Glasgow Math. J. **55** (2013) 511–532.
doi:10.1017/S0017089512000699.

TAUBERIAN THEOREMS AND SPECTRAL THEORY IN TOPOLOGICAL VECTOR SPACES

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(Received 1 February 2012; revised 9 August 2012; accepted 28 November 2012;
first published online 25 February 2013)

Abstract. We investigate the spectral theory of integrable actions of a locally compact abelian group on a locally convex vector space. We start with an analysis of various spectral subspaces induced by the action of the group. This is applied to analyse the spectral theory of operators on the space, generated by measures on the group. We apply these results to derive general Tauberian theorems that apply to arbitrary locally compact abelian groups acting on a large class of locally convex vector spaces, which includes Fréchet spaces. We show how these theorems simplify the derivation of Mean Ergodic theorems.

2010 *Mathematics Subject Classification.* 37A30, 40E05, 43A45

1. Introduction. The aim of this paper is to develop enough spectral theory of integrable group actions on locally convex vector spaces to prove Tauberian theorems, which are applicable to ergodic theory. The Tauberian theorems proved in Section 5 apply to the situation where a general locally compact abelian group acts on certain types of barrelled spaces, and in particular all Fréchet spaces. This generalises the Tauberian theorem shown in [**4**], which applies only to the action of the integers on a Banach space. We use these theorems to simply derive Mean Ergodic theorems in a rather general context.

The bulk of this paper consists of using spectral theory to derive dynamical properties of the action of a locally compact abelian group G on the topological vector space E, from harmonic analytic considerations on the group itself.

The plan of this paper is as follows: Section 2 contains some basic material on harmonic analysis and topological vector spaces. First, there is a brief discussion on the harmonic analysis required and includes extensions of known results, most notably Theorem 2.1. There follows some work on locally convex topological vector spaces and vector-valued measures. These results form the core of the techniques used to transfer information from the group to the topological vector space upon which it acts.

In Section 3, we discuss integrable actions of G on E. We shall do so using general topological considerations and employing a little measure theory of vector-valued measures in the hope that it will bring some clarity to the idea (Definition 3.1). This definition elaborates on an idea introduced in [**1**] and is discussed elsewhere, such as in [**7**, **12**, **13**]. In this work, we stress the continuity properties that a group action may have, and how such continuity properties can be analysed using vector-valued measure theory. Next we introduce *spectral subspaces* by providing the definitions that appear in [**1**, **6**, **7**], namely Definitions 3.6 and 3.8. We demonstrate that they are in fact the same. There is a third kind of spectral subspace given in Definition 3.10. It is important because it is directly related to a given finite Radon measure and provides a link to the associated operator. We show how this type of spectral subspace is related to the first two mentioned. Finally, we show how to employ the tool of spectral synthesis in harmonic analysis to analyse spectral subspaces. Here the highlight is Theorem 3.15.

In Section 4 we discuss properties of operators on E induced by finite Radon measures on G. A major theme is how properties of the Fourier transform of a measure determine how the associated operator will act on spectral subspaces. This underlines the intuition that the Fourier transform on the group side of an action corresponds to spectral spaces on the vector space side. For the development of the Tauberian theorems, we need to know how to transfer convergence properties of sequences of measures to convergence properties of sequences of operators. This is done in Proposition 4.3. We prove these results by applying our knowledge of the relationship between a convergent sequence of measures and its sequence of Fourier transforms as set out in Section 2, as well as the link between spectral synthesis and spectral subspaces.

Having developed enough spectral theory, we come to the highlight of this work: the Tauberian theorems 5.1 and 5.2. Apart from being generally applicable to situations where a locally compact abelian group G acts on a Fréchet space X, it also handles general topologies of the action – where the action is continuous in the weak or strong operator topologies as well as intermediate topologies. We also discuss some general cases in Remark 5.3 where the hypotheses of the Tauberian theorems are automatically satisfied.

In Section 6 we show how, from the Tauberian results, we can quickly deduce Mean Ergodic theorems for general locally compact abelian groups acting on Fréchet spaces.

2. Harmonic analysis and locally convex vector spaces. We develop here the harmonic analysis of abelian locally compact Hausdorff groups that we shall require. Thereafter, we discuss some locally convex topologies on vector spaces.

By $M(G)$ we shall mean the Banach*-algebra of all finite Radon measures on G, where the multiplication of measures is given by their convolution. The closed ideal of all those measures absolutely continuous with respect to the Haar measure is the Banach algebra $L^1(G)$. By $\widehat{G}$ we shall mean the Pontryagin dual of G consisting of all continuous characters of G; we will call continuous characters simply 'characters' in what follows. We denote by $\widehat{\mu}$ the Fourier transform of a measure $\mu \in M(G)$ and by $\nu(\mu) = \{\xi \in \widehat{G} : \widehat{\mu}(\xi) = 0\}$ the *null-set* of μ.

We will need some results concerning the convergence of measures given the convergence of their Fourier series. We recall some elements of the representation theory of groups, as presented in Chapter 3 of [**5**]. We denote by $\mathcal{P} \subset C_b(\widehat{G})$ the set of continuous functions of positive type. Such functions are also known as positive-definite functions. (See [**5**] for the theory of functions of positive type, and both [**5**] and [**9**] for material on positive-definite functions).

Set $\mathcal{P}_0 = \{\phi \in \mathcal{P} : \|\phi\|_\infty \leq 1\}$. This set, viewed as a subset of the unit ball of $L^1(G)^*$, is weak*-compact.

We have the following extension of [**9**, Theorem 1.9.2].

THEOREM 2.1. *Let (μ_n) be a bounded sequence of Radon measures on G and let K be a closed subset of $\widehat{G}$ such that K is the closure of its interior. If $(\widehat{\mu}_n)$ converges uniformly on compact subsets of K to a function ϕ, then there is a bounded Radon measure μ such that $\widehat{\mu} = \phi$ on K.*

Proof. Without loss of generality, we may assume that $(\mu_n) \subset M_1^+(G)$, the set of positive Radon measures of norm no greater than 1. Also, for those closed $K \subset \widehat{G}$ as in the hypotheses, the space $C_b(K)$ may be identified with a norm-closed subspace of $L^\infty(K, m)$, where m is the Haar measure on $\widehat{G}$. In the sequel, all L^∞-spaces will be taken with respect to the Haar measure and so we shall simply write $L^\infty(K)$ for $L^\infty(K, m)$.

First we prove the result for compact K. Note that $\mathcal{P}_0 \subset C_b(\widehat{G})$, which can be identified with a closed subset of $L^\infty(\widehat{G})$. Furthermore, $\mathcal{P}_0$ is absolutely convex and closed in the weak*-topology on $L^\infty(\widehat{G})$ and hence closed in the finer norm topology. Now consider the restriction map $R : L^\infty(G) \to L^\infty(K)$. This map is not only norm-continuousbut also weak*-continuous. Hence, $R(\mathcal{P}_0)$ is weak*-compact and absolutely convex in $L^\infty(K)$, which implies that it is also norm-closed. Of course, $R(\mathcal{P}_0) \subset C(K)$, which can be identified with a closed subset of $L^\infty(K)$.

By Bochner's Theorem (cf **[9]** or **[5]**), the Fourier transform gives a bijection between $M_1^+(G)$ and $\mathcal{P}_0(\widehat{G})$. This means that $(\widehat{\mu}_n|_K) \subset R(\mathcal{P}_0)$. By hypothesis, this sequence converges uniformly to some $\phi \in R(\mathcal{P}_0)$ and so there is a $\tilde{\phi} \in \mathcal{P}_0$ such that $R(\tilde{\phi}) = \phi$. Consequently, there is a $\mu \in M_1^+(G)$ such that $\widehat{\mu} = \tilde{\phi}$ and so $\widehat{\mu}|_K = \phi$.

This proves the result when K is compact. To prove it in the general case when K is closed, we use the above result as well as the fact that Bochner's Theorem states that the Fourier transform is in fact a homeomorphism when $M_1^+(G)$ and $\mathcal{P}_0(\widehat{G})$ are each given their weak*-topologies.

As $(\widehat{\mu}_n|_K)$ is bounded and converges uniformly on compact subsets of K, its limit ϕ is continuous and bounded on K. Let $\mathcal{C}$ be the collection of all compact subsets of K, which are the closures of their interiors. For any $C \in \mathcal{C}$, define

$$S(C, \phi) = \{\mu \in M_1^+(G) : \widehat{\mu}|_C = \phi|_C\} \subset M_1^+(G).$$

As proved above, $S(C, \phi)$ is non-empty. Furthermore, $S(C, \phi)$ is a weak*-compact subset of $M_1^+(G)$. To see this, set

$$B_C(\phi) = \{f \in L^\infty(G) : f|_C = \phi|_C \ a.e.\}$$

and note that $B_C(\phi)$ is weak*-closed. Hence, $\mathcal{P}_0 \cap B_C(\phi)$ is weak*-compact. Finally $\mathcal{P}_0 \cap B_C(\phi)$ is the image of $S(C, \phi)$ under the Fourier transform, so $S(C, \phi)$ must be weak*-compact as well, by Bochner's Theorem.

Now the collection $\{S(C, \phi) : C \in \mathcal{C}\}$ is a collection of non-empty weak*-compact subsets of $M_1^+(G)$. This collection has the finite intersection property, because if $C_1, C_2, \ldots, C_n \in \mathcal{C}$, then

$$S(C_1, \phi) \cap S(C_2, \phi) \cap \ldots \cap S(C_n, \phi) = S(C_1 \cup C_2 \cup \ldots \cup C_n, \phi),$$

which is, of course, also non-empty. Hence, the intersection of all the sets $S(C, \phi)$ is non-empty. Let μ be in this intersection.

Then $\widehat{\mu}|_C = \phi|_C$ for every $C \in \mathcal{C}$; hence $\widehat{\mu}|_K = \phi$. □

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There is an ideal of $L^1(G)$ that will be important for our purposes: $K(G)$, the set of all functions in $L^1(G)$ whose Fourier transforms have compact support. It is shown in [**5**, **9**] that the ideal $K(G)$ is a norm-dense subset of $L^1(G)$.

