Forum of Mathematics, Sigma (2022), **Vol. 10**:e108 1–29
doi:10.1017/fms.2022.99

RESEARCH ARTICLE

Riemann–Roch for stacky matrix factorizations

Dongwook Choa[1], Bumsig Kim*[2] and Bhamidi Sreedhar[3]

[1]Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul, 02455, Republic of Korea;
E-mail: dwchoa@kias.re.kr.
[2]Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul, 02455, Republic of Korea.
[3]Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang 37673, Republic of Korea; current address: Harish-Chandra Research Institute, A CI of Homi Bhabha National Institute, Chhatnag Road, Jhunsi, Prayagraj, 211019, India
E-mail: bhamidisreedhar@hri.res.in.

Received: 29 March 2022; **Revised:** 27 September 2022; **Accepted:** 3 November 2022

2020 Mathematics Subject Classification: *Primary* – 14A22; *Secondary* – 16E40, 18G80

Abstract
We establish a Hirzebruch–Riemann–Roch-type theorem and a Grothendieck–Riemann–Roch-type theorem for matrix factorizations on quotient Deligne–Mumford stacks. For this, we first construct a Hochschild–Kostant–Rosenberg-type isomorphism explicit enough to yield a categorical Chern character formula. Then, we find an expression of the canonical pairing of Shklyarov under the isomorphism.

Contents

1. Introduction

1.1. Main results

Let k be an algebraically closed field of characteristic zero. The main interest of this paper is a *Landau–Ginzburg model*, $(\mathcal{X}, w)$, where $\mathcal{X}$ is a smooth separated Deligne–Mumford stack of finite type over k and a regular function w with no other critical values but zero.

By a matrix factorization for $(\mathcal{X}, w)$ we mean a pair (P, δ_P) of a locally free coherent $\mathbb{G}$-graded sheaf P on $\mathcal{X}$ and a curved differential δ_P whose square is $w \cdot \mathrm{id}_P$. Here, $\mathbb{G}$ can be either the group $\mathbb{Z}$ or

*: The second author, Professor Bumsig Kim (1968–2021), sadly passed away during the final stages of the preparation of this manuscript. The first and third authors were both postdocs under his mentorship at Korea institute for advanced study when this project started. We thank Prof. Bumsig Kim for his mathematical generosity and kindness. We shall eternally be grateful for the opportunity to have known him not only as a great mathematician but also as a wonderful human being.

$\mathbb{Z}/2$ depending on w. There is the notion of the coderived category of matrix factorizations $\mathrm{DMF}(\mathcal{X}, w)$ and its differential graded (dg) enhancement defined as the dg quotient of the dg category of matrix factorizations by the subcategory of coacyclic or equivalently locally contractible matrix factorizations. Later, we will introduce a Čech-type dg enhancement of $\mathrm{DMF}(\mathcal{X}, w)$ denoted by $\mathrm{MF}_{dg}(\mathcal{X}, w)$ which we are going to use throughout the paper; see [15, 34] and also Definition 4.8.

The goal of this text is to firstly prove a Hochschild–Kostant–Rosenberg (HKR) for the category of matrix factorizations. The HKR theorem allows us to prove a Hirzebruch–Riemann–Roch (HRR) theorem and a Grothendieck–Riemann–Roch(GRR) theorem under the HKR-type map. In the rest of this section, we give an outline of the results, proofs and relations to previous works.

1.1.1. HKR and a Chern character formula

To the dg category $\mathrm{MF}_{dg}(\mathcal{X}, w)$, one can associate a Hochschild chain complex

$$\mathrm{MC}(\mathrm{MF}_{dg}(\mathcal{X}, w)).$$

which is a mixed complex equipped with the bar differential with Conne's B operator. It has been expected that $\mathrm{MC}(\mathrm{MF}_{dg}(\mathcal{X}, w))$ should be quasi-isomorphic to the dw-twisted de Rham mixed complex $(\Omega^{\bullet}_{I\mathcal{X}}, -dw|_{I\mathcal{X}}, d)$ of the inertia stack $I\mathcal{X}$ of $\mathcal{X}$. However, only particular cases have been proven so far. In this paper, we verify that the expectation is indeed true.

We first introduce some notations. Let P be a matrix factorizations and $\rho_{\mathcal{X}} : I\mathcal{X} \to \mathcal{X}$ be the natural morphism. Write $P|_{I\mathcal{X}}$ and $w|_{I\mathcal{X}}$ for $\rho^*_{\mathcal{X}}P$ and $\rho^*_{\mathcal{X}}w$ respectively. Let

$$\mathfrak{can}_{P|_{I\mathcal{X}}} \in \mathrm{Hom}_{\mathrm{MF}_{dg}(I\mathcal{X}, w|_{I\mathcal{X}})}(P|_{I\mathcal{X}}, P|_{I\mathcal{X}})$$

be the canonical automorphism of $P|_{I\mathcal{X}}$; see §3. Next, let

$$\hat{\mathrm{at}}(P|_{I\mathcal{X}}) \in \mathrm{Ext}^1(P|_{I\mathcal{X}}, P|_{I\mathcal{X}} \otimes \Omega^{-dw|_{I\mathcal{X}}}_{I\mathcal{X}})$$

denote the Atiyah class of the matrix factorization $P|_{I\mathcal{X}}$ for $(I\mathcal{X}, w|_{I\mathcal{X}})$; see [16, 24, 25] and §5.2. Finally, tr denotes the supertrace morphism

$$\mathrm{tr} : \mathbb{R}\mathrm{Hom}(P|_{I\mathcal{X}}, P|_{I\mathcal{X}} \otimes (\Omega^{\bullet}_{I\mathcal{X}}, -dw|_{I\mathcal{X}})) \to \mathbb{H}^*(I\mathcal{X}, (\Omega^{\bullet}_{I\mathcal{X}}, -dw|_{I\mathcal{X}})).$$

Theorem 1.1. *Suppose $\mathcal{X}$ is smooth and has the resolution property. Then there is an isomorphism*

$$\mathrm{MC}(\mathrm{MF}_{dg}(\mathcal{X}, w)) \cong \mathbb{R}\Gamma(\Omega^{\bullet}_{I\mathcal{X}}, -dw|_{I\mathcal{X}}, d)$$

in the derived category of mixed complexes. Under the isomorphism, the Hochschild homology valued Chern character $\mathrm{ch}_{HH}(P)$ is representable by

$$\mathrm{tr}\big(\mathfrak{can}_{P|_{I\mathcal{X}}} \exp(\hat{\mathrm{at}}(P|_{I\mathcal{X}}))\big)$$

in $\mathbb{H}^*(I\mathcal{X}, (\Omega^{\bullet}_{I\mathcal{X}}, -dw|_{I\mathcal{X}}))$ after the appropriate sense of the exponential operation exp *is taken into account; see §5.2.*

The history of related works is very rich. Here, we mention only the case of stacky matrix factorizations. In the local case Theorem 1.1 was proved by Polishchuk and Vaintrob [35]. There are works of Căldărau, Tu and Segal [8, 40] for HKR-type isomorphisms in affine cases with a finite group action. The paper [3] of Ballard, Favero and Katzarkov show an HKR-type isomorphism for the graded cases on linear spaces. This result has also been obtained by Halpern–Leistner and Pomerleano [18, Remark 3.20] and [17, Corollary 4.6]. We note that there is a difference in the map constructed in the current text and [17] (see equation (4.5) and Remark 4.15). Theorem 1.1 is proven by Kuerak Chung, Taejung Kim and the second author in [10], when one considers quotient stacks of the form $[X/G]$,

https://doi.org/10.1017/fms.2022.99 Published online by Cambridge University Press

where X is a smooth variety with a finite group action. In [25], an HKR-type isomorphism and Chern character formula including the case for the graded matrix factorizations are obtained by the universal Atiyah class.

1.1.2. HRR and GRR

Further, assume that the smooth separated Deligne-Mumford stack $\mathcal{X}$ is a stack quotient of a smooth variety by an action of an affine algebraic group and the critical locus of w is proper over k. When $\mathbb{G} = \mathbb{Z}/2$, we shall further assume that the morphism $w : \mathcal{X} \to \mathbb{A}^1_k$ is flat. We call the pair $(\mathcal{X}, w)$ a proper Landau-Ginzburg (LG) model. We define the Euler characteristic $\chi(P, Q)$ of the pair (P, Q) by the alternating sum of the dimensions of higher sheaf cohomology:

$$\chi(P, Q) := \sum_{i \in \mathbb{G}} (-1)^i \dim \mathbb{R}^i \mathrm{Hom}(P, Q).$$

For a vector bundle E on $I\mathcal{X}$, let $\mathrm{at}(E) \in \mathrm{Ext}^1(E, E \otimes \Omega^1_{I\mathcal{X}})$ denote the usual Atiyah class of E. Let

$$\mathrm{ch}_{tw}(E) := \mathrm{tr}(\mathfrak{can}_E \exp(\mathrm{at}(E)) \in \mathbb{H}^*(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, 0)).$$

For a virtual vector bundle E, define $\mathrm{ch}_{tw}(E)$ by linearity. We define the Todd class $\mathrm{td}(T_{I\mathcal{X}})$ of $T_{I\mathcal{X}}$ by the formulation of Todd class in terms of the Chern character $\mathrm{ch}_{tw}(T_{I\mathcal{X}})$; see §5.2.

