because it implies strong separability. Table 1.2 uses arrows to indicate entailments. Broome (1991: 70, 82–89) states the theorems of separability and sketches proofs.[16]

As Chapter 2 introduces utility, a utility function indicates strength of desire. The function represents preferences, but preferences do not define

[16] Gorman ([1968] 1995: chap. 12) shows that the overlapping or crosscutting condition establishes additive separability. Krantz et al. (1971: sec. 6.11) state Debreu's theorem: strong separability implies additive separability. Keeney and Raiffa ([1976] 1993: chap. 3) state Debreu's and Gorman's theorems on additive separability.

Table 1.2 *Properties of an order of composites*

Additive Separability	↔	Strong Separability	→	Weak Separability	→	Compositionality
		↑				
		Overlapping Separability				

the function. Given utility's definitional independence from preferences, utility's additivity means more than the possibility of an additive utility-representation of preferences concerning values of a set of variables. The utility function's additivity means that the utility of a vector of values equals the sum of the utilities of the values (given a common scale for their utilities). The utilities of vectors and subvectors may generate preference orders for the vectors and subvectors. A unique additive utility-representation of a rational ideal agent's preferences, given a choice of utility scale, reveals utilities in the sense of strengths of desire (on the same scale) if the utilities are additive.

A strategy for establishing an additive utility-representation of preferences over a set of composites uses, first, the theorem of overlapping sets to extend separation of some subsets of variables to separation of all subsets from their complements and, second, the theorem of strong separability to transform separability of all subsets from their complements into additive separability. However, the strategy yields only an additive utility-representation and not the additivity of utility taken as strength of desire.

Utility's additivity entails that for some wholes and divisions into parts, the degree of desire for a whole equals the sum of the degrees of desire for the whole's parts. Utility's additivity is not necessary for an additive utility-representation of preferences. A preference order may be additively separable without utilities being additive, if the order has just one vector, because in this case separability holds trivially whereas additivity is substantive. Also, if the order of vectors defines the order of subvectors, the order of a variable's values according to utility may run counter to the order of vectors so that in a utility function for the vectors, the variable is not separable from its complement, and utility is not additive. Establishing additivity for utilities taken as quantitative representations of desires differs from using the additive separability of a preference order to ground an additive utility-representation of preferences. In rational ideal agents, additive separability is easier to establish for a preference order than is additivity for a utility function because utility's additivity entails an additively separable preference order, whereas an additively separable preference order does not entail utility's additivity. An additively separable preference order of composites has an additive utility-representation. However, the representation may not accurately represent utility taken as strength of desire and so may not establish utility's additivity.

In some quantitative functions, arguments are additively separable although not additively aggregated. According to probability theory, the probability of a pair of independent events equals the product of the events' probabilities: $P(A \,\&\, B) = P(A)P(B)$ if A and B are independent events. Take this equality as a principle of probability governing informationally independent events rather than as a definition of independent events. Then, for independent event-variables A and B, composites formed with binary conjunction, and a probability function P that represents comparative probability, the multiplication principle makes $P(A \,\&\, B)$ an additively separable function, but not an additive function, of $P(A)$ and $P(B)$.[17]

Using the theorems of separability to argue for utility's additivity faces two shortcomings. First, although utility's additivity holds generally, the assumptions under which the theorems show that a preference order has a unique additive utility-representation do not hold generally. Second, demonstrating that given the assumptions, a preference order has a unique additive utility-representation does not establish utility's additivity for the composites in the order. Therefore, Chapters 2–6 do not use the theorems of separability to argue for utility's additivity.

[17] A common *return-risk* method of evaluating an investment uses the investment type's coefficient of variation s/m, where s is the standard deviation for returns from the investment type, and m is the mean of returns from the investment type, or the investment's expected return. The smaller the coefficient of variation, the better the investment. Hence, the greater m/s, the better the investment. The reciprocal of the coefficient of variation yields a measure of an investment's value that uses assessments of an investment with respect to return and with respect to risk (taken in a technical sense). An investment's value, according to the evaluation, is proportional to return given a fixed risk. Also, an investment's value is inversely proportional to risk given a fixed return. Multiplying return and the reciprocal of risk yields a quantity proportional to an investment's value. The reciprocal of risk is separable from return, and return from the reciprocal of risk. Multiplication is a function besides addition with separable factors (although it may be transformed into addition using logarithms). Gorman ([1968] 1995: chap. 14) notes that addition and its transformations are pretty much the only associative functions that make factors separable. Addition is the most common way of representing factors' separability.