Turning to closed ideals $\mathcal{I}$ of $L^1(G)$, we can define the null-set $\nu(\mathcal{I})$ as we did for individual functions and measures:

$$\nu(\mathcal{I}) = \{\xi \in \widehat{G} : \widehat{f}(\xi) = 0, \text{ for all } f \in \mathcal{I}\}.$$

Thus, to each closed ideal in $L^1(G)$ we can assign a unique closed subset of $\widehat{G}$. However, the converse is not true in general: For a closed subset K of $\widehat{G}$, there is usually more than one ideal whose null-set is K. Among all such ideals, two can be singled out: the largest, $\iota_+(K)$ consisting of all $f \in L^1(G)$ such that $\widehat{f}$ is 0 on K, and the smallest, $\iota_-(K)$, consisting of all $f \in L^1(G)$ such that $\widehat{f}$ is 0 on some *open neighbourhood* of K. From the definitions, it is clear that $\iota_-(K) \subseteq \iota_+(K)$. It is proved in [**5**, **9**] that if $\mathcal{I}$ is a closed ideal in $L^1(G)$ with $\nu(\mathcal{I}) = K$, then

$$\iota_-(K) \subseteq \mathcal{I} \subseteq \iota_+(K).$$

There are some sets K for which there is only one associated ideal. Such closed sets are called *sets of synthesis*, or *S-sets* for short. In this case, we shall call $\iota(K)$ the unique ideal associated with the S-set K. The fact that such sets have only one closed ideal in $L^1(G)$ associated with them will be used often in what follows.

Spectral synthesis will play a large part in the sequel. References for this material are [**9**, Section 7.8] and [**5**, Section 4.6]. To fix notation, we make a few remarks here. Any weak*-closed translation-invariant subset T of $L^\infty(G)$ has a *spectrum*, denoted by $\sigma(T)$, consisting of all characters contained in T. The spectrum is always closed in $\widehat{G}$.

The following theorem is crucial in the use of spectral synthesis. It is a slight restatement of [**6**, Théorème F, p. 132]. The second part is proved in [**9**, Theorem 7.8.2e)] (a special case is shown in [**5**, Proposition 4.75]).

THEOREM 2.2 **(Spectral Approximation Theorem)**. *Let V be a weak*-closed translation-invariant subspace of $L^\infty(G)$ with spectrum $\sigma(V) = \Lambda$. Then for any open set U containing Λ, any $f \in V$ can be weak*-approximated by trigonometric polynomials formed from elements of U.*

Furthermore, if Λ is an S-set, then any $f \in V$ can be weak-approximated by trigonometric polynomials formed from elements of Λ.*

We need the following modification of [**6**, Théorème A, p. 124].

THEOREM 2.3. *Let K be a compact subset of $\widehat{G}$ and let μ be a measure in $M(G)$ whose Fourier transform $\widehat{\mu}$ does not vanish on K. Then there exists a $g \in L^1(G)$ satisfying $\widehat{g}(\xi) = \frac{1}{\widehat{\mu}(\xi)}$ for all $\xi \in K$.*

Proof. By [**5**, Lemma 4.50] or [**9**, Theorem 2.6.2], there exists a summable h on G such that $\widehat{h} = 1$ on K. As $\mu * h \in L^1(G)$, we can apply [**6**, Théorème A, p. 124] to obtain a $g \in L^1(G)$ such that

$$\widehat{g}(\xi) = \frac{1}{\widehat{\mu * h}(\xi)}$$

for all $\xi \in K$. From the above equation and the fact that $\widehat{h} = 1$ on K, we see that this g is the one required. $\square$

As with Theorem 2.1, we will often be in a position where we must infer properties of a summable function μ from knowledge of its Fourier transform on a compact subset $K \subset \widehat{G}$. Of course, there will be in general many functions whose Fourier transforms agree on K.

To clarify the situation, we make use of a quotient space construction. Using $\iota_+(K)$, the largest ideal with null-set K, we can form the quotient $L^1(G)/\iota_+(K)$. Let $[\mu]$ denote an element of this quotient; it is an equivalence class consisting of all functions ν such that $\widehat{\mu}|_K = \widehat{\nu}|_K$.

The following lemma exemplifies some of the techniques employed when working with quotient spaces of $L^1(G)$, and will come in handy when proving the main result, Theorem 5.1.

LEMMA 2.4. *Let (φ_n) be a sequence in $L^1(G)$ and let K be a compact subset of $\widehat{G}$. If $([\varphi_n]) \subset L^1(G)/\iota_+(K)$ is a relatively weakly compact sequence and $\lim_{n\to\infty} \widehat{\varphi}_n(\xi)$ exists for each $\xi \in K$, then $([\varphi_n])$ is weakly convergent.*

Proof. Suppose $([\varphi_n])$ was not weakly convergent. Being relatively weakly compact, it must then contain two weakly convergent subsequences $([\varphi_{n_k}])$ and $([\varphi_{n_\ell}])$ with different limits $[\mu]$ and $[\nu]$ respectively.

The Gelfand transform $\mathcal{F}_K : L^1(G)/\iota_+(K) \to C(K)$, given by $\varphi \mapsto \widehat{\varphi}|_K$, is norm-continuous and hence weakly continuous. Therefore, $(\widehat{\varphi}_{n_k})$ and $(\widehat{\varphi}_{n_\ell})$ are weakly convergent in $C(K)$. By [**2**, Ch. VII, Theorem 2], this means that $\lim_{k\to\infty} \widehat{\varphi}_{n_k}(\xi) = \widehat{\mu}(\xi)$ and $\lim_{\ell\to\infty} \widehat{\varphi}_{n_\ell}(\xi) = \widehat{\nu}(\xi)$. By hypothesis, then $\widehat{\mu} = \widehat{\nu}$ on K and so $[\mu] = [\nu]$ in $L^1(G)/\iota_+(K)$, a contradiction. Hence, $([\varphi_n])$ is weakly convergent. $\square$

We now mention some aspects of the theory of locally convex topological vector spaces. A pair of complex vector spaces (E, E') is said to be a *dual pair* if E' can be viewed as a separating set of functionals on E and vice versa. For example, a Banach space X and its dual X^* are in duality.

The spaces E and E' induce upon each other certain topologies via their duality. We denote the smallest such topology, the weak topology, by $\sigma(E, E')$, and the largest, the strong topology, by $\beta(E, E')$. There is also the Mackey topology, called $\tau(E, E')$, which is the finest locally convex topology on E such that under this topology, E' is exactly the set all continuous linear functionals on E.

Suppose (E, E') and (F, F') are dual pairs. The set of all $\sigma(E, E') - \sigma(F, F')$-continuous linear mappings between topological vector spaces E and F is denoted by $\mathcal{L}_\omega(E, F)$). The set of all $\beta(E, E') - \beta(F, F')$-continuous linear mappings between topological vector spaces E and F is denoted by $\mathcal{L}_\sigma(E, F)$).

Now any linear map $T : E \to F$ is $\sigma(E, E') - \sigma(F, F')$-continuous if and only if it is $\tau(E, E') - \tau(F, F')$-continuous. Also, if $T : E \to F$ is $\sigma(E, E') - \sigma(F, F')$-continuous, then it is $\beta(E, E') - \beta(F, F')$-continuous. Hence, $\mathcal{L}_\omega(E, F) \subset \mathcal{L}_\sigma(E, F)$.

A linear map from a Fréchet space X to a locally convex topological vector space is continuous if and only if it is bounded. Hence, the set $B(X)$ of all bounded linear mappings from X to itself is precisely the set of all $\tau - \tau$ and hence $\sigma - \sigma$-continuous linear mappings. On a Fréchet space, the τ and β topologies are the same, so in this case we have $\mathcal{L}_\omega(X) = \mathcal{L}_\sigma(X) = B(X)$.

The weak operator topology (WOT) and the strong operator topology (SOT) can be described in terms of dual pairs. The pair $(\mathcal{L}_\omega(E), E \otimes E')$ is in duality via the bilinear

https://doi.org/10.1017/S0017089512000699 Published online by Cambridge University Press

form

$$\langle T, x \otimes y \rangle := y(T(x)).$$

Then the WOT on $\mathcal{L}_\omega(E)$ is generated by the polars of all finite subsets of $E \otimes E'$. The SOT is generated by polars of the form $A \otimes B$, where A is a finite subset of E and B is a ξ-equicontinuous subset of E'.

We make some remarks on vector-valued functions on a measure space (Ω, μ). Here the theory and proofs closely follow the standard treatments for Banach space-valued functions, such as [**11**] or [**3**]. A *μ-simple measurable function* $f : \Omega \to E$ is given by $f = \sum_{i=1}^{N} \chi_{E_i} x_i$, where $E_1, \ldots, E_N$ are μ-measurable subsets of Ω and $x_1, \ldots, x_N \in E$. A function $f : \Omega \to E$ is said to be *μ-measurable* if there is a sequence of μ-simple measurable functions (f_n) that converges μ-almost everywhere to f. This means that for any neighbourhood U of 0 in E and $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n > N$,

$$\mu\big(\{x : f_n(x) - f(x) \notin U\}\big) < \epsilon.$$

A function $f : \Omega \to E$ is said to be *μ-weakly measurable* if the scalar-valued function $e'f$ is μ-measurable for every $e' \in E'$. Finally, f is *μ-essentially $\langle$separably/metrisably$\rangle$ valued* if there is a μ-measurable subset A of Ω whose complement has measure 0 such that $f(A)$ is contained in a $\langle$separable/metrisable$\rangle$ subspace of E.

It is worth noting that there are convex separable vector spaces that are not metrisable. The strict inductive limit topology discussed in [**8**, Section VII.1] can be used to construct such topologies.

THEOREM 2.5 (Pettis Measurability Theorem). *For a σ-finite measure space (Ω, μ) and dual pair (E, E') under a topology ξ, the following are equivalent for a μ-essentially metrisably valued function $f : \Omega \to E$:*

(1) *f is μ-measurable.*

(2) *f is μ-weakly measurable and essentially separably valued.*

The proof of this theorem is a straightforward adaptation of the proof of the Banach space-valued proof presented in [**11**]. In particular, if E is separable and metrisable, the measurability and weak measurablility of a function are equivalent.