Theorem 1.2. *Let $P^\vee$ denote the matrix factorization $(P^\vee, \delta_P^\vee)$ for $(\mathcal{X}, -w)$ dual to (P, δ_P), let $N_{I\mathcal{X}/\mathcal{X}}$ denote the normal bundle of $I\mathcal{X}$ to $\mathcal{X}$ via $\rho_\mathcal{X}$, let $\dim_{I\mathcal{X}}$ be the locally constant function for local dimensions of $I\mathcal{X}$ and let $\lambda_{-1}(N^\vee_{I\mathcal{X}/\mathcal{X}})$ be the alternating sum of exterior powers of $N^\vee_{I\mathcal{X}/\mathcal{X}}$. Then*

$$\chi(P, Q) = \int_{I\mathcal{X}} (-1)^{\binom{\dim_{I\mathcal{X}}+1}{2}} \mathrm{ch}_{HH}(Q) \wedge \mathrm{ch}_{HH}(P^\vee) \wedge \frac{\mathrm{td}(T_{I\mathcal{X}})}{\mathrm{ch}_{tw}(\lambda_{-1}(N^\vee_{I\mathcal{X}/\mathcal{X}}))}. \tag{1.1}$$

Here, the right-hand side is the composition of the following operations:

$$- \wedge - : \mathbb{H}^*(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, dw|_{I\mathcal{X}})) \otimes \mathbb{H}^*(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, -dw|_{I\mathcal{X}})) \to \mathbb{H}^*_Z(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, 0));$$

$$- \wedge \frac{\mathrm{td}(T_{I\mathcal{X}})}{\mathrm{ch}_{tw}(\lambda_{-1}(N^\vee_{I\mathcal{X}/\mathcal{X}}))} : \mathbb{H}^*_Z(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, 0)) \to \mathbb{H}^*_Z(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, 0));$$

$$\int_{I\mathcal{X}} : \oplus_{p \in \mathbb{Z}} H^*_c(I\mathcal{X}, \Omega^p_{I\mathcal{X}}[p]) \xrightarrow{projection} H^0_c(I\mathcal{X}, \Omega^n_{I\mathcal{X}}[n]) \xrightarrow{\mathrm{tr}_{I\mathcal{X}}} k,$$

where Z denotes the critical locus of $w|_{I\mathcal{X}}$ which is proper, and H^*_c denotes compactly supported cohomology.

Let $(\mathcal{Y}, v)$ be another proper LG model. Consider a proper morphism $f : \mathcal{X} \to \mathcal{Y}$ with $f^* v = w$. Let $K_0(\mathcal{A})$ be the Grothendieck group of the homotopy category of a pretriangulated dg category $\mathcal{A}$, and let $f_! : K_0(\mathrm{MF}_{dg}(\mathcal{X}, w)) \to K_0(\mathrm{MF}_{dg}(\mathcal{Y}, v))$ be the pushforward induced by f; see [9, §2]. Let $\widetilde{\mathrm{td}}(T_{If}) := \widetilde{\mathrm{td}}(T_{I\mathcal{X}})/If^*\widetilde{\mathrm{td}}(T_{I\mathcal{Y}})$, where

$$\widetilde{\mathrm{td}}(T_{I\mathcal{X}}) = \frac{\mathrm{td}(T_{I\mathcal{X}})}{\mathrm{ch}_{tw}(\lambda_{-1}(N^\vee_{I\mathcal{X}/\mathcal{X}}))}.$$

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Let $\dim_{If}$ be the function on $I\mathcal{X}$ for relative local dimensions of $If : I\mathcal{X} \to I\mathcal{Y}$, and let

$$\int_{If} : \mathbb{H}^*(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, -dw)) \to \mathbb{H}^*(I\mathcal{Y}, (\Omega^\bullet_{I\mathcal{Y}}, -dv))$$

be the pushforward; see §7.0.1.

Theorem 1.3 (=Theorem 6.10). *The following diagram is commutative:*

$$\begin{array}{ccc} K_0(\mathrm{MF}_{dg}(\mathcal{X}, w)) & \xrightarrow{\quad f_! \quad} & K_0(\mathrm{MF}_{dg}(\mathcal{Y}, v)) \\ \Big\downarrow \mathrm{ch}_{HH} & & \Big\downarrow \mathrm{ch}_{HH} \\ \mathbb{H}^0(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, -dw|_{I\mathcal{X}})) & \xrightarrow[\int_{If}(-1)^{\dim_{If}} \cdot \wedge \widetilde{\mathrm{td}}(T_{If})]{} & \mathbb{H}^0(I\mathcal{Y}, (\Omega^\bullet_{I\mathcal{Y}}, -dv|_{I\mathcal{Y}})). \end{array}$$

When $\mathbb{G} = \mathbb{Z}$ (and hence $w = 0$), various versions of Riemann–Roch theorems on DM stacks are proven by Kawasaki [20], Toën [43] and Edidin and Graham [11, 12, 13]. In the context of Hochschild homology of schemes (and $w = 0$), Theorem 1.3 was proved by Ramadoss [38]. When w has one critical point, Theorem 1.2 is proven by Polishchuk and Vaintrob [35].

1.2. On the proofs and pertinent works

1.2.1. HKR

For the computation of Hochschild homology of the category of matrix factorizations, there are at least three known approaches by (1) finding a suitable flat resolution of the diagonal module [3, 27, 35], (2) using the quasi-Morita equivalence [5, 8, 10, 14, 40] and (3) using the universal Atiyah classes [24, 25] which goes back to [7, 31]. In this paper, we take the second approach by constructing a globalization of Baranovsky's map [4] closely following the proof of Proposition 2.13 of [18] and [17, Corollary 4.6]. Combining this with a chain-level map from [5, 10], we obtain a boundary-bulk map formula as well as a Chern character formula; see §5.

1.2.2. HRR

For any proper smooth dg category $\mathcal{A}$, there is a categorical HRR theorem by Shklyarov [41]. Let $\mathcal{A}^{op}$ denote the opposite category of $\mathcal{A}$. Let $\langle , \rangle_{can}$ be the canonical pairing (or the Mukai pairing in [7]):

$$\langle , \rangle_{can} : HH_*(\mathcal{A}) \otimes HH_*(\mathcal{A}^{op}) \to k.$$

Then the categorical HRR theorem is the equality

$$\chi(P, Q) = \langle \mathrm{Ch}_{HH}(Q), \mathrm{Ch}_{HH}(P^\vee) \rangle_{can} \ \forall P, Q \in \mathcal{A}.$$

There is a characteristic property of the canonical pairing in terms of the Chern character of diagonal bimodule; see, for example, §6.1.2. Let $\mathcal{A}$ be the dg category of matrix factorizations for $(\mathcal{X}, w)$ localized by coacyclic matrix factorizations. When $\mathcal{X}$ is local, using the characteristic property Polishchuk and Vaintrob [35] show that the canonical pairing becomes up to sign the residue pairing under their HKR type isomorphism. In the nonstacky local case, there is also a work of Brown and Walker [6] identifying the canonical pairing with the residue pairing under the HKR type isomorphism.

When $\mathcal{X}$ is a smooth variety, using the deformation to the normal cone as well as the characteristic property of the canonical pairing, the second author [23] shows that the canonical pairing becomes a trace map under the HKR-type isomorphism up to a Todd correction term. When $\mathcal{X}$ is stacky, furthermore using the deformation to the normal cone for *local immersions* [26, 45] and the Chern character formula in Theorem 1.1, we are able to prove Theorem 1.2.

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1.2.3. GRR

The proper morphism f in Theorem 1.3 induces a dg functor from $\mathrm{MF}_{dg}(\mathcal{X}, w)$ to $\mathrm{MF}_{dg}(\mathcal{Y}, v)$. The induced homomorphism

$$HH_*(\mathrm{MF}_{dg}(\mathcal{X}, w)) \to HH_*(\mathrm{MF}_{dg}(\mathcal{Y}, v))$$

in Hochschild homology has a description §6.1.1 in terms of the canonical pairing of $\mathrm{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, -w \otimes 1 + 1 \otimes v)$ and the categorical Chern character of the matrix factorization associated to the graph morphism $\Gamma_f : \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$.

Under the HKR isomorphisms, the deformation to the normal cone allows us to interpret the description as the pushforward by f on $\mathbb{H}^0(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, -dw|_{I\mathcal{X}})$ up to a Todd correction term.

1.3. Conventions and notations

Let the ground field k be an algebraically closed field of characteristic zero. Let μ_r denote the group of r-th roots of unity over the field k. Throughout this paper, let $\mathcal{X}$ be a finite type separated DM stack over k. We denote by $I\mathcal{X}$ the inertia stack of $\mathcal{X}$. Let

$$\rho_{\mathcal{X}} : I\mathcal{X} \to \mathcal{X}$$

denote the natural representable morphism, which is finite and unramified [1, §3]. For a group G, $\widehat{G}$ shall denote its character group $\mathrm{Hom}(G, \mathbb{G}_m)$. Let G be a finite group which acts on a scheme Y of finite type over k. For a quotient stack $[Y/G]$, let

$$IY := \{(g, y) \in G \times Y : gy = y\} := G \times Y \times_{Y \times Y} \Delta_Y$$

so that $I[Y/G] = [IY/G]$.

For a local immersion (i.e., an unramified representable morphism) $f : \mathcal{X} \to \mathcal{Y}$ between DM stacks, we denote $C_{\mathcal{X}/\mathcal{Y}}$ be the normal cone to $\mathcal{X}$ in $\mathcal{Y}$; see [26, 45]. If f is a regular local immersion, then we write N_f or $N_{\mathcal{X}/\mathcal{Y}}$ for the vector bundle $C_{\mathcal{X}/\mathcal{Y}}$ on $\mathcal{X}$.