Turning to the question of the integrability of vector-valued functions, we shall require two different integrals. We define the integral of a μ-simple measurable function $f = \sum_{i=1}^{N} \chi_{E_i} x_i$ to be

$$\int_\Omega f \, d\mu = \int_\Omega \sum_{i=1}^{N} x_i \chi_i \, d\mu = \sum_{i=1}^{N} \mu(E_i) x_i. \tag{1}$$

DEFINITION 2.6. Consider a σ-finite measure space (Ω, μ) and dual pair (E, E') under a topology ξ and let $f : \Omega \to E$ be a vector valued function.

(1) If f is μ-weakly measurable, we say that it is μ-**Pettis integrable** if for every μ-measurable subset $A \subset \Omega$ there is an element $\int_A f \, d\mu \in E$ such that for all $e' \in E'$,

$$\left\langle \int_A f \, d\mu, e' \right\rangle = \int_A \langle f(\omega), e' \rangle \, d\mu(\omega). \tag{2}$$

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(2) If f is μ-measurable and Pettis integrable, we say that it is **μ-Bochner integrable** if there exists a sequence (f_n) of μ-simple measurable functions converging *a.e.* to f such that for every equicontinuous subset $\mathcal{A} \subset E'$

$$\int_\Omega \sup_{e' \in \mathcal{A}} |\langle f(x) - f_n(x), e' \rangle| \, d|\mu|(x) \longrightarrow 0 \tag{3}$$

as $n \to \infty$.

In the definition of the Bochner integral, the hypothesis that the function is Pettis integrable ensures that the sequence $(\int_\Omega f_n d\mu)$ is not only Cauchy but also convergent. We define the Bochner integral of a μ-Bochner integrable function by the limit

$$\int_\Omega f(x) \, d\mu(x) = \lim_{n \to \infty} \int_\Omega f_n(x) \, d\mu(x).$$

The proof of the existence of this limit and its independence from the particular sequence of μ-simple measurable functions chosen works exactly as in the Banach-valued case.

LEMMA 2.7. *If f is μ-Bochner integrable from the measure space (Ω, μ) into the convex vector space E, then for any equicontinuous $\mathcal{A} \subset E'$ we have*

$$\sup_{e' \in \mathcal{A}} \left| \left\langle \int_\Omega f \, d\mu, e' \right\rangle \right| \leq \int_\Omega \sup_{e' \in \mathcal{A}} |\langle f(x), e' \rangle| \, d|\mu|(x). \tag{4}$$

These integrability concepts will be crucial to understanding continuity properties of the action of a group on a convex vector space and will be used in Definition 3.1.

3. Integrable actions and spectral subspaces. We first describe the general type of Group Actions that shall concern us. In [**7**], for example, the author uses the central concept of an integrable action. Earlier Godement in [**6**] considered bounded group actions on Banach spaces to study Tauberian theorems. However, we shall work more generally, considering actions on locally convex vector spaces. Many of these ideas are important in Operator Theory and so expositions of various aspects of this material can be found in [**7**] and [**12**]. We differentiate between two types of integrability – weak and strong – and express our definition in the language of vector-valued integration theory. We use [**8**] as our reference for the theory of locally convex topological vector spaces.

We take as our starting point the concept of an integrable action, given in Definition 3.1. We pay special attention to the different topologies on $\mathcal{L}_\omega(E)$ and how this affects the continuity properties of α. From there, as in Arveson's work [**1**], we define various kinds of spectral subspaces in Definitions 3.6 and 3.8, stressing their equivalence. Other kinds of spectral subspaces are considered in Definition 3.10. In this section, we will stress the importance of S-sets, a condition from harmonic analysis that fruitfully links all these different kinds of spectral subspaces. One advantage of this is that, depending on the situation, it will be easier to recognise invariant subspaces as being spectral subspaces of one of these types; the general theory presented here will show how to view each of these subspaces in the light of the other, complementary definitions.

Definition 3.1. An **action** α of a locally compact group G on a dual pair of topological vector spaces (E, E') is a homomorphism $t \mapsto \alpha_t$ from G into $\mathcal{L}_\omega(E)$.

The action α is a **weak action** if it is bounded and continuous when $\mathcal{L}_\omega(E)$ has the WOT. The action α is a **strong action** if it is bounded and continuous when $\mathcal{L}_\omega(E)$ has the SOT.

We call a weak action α a **weak integrable action** if for each $x \in E$, the function $t \mapsto \alpha_t(x)$ is μ-Pettis integrable for every finite Radon measure μ on G.

We call a strong action α a **strong integrable action** if for each $x \in E$, the function $t \mapsto \alpha_t(x)$ is μ-Bochner integrable for every finite Radon measure μ on G.

From the definitions it is immediate that the transposed map $t \mapsto \alpha'_t$ of an action on E is an action on E' and that α' is weak or strong integrable if and only if α has that property. Indeed, we work with the space $\mathcal{L}_\omega(E)$ because it contains an operator T if and only if the transpose T' lies in $\mathcal{L}_\omega(E')$. With a view to our applications in the final section, recall from Section 2 that if E is a Fréchet space, then $\mathcal{L}_\omega(E) = \mathcal{L}_\sigma(E)$.

It is also clear that an integrable action yields a map, also called α, from $M(G)$ to $\mathcal{L}_\omega(E)$, sending μ to α_μ, where α_μ is the Pettis integral of equation (2) in Definition 2.6:

$$\langle \alpha_\mu(x), y\rangle = \int_G \langle \alpha_t(x), y\rangle \, d\mu(t).$$

The validity of this equation for the action α is the definition of an integrable action in [**1**, **7**].

Definition 3.2. For each $x \in E$ and $y \in E'$, we define the function $\eta_{x,y} : G \to \mathbb{C}$ by

$$\eta_{x,y} : t \mapsto \langle \alpha_t(x), y\rangle. \tag{5}$$

Note that each $\eta_{x,y}$ is in $C_b(G) \subset L^\infty(G)$.

For each $x \in E$, we also define a weak*-closed subspace E_x of $L^\infty(G)$ by

$$E_x = \{\eta_{x,y} : y \in E'\}^{-wk*}. \tag{6}$$

Note that E_x is translation-invariant. Indeed, for any $x \in G$,

$$\eta_{x,y}(t+s) = \langle \alpha_{t+s}(x), y\rangle = \langle \alpha_t(x), \alpha'_s(y)\rangle$$

and so the function $t \mapsto \eta_{x,y}(t+s) \in E_x$.

Part of the importance of the above definition stems from the fact that the well-defined map $\eta : E \otimes E' : x \otimes y \mapsto \eta_{x,y}$ is the transpose of $\alpha : M(G) \to \mathcal{L}_\omega(E) : \mu \mapsto \alpha_\mu$.

Lemma 3.3. *Let E be a convex vector space with topology ξ and dual E'. Let α be an action of G on E.*

(1) *If α is weak integrable then the map $M(G) \to \mathcal{L}_\omega(E)$ defined by $\mu \mapsto \alpha_\mu$ is weak-*WOT *and norm-*SOT *continuous.*

(2) *If α is strong integrable then the map $M(G) \to \mathcal{L}_\omega(E)$ defined by $\mu \mapsto \alpha_\mu$ is weak-*SOT *and norm-*SOT *continuous.*

Proof. For the first part, define as above $\eta : E \otimes E' \to C_b(G)$ by $\eta(x \otimes y) = \eta_{x,y}$. By definition of the Pettis integral, η is the transpose of $\alpha : M(G) \to \mathcal{L}_\omega(E)$. As

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$C_b(G)$ may be identified with a subspace of $M(G)^*$ by [**8**, Ch. II, Prop. 12 p. 38], α is $\sigma(M(G), M(G)^*) - \sigma(\mathcal{L}_\omega(E), E \otimes E')$-continuous, that is, weak-WOT continuous. As noted in Section 2, this means that α is also $\beta(M(G), M(G)^*) - \beta(\mathcal{L}_\omega(E), E \otimes E')$-continuous, that is, norm-SOT continuous.

For the second part, recall that a neighbourhood base of the SOT topology on $\mathcal{L}_\omega(E)$ is given by sets of the form $W(A, V)$, where A is a finite subset of E, V is an absolutely convex ξ-neighbourhood in E and $W(A, V) = \{T \in \mathcal{L}_\omega(E) : T(A) \subseteq V\}$. To prove the result, we must show that for every such $W(A, V)$ there is a weak neighbourhood U of $M(G)$ such that U is mapped into $W(A, V)$.

As α is a strong bounded action, for each $x \in E$ and V as above, $f_{x,V} : t \mapsto \sup_{e' \in V^\circ} |\langle \alpha_t(x), e' \rangle|$ is bounded and continuous. In fact, the strong boundedness of α implies that for each $x \in E$, there is an $M \in \mathbb{R}^+$ such that $\{\alpha_t(x) : t \in G\} \subset M.W(A, V)$. The polar of the finite subset $\{f_{x,V} : x \in A\} \subset C_b(G) \subset M(G)^*$ is a weak-neighbourhood in $M(G)$. Call this set U. Then by (4) of Lemma 2.7, $(1/M)U \subset W(A, V)$.

From the above, the norm-SOT continuity is trivial. □

For an action to be integrable, E in the topology $\tau(E, E')$ must possess a fair degree of completeness – complete enough for the action to be weak or strong integrable. We show that this is the case for Fréchet spaces. Recall that a Fréchet space X with dual X^* has the Mackey topology $\tau(X, X^*)$ and is metrisable and complete.