For a vector bundle E, we often write 1_E for the identity morphism id_E of E. For a dg category $\mathcal{A}$, its homotopy category is denoted by $[\mathcal{A}]$.

The label $\sim$ on an arrow indicates the arrow is a quasi-isomorphism.

2. Mixed Hochschild complexes and Chern characters

Unless otherwise stated, we follow notation and conventions of [5, 10] for curved dg (in short CDG) categories $\mathcal{A}$, the mixed Hochschild complexes of $\mathcal{A}$ and the category of mixed complexes. We briefly recall the notations therein and some foundational facts which we are going to use later.

2.1. Mixed Hochschild complexes

For a CDG category $\mathcal{A}$, we use the following notation:

- $C(\mathcal{A})$ $(\mathrm{MC}(\mathcal{A}))$: (mixed) Hochschild complex.
- $\overline{C}(\mathcal{A})$ $(\overline{\mathrm{MC}}(\mathcal{A}))$: (mixed) normalized Hochschild complex.
- $C^{II}(\mathcal{A})$ $(\mathrm{MC}^{II}(\mathcal{A}))$: (mixed) Hochschild complex of the second kind.
- $\overline{C}^{II}(\mathcal{A})$ $(\overline{\mathrm{MC}}^{II}(\mathcal{A}))$: (mixed) normalized Hochschild complex of the second kind.

For notational convenience, we let C' denote either $C, \overline{C}, C^{II}$ or $\overline{C}^{II}$ and MC' denote either MC, $\overline{\mathrm{MC}}$, MC^{II}, or $\overline{\mathrm{MC}}^{II}$. The normalized negative cyclic complex is denoted by $\overline{C}(\mathcal{A})[[u]]$, where u is a

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formal variable of degree 2. For a mixed complex (C, b, B), we simply write $C[[u]]$ for the complex $(C[[u]], b + uB)$.

2.2. Foundational facts frequently in use

2.2.1. Invariance under the natural projections

The projection $\mathrm{MC}(\mathcal{D}) \to \overline{\mathrm{MC}}(\mathcal{D})$ for a dg category $\mathcal{D}$ and the projection $\mathrm{MC}^{II}(\mathcal{A}) \to \overline{\mathrm{MC}}^{II}(\mathcal{A})$ for a CDG category $\mathcal{A}$ are quasi-isomorphisms [5, 36].

2.2.2. (Quasi-)Morita invariance

If a dg functor $\mathcal{D} \to \mathcal{D}'$ is Mortia-equivalent (i.e., the induced functor $D(\mathcal{D}) \to D(\mathcal{D}')$ of derived categories of $\mathcal{D}$ and $\mathcal{D}'$ is an equivalence), then the induced morphism $\mathrm{MC}(\mathcal{D}) \to \mathrm{MC}(\mathcal{D}')$ of mixed complexes is a quasi-isomorphism [21]. If $\mathcal{A} \to \mathcal{A}'$ is a pseudo-equivalence of CDG categories, then the induced morphism $\overline{\mathrm{MC}}^{II}(\mathcal{A}) \to \overline{\mathrm{MC}}^{II}(\mathcal{A}')$ is a quasi-isomorphism [36]. This invariance is dubbed as quasi-Morita invariance.

2.2.3. Localization in cyclic homology

If $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be an exact sequence of exact dg categories, then it induces an exact triangle of mixed complexes

$$MC(\mathcal{A}) \to \mathrm{MC}(\mathcal{B}) \to \mathrm{MC}(\mathcal{C}) \to \mathrm{MC}(\mathcal{A})[1];$$

see [22, §5.6].

2.2.4. Local description of the inertia stack

Let G be a finite group acting on a k-scheme X, and let G/G denote the set of conjugacy classes of G. Then the following holds $I[X/G] \cong \sqcup_{g \in G/G}[X^g/\mathrm{C}_G(g)]$.

2.2.5. Invariants and coinvariants

Let G be a group with a linear action upon V a not-necessarily finite-dimensional vector space. Define the invariant space $V^G := \mathrm{Hom}_G(k, V)$ and the coinvariant space $V_G := k \otimes_G V$. If G is a finite group, then the composition of the natural homomorphisms $V^G \to V \to V_G$ is an isomorphism.

2.3. Categorical Chern characters

For $P \in \mathcal{A}$, the cycle class represented by the identity morphism 1_P in the normalized Hochschild complex $\overline{C}(\mathcal{A})$ (resp. the normalized negative cyclic complex $\overline{C}(\mathcal{A})[[u]]$) of $\mathcal{A}$ is denoted by $\mathrm{Ch}_{HH}(P)$ (resp. $\mathrm{Ch}_{HN}(P)$).

3. The canonical central automorphism

3.1. The central embedding

The inertia stack has a decomposition:

$$I\mathcal{X} = \sqcup_{r=1}^{\infty} I_{\mu_r}\mathcal{X}.$$

An object of $I_{\mu_r}\mathcal{X}$ over a k-scheme T is a pair (ξ, α) of an object $\xi \in \mathcal{X}(T)$ and an injective morphism of group-schemes $\alpha : \mu_r \times T \to \mathcal{A}ut_T(\xi)$; see [1, §3]. Note that, for all but finitely many r, $I_{\mu_r}\mathcal{X}$ is empty.

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An automorphism of the pair (ξ, α) in $I_{\mu_r}\mathcal{X}$ is by definition an automorphism $f \in \mathcal{A}ut_T(\xi)$ such that

$$f \circ \alpha \circ f^{-1} = \alpha.$$

In other words, the automorphism group-scheme $\mathcal{A}ut_T(\xi, \alpha)$ of (ξ, α) over T is the centralizer of α in $\mathcal{A}ut_T(\xi)$. We have a canonical central embedding $\mathfrak{c} : \mu_r \times T \to \mathcal{A}ut_T(\xi, \alpha)$. This gives a natural morphism

$$\mu_r \times T \to T \times_{I_{\mu_r}\mathcal{X}} T; (\xi, t) \mapsto (t, t, \mathfrak{c}(\xi, t)).$$

3.2. *The central automorphism*

Let $T \to I_{\mu_r}\mathcal{X}$ be an étale surjection, and let $pr_i : T \times_{I_{\mu_r}\mathcal{X}} T \to T$ be the i-th projection. A vector bundle E on $I_{\mu_r}\mathcal{X}$ amounts to a vector bundle F on T with an isomorphism $\phi^F \in \mathrm{Isom}_T(pr_1^*F, pr_2^*F)$ satisfying the cocycle condition. By pulling back the isomorphism ϕ^F to $\mu_r \times T$, we obtain a morphism of group-schemes $\mu_r \times T \to \mathrm{Aut}_T(F)$. Here, $\mathrm{Aut}_T(F)$ denotes the group of automorphisms of F fixing T. Since $\mathfrak{c}$ is central, the homomorphism descends to a homomorphism $\mu_r \to \mathrm{Aut}_{I_{\mu_r}\mathcal{X}}(E)$. Denote by

$$\mathfrak{can}_E \in \mathrm{Aut}_{I_{\mu_r}\mathcal{X}}(E)$$

the image of the chosen r-th root $e^{2\pi i/r}$ of unity.

According to the action μ_r upon E, the bundle E is decomposable into eigenbundles

$$\bigoplus_{\chi \in \widehat{\mu_r}} E_\chi,$$

where $\widehat{\mu_r}$ is the character group of μ_r. Then we have

$$\mathfrak{can}_E = \bigoplus_{\chi \in \widehat{\mu_r}} \chi(e^{2\pi i/r})\mathrm{id}_{E_\chi} \in \mathrm{Hom}_{I_{\mu_r}\mathcal{X}}(E, E).$$

3.3. *The local description*

Locally the central automorphisms $\mathfrak{can}$ can be described as follows. Suppose that $\mathcal{X} = [X/G]$, where G is a finite group and X is a scheme. Let $g \in G$ with order r, and write $\mathrm{C}_G(g)$ for the centralizer of g in G. For the component $[X^g/\mathrm{C}_G(g)]$ of $I_{\mu_r}\mathcal{X}$ and a $\mathrm{C}_G(g)$-equivariant sheaf E on X^g, we have an isomorphism

$$(g^{-1})^*E \xrightarrow{\varphi_g^E} E$$

from the equivariant structure of E. Since g acts trivially on X^g, $(g^{-1})^*E = E$. And hence, φ_g^E is an automorphism of E, which is the automorphism $\mathfrak{can}_E$. Since any element of $\mathrm{C}_G(g)$ commutes with g, the homomorphism φ_g^E is $\mathrm{C}_G(g)$-equivariant. Thus, $\varphi_g^E \in \mathrm{Hom}_{I_{\mu_r}\mathcal{X}}(E, E)$. For any $\mathrm{C}_G(g)$-equivariant sheaf E' on X^g and any $\mathrm{C}_G(g)$-equivariant $\mathcal{O}_{X^g}$-module homomorphism $a : E \to E'$, note that $\varphi_g^{E'} \circ a = a \circ \varphi_g^E$.

3.4. $T_{I\mathcal{X}} \cong IT_{\mathcal{X}}$

In this subsection, let $\mathcal{X}$ be smooth over k. We prove that there is a natural isomorphism $T_{I\mathcal{X}} \cong IT_{\mathcal{X}}$.

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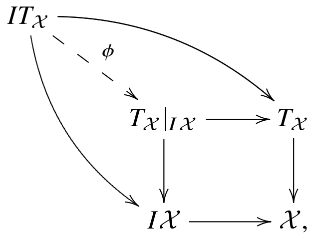
Consider a commuting diagram of natural morphisms



where the square is a fiber square.

Lemma 3.1. *The morphism ϕ induces an isomorphism $IT_{\mathcal{X}} \cong T_{I\mathcal{X}}$.*

Proof. First, note that it is enough to check the isomorphism over the étale site of the coarse moduli space of $\mathcal{X}$. Since $\mathcal{X}$ is separated, then $\mathcal{X}$ is étale locally a quotient of a nonsingular variety Y by a finite group G action. Hence, we may assume that $\mathcal{X} = [Y/G]$. Since

$$T_{[Y/G]} \cong [T_Y/G] \text{ and } I[Y/G] \cong \coprod_{g \in G/G} [Y^g/\mathrm{C}_G(g)],$$

we have

$$IT_{[Y/G]} \cong \coprod_{g \in G/G} [(T_Y)^g/\mathrm{C}_G(g)] \text{ and } T_{I[Y/G]} \cong \coprod_{g \in G/G} [T_{Y^g}/\mathrm{C}_G(g)].$$

Since $(T_Y)^g \cong T_{Y^g}$, we conclude the proof. □

Note first that there is a natural short exact sequence of vector bundles

$$0 \to T_{I\mathcal{X}} \to \rho_{\mathcal{X}}^* T_{\mathcal{X}} \to N_{\rho_{\mathcal{X}}} \to 0.$$

In fact, this sequence splits. The reason is that according to the canonical automorphism of $\rho_{\mathcal{X}}^* T_{\mathcal{X}}$ there is a decomposition of $\rho_{\mathcal{X}}^* T_{\mathcal{X}}$ into the fixed part and the moving part, which are naturally isomorphic to $T_{I\mathcal{X}}$ and $N_{\rho_{\mathcal{X}}}$, respectively. $N_{\rho_{\mathcal{X}}}$ is in particular a vector bundle. For a detailed discussion on local embeddings, see [45, §1.20]).

Remark 3.2. When $\mathcal{X}$ is the global quotient $[Y/G]$ by a finite group G, $N_{\rho_{\mathcal{X}}} \cong [N_{Y^g/Y}/\mathrm{C}_G(g)]$.

4. Hochschild–Kostant–Rosenberg for stacky matrix factorizations

We introduce the main object of this paper.

Definition 4.1. A *LG model* is a pair $(\mathcal{X}, w)$ of a smooth separated DM stack $\mathcal{X}$ over k and a regular function w on $\mathcal{X}$. We further assume that $\mathcal{X}$ satisfies the resolution property. We assume that the critical value is over 0. If $\mathbb{G} = \mathbb{Z}$, then $w = 0$. If $\mathbb{G} = \mathbb{Z}/2$, then we furthermore assume that $w : \mathcal{X} \to \mathbb{A}^1$ is flat. The pair $(\mathcal{X}, w)$ will be called a *proper LG model* if the critical locus of w is proper over k.

Remark 4.2. We note that if $\mathcal{X}$ is a smooth quotient stack which satisfies the resolution property, it follows from [44, Theorem 1.1] that $\mathcal{X}$ is a quotient stack.

4.1. Matrix factorizations and their derived categories

Whenever an LG model $(\mathcal{X}, w)$ is given, we consider a sheaf of curved differential graded (CDG for short) algebra $(\mathcal{O}_{\mathcal{X}}, -w)$ over $\mathcal{X}$. It is concentrated in degree 0 with zero differential and a curvature $-w$.

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Definition 4.3. A *quasi-differential graded module (QDG-module for short)* over $(\mathcal{O}_{\mathcal{X}}, -w)$ is a pair (P, δ_P) of an $\mathcal{O}_{\mathcal{X}}$-module P and an $\mathcal{O}_{\mathcal{X}}$-linear degree 1 endomorphism δ_P. We say a QDG module is

- *(quasi-)coherent* if P is (quasi-)coherent,
- *locally free* if P is locally free,
- *matrix quasi-factorization* if P is locally free of finite rank.

We denote the category of QDG modules over $(\mathcal{O}_{\mathcal{X}}, -w)$ by $q\mathrm{Mod}(\mathcal{X}, w)$. It is a CDG category whose morphisms and differentials are

$$\mathrm{Hom}_{q\mathrm{Mod}}((P, \delta_P), (Q, \delta_Q)) = \big(\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(P, Q), \delta\big),$$
$$\delta(f) = \delta_Q \circ f - (-1)^{|f|} f \circ \delta_P.$$

The curvature element $h_{(p,\delta_P)}$ of (P, δ_P) is defined as $\delta_P^2 + \rho_{-w} \in \mathrm{End}(P)$, where ρ_{-w} is the multiplication map by $-w$.

Definition 4.4. A QDG-module (P, δ_P) is called *a factorization* if its curvature is zero. We define (quasi-)coherent or locally free factorizations as in 4.3. In particular, we call it a *matrix factorization* if P is locally free of finite rank.

By definition, factorizations form a dg subcategory inside $q\mathrm{Mod}(\mathcal{X}, w)$ denoted by $\mathrm{Mod}(\mathcal{X}, w)$. We denote a full dg subcategory of (quasi-) coherent and matrix factorizations by $\mathrm{QCoh}(\mathcal{X}, w)$, $\mathrm{Coh}(\mathcal{X}, w)$ and $\mathrm{MF}(\mathcal{X}, w)$, respectively.

We recall constructions of the derived category of factorizations following [37]. Let $[\mathrm{QCoh}(\mathcal{X}, w)]$ be the homotopy category of $\mathrm{QCoh}(\mathcal{X}, w)$. Denote by $\mathrm{AbsAcyc}(\mathcal{X}, w)$ the smallest triangulated subcategory containing the totalizations of all short exact sequences in $Z^0\mathrm{QCoh}(\mathcal{X}, w)$. Its object is called *absolutely acyclic factorizations.* Also, denote by $\mathrm{CoAcyc}(\mathcal{X}, w)$ the smallest triangulated subcategory containing the totalizations of all acyclic factorizations which is closed under infinite direct sum; its object are called *a coacyclic factorizations*.

Definition 4.5. The *absolute derived category of* $\mathrm{QCoh}(\mathcal{X}, w)$ is the Verdier quotient

$$\mathrm{D}^{\mathrm{abs}}(\mathrm{QCoh}(\mathcal{X}, w)) := [\mathrm{QCoh}(\mathcal{X}, w)]/\mathrm{AbsAcyc}(\mathcal{X}, w).$$

The *coderived category of* $\mathrm{QCoh}(\mathcal{X}, w)$ is the Verdier quotient

$$\mathrm{D}^{\mathrm{co}}(\mathrm{QCoh}(\mathcal{X}, w)) := [\mathrm{QCoh}(\mathcal{X}, w)]/\mathrm{CoAcyc}(\mathcal{X}, w).$$

We define an absolute/coderived category of $\mathrm{Coh}(\mathcal{X}, w)$ and $\mathrm{MF}(\mathcal{X}, w)$.

Definition 4.6. The derived category of of matrix factorizations denoted by $\mathrm{DMF}(\mathcal{X}, w)$ is the smallest full triangulated subcategory of $\mathrm{D}^{\mathrm{co}}(\mathrm{QCoh}(\mathcal{X}, w))$ which contains $\mathrm{D}^{\mathrm{abs}}(\mathrm{MF}(\mathcal{X}, w))$.

Remark 4.7. Relations between various categories are well known. We only recall a few facts we will be going to use later. If $\mathcal{X}$ is smooth, then it is known that Verdier localization $\mathrm{D}^{\mathrm{abs}}(\mathrm{QCoh}(\mathcal{X}, w)) \to \mathrm{D}^{\mathrm{co}}(\mathrm{QCoh}(\mathcal{X}, w))$ is an equivalence, and an image of $\mathrm{D}^{\mathrm{abs}}(\mathrm{Coh}(\mathcal{X}, w))$ consists of compact generators inside $\mathrm{D}^{\mathrm{co}}(\mathrm{QCoh}(\mathcal{X}, w))$ (see [37, §3.6]). If $\mathcal{X}$ has the resolution property, then $\mathrm{D}^{\mathrm{abs}}(\mathrm{MF}(\mathcal{X}, w)) \to \mathrm{D}^{\mathrm{abs}}(\mathrm{Coh}(\mathcal{X}, w))$ is an equivalence. (See [34].)

4.2. Čech model

In this subsection, we recall the Čech type dg enhancement of $\mathrm{DMF}(X, w)$ as in [10].

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Fix an affine étale surjective morphism $\mathfrak{p} : \mathfrak{U} \to \mathcal{X}$ from a k-scheme $\mathfrak{U}$. Let $\mathfrak{U}^r$ denote the r-th fold product of $\mathfrak{U}$ over $\mathcal{X}$, and let $\mathfrak{p}_r : \mathfrak{U}^r \to \mathcal{X}$ denote the projection. For a vector bundle E on $\mathcal{X}$, let $\check{C}^r(E) = \mathfrak{p}_{r*}\mathfrak{p}_r^* E$ and

$$\check{C}(E) := \left(\bigoplus_{r\geq 1} \check{C}^r(E), d_{\check{\mathrm{C}}\mathrm{ech}}\right) = \left[0 \to p_{1*}p_1^*E \to p_{2*}p_2^*E \to \cdots\right]$$

a Čech complex.