PROPOSITION 3.4. *Let G be a locally compact σ-compact abelian group and X is a Fréchet space with dual X^*. If α_t is a continuous isomorphism from X to itself such that the mapping $t \mapsto \alpha_t$ from G into $B(X)$ is continuous and bounded when $B(X)$ has the* WOT, *then α is a weak integrable action of G on the dual pair (X, X^*).*

Proof. In the sequel, fix an $x \in X$. From the hypotheses, the map $t \mapsto \alpha_t(x)$ is continuous when X has its weak topology. Hence, if K is a compact subset of G, the set $\alpha_K(x) = \{\alpha_t(x) : t \in K\}$ is weakly compact. As a Fréchet space is barrelled, by [**8**, Ch. IV, Corollary 3, p. 66], the closed convex hull of $\alpha_K(x)$, denoted by $\overline{co}(\alpha_K(x))$, is also weakly compact. Suppose that μ is a Radon probability measure on K. By [**2**, Theorem 1 p. 148], there is a unique $x_{K,\mu} \in \overline{co}(\alpha_K(x))$ such that

$$\langle x_{K,\mu}, y \rangle = \int_K \langle \alpha_t(x), y \rangle \, d\mu(t)$$

for all $y \in X^*$. By the same token, if μ is not a probability measure, there exists a unique $x_{K,\mu} \in \|\mu\| \overline{co}(\alpha_K(x))$.

Now fix a $\mu \in M(G)$ and a sequence of compact sets $K_n \subset G$ whose union is all of G. Write $x_n = x_{K_n,\mu}$ for each $n \in \mathbb{N}$. We will show that the sequence (x_n) is Cauchy in X under the $\tau(X, X^*)$-topology and hence convergent.

A neighbourhood base of 0 in the Mackey topology is by definition given by the polar sets Y°, where $Y \subset X^*$ is $\sigma(X^*, X)$-compact and absolutely convex.

Now as the map $t \mapsto \alpha_t(x)$ is bounded, the orbit set $\{\alpha_t(x) \ : \ t \in G\}$ is bounded in all topologies of the dual pair (X, X^*), including the Mackey topology. Hence, there is an $M \in \mathbb{R}$ such that

$$|\langle \alpha_t(x), y \rangle| \leq M$$

for all $t \in G$ and $y \in Y$.

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Take $N \in \mathbb{N}$ such that $|\mu|(G\backslash K_N) \leq 1/M$. Then for $n > m > N$,

$$
\begin{aligned}
|\langle x_n - x_m, y\rangle| &= \left|\int_{K_n} \langle \alpha_t(x), y\rangle \, d\mu(t) - \int_{K_m} \langle \alpha_t(x), y\rangle \, d\mu(t)\right| \\
&= \left|\int_{K_n\backslash K_m} \langle \alpha_t(x), y\rangle \, d\mu(t)\right| \\
&\leq \int_{K_n\backslash K_m} |\langle \alpha_t(x), y\rangle| \, d|\mu|(t) \\
&\leq \int_{G\backslash K_m} |\langle \alpha_t(x), y\rangle| \, d|\mu|(t) \leq 1.
\end{aligned}
$$

Hence, $x_n - x_m \in Y^\circ$ and (x_n) is Cauchy in X under the Mackey topology. As the Fréchet space is complete in this topology, the sequence has a limit. Call its limit $\alpha_\mu(x)$. We have shown that

$$
\langle \alpha_\mu(x), y\rangle = \lim_{n\to\infty} \int_{K_n} \langle \alpha_t(x), y\rangle \, d\mu(t) = \int_G \langle \alpha_t(x), y\rangle \, d\mu(t).
$$

Therefore, the action is weakly integrable. □

PROPOSITION 3.5. *Let α be a strong action of a locally compact σ-compact abelian group G on a Fréchet space X. Then α is a strong integrable action for any finite Radon measure.*

Proof. Let μ be a finite Radon measure and $x \in X$. We are going to show that $f(t) = \alpha_t(x)$ is μ-measurable by constructing a sequence of μ-simple measurable functions converging *a.e.* to it. Fix an $\epsilon > 0$ in all the constructions that follow. As G is σ-compact, there is a compact $K \subset G$ such that $\mu(G\backslash K) < \epsilon$. Because α is strongly continuous, $\alpha_K(x) = \{\alpha_t(x) \ : t \in K\}$ is compact and so for any open neighbourhood U of 0, there is a finite set $t_1, \ldots, t_n \in G$ such that the sets $\alpha_{t_1}(x) + U, \ldots, \alpha_{t_n}(x) + U$ cover $\alpha_K(x)$. Let $E_1 = \alpha_K(x) \cap (\alpha_{t_1}(x) + U)$ and $E_i = \big(\alpha_K(x) \cap (\alpha_{t_i}(x) + U)\big)\backslash E_{i-1}$ for $i = 2, \ldots, n$. Define the μ-simple function

$$
f_{U,K}(t) = \sum_{i=1}^{n} \alpha_{t_i}(x)\chi_{E_i}(t).
$$

Then $\mu(\{t \in G : f(t) - f_{U,K}(t) \notin U\}) < \epsilon$. As X is metrisable, we may choose a decreasing sequence of open neighbourhoods of 0, say (U_i), that generate the topology. Owing to the σ-compactness of G, we can choose an increasing sequence of compact subsets of G, say (K_i), whose union is all of G and such that $\mu(G\backslash K_i) < 1/i$. Define

$$
f_i := f_{U_i,K_i}.
$$

This sequence of μ-measurable functions converges *a.e.* to f. (Note that the functions f_i do not depend on ϵ for their construction.)

Next we show that f is μ-Bochner integrable. Take any equicontinuous set $\mathcal{A} \subset X^*$. Its polar $\mathcal{A}^\circ$ is a neighbourhood of 0 in X and so there is an $N_1 \in \mathbb{N}$ such that $U_n \subset (\epsilon/2|\mu|(G))\mathcal{A}^\circ$ for all $n \geq N_1$. Let $M = \sup_{t\in G} \sup_{e'\in\mathcal{A}} |\langle f(t), e'\rangle|$. This value is finite because the boundedness of the action ensures that f is bounded too. There is an $N_2 \in \mathbb{N}$ such that $M/n < \epsilon/2$ for any $n \geq N_2$.

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For any $n \geq N = \max\{N_1, N_2\}$, we compute:

$$
\begin{aligned}
&\int_{\Omega} \sup_{e' \in \mathcal{A}} |\langle f(t) - f_n(t), e' \rangle| \, d|\mu|(t) \\
&\quad \leq \int_{K_i} \sup_{e' \in \mathcal{A}} |\langle f(t) - f_n(t), e' \rangle| \, d|\mu|(t) + \int_{G \setminus K_i} \sup_{e' \in \mathcal{A}} |\langle f(t) - f_n(t), e' \rangle| \, d|\mu|(t) \\
&\quad \leq \int_{K_i} \frac{\epsilon}{2|\mu|(G)} \, d|\mu|(t) + \int_{G \setminus K_i} \sup_{e' \in \mathcal{A}} |\langle f(t), e' \rangle| \, d|\mu|(t) + \int_{G \setminus K_i} \sup_{e' \in \mathcal{A}} |\langle f_n(t), e' \rangle| \, d|\mu|(t) \\
&\quad \leq \frac{\epsilon}{2} + \frac{M}{n} + 0 \\
&\quad \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

As ϵ is arbitrary throughout the above constructions, we see that (3) of Definition 2.6 is satisfied. Hence, α is a strong integrable action. $\square$

Now we turn to the definition and elementary characteristics of *spectral subspaces*. In the sequel, α will denote a weak integrable action, unless otherwise specified. We define certain closed subspaces of E and E', which are α-invariant.

DEFINITION 3.6 (Arveson [**1**]). For each *open* subset $\Omega \in \widehat{G}$ define the **spectral *R*-subspace** $R^{\alpha}(\Omega)$ as the $\sigma(E, E')$-closure in E of the linear span of the elements $\alpha_f(x)$, where $x \in E, f \in K(G)$ and $\operatorname{supp} \widehat{f} \subset \Omega$. Similarly we define $R^{\alpha'}(\Omega)$ in E'.

For each *closed* subset Λ of $\widehat{G}$ we define the **spectral M-subspace** $M^{\alpha}(\Lambda)$ as the polar (or annihilator) of $R^{\alpha'}(\widehat{G} \setminus \Lambda)$. Similarly we define $M^{\alpha'}(\Lambda)$ in E'.

If we need to emphasise the space on which G acts, we will write $M_E^{\alpha}(\Lambda)$ and $R_E^{\alpha}(\Omega)$.

Looking at the definition of $M^{\alpha}(\Lambda)$ given above, $x \in M^{\alpha}(\Lambda)$ if and only if $\langle x, \alpha'_f(y) \rangle = 0$ for all $y \in E'$ and all f in $K(G)$ with $\operatorname{supp} \widehat{f} \subset \widehat{G} \setminus \Lambda$; i.e. if $\alpha_f(x) = 0$.

In [**7**, Theorem 8.1.4], the author lists several elementary properties of these subspaces, several of which also appear in [**1**, **13**].

DEFINITION 3.7. Let V be a $\sigma(E, E')$-closed subspace of E. We define $\gamma(V)$ to be the weak*-closure in $L^{\infty}(G)$ of the subspace

$$\{\eta_{x,y} : x \in V, y \in E'\}.$$

Furthermore, we define the **spectrum of V** to be $\sigma(V)$, where $\sigma(V) = \sigma(\gamma(V))$.

DEFINITION 3.8 (Godement [**6**]). For each closed subset $\Lambda \subset \widehat{G}$, we define $\Gamma(\Lambda)$ to be the set of all $x \in E$ such that the spectrum of the set E_x defined in Definition 3.2 is contained in Λ.

The subspace $\Gamma(\Lambda)$ is invariant under the action α. This is because if $x \in \Gamma(\Lambda)$ then the spectrum of E_x is contained in Λ and $E_{\alpha_t(x)} = E_x$.

Again it is clear that $\gamma(V)$ is a translation-invariant subspace of $L^{\infty}(G)$.

The definitions of $\Gamma(\Lambda)$ and $M^{\alpha}(\Lambda)$ for a given closed subset Λ of $\widehat{G}$ are in fact equal. We have produced both here because their constructions provide a slightly different emphasis, which will be useful when proving the main theorems and discussing examples.

PROPOSITION 3.9. *For a given closed subset Λ of $\widehat{G}$, $\Gamma(\Lambda) = M^{\alpha}(\Lambda)$.*

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Proof. Let $x \in \Gamma(\Lambda)$. Take any $g \in K(G)$ such that $\operatorname{supp}\widehat{g}$ is in $\widehat{G}\backslash\Lambda$, and any $y \in E'$. We show that

$$\langle x, \alpha'_g(y)\rangle = \int_G \langle \alpha_t(x), y\rangle g(t)\, dt = 0. \tag{7}$$

From this equation, we see at once that $x \in (R^{\alpha'}(\widehat{G}\backslash\Lambda))^\circ = M^\alpha(\Lambda)$ and so $\Gamma(\Lambda) \subseteq M^\alpha(\Lambda)$. Now let U be an open subset containing Λ such that $\widehat{g}$ vanishes on U. As the space of functions in $L^\infty(G)$ vanishing on Λ is weak*-closed, we can apply the Spectral Approximation Theorem (Theorem 2.2). For any $\epsilon > 0$ we can find a trigonometric polynomial $\sum_{n=0}^N a_n\langle t, \xi_n\rangle$, where $\xi_1, \dots, \xi_n \in U$ such that

$$\left| \int_G \langle \alpha_t(x), y\rangle g(t)\, dt - \int_G \sum_{n=0}^N a_n \langle t, \xi_n\rangle g(t)\, dt \right| < \epsilon$$

and as $\int_G \sum_{n=0}^N a_n\langle t, \xi_n\rangle g(t)\, dt = \sum_{n=0}^N a_n \widehat{g}(\xi_n) = 0$, we can conclude that

$$\left| \int_G \langle \alpha_t(x), y\rangle g(t)\, dt \right| < \epsilon.$$

As ϵ is arbitrary, (7) is proved.

For the reverse inclusion, take any $x \in M^\alpha(\Lambda)$ and $y \in E'$. If $x \notin \Gamma(\Lambda)$, then $\sigma(E_x) \not\subseteq \Lambda$. This means that there is a character $\xi \in \sigma(E_x)\backslash\Lambda$ which can be weak*- approximated by a finite combination of functions $\eta_{x,y_1}, \dots, \eta_{x,y_n}$. Any such combination is again of the form $\eta_{x,y}$, where y is a linear combination of $y_1, \dots, y_n$. We can in fact find a net y'_i in E' such that η_{x,y'_i} converges in the weak* topology to ξ . Hence, $\langle \alpha_t(x), y'_i\rangle \to \xi(t)$ as $i \to \infty$ and for any $f \in L^1(G)$,

$$\langle x, \alpha'_f(y'_i)\rangle = \int_G \langle \alpha_t(x), y'_i\rangle f(t)\, dt \to \overline{\overline{\widehat{f}}}(\xi)$$

as $i \to \infty$. But we can find an $f \in K(G)$ such that $\overline{\overline{\widehat{f}}}(\xi) \neq 0$ and $\overline{\overline{\widehat{f}}}$ is 0 on an open neighbourhood of Λ not containing ξ. Thus, $\langle x, \alpha'_f(y_i)\rangle \neq 0$, contradicting the fact that $x \in M^\alpha(\Lambda) = (R^{\alpha'}(\widehat{G}\backslash\Lambda))^\circ$.

This contradiction shows that $M^\alpha(\Lambda) \subseteq \Gamma(\Lambda)$, and the proof is complete. □

Apart from the invariant subspaces described above, there are other invariant subspaces that will be useful to us.

DEFINITION 3.10. Let $\mu \in M(G)$. The *null-space* of μ is given by

$$N(\mu) = \{x \in E : \alpha_\mu(x) = 0\}.$$

Similarly,

$$N'(\mu) = \{x \in E' : \alpha'_\mu(x) = 0\}.$$

The *range space* of μ is given by

$$R(\mu) = \{\alpha_\mu(x) : x \in E\}^{-\sigma},$$

where $^{-\sigma}$ denotes the $\sigma(E, E')$-closure of the space. One can likewise define

$$R'(\mu) = \{\alpha'_\mu(x) : x \in E'\}^{-\sigma}.$$

Again, it is easy to see that these spaces are invariant under the action α of G. We will have the need of the following straightforward relations between these four sets.

LEMMA 3.11. *The following equalities hold between the spaces $N(\mu)$, $R(\mu)$, $N'(\mu)$ and $R'(\mu)$ defined above:*

$$N(\mu) = \big(R'(\mu)\big)^\circ,$$
$$R(\mu) = \big(N'(\mu)\big)^\circ.$$

Proof. If $x \in \big(R'(\mu)\big)^\circ$, then by definition, for any $y \in E'$,

$$0 = \langle x, \alpha'_\mu(y)\rangle = \langle \alpha_\mu(x), y\rangle.$$

Hence, $\alpha_\mu(x) = 0$ and $x \in N(\mu)$. Thus, $\big(R'(\mu)\big)^\circ \subseteq N(\mu)$.

On the other hand, if $x \in N(\mu)$, then for any $y \in E'$, $\langle x, \alpha'_\mu(y)\rangle = 0$. As the set $\{\alpha'_\mu(y) : y \in E'\}$ is by definition $\sigma(E', E)$-dense in $R'(\mu)$, we see that for any $z \in R'(\mu)$, $\langle x, z\rangle = 0$. Hence, $N(\mu) \subseteq \big(R'(\mu)\big)^\circ$.

Putting the two inclusions together, $N(\mu) = \big(R'(\mu)\big)^\circ$.

The second equation is proved in the same manner as the first. $\square$

Although the subspaces given in Definition 3.10 are similar to the spectral subspaces specified in Definition 3.6, they are not in general the same. Whether or not they are equal depends on the structure of the null-set $\nu(\mu)$ of the measure. To prove the results linking the two types of subspaces, we first define certain types of ideals.

DEFINITION 3.12. Let $x \in E$ and set $\mathcal{I}_x = \{f \in L^1(G) : \alpha_f(x) = 0\}$. Similarly for $x \in E'$ we define $\mathcal{I}^x$.

These closed ideals are called the *isotropy ideals*, to borrow a term from the study of group actions on sets. Takesaki uses them in **[12]** as the basis for his analysis of spectral subspaces.

LEMMA 3.13. *Let μ be a measure in $M(G)$. The inclusion $N(\mu) \subseteq M^\alpha(\nu(\mu))$ always holds. If, furthermore, the null-set $\nu(\mu)$ is an S-set, then $N(\mu) = M^\alpha(\nu(\mu))$.*

Proof. Let $x \in N(\mu)$: this means $\alpha_\mu(x) = 0$. Now take any $\alpha'_f(y) \in R^{\alpha'}(\widehat{G}\backslash\nu(\mu))$ and set $K = \operatorname{supp}\widehat{f} \subset \{\xi : \widehat{\mu}(\xi) \neq 0\}$. By Theorem 2.3, there is an $h \in L^1(G)$ such that $\widehat{\mu}\widehat{h} = 1$ on K. Hence, $(h * \mu * f)\widehat{\ } = \widehat{f}$ and so by the Fourier Uniqueness Theorem, $h * \mu * f = f$. We conclude that

$$\langle x, \alpha'_f(y)\rangle = \langle \alpha_f(x), y\rangle = \langle \alpha_h\alpha_\mu\alpha_f(x), y\rangle = 0,$$

which shows that $x \in M^\alpha(\nu(\mu))$.

To prove the second part of the lemma, we must prove the reverse inclusion under the additional hypothesis that $\nu(\mu)$ is an S-set. Let $x \in M^\alpha(\nu(\mu))$. Consider $f \in K(G)$

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such that f has compact support in $\widehat{G}\backslash\nu(\mu)$. By definition, $M^\alpha(\nu(\mu)) = \left(R^{\alpha'}(\widehat{G}\backslash\nu(\mu))\right)^\circ$, so

$$0 = \langle x, \alpha'_f(y)\rangle = \langle \alpha_f(x), y\rangle$$

for all $y \in E'$. Hence, it must be that $\alpha_f(x) = 0$; so $f \in \mathcal{I}_x$. Now because $\nu(\mu)$ is an S-set, the set of all $f \in K(G)$ with $\operatorname{supp}\widehat{f} \subset \widehat{G}\backslash\nu(\mu)$ generates $\iota(\nu(\mu))$, the unique ideal with null-set $\nu(\mu)$. Therefore, $\mathcal{I}_x \supseteq \iota(\nu(\mu))$.

Let $(g_i)_{i\in\Gamma}$ be an approximate identity for $L^1(G)$. We have that $\mu * g_i \in \mathcal{I}_x$ for each $i \in \Gamma$. As $\mu * g_i \to \mu$ in norm, $\alpha_\mu = \lim_{i\to\infty} \alpha_{\mu * g_i}$ in the SOT and in fact

$$\alpha_\mu(x) = \lim_{i\to\infty} \alpha_{\mu * g_i}(x) = 0.$$

Therefore, $x \in N(\mu)$ and $M^\alpha(\nu(\mu)) \subseteq N(\mu)$. □

In discussing the properties of operators induced by measures in the next section, we will need to know how to approximate the functions $\eta_{x,y}$ by trigonometric polynomials. This is presented in Theorem 3.15. To prove this theorem, we proceed via the following calculation of the spectra of certain invariant subspaces.

LEMMA 3.14. *Let μ be a finite Radon measure on G and Λ be a closed subset of $\widehat{G}$. The spectra of the subspaces $M^\alpha(\Lambda)$ and $N(\nu(\mu))$ are given by*

$$\sigma(M^\alpha(\Lambda)) = \Lambda, \tag{8}$$
$$\sigma(N(\mu)) \subseteq \nu(\mu). \tag{9}$$

Proof. Equation (8) is derived directly from Definitions 3.8, 3.7 and Proposition 3.9.

For (9), note that by Lemma 3.13 $N(\mu) \subseteq M^\alpha(\nu(\mu))$ and so by (8), $\sigma(N(\mu)) \subseteq \nu(\mu)$. □

THEOREM 3.15. *Let μ be a finite Radon measure on G, $x \in N(\mu)$ and y any element in E'.*

Then for any open neighbourhood U containing $\nu(\mu)$, $\eta_{x,y}$ can be weak-approximated by a finite linear combination of characters in U.*

Furthermore, if $\nu(\mu)$ is an S-set, each $\eta_{x,y}$ can be weak-approximated by a finite linear combination of characters in $\nu(\mu)$.*

Proof. If $x \in N(\mu)$ then $E_x \subset \gamma(N(\mu))$ and $\sigma(E_x) \subset \sigma(N(\mu))$ by Definitions 3.7 and 3.8. So by Lemma 3.14, $\sigma(E_x) \subset \nu(\mu)$. Hence, by the Spectral Approximation Theorem 2.2, for any $y \in E'$, $\eta_{x,y}$ can be approximated by finite linear combinations of characters from U.