Now, let (E, δ_E) be a matrix quasi-factorization over $(\mathcal{O}_{\mathcal{X}}, -w)$. Observe that $\check{C}(\mathcal{O}_{\mathcal{X}})$ can be viewed as a sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras equipped with an Alexander–Whitney product. By projection formula $\check{C}(E) = E \otimes_{\mathcal{O}_{\mathcal{X}}} \check{C}(\mathcal{O}_{\mathcal{X}})$ and $\check{C}(E)$ carries a natural $\check{C}(\mathcal{O}_{\mathcal{X}})$-module structure. We equip $\check{C}(E)$ with a curved differential

$$\delta_{\check{C}(E)} := \delta_P \otimes 1 + 1 \otimes d_{\check{\mathrm{C}}\mathrm{ech}}.$$

We regard $(\check{C}(E), \delta_{\check{C}(E)})$ as a QDG-module over $(\check{C}(\mathcal{O}_{\mathcal{X}}), w)$. Notice that the curvature of $(\check{C}(E), \delta_{\check{C}(E})$ coincides with the curvature of E under the canonical map $\mathrm{End}_{\mathcal{O}_{\mathcal{X}}}(E) \to \mathrm{End}_{\check{C}(\mathcal{O}_{\mathcal{X}})}(\check{C}(E))$.

Definition 4.8. For a fixed affine étale open cover $\mathfrak{U}$, a *Čech model CDG category* $q\mathrm{MF}_{dg}(\mathcal{X}, w)$ is a CDG category whose objects are matrix quasi-factorizations and its $\mathbb{G}$-graded Hom spaces are defined as usual

$$\mathrm{Hom}_{q\mathrm{MF}_{dg}}(P, Q) := \left(\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\check{C}(P), \check{C}(Q)), \delta\right) \tag{4.1}$$

$$\delta = \delta_{\check{C}(P')} \circ f - (-1)^{|f|} f \circ \delta_{\check{C}(P)}. \tag{4.2}$$

Similarly, we define a *Čech model dg category* $\mathrm{MF}_{dg}(\mathcal{X}, w)$ of matrix factorizations for $(\mathcal{X}, w)$ as a full dg subcategory of $q\mathrm{MF}_{dg}(\mathcal{X}, w)$ consisting of matrix factorizations for $(\mathcal{X}, w)$. When it is necessary to specify the covering $\mathfrak{U}$, we write $\mathrm{MF}_{dg}(\mathcal{X}, w; \mathfrak{U})$ for $\mathrm{MF}_{dg}(\mathcal{X}, w)$.

In the following lemma, we show that $\mathrm{MF}_{dg}(\mathcal{X}, w)$ is equivalent to the dg-quotient dg enhancement of $\mathrm{DMF}(\mathcal{X}, w)$.

Lemma 4.9. *The natural functor*

$$\begin{aligned} \epsilon : [\mathrm{MF}_{dg}(\mathcal{X}, w)] &\to \mathrm{D}^{\mathrm{co}}(\mathrm{QCoh}(\mathcal{X}, w)) \\ P &\mapsto \check{C}(P) \end{aligned}$$

is fully faithful, and its essential image is equal to $\mathrm{DMF}(\mathcal{X}, w)$.

Proof. To show that the essential image of ϵ is as claimed, one can check $\mathrm{Cone}(P \to \check{C}(P))$ is coacylic so that P is isomorphic to $\check{C}(P)$ in $\mathrm{DMF}(\mathcal{X}, w)$ (see [9, equation 2.1]).

To prove fully faithfulness, we need to show $\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{X},w)}(\check{C}(P), \check{C}(Q)) \simeq \mathrm{Hom}_{\mathrm{DMF}(\mathcal{X},w)}(\check{C}(P), \check{C}(Q))$. Using Cech filtration on the second argument, it is enough to show that

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{X},w)}(\check{C}(P), j_*j^*Q) \simeq \mathrm{Hom}_{\mathrm{D^{co}QCoh}(\mathcal{X},w)}(\check{C}(P), j_*j^*Q)$$

for any open étale affine open $j : V \to \mathcal{X}$. Consider the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{QCoh}(\mathcal{X},w)}(\check{C}(P), j_*j^*Q) & \longrightarrow & \mathrm{Hom}_{\mathrm{D^{co}QCoh}(\mathcal{X},w)}(\check{C}(P), j_*j^*Q) \\ \downarrow p_1 & & \downarrow p_2 \\ \mathrm{Hom}_{\mathrm{QCoh}(\mathcal{X},w)}(P, j_*j^*Q) & \xrightarrow{q} & \mathrm{Hom}_{\mathrm{D^{co}QCoh}(\mathcal{X},w)}(P, j_*j^*Q). \end{array} \tag{4.3}$$

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The right vertical p_2 is an isomorphism because $P \to \check{C}(P)$ is an isomoprhism in $\mathrm{D^{co}QCoh}$. The bottom horizontal q becomes

$$\mathrm{Hom}_{\mathrm{MF}(V,w|_V)}(j^*P, j^*Q) \to \mathrm{Hom}_{\mathrm{D^{co}QCoh}(V,w|_V)}(j^*P, j^*Q).$$

This is known to be an isomorphism for all affine V. To show that the left vertical p_1 is an isomorphism, it is enough to show that

$$\left(\mathrm{Hom}_{\mathrm{QCoh}(V)}(\check{C}^\bullet(j^*P^m), j^*Q^n), \delta_{Cech}\right) \to \mathrm{Hom}_{\mathrm{QCoh}(V)}(j^*P^m, j^*Q^n)$$

for all $m, n \in \mathbb{Z}/2$. This is nothing but an (unordered) Cech resolution of $\mathrm{Hom}_{\mathrm{QCoh}(V)}(j^*P^m, j^*Q^n)$ associated to the open cover $\mathfrak{U} \times_{\mathcal{X}} V$ on V, which is an isomorphism because V is affine. The claim follows. □

Lemma 4.9 is an analogue of the Čech enhancement of [30, §4.1] for Deligne–Mumford stacks. It follows from the proof of the lemma that the dg-quotient enhancement and the Čech model dg-enhancement of Definition 4.8 coincide (see [29, §2]).

Remark 4.10. We view $\mathrm{Hom}_{\mathrm{MF}_{dg}}(P, Q)$ as a $\mathbb{Z} \times \mathbb{Z}/2$ -graded bicomplex. This complex is not bounded above in $\mathbb{Z}$-direction because a Čech cover $\mathfrak{U}$ of a stack is genuinely unordered. One can go around the subtleties by taking a suitable truncation. Suppose (E, δ_E) is a matrix factorization. Define

$$\tau\check{C}(E) := \tau_{\leq \dim \mathcal{X}}\left(\bigoplus_{r \geq 1} \check{C}^r(E), d_{\check{\mathrm{C}}\mathrm{ech}}\right),$$

where $\tau_{r \leq \dim \mathcal{X}}$ denotes the truncation. Note that under the assumptions in this text all stacks have finite cohomological dimension, hence the truncated one does compute the sheaf cohomology of the quasi-coherent sheaf E since the map to the coarse moduli $\mathcal{X} \to \underline{\mathcal{X}}$ is cohomologically affine. Therefore, the induced map between the spectral sequences associated to Čech filtrations

$$\mathrm{Hom}(\check{C}(P), \check{C}(Q)) \to \mathrm{Hom}(\tau\check{C}(P), \check{C}(Q)) \leftarrow \mathrm{Hom}(\tau\check{C}(P), \tau\check{C}(Q))$$

are isomorphisms on the first page and, hence, are quasi-isomorphisms themselves. Also, notice that $\tau\check{C}(E) \to \check{C}(E)$ is a quasi-isomorphism in $\mathrm{DMF}(\mathcal{X}, w)$.

Remark 4.11. In literature, $\mathrm{DMF}(\mathcal{X}, w)$ is defined as a dg-quotient of DCoh by *locally contractible*; see [34] or [18]. Here, P is called *locally contractible* if there is an open covering $\mathfrak{V}$ in smooth topology of $\mathcal{X}$ such that $P|_{\mathfrak{V}}$ is contractible. We note that

1. P is coacyclic if and only if $P|_{\mathfrak{U}}$ is coacyclic since the natural morphism $P \to \check{C}(P)$ is termwise exact. See [9, Proposition 2.2.6].
2. TFAE: $P|_{\mathfrak{U}}$ is coacyclic, $P|_{\mathfrak{U}}$ is absolutely acyclic and $P|_{\mathfrak{U}}$ is contractible by [37, §3.6];
3. TFAE: $P|_{\mathfrak{U}}$ is coacyclic, P is locally contractible by [9, Proposition 2.2.6] and (1).

Therefore, if $P \in \mathrm{MF}_{dg}(\mathcal{X}, w)$ represents a coacyclic object in $\mathrm{DMF}(\mathcal{X}, w)$ if and only if P is locally contractible.

Remark 4.12. Consider another affine covering $\mathfrak{U}' \to \mathcal{X}$. Let $\mathfrak{U}'' = \mathfrak{U}' \times_{\mathcal{X}} \mathfrak{U}$. Then there is a natural dg functor $\mathrm{MF}_{dg}(\mathcal{X}, w, \mathfrak{U}') \to \mathrm{MF}_{dg}(\mathcal{X}, w, \mathfrak{U}'')$, which is a quasi-equivalence. The induced chain map $\mathrm{MC}'(\mathrm{MF}_{dg}(\mathcal{X}, w, \mathfrak{U})) \to \mathrm{MC}'(\mathrm{MF}_{dg}(\mathcal{X}, w, \mathfrak{U}''))$ between mixed complexes are quasi-isomorphism. For the mixed complex of the first kind, it follows from Morita invariance. For that of either kind, consider the Hochschild complex $C'(\mathrm{MF}_{dg}(\mathcal{X}, w))$ filtered by Čech degree. Since the filtration is bounded (see 4.10), we may apply the Eilenberg–Moore comparison theorem ([46, Theorem 5.5.11]) to check that the induced chain map is a quasi-isomorphism. A similar argument shows that the natural chain map

$$\mathrm{MC}'(q\mathrm{MF}_{dg}(\mathcal{X}, w, \mathfrak{U})) \to \mathrm{MC}'(q\mathrm{MF}_{dg}(\mathcal{X}, w, \mathfrak{U}''))$$

is also a quasi-isomorphism.