For the second part, if $\nu(\mu)$ is an S-set, then reasoning as above but appealing to the second part of Theorem 2.2, the result follows at once. □

4. Operators on spectral subspaces. A large class of operators on a vector space can be induced via the integrable action by finite Radon measures on the group. In this section we discuss how properties of the measures relate to properties of the corresponding operators. In particular, we are interested in what can be gleaned from the Fourier transform of the measures and how to handle sequences of measures and their associated operators.

Let us take a sequence of bounded $L^1(G)$-functions (φ_n). We are looking for conditions on the functions φ_n, $n \in \mathbb{N}$, which cause the corresponding operators α_{φ_n} to converge.

LEMMA 4.1. *Let U be an open subset of $\widehat{G}$ and μ, ν be finite Radon measures such that $\widehat{\mu} = \widehat{\nu}$ on U. Then the operators α_μ and α_ν are equal on $M^\alpha(K)$ for any compact subset K of U.*

In particular, let $\mu \in M(G)$ such that $\widehat{\mu} \equiv 1$ on U. If $x \in M^\alpha(K)$ then $\alpha_\mu(x) = x$.

Proof. First of all, there is an open subset V of U with compact closure such that $K \subset V \subset \overline{V} \subset U$ and an $h \in L^1(G)$ such that $\widehat{h} = 1$ on $\overline{V}$. Then $h * \mu$ and $h * \nu$ are in $L^1(G)$ and the Fourier transform of $h * \mu - h * \nu$ is zero on $V \supset K$.

Now fix an $x \in M^\alpha(K)$ and a $y \in E'$. By Theorem 3.15, for any $\epsilon > 0$ and any $f \in L^1(G)$, there is a trigonometric polynomial $\sum_{i=0}^n c_i\overline{\langle t, \xi_i\rangle}$ with $\xi_i \in V$ such that

$$\left| \int_G \eta_{x,y}(t) f(t)\, dt - \int_G \sum_{i=0}^n c_i \overline{\langle t, \xi_i\rangle} f(t)\, dt \right| < \epsilon.$$

Taking $f = h * \mu - h * \nu$, we have

$$\begin{aligned} \big|\langle \alpha_\mu(x), y\rangle - \langle \alpha_\nu(x), y\rangle\big| &= \left| \int_G \langle \alpha_t(x), y\rangle\, (h * \mu - h * \nu)(t)\, dt \right| \\ &< \epsilon + \left| \sum_{i=0}^n c_i \big(\widehat{h * \mu} - \widehat{h * \nu}\big)(\xi_i) \right| = \epsilon \end{aligned}$$

due to the equality of $\widehat{h * \mu}$ and $\widehat{h * \nu}$ on V. As ϵ and y are arbitrary, we have shown that for any $x \in M^\alpha(K)$, $\alpha_\mu(x) = \alpha_\nu(x)$. □

COROLLARY 4.2. *Let K be a compact set in $\widehat{G}$ and let U be an open set containing K. Furthermore, let $\mu \in M(G)$ such that $\widehat{\mu}$ is never 0 on U. Then α_μ is invertible on $M^\alpha(K)$ and its inverse is continuous.*

Proof. There is an open set V with compact closure such that $K \subset V \subset \overline{V} \subset U$ and $\widehat{\mu}$ is never 0 on $\overline{V}$. By Theorem 2.3, there is a function $g \in L^1(G)$ such that $\widehat{g}\widehat{\mu} = 1$ on $\overline{V}$. By Lemma 4.1, $\alpha_{g*\mu}(x) = x$ for all $x \in M^\alpha(K)$. Because $\alpha_{g*\mu} = \alpha_g\alpha_\mu$, the inverse of α_μ on $M^\alpha(K)$ is α_g. □

PROPOSITION 4.3. *Let α be an action of a locally compact abelian Hausdorff group G on the dual pair (E, E'), where E has the topology ξ. Let $K \subset \widehat{G}$ be a compact set and let (μ_n) be a sequence of functions in $L^1(G)$ such that the sequence $([\mu_n]) \subset L^1(G)/\iota_-(K)$ is weakly convergent.*

If α is weak integrable, then there is a function Φ in $L^1(G)$ such that the sequence (α_{μ_n}) converges to α_Φ on $M^\alpha(K)$ in the WOT.

If α is strong integrable, then (α_{μ_n}) converges to α_Φ on $M^\alpha(K)$ in the SOT.

Proof. Suppose that α is weak integrable. We shall show that the map

$$\alpha_K : L^1(G)/\iota_-(K) \to \mathcal{L}_\omega(M^\alpha(K)) : [\mu] \mapsto \alpha_\mu\big|_{M^\alpha(K)}$$

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is well defined and weak-WOT continuous. If μ and ν are in $L^1(G)$ such that $[\mu] = [\nu] \in L^1(G)/\iota_-(K)$, then $\widehat{\mu} = \widehat{\nu}$ on some open set containing K and so by Lemma 4.1, $\alpha_\mu = \alpha_\nu$ on $M^\alpha(K)$. Hence, α_K is well defined.

Now by Definition 3.6, the dual of $M^\alpha(K)$ is the quotient space $E'/R^{\alpha'}(\widehat{G}\backslash K)$. For any $y \in E'$, let $[y]$ denote the equivalence class of y in $E'/R^{\alpha'}(\widehat{G}\backslash K)$. The spaces $L_\omega(M^\alpha(K))$ and $M^\alpha(K) \otimes E'/R^{\alpha'}(\widehat{G}\backslash K)$ are in duality via the bilinear form $\langle T, x \otimes [y]\rangle = \langle Tx, [y]\rangle$ and the map

$$\eta_K : M^\alpha(K) \otimes E'/R^{\alpha'}(\widehat{G}\backslash K) \to C_b(G) : x \otimes [y] \mapsto \eta_{x,y}$$

is well defined. By Proposition 3.9, $\eta_{x,y} \in \iota_+(K)^\circ \subseteq \iota_-(K)^\circ$, the dual of $L^1(G)/\iota_-(K)$. From this we see that η_K is the transpose of α_K. Hence, by [**8**, Ch. II, Prop. 12, p. 38] and the fact that η_K is the transpose of α_K, the map α_K is weak-WOT continuous.

As $([\mu_n])$ is weakly convergent to $[\Phi]$ say, the sequence $(\alpha_{\mu_n}|_{M^\alpha(K)})$ is convergent to $\alpha_\Phi|_{M^\alpha(K)}$ in the WOT on $L_\omega(M^\alpha(K))$.

If α is strong integrable and $\mathcal{A} \subset E'/R^{\alpha'}(\widehat{G}\backslash K)$ is ξ-equicontinuous, for any $x \in M^\alpha(K)$, the maps $t \mapsto \eta_{x,y}(t)$ where $[y] \in \mathcal{A}$ are uniformly bounded and so is $t \mapsto \sup_{[y]\in\mathcal{A}} |\langle \alpha_t(x), [y]\rangle|$. As in the proof of Lemma 3.3, this implies that α_K is weak-SOT continuous, and hence that (α_{μ_n}) converges to α_Φ in the SOT. □

LEMMA 4.4. *Let $K \subset \widehat{G}$ be a compact S-set and let μ, ν be finite Radon measures such that $\widehat{\mu} = \widehat{\nu}$ on K. Then the operators α_μ and α_ν are equal on $M^\alpha(K)$.*

Proof. First of all, note that there is an $h \in L^1(G)$ such that $\widehat{h} = 1$ on a neighbourhood of K. Thus, $\widehat{\mu * h} = \widehat{\nu * h}$ on K. Hence, the lemma is proved for all finite Radon measures μ, ν if it is proved whenever μ and ν are functions in $L^1(G)$.

Now fix an $x \in M^\alpha(K)$ and a $y \in E'$. As K is an S-set, by Theorem 3.15, for any $\epsilon > 0$ and any $f \in L^1(G)$, there is a trigonometric polynomial $\sum_{i=0}^n c_i\overline{\langle t, \xi_i\rangle}$ with $\xi_i \in K$ such that

$$\left| \int_G \eta_{x,y}(t) f(t)\, dt - \int_G \sum_{i=0}^n c_i \overline{\langle t, \xi_i\rangle} f(t)\, dt \right| < \epsilon.$$

Taking $f = \mu - \nu$, we have

$$\begin{aligned} \big|\langle \alpha_\mu(x), y\rangle - \langle \alpha_\nu(x), y\rangle\big| &= \left| \int_G \langle \alpha_t(x), y\rangle (\mu - \nu)(t)\, dt \right| \\ &< \epsilon + \left| \sum_{i=0}^n c_i \big(\widehat{\mu} - \widehat{\nu}\big)(\xi_i) \right| = \epsilon \end{aligned}$$

due to the equality of $\widehat{\mu}$ and $\widehat{\nu}$ on K. As ϵ and y are arbitrary, we have shown that for any $x \in M^\alpha(K)$, $\alpha_\mu(x) = \alpha_\nu(x)$. □

COROLLARY 4.5. *Let $K \subset \widehat{G}$ be a compact S-set and let μ be a finite Radon measure such that $\widehat{\mu}$ never vanishes on K. Then the restriction of α_μ to $M^\alpha(K)$ is invertible.*

Proof. By Theorem 2.3, there is a $g \in L^1(G)$ such that $\widehat{g}\widehat{\mu} = 1$ on K, and so by Lemma 4.4, $\alpha_{g*\mu}(x) = x$ for $x \in M^\alpha(K)$. Thus, α_g is the inverse of α_μ on $M^\alpha(K)$. □

REMARK 4.6. One virtue of proving these results for the abstract dual pair (E, E') is that all these results remain true if E and E' are swapped around. In particular,

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Proposition 4.3 now states that $\alpha'_{\varphi_n} \to \alpha'_{\Phi}$ on $M^{\alpha'}(K)$ in the WOT or SOT, accordingly as the action α is weak or strong integrable.

5. Tauberian theorems for ergodic theory. The Tauberian theorems in this section are the culmination of our development of the spectral theory of integrable actions given in the previous sections.

THEOREM 5.1. *Let α be a weak integrable action of a locally compact abelian Hausdorff group G on a barrelled space E with dual E'. Let $\mu \in M(G)$ such that $\nu(\mu)$ is an S-set and let (φ_n) be a sequence in $M(G)$ such that*

(1) $\{\alpha_{\varphi_n}(x)\}$ *is relatively weakly compact,*
(2) $([\varphi_n]) \subset L^1(G)/\iota(\nu(\mu))$ *is relatively weakly compact,*
(3) $\lim_{n\to\infty} \widehat{\varphi}_n(\xi) > A$ *for some $A > 0$ and all $\xi \in \nu(\mu)$,*
(4) $\alpha_{\mu*\varphi_n} \to 0$ *in the* WOT.

Then we have that

(1′) (α_{φ_n}) *converges in the* WOT *to an invertible operator on $N(\mu)$, and 0 on $R(\mu)$,*
(2′) $E = R(\mu) \oplus N(\mu)$.

Note that because $\nu(\mu)$ is an S-set, $\iota_+(\nu(\mu)) = \iota_-(\nu(\mu))$, so we write $\iota(\nu(\mu))$ for this ideal, as explained in our introduction to S-sets before Theorem 2.2.

Proof. Because $\nu(\mu)$ is compact, without loss of generality, we may assume that $(\varphi_n) \subset L^1(G)$, by replacing it, if necessary, by the sequence $(\varphi_n * h)$, where $h \in L^1(G)$ such that $\widehat{h}$ is identically 1 on $\nu(\mu)$. We prove the result in the following three steps:

(1) (α_{φ_n}) converges weakly to an invertible operator on $N(\mu)$.
(2) $R(\mu) \cap N(\mu) = \{0\}$ and (α_{φ_n}) converges weakly to 0 on $R(\mu)$.
(3) (α_{φ_n}) converges weakly to an operator on E and $R(\mu) \oplus N(\mu) = E$.

Step 1. By Lemma 2.4, hypotheses (2) and (3) imply that $([\varphi_n])$ is weakly convergent. So by Proposition 4.3, the sequence (α_{φ_n}) converges on $M^{\alpha}(\nu(\mu))$ in the WOT to an operator α_Φ, where $\Phi \in L^1(G)$. By hypothesis (3), $\widehat{\Phi}$ does not vanish on $\nu(\mu)$, so by Corollary 4.5, α_Φ is invertible on $M^{\alpha}(\nu(\mu))$. Finally, note that $N(\mu) = M^{\alpha}(\nu(\mu))$ by Lemma 3.13.

Step 2. As both $R(\mu)$ and $N(\mu)$ are α-invariant subspaces of E, so is $R(\mu) \cap N(\mu)$. As noted above, α_Φ restricted to $N(\mu)$ is invertible. Hence, α_Φ restricted to $R(\mu) \cap N(\mu)$ is also invertible. Pick any $x \in R(\mu) \cap N(\mu)$ and let $y \in R(\mu) \cap N(\mu)$ such that $\alpha_\Phi(y) = x$.

The sequence (α_{φ_n}) is point-wise bounded on $N(\mu)$ and each one is continuous in the $\tau(E, E')$-topology on E. As E is barrelled, this is exactly the strong topology on E and we may use the Banach–Steinhaus theorem [**8**, Ch. IV, Theorem 3, p. 69] to conclude that (α_{φ_n}) is equicontinuous on $N(\mu)$. In other words, for any weak neighbourhood V of 0 in $N(\mu)$, there is a $\tau(E, E')$-neighbourhood U such that

$$\alpha_{\varphi_n}(U) \subset V/3$$

for all $n \in \mathbb{N}$. Furthermore, as $R(\mu)$ is the closure of the space of elements of the form $\alpha_\mu(e)$ for $e \in E$, we can find a $y' \in E$ such that $\alpha_\mu(y') - y \in U$.

Hence, $\alpha_{\varphi_n}\alpha_\mu(y') - \alpha_{\varphi_n}(y) \in V/3$ for all $n \in \mathbb{N}$. By hypothesis (4), there exists an N_1 such that $\alpha_{\varphi_n}\alpha_\mu(y') \in V/3$ for all $n \geq N_1$. Because $\alpha_{\varphi_n} \to \alpha_\Phi$ in the WOT on $N(\mu)$, there exists an N_2 such that $\alpha_{\varphi_n}(y) - \alpha_\Phi(y) = \alpha_{\varphi_n}(y) - x \in V/3$ for all $n \geq N_2$.

Hence,

$$x = \alpha_{\varphi_n}\alpha_\mu(y) - \big(\alpha_{\varphi_n}\alpha_\mu(y) - \alpha_{\varphi_n}(y)\big) - \big(\alpha_{\varphi_n}(y) - x\big) \in V/3 + V/3 + V/3 = V$$

for all $n \geq \max\{N_1, N_2\}$. As V is arbitrary, $x = 0$ and $R(\mu) \cap N(\mu) = \{0\}$.

The same technique shows that $\alpha_{\varphi_n} \to 0$ in the WOT on $R(\mu)$. For any weak neighbourhood V, there is a $\tau(E, E')$-neighbourhood U such that $\alpha_{\varphi_n}(U) \subset V/2$, as we have seen above. Furthermore, there is a $y' \in E$ such that $\alpha_\mu(y') - y \in U$. As $\alpha_{\varphi_n * \mu} \to 0$ weakly, there is an $N \in \mathbb{N}$ such that $\alpha_{\varphi_n}\alpha_\mu(y) \in V/2$ for all $n \geq N$. Hence,

$$\begin{aligned}\alpha_{\varphi_n}(y) &= \alpha_{\varphi_n}(y) - \alpha_{\varphi_n}(\alpha_\mu(y)) + \alpha_{\varphi_n}(\alpha_\mu(y)) \\ &\in V/2 + V/2 = V\end{aligned}$$

for all $n \geq N$; hence, $\alpha_{\varphi_n}(y) \to 0$ as $n \to \infty$.

Step 3. First we show that $(\alpha_{\varphi_n}(x))$ converges weakly for every $x \in E$. As this sequence is relatively weakly compact, if it is not convergent, we can find two subsequences with different limits:

$$\alpha_{\varphi_{n_i}}(x) \to x_0 \text{ and } \alpha_{\varphi_{n_j}}(x) \to x_1$$

with $x_0 \neq x_1$. As $\lim_{i\to\infty} \alpha_\mu\alpha_{\varphi_{n_i}}(x) = 0 = \lim_{j\to\infty} \alpha_\mu\alpha_{\varphi_{n_j}}(x)$ by hypothesis, x_0 and x_1 are in $N(\mu)$.

So $x_0 - x_1 \notin R(\mu)$ because $R(\mu) \cap N(\mu) = \{0\}$. This means that there is a y in $R(\mu)^\circ$ such that $\langle x_0 - x_1, y\rangle \neq 0$. By Lemma 3.11, $R(\mu)^\circ = N'(\mu)$ and

$$\begin{aligned}\langle x_0, y\rangle &= \lim_{n\to\infty} \langle \alpha_{\varphi_{n_i}}(x), y\rangle \\ &= \lim_{n\to\infty} \langle x, \alpha'_{\varphi_{n_i}}(y)\rangle \\ &= \langle x, \alpha'_\Phi(y)\rangle\end{aligned}$$

where in the last equality we invoked the Remark 4.6 at the end of the previous section. Similarly,

$$\langle x_1, y\rangle = \langle x, \alpha'_\Phi(y)\rangle$$

and so $\langle x_0, y\rangle = \langle x_1, y\rangle$, which is a contradiction. Hence, $(\alpha_{\varphi_n}(x))$ is weakly convergent for all $x \in E$.

We define $T(x) = \lim_{n\to\infty} \alpha_{\varphi_n}(x)$ so that T is continuous by the Banach–Steinhaus Theorem and the range of T is $N(\mu)$ by hypothesis (4). Furthermore, $\ker(T) = R(\mu)$, for if $x \in \ker(T)$, then for all $y \in E'$,

$$0 = \langle Tx, y\rangle = \langle x, T'y\rangle.$$

As $T'y \in N'(\mu)$, $x \in \big(N(\mu)\big)^\circ = R(\mu)$, so $\ker(T) \subseteq R(\mu)$. By the hypotheses of the theorem and the definition of T, $R(\mu) \subseteq \ker(T)$.

Now if $\rho \in L^1(G)$ such that $\widehat{\rho}\widehat{\Phi} = 1$ on $\nu(\mu)$, then α_ρ and α_Φ are inverses on $N(\mu)$ and so the operator $P = \alpha_\rho T$ is a projection whose range is $N(\mu)$ and whose kernel is $\ker(T) = R(\mu)$. This proves that $E = R(\mu) \oplus N(\mu)$. □

We can prove Theorem 5.1 for other topologies on $L_\omega(E)$.

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THEOREM 5.2. *Let α be a strong integrable action of a locally compact abelian Hausdorff group G on a barrelled space E with dual E'. Let $\mu \in M(G)$ such that $\nu(\mu)$ is an S-set and let (φ_n) be a sequence in $M(G)$ such that*

(1) $\{\alpha_{\varphi_n}(x)\}$ *is relatively weakly compact,*
(2) $([\varphi_n]) \subset L^1(G)/\iota(\nu(\mu))$ *is relatively weakly compact,*
(3) $\lim_{n\to\infty} \widehat{\varphi}_n(\xi) > A$ *for some* $A > 0$ *and all* $\xi \in \nu(\mu)$,
(4) $\alpha_{\mu*\varphi_n} \to 0$ *in the* SOT.

Then we have that

(1′) (α_{φ_n}) *converges in the* SOT *to an invertible operator on* $N(\mu)$, *and* 0 *on* $R(\mu)$,
(2′) $E = R(\mu) \oplus N(\mu)$.