4.3. Twisting

For a morphism from a scheme U to $\mathcal{X}$, write IU for the fiber product $U \times_{\mathcal{X}} I\mathcal{X}$. Following Toën, Halpern–Leistner and Pomerleano [18], we consider the assignments

$$U \mapsto C'(q\mathrm{MF}_{dg}(IU, w|_{IU}, \mathfrak{U} \times_{\mathcal{X}} IU))$$

for all étale morphisms $U \to \mathcal{X}$. They form a presheaf of mixed complexes on the small étale site $\mathcal{X}$, which we denote by

$$\mathrm{MC}'(q\mathrm{MF}_{I\mathcal{X},w}(-)).$$

Denote the associated cochain complex by $C'(q\mathrm{MF}_{I\mathcal{X},w}(-))$. There is a natural morphism of mixed complexes

$$nat : \mathrm{MC}'(q\mathrm{MF}_{dg}(I\mathcal{X}, w)) \to (\Gamma(\mathcal{X}, \mathrm{MC}'(q\mathrm{MF}_{I\mathcal{X},w}(-))),$$

where the Čech model $q\mathrm{MF}_{dg}(I\mathcal{X}, w)$ uses the affine cover $\mathfrak{U} \times_{\mathcal{X}} I\mathcal{X} \to I\mathcal{X}$.

In Section 3.2, we defined a canonical twist for every coherent sheaf on $I\mathcal{X}$. In turn, this gives an automorphism $\mathfrak{can}_P$ of P for every object P of $\mathrm{MF}_{dg}(I\mathcal{X}, w)$. It is a morphism in the category $\mathrm{MF}_{dg}(I\mathcal{X}, w)$.

Note that

$$\mathfrak{can}_{P'} \circ a = a \circ \mathfrak{can}_P \quad \forall a \in \mathrm{Hom}_{I\mathcal{X}}(P, P'). \tag{4.4}$$

Hence, the assignments $P \mapsto \mathfrak{can}_P$ yield a natural transformation $\mathfrak{can}$ between the identity functor $\mathrm{id} : \mathrm{MF}_{dg}(I\mathcal{X}, w) \to \mathrm{MF}_{dg}(I\mathcal{X}, w)$. We will often drop the subscript P in $\mathfrak{can}_P$ for simplicity.

Define a k-linear map $\mathfrak{tw} : C'(q\mathrm{MF}_{dg}(I\mathcal{X}, w)) \to C'(q\mathrm{MF}_{dg}(I\mathcal{X}, w))$ associated to $\mathfrak{can}$ by

$$a_0[a_1|\cdots|a_n] \mapsto a_0[a_1|\cdots|\mathfrak{can} \circ a_n].$$

Note that by equation (4.4) $b \circ \mathfrak{tw} = \mathfrak{tw} \circ b$, that is, $\mathfrak{tw}$ is a chain automorphism of the Hochschild complex $C'(q\mathrm{MF}_{dg}(I\mathcal{X}, w))$.

Consider the composition τ' of a sequence of chain maps

$$C'(q\mathrm{MF}_{dg}(\mathcal{X}, w)) \xrightarrow{pullback} C'(q\mathrm{MF}_{dg}(I\mathcal{X}, w)) \\ \xrightarrow{\mathfrak{tw}} C'(q\mathrm{MF}_{dg}(I\mathcal{X}, w)) \xrightarrow{nat} \Gamma(\mathcal{X}, C'(q\mathrm{MF}_{I\mathcal{X},w}(-))). \tag{4.5}$$

Proposition 4.13. *The chain map $\overline{\tau}^{II}$ is a quasi-isomorphism when $\mathcal{X}$ is of form $[\mathrm{Spec} A/G]$ for some commutative k-algebra A with a finite group G action.*

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A proof of the above proposition will be given §4.4 and §4.5. For simplicity, we will often write τ for τ' when there is no risk of confusion.

4.4. Local case

Let $\mathcal{X} = [\mathrm{Spec}A/G]$, and let $w \in A^G$ a G-invariant element of A as in Proposition 4.13. Let $\mathrm{MF}^G_{dg}(A, w)$ denote the dg category of G-equivariant factorizations P for (A, w) which are projective as A-modules. The Hom space from P to Q is the G-invariant part of $\mathrm{Hom}_A(P, Q)$ of $\mathbb{G}$-graded A-module homomorphisms. Likewise, we have the CDG category $q\mathrm{MF}^G_{dg}(A, w)$ of G-equivariant quasi-modules for (A, w) which are projective as A-modules. In fact, these coincide with the Čech models $\mathrm{MF}_{dg}(\mathcal{X}, w)$ and $q\mathrm{MF}_{dg}(\mathcal{X}, w)$ with respect to the natural choice of an affine cover: $\mathrm{Spec}A \to \mathcal{X}$.

Let I_g be the ideal of A generated by $a - ga$ for all $a \in A$. Denote $A_g := A/I_g$ and $w_g := w|_{A_g} \in A_g$. We regard the pair (A, w) (resp. (A_g, w_g)) as a CDG algebra A (resp. A_g) with zero differential and curvature w (resp. w_g).

The algebra A has the induced left G-action. Note that for $a, b \in A$ and $g, h \in G$,

$$g(a\ h(b)) = g(a)\ gh(b).$$

The cross product algebra $A \rtimes G := A \otimes k[G]$ has the multiplication defined by $(a \otimes g) \cdot (b \otimes h) = ag(b) \otimes gh$. We also view $A \rtimes G$ as a right A-module with a left G-action by

$$(a \otimes g) \cdot b = ag(b) \otimes g \text{ and } h \cdot (a \otimes g) = a \otimes gh^{-1}.$$

Equivalently, $A \rtimes G$ is a right $A \rtimes G$-module by the multiplication

$$(a \otimes g) \cdot (b \otimes h') = ag(b) \otimes gh'.$$

Note that the curvature of $A \rtimes G$ with zero differential as a right quasi-module over $(A \rtimes G, w \otimes 1)$ is $-w \otimes 1$.

Denote by $\{(A \rtimes G, -w \otimes 1)\}$ the full subcategory of $q\mathrm{MF}_{dg}(A \rtimes G, w \otimes 1)$ consisting of the indicated object $A \rtimes G$ with zero differential and curvature $-w \otimes 1$. The embedding $\{(A \rtimes G, -w)\} \hookrightarrow q\mathrm{MF}_{dg}(A \rtimes G, w \otimes 1)$ is called a quasi-Yoneda embedding. It is a pseudo-equivalence; see [36]. Consider the embedding $q\mathrm{MF}_{dg}(A \rtimes G, w \otimes 1) \hookrightarrow q\mathrm{MF}^G_{dg}(A, w)$, which is also a pseudo-equivalence; see [10]. Hence, by the quasi-Morita invariance the induced morphism of mixed complexes

$$\overline{\mathrm{MC}}^{II}(A \rtimes G, -w) \to \overline{\mathrm{MC}}^{II}(q\mathrm{MF}^G_{dg}(A, w))$$

is a quasi-isomorphism.

Consider an embedding $\{(A_g \rtimes G, -w_g \otimes 1)\} \hookrightarrow q\mathrm{MF}_{dg}(A_g, w_g)$. This is a pseudo-equivalence since $(A_g, -w_g)$ is a direct summand of $(A_g \rtimes G, -w_g \otimes 1)$ as a right quasi-module over $(A_g, -w_g)$. Hence, by the quasi-Morita invariance the induced morphism of mixed complexes

$$\overline{\mathrm{MC}}^{II}(\mathrm{End}_{A_g}(A_g \rtimes G), -w_g \otimes 1) \to \overline{\mathrm{MC}}^{II}(q\mathrm{MF}_{dg}(A_g, w_g))$$

is a quasi-isomorphism.

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We have a diagram of chain maps

$$\begin{array}{ccccc} \overline{C}^{II}(q\mathrm{MF}^G_{dg}(A,w)) & \xrightarrow{\tau} & (\oplus_g \overline{C}^{II}(q\mathrm{MF}_{dg}(A_g,w_g)))^G & \xrightarrow[\cong]{natural} & (\oplus_g \overline{C}^{II}(q\mathrm{MF}_{dg}(A_g,w_g)))_G \\ \uparrow\sim & & & & \uparrow\sim \\ \overline{C}^{II}(A\rtimes G,-w\otimes 1) & & \xrightarrow{\tau|_{A\rtimes G}} & & (\oplus_g \overline{C}^{II}(\mathrm{End}_{A_g}(A_g\rtimes G),-w_g\otimes 1))_G \\ & \searrow^{\sim}_{|G|\cdot\psi_A} & & \swarrow^{\sim}_{\mathrm{Tr}} & \\ & & (\oplus_g \overline{C}^{II}(A_g,-w_g))_G & & \end{array}, \tag{4.6}$$

where:

- ∘ two vertical chain maps are quasi-isomorphisms as explained already;
- ∘ the middle horizontal map $\tau|_{A\rtimes G}$ is induced from the composition $natural \circ \tau$;
- ∘ ψ_A is the quasi-isomorphic chain map defined by Baranovsky [4, page 799], Segal[40], and Căldăraru and Tu [8, Proof of 6.3];
- ∘ Tr is the generalized trace map.