Proof. All parts of Theorem 5.2 but the strong convergence of (α_{φ_n}) to T follow immediately from Theorem 5.1. But $E = R(\mu) \oplus N(\mu)$ and by Proposition 4.3 the convergence is strong on $N(\mu)$ and by hypothesis it is also strong on $R(\mu)$. □

REMARK 5.3. We now make some remarks on further generalisations as well as specific situations where the hypotheses of the Tauberians theorems can always be shown to hold.

Different operator topologies: The above theorem remains true when the SOT on $L_\omega(E)$ is replaced by any weaker topology in the following sense. If $\mathcal{A}$ is a collection of $\sigma(E', E)$-bounded subsets of E', we can form the topology of $\mathcal{A}$-convergence on $L_\omega(E)$ given by the neighbourhood base

$$W_{A,V} = \{L \in L_\omega(E) : L(A^\circ) \subseteq V\},$$

where $A \in \mathcal{A}$ and V is a bounded set in E. Then (α_{φ_n}) will converge in the topology of $\mathcal{A}$-convergence to an operator invertible on $N(\mu)$ and 0 on $R(\mu)$.

Reflexive spaces: The condition that $\{\alpha_{\varphi_n}(x)\}$ be relatively weakly compact is routinely satisfied in a number of general cases: for instance, if $(\varphi_n) \in M(G)$ is bounded, then $\{\alpha_{\varphi_n}(x)\}$ is weakly bounded for all $x \in E$. If E is in fact reflexive, then $\{\alpha_{\varphi_n}(x)\}$ is automatically relatively weakly compact.

Relatively weakly compact sequences: If in the above theorems the sequence $(\varphi_n) \subset L^1(G)$ is relatively weakly compact, by Lemma 3.3 the set $\{\alpha_{\varphi_n}\}$ is relatively weakly compact in the WOT and so for any $x \in E$, $\{\alpha_{\varphi_n}(x)\}$ is relatively weakly compact in E. Also, as the quotient map from $L^1(G)$ to $L^1(G)/\iota_+(K)$ is weakly continuous, the sequence $([\varphi_n])$ is also relatively weakly compact in $L^1(G)/\iota_+(K)$. Thus, the relative weak compactness of (φ_n) ensures that the first two hypotheses of the Tauberian theorems are satisfied.

6. Applications to ergodic theorems. In this section, we show how to use the Tauberian theorems 5.1 and 5.2 to prove results in ergodic theory. By a judicious choice of the measures μ and φ_n, we can quickly prove several Mean Ergodic theorems.

By $F(E)$ we mean the (closed) subspace of all α-invariant elements in E. By Lemma 4.4, the elements of $M^\alpha(\{0\})$ are fixed because $\widehat{\delta_t} = \widehat{\delta_0}$ on $\{0\}$, so $\alpha_t = \alpha_0 = id$ on $M^\alpha(\{0\})$ for all $t \in G$. Hence, $M^\alpha(\{0\}) \subseteq F(E)$. On the other hand, if $x \in F(E)$, then for any $t \in G$ and $y \in E'$, $\langle \alpha_t(x), y\rangle = \langle x, y\rangle$, so by Definition 3.8, $x \in \Gamma(\{0\})$, which equals $M^\alpha(\{0\})$ by Proposition 3.9. Hence, $F(E) = M^\alpha(\{0\})$.

Let us now discuss the generalisation of the classical Mean Ergodic theorem in the context of Fréchet spaces. Suppose that X is a Fréchet space and T is a power-bounded automorphism of X – that is, for any bounded subset C of X, there is a bounded subset B such that $T^n(B) \subseteq C$ for all n. By Proposition 3.5, T induces a strong integrable action α of the group $\mathbb{Z}$ on X. Suppose that the convex hull of $\{T^n(x) : n \in \mathbb{Z}\}$ is weakly relatively compact for each $x \in X$. (This is always true if X is reflexive, for example, because then every weakly bounded set is weakly relatively compact, as shown in [**8**]).

Then the Mean Ergodic theorem states that there is a projection P_F of X onto $F(X)$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} T^i \longrightarrow P_F \tag{10}$$

in the SOT. To prove this, on $\mathbb{Z}$ define the measures $\mu = \delta_0 - \delta_1$ and $\varphi_n = \frac{1}{n}\chi_{[0,n-1]}$ for each $n \in \mathbb{N}$, where $\chi_{[0,n-1]}$ is the characteristic function of the set $\{0, 1, \dots, n-1\}$. As $\mu * \varphi_n$ converges to 0 in norm, by Lemma 3.3 we conclude that $\alpha_{\mu*\varphi_n} \to 0$ in the SOT.

Now $\nu(\mu) = \{1\}$, where 1 is the identity element of $\mathbb{T}$. Being a singleton, $\{1\}$ is an S-set. (The fact that a singleton is an S-set is a direct consequence of [**5**, Corollary 4.67]). Furthermore, on $\{1\}$, we see that obviously $\lim_{n\to\infty} \widehat{\varphi}_n(1) = 1$, and $[\varphi_n] = [\varphi_m]$ in $L^1(\mathbb{Z})/\iota(\{0\})$, which is one-dimensional, for all $n, m \in \mathbb{N}$.

As the convex hulls of the orbits $\{T^n(x) : n \in \mathbb{Z}\}$ are weakly relatively compact, so are the sets $\{\alpha_{\varphi_n}(x) : n \in \mathbb{N}\}$ for all $x \in X$. Indeed, by the theory of vector-valued integration outlined in [**10**], because $\|\varphi_n\| \le 1$ for all $n \in \mathbb{N}$, $\alpha_{\varphi_n}(x)$ lies in the closure of the convex hull of $\{T^n(x) : n \in \mathbb{Z}\}$. Hence, all the hypotheses of Theorem 5.2 are satisfied; hence this theorem establishes the validity of (10).

Similarly, for actions of $\mathbb{R}$ on X, we obtain the formula

$$\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} \alpha_t(x)\, dt \longrightarrow P_F(x).$$

Here we set $\mu(x) = xe^{-x^2}$ and $\varphi_n = \frac{1}{2n}\chi_{[-n,n]}$ and follow the same steps as in the proof of (10).

It is possible to extend this technique to all projections onto eigenspaces of the group action. Recall that $x \in X$ is an *eigenvector* corresponding to the eigenvalue $\xi \in \widehat{G}$ if $\alpha_t(x) = \langle t, \xi\rangle x$ for all $t \in G$. Using the same arguments as where we showed that $M^\alpha(\{0\})$ is the fixed point space of the action, it is possible to show that $M^\alpha(\{\xi\})$ is the eigenspace with eigenvalue ξ.

We shall prove that it is a consequence of our Tauberian theorem that there is a projection P_ξ of X onto $M^\alpha(\{\xi\})$, and that it can be computed by an ergodic limit in the SOT. In the case of an action of $\mathbb{Z}$ given by a power-bounded automorphism as above, the formula can be determined explicitly:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \langle i, \xi\rangle T^i(x) \longrightarrow P_\xi(x).$$

To prove it, we take $\mu = \delta_0 - \langle 1, \xi\rangle\delta_1$ and $\varphi_n(i) = \frac{1}{n}\langle i, \xi\rangle\chi_{[0,n-1]}(i)$ for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Using an approximation result in harmonic analysis, we can prove these ideas in full generality.

PROPOSITION 6.1. *Let G be a σ-compact locally compact abelian group and α be a weak integrable action of G on the dual pair (E, E'), where E is a Fréchet space such that the convex hulls of the orbits $\{\alpha_t(x) : t \in G\}$ are weakly relatively compact for each $x \in X$.*

If $\xi \in \widehat{G}$, then there is a projection P_ξ of E onto $M^\alpha(\{\xi\})$ and a bounded sequence φ_n of functions in $L^1(G)$ such that

$$\alpha_{\varphi_n} \to P_\xi$$

in the WOT. *In particular, each $M^\alpha(\{\xi\})$ is a complemented subspace of E.*

Proof. Let $\mu = \delta_0 - \delta_\xi$ and W_n, $n \in \mathbb{N}$ a sequence of open neighbourhoods of $\{\xi\}$ with compact closure such that $\cap W_n = \{\xi\}$. By [**9**, Theorem 2.6.3, p. 49], we can choose a bounded sequence $(\varphi_n) \subset L^1(G)$ such that $\|\varphi_n * \mu\|_1 < 1/n$, $\widehat{\varphi_n}(\xi) = 1$ and $\operatorname{supp} \widehat{\varphi_n} \subset W_n$ for all $n \in \mathbb{N}$.

We see that $(\varphi_n * \mu)$ converges to 0 in norm and hence that $\alpha_{\varphi_n * \mu} \to 0$ in the SOT (and hence certainly in the WOT). Clearly $\widehat{\varphi_n}$ is convergent on $\nu(\mu) = \{\xi\}$, as $\widehat{\varphi_n}(\xi) = 1$. Furthermore, $[\varphi_n] = [\varphi_m] \in L^1(G)/\iota(\{\xi\})$, which is one-dimensional, for all $n, m \in \mathbb{N}$.

Because $(\varphi_n) \subset L^1(G)$ is bounded and the convex hull of $\{\alpha_t(x) : t \in G\}$ is relatively weakly compact for each $x \in X$, $\alpha_{\varphi_n}(x)$ lies in the compact closure of the convex hull of $\|\varphi_n\|_1\{\alpha_t(x) : t \in G\}$. Hence, $\{\alpha_{\varphi_n}(x) : n \in \mathbb{N}\}$ is also relatively weakly compact for each $x \in E$.

Applying Theorem 5.1, the result follows. □

This result is stated using the WOT. The analogous result for the SOT is also true and can be proved in the same way.

The fact that $M^\alpha(\{\xi\})$ is complemented in E is already known; it may be found, for example, in [**13**]. It can be easily seen that the Mean Ergodic theorem for the fixed point space is just a special consequence of this result. Indeed, the fixed point subspace is just the eigenspace corresponding to the eigenvalue 1.

ACKNOWLEDGEMENTS. I would like to express my deep appreciation to Prof. Louis Labuschagne for his valuable insights and encouragement, as well as to an anonymous referee. Their comments and suggestions have greatly improved this paper.

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