Lemma 4.14. *The triangle in equation (4.6) is commutative. Hence, $\tau|_{A\rtimes G}$ is a quasi-isomorphism.*

Proof. This will be clear since g acts on A_g trivially. Note that the following diagram commutes

$$\begin{array}{ccc} a_0\otimes g_0[a_1\otimes g_1|\cdots|a_n\otimes g_n] & \xmapsto{\tau|_{A\rtimes G}} & \bigoplus_g \overline{a}_0\otimes g_0[\overline{a}_1\otimes g_1|\cdots|\overline{a}_n\otimes g_n g^{-1}] \\ \downarrow{\scriptstyle |G|\cdot\psi_A} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ |G|\cdot\overline{a}_0[g_0(\overline{a}_1)|\cdots|g_0\cdots g_{n-1}(\overline{a}_n)] & = & \bigoplus_{g:\ g=g_0\cdots g_n} |G|\cdot\overline{a}_0[g_0(\overline{a}_1)|\cdots|g_0\cdots g_{n-1}g^{-1}(\overline{a}_n)], \end{array} \tag{4.7}$$

where $\overline{a}$ denotes the element in A_g associated to a. Here, the right-bottom corner is meant to have the g-th component

$$\begin{cases} |G|\cdot\overline{a}_0[g_0\overline{a}_1|\cdots|g_0\cdots g_{n-1}g^{-1}\overline{a}_n] & \text{if } g=g_0\cdots g_n \\ 0 & \text{otherwise.} \end{cases}$$

□

Remark 4.15. Note that for for example, when $A = k$ with a nontrivial G, diagram (4.7) is not commutative without the twisting $\mathfrak{tw}$ insertion in the definition of τ.

4.5. Proof of Proposition 4.13

Due to Remark 4.12 and the compatibility of the map τ with Čech differentials, the proof follows from Lemma 4.14.

4.6. The role of $\mathfrak{can}$ *at the level of category*

As we have already remarked, the insertion of central automorphism $\mathfrak{tw}$ was essential. We would like to sketch how it appears naturally in the computation of Hochschild invariants to clarify its role. The proof of Proposition 4.13 can be interpreted as a two-step process.

The first step is purely categorical. Suppose a k-linear category $\mathcal{C}$ carries a strict G-action of a finite group G. We still denote corresponding endofunctors by $g : \mathcal{C} \to \mathcal{C},\quad g \in G$. We also denote its category of G-equivariant objects by $\mathcal{C}_G$. Its object consists of a pair (E, ϕ^E_g), where E is an object of $\mathcal{C}$ and ϕ^E_g is an isomorphism $\phi^E_g : E \simeq gE$ satisfying a cocycle condition. The morphism is defined as usual.

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There is a natural functor

$$\widetilde{\ }: \mathcal{C} \to \mathcal{C}_G$$

called *linearization* which is defined on objects as

$$\widetilde{E} = \left(\bigoplus_{h\in G} hE, \phi_g^{\widetilde{E}}\right), \quad \phi_g^{\widetilde{E}} : \bigoplus_{h\in G} hE = \bigoplus_{h\in G} g(hE) \simeq \bigoplus_{h\in G} (gh)E.$$

It is not hard to see that the linearization is a both left and right adjoint to the forgetful functor. Its essential image generates $\mathcal{C}_G$ and

$$\mathrm{Hom}_{\mathcal{C}_G}(\widetilde{E}_1, \widetilde{E}_2) \simeq \mathrm{Hom}_{\mathcal{C}}(\widetilde{E}_1, \widetilde{E}_2)^G.$$

This fact leads to the following simple description of Hochschild homology of $\mathcal{C}_G$. (See [33].)

$$HH_*(\mathcal{C}_G) \simeq (\bigoplus_{g\in G} HH_*(\mathcal{C}, g))^G \simeq \bigoplus_{g\in \mathrm{Conj}(G)} HH_*(\mathcal{C}_{C(g)}, g). \tag{4.8}$$

Here, $HH_*(\mathcal{C}, g)$ denotes Hochschild homology with coefficient g, where the endofunctor g is considered as a bimodule.

The second observation is geometric. For simplicity, let $\mathcal{C} = D(X)$ be a dg category of coherent sheaves on a smooth affine scheme $X = \mathrm{Spec}(A)$ acted on by a finite group G. One can easily extend the discussion to the case of matrix factorizations. Each component of equation (4.8) has a simpler description:

$$HH_*(D_{C(g)}(X), g) \xrightarrow[\mathrm{res}]{\sim} HH_*(D_{C(g)}(X^g), g). \tag{4.9}$$

Notice that the action of g on X^g is trivial. If $(E, \{\phi_h^E\}_{h\in G})$ is a G-equivariant sheaves on X, then $\varphi_g^E = \left(\phi_g^E\right)^{-1}$ restricts to the central automorphism $\mathfrak{can}_{E|_{X_g}}$ of $E|_{X^g}$. In fact, any $C(g)$-equivariant object $(F, \{\phi_h^F\}_{h\in C(g)})$ on X^g carries a distinguished automophism $\mathfrak{can}_F = \left(\phi_g^F\right)^{-1}$. This assignment is viewed as a natural transformation between identity functors or an element of zeroth Hochschild cohomology;

$$[\mathfrak{can}] \in HH^0(D_{C(g)}(X^g)).$$

The map $\mathfrak{tw}$ on Hochschild chains is a cap product with $[\mathfrak{can}]$.

Lastly, observe that $D_{C(g)}(X^g)$ is generated by $\mathcal{O}_{X^g}$. Notice that g-action on $\mathcal{O}_{X^g}$ is trivial, so $\mathfrak{can}$ could be ignored. This implies

$$HH_*(A_g)^{C(g)} \xrightarrow[\mathrm{inc}]{\sim} HH_*(D_{C(g)}(X^g), g). \tag{4.10}$$

4.7. Mixed complex case

In general, the map τ is not a morphism of mixed complexes. In this subsection, we modify τ to get a morphism of mixed complexes.

For $\chi \in \hat{\mu_r}$, let $q\mathrm{MF}_{dg}^{\chi}(I_{\mu_r}\mathcal{X}, w)$ be the full subcategory of $q\mathrm{MF}_{dg}(I_{\mu_r}\mathcal{X}, w)$ consisting χ-eigenobjects of $q\mathrm{MF}_{dg}(I_{\mu_r}\mathcal{X}, w)$. The map $\mathfrak{tw}$ restricted to the subcomplex $C(q\mathrm{MF}_{dg}^{\chi}(I_{\mu_r}\mathcal{X}, w))$,

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denoted by $\mathfrak{tw}_\chi$, is nothing but the multiplication by $\chi(e^{2\pi i/r})$. Hence,

$$\mathfrak{tw}_\chi : \mathrm{MC}(q\mathrm{MF}^{\chi}_{dg}(I_{\mu_r}\mathcal{X})) \to \mathrm{MC}(q\mathrm{MF}^{\chi}_{dg}(I_{\mu_r}\mathcal{X}))$$

is an automorphism of the mixed Hochschild complex.

Consider the composition τ_m of a sequence of morphisms of mixed complexes

$$\tau_m : \overline{\mathrm{MC}}^{II}(q\mathrm{MF}_{dg}(\mathcal{X}, w)) \xrightarrow{pullback} \oplus_{r,\chi}\overline{\mathrm{MC}}^{II}(q\mathrm{MF}^{\chi}_{dg}(I_{\mu_r}\mathcal{X}, w))$$
$$\xrightarrow{\oplus \mathfrak{tw}_\chi} \oplus_{r,\chi}\overline{\mathrm{MC}}^{II}(q\mathrm{MF}^{\chi}_{dg}(I_{\mu_r}\mathcal{X}, w)) \xrightarrow{nat} \oplus_r \Gamma(I_{\mu_r}\mathcal{X}, \overline{\mathrm{MC}}^{II}(q\mathrm{MF}_{I_{\mu_r}\mathcal{X},w}(-))),$$

where the natural map *nat* is defined by setting

$$nat\left(\sum_\chi a_0^\chi[\overline{a}_1^\chi|\dots|\overline{a}_n^\chi]\right) = \left(U \mapsto \sum_\chi a^\chi_{0|_{IU}}[\overline{a}^\chi_{1|_{IU}}|\dots|\overline{a}^\chi_{n|_{IU}}]\right).$$

Remark 4.16. While the cochain map

$$C^{II}(q\mathrm{MF}_{dg}(I_{\mu_r}\mathcal{X}, w)) \xrightarrow{\mathrm{Tr}} \oplus_\chi C^{II}(q\mathrm{MF}^{\chi}_{dg}(I_{\mu_r}\mathcal{X}, w))$$

is an isomorphism, $\overline{C}^{II}(q\mathrm{MF}_{dg}(I_{\mu_r}\mathcal{X}, w)) \xrightarrow{\overline{\mathrm{Tr}}} \oplus_\chi \overline{C}^{II}(q\mathrm{MF}^{\chi}_{dg}(I_{\mu_r}\mathcal{X}, w))$ is not an isomorphism in general but a quasi-isomorphism from the facts that $C^{II} \to \overline{C}^{II}$ is a quasi-isomorphism and the above Tr is an isomorphism.

Proposition 4.17. *Suppose that $\mathcal{X}$ is of form $[\mathrm{Spec} A/G]$ as in Proposition 4.13. The morphism τ_m in the category of mixed complexes is a quasi-isomorphism.*

Proof. By the definition, we need to show that τ_m is a quasi-isomorphism between Hochschild-type chain complexes. Replacing τ by τ_m and $\overline{C}^{II}$ by $\overline{\mathrm{MC}}^{II}$ in diagram (4.6) we conclude the proof. □

4.8. Global case

Let $\underline{\mathcal{X}}$ denote the coarse moduli space of $\mathcal{X}$. For an étale map $V \to \underline{\mathcal{X}}$, let $\mathcal{X}_V := V \times_{\underline{\mathcal{X}}} \mathcal{X}$. We take the sheafification $\underline{\overline{\mathrm{MC}}}^{II}(q\mathrm{MF}_{dg}(\mathcal{X}, w))$ (resp. $\underline{\overline{\mathrm{MC}}}^{II}(q\mathrm{MF}_{I\mathcal{X},w})$) of the presheaf

$$V \mapsto \overline{\mathrm{MC}}^{II}(q\mathrm{MF}_{dg}(\mathcal{X}_V, w)) \text{ resp. } (V \mapsto \Gamma(I\mathcal{X}_V, \overline{\mathrm{MC}}^{II}(q\mathrm{MF}_{I\mathcal{X},w}(-)))$$

both on the étale site of $\underline{\mathcal{X}}$. We take the sheaf homomorphism $\underline{\tau}_m$ induced from τ_m

$$\underline{\tau}_m : \underline{\overline{\mathrm{MC}}}^{II}(q\mathrm{MF}_{dg}(\mathcal{X}, w)) \to \underline{\overline{\mathrm{MC}}}^{II}(q\mathrm{MF}_{I\mathcal{X},w}(-)).$$

Lemma 4.18. *Suppose that $\mathcal{X}$ is smooth over k. Then the induced morphism $\mathbb{R}\Gamma(\underline{\tau}_m)$ fits into a diagram of isomorphisms in the derived category of mixed complexes:*

$$\begin{array}{ccc} \overline{\mathrm{MC}}(\mathrm{MF}_{dg}(\mathcal{X}, w)) & & \mathbb{R}\Gamma(\underline{\overline{\mathrm{MC}}}^{II}(\mathcal{O}_{I\mathcal{X}}, -w|_{I\mathcal{X}})) \\ \downarrow \wr & & \downarrow \wr \\ \mathbb{R}\Gamma\underline{\overline{\mathrm{MC}}}^{II}(q\mathrm{MF}_{dg}(\mathcal{X}, w)) & \xrightarrow[\mathbb{R}\Gamma(\underline{\tau}_m)]{\sim} & \mathbb{R}\Gamma\underline{\overline{\mathrm{MC}}}^{II}(q\mathrm{MF}_{I\mathcal{X},w}(-)). \end{array} \tag{4.11}$$

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Proof. The right vertical map is a quasi-isomorphism by the quasi-Morita invariance and the fact that for each étale morphism $U \to I\mathcal{X}$ the Yoneda embedding $(\mathcal{O}_{I\mathcal{X}}(U), -w) \to q\mathrm{MF}_{I\mathcal{X},w}(U)$ is a pseudo-equivalence; see [5, Proposition 3.25] and [36]. It remains to show that the left vertical map is a quasi-isomorphism. Let $\pi : \mathcal{X} \to \underline{\mathcal{X}}$ be the coarse moduli space. By [17, Corollary 4.6], the presheaf $(V \mapsto \overline{C}(\mathrm{MF}_{dg}(V \times_{\underline{\mathcal{X}}} \mathcal{X}, w))$ is a sheaf on the étale site of $\underline{\mathcal{X}}$. It is thus enough to show that the left vertical map is a quasi-isomorphism when $\mathcal{X} = [X/G]$ for a smooth variety X and a finite group G which follows from [10, Theorem 6.9]. □

Theorem 4.19. *Suppose that $\mathcal{X}$ is smooth over k. Then the isomorphism*

$$\overline{\mathrm{MC}}(\mathrm{MF}_{dg}(\mathcal{X}, w)) \cong \mathbb{R}\Gamma(\underline{\overline{\mathrm{MC}}}^{II}(\mathcal{O}_{I\mathcal{X}}, -w|_{I\mathcal{X}})) \tag{4.12}$$

in the derived category of mixed complexes induces an isomorphism

$$\overline{\mathrm{MC}}(\mathrm{MF}_{dg}(\mathcal{X}, w)) \cong \mathbb{R}\Gamma(\Omega^\bullet_{I\mathcal{X}}, -dw|_{I\mathcal{X}}, ud).$$

Proof. The proof follows from Lemma 4.18 and the HKR-type isomorphism ([8, 40]) for affine orbifolds. □

5. Chern character formulae

Let $\mathcal{X}$ be a smooth separated finite-type DM stack over k, and let $w : \mathcal{X} \to \mathbb{A}^1_k$ be an algebraic function on $\mathcal{X}$ with only critical value 0.

5.1. *A formula via Čech model and Chern–Weil theory*

We fix an affine étale surjective morphism $\mathfrak{p} : \mathfrak{U} \to \mathcal{X}$ from a k-scheme $\mathfrak{U}$ as in §4.2. Since $\mathfrak{U}$ is an affine scheme over k, every $E|_{\mathfrak{U}}$ has a connection

$$\nabla_{E|_{\mathfrak{U}}} : E|_{\mathfrak{U}} \to E|_{\mathfrak{U}} \otimes \Omega^1_{\mathfrak{U}}.$$

Define a connection

$$\nabla_{E|_{\mathfrak{U}^r}} : E|_{\mathfrak{U}^r} \to E|_{\mathfrak{U}^r} \otimes \Omega^1_{\mathfrak{U}^r}$$

by letting $\nabla_{E|_{\mathfrak{U}^r}} = p_1^*\nabla_{E|_{\mathfrak{U}}}$, where p_1 is the first projection $\mathfrak{U}^r \to \mathfrak{U}$. This gives rise to a connection

$$\nabla_E : \check{\mathrm{C}}(E) \to \Omega^1_{\mathcal{X}} \otimes \check{\mathrm{C}}(E),$$

where $\check{\mathrm{C}}(E) := (\bigoplus_{r\geq 0} \check{\mathrm{C}}^r(E), d_{\text{Čech}})$ and $\check{\mathrm{C}}^r(E) = \mathfrak{p}_{r*}\mathfrak{p}_r^*E$. For every E, fix such a connection once and for all.

Let $I\mathfrak{U}$ denote the affine scheme $\mathfrak{U} \times_{\mathcal{X}} I\mathcal{X}$. Using this affine covering of $I\mathcal{X}$, we have the Čech resolution $\check{\mathrm{C}}(E|_{I\mathcal{X}})$ and the connection

$$\nabla_{E|_{I\mathcal{X}}} : \check{\mathrm{C}}(E|_{I\mathcal{X}}) \to \Omega^1_{I\mathcal{X}} \otimes \check{\mathrm{C}}(E|_{I\mathcal{X}}).$$

In general, for every vector bundle F on $I\mathcal{X}$, we can choose a connection $\nabla_F : \check{\mathrm{C}}(F) \to \Omega^1_{I\mathcal{X}} \otimes \check{\mathrm{C}}(F)$.

For each $P \in q\mathrm{MF}_{dg}(I\mathcal{X}, w))$, choose a connection ∇_P as above once and for all. Let $R = u\nabla^2_\nabla + [\nabla_P, \delta_{\check{\mathrm{C}}(P)}]$ a kind of the total curvature of ∇_P. By a straightforward generalization of the definition of a chain map tr_∇ in [10] to the stacky case, we obtain a $k[[u]]$-linear map

$$\mathrm{tr}_{\nabla, I\mathcal{X}} : C(q\mathrm{MF}_{dg}(I\mathcal{X}, w))[[u]] \to \Gamma(\check{\mathrm{C}}(\Omega^\bullet_{I\mathcal{X}}))[[u]]$$

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mapping $a_0[a_1|\cdots|a_n]$ for $a_i \in \mathrm{End}_{\check{C}(P)}(P)$, $P \in q\mathrm{MF}_{dg}(I\mathcal{X}, w))$ to

$$\sum_{(j_0,\ldots,j_n):j_i\in\mathbb{Z}_{\geq 0}} \frac{(-1)^{|j_0+\cdots+j_n|}\mathrm{tr}(a_0R^{j_0}[\nabla_P, a_1]R^{j_1}[\nabla_P, a_2]\cdots[\nabla_P, a_n]R^{j_n})}{(n+|j_0+\cdots+j_n|)!}.$$

It is clear how to map an arbitrary element of $C(q\mathrm{MF}_{dg}(I\mathcal{X}, w))[[u]]$. By the same proof in [10, Appendix B], the map $\mathrm{tr}_{\nabla, I\mathcal{X}}$ is a chain map. Likewise, we have a chain map

$$\mathrm{tr}_\nabla : \underline{\overline{C}}^{II}(q\mathrm{MF}_{I\mathcal{X},w})[[u]] \to p_*\check{\mathrm{C}}(\Omega^\bullet_{I\mathcal{X}})[[u]],$$

where $p : I\mathcal{X} \to \underline{\mathcal{X}}$ is the natural morphism.

Consider a diagram of chain maps in negative cyclic type complexes

$$\begin{array}{ccccc} C(q\mathrm{MF}_{dg}(\mathcal{X}, w))[[u]] & \xrightarrow{\mathfrak{tw}\circ\rho^*_{\mathcal{X}}} & C(q\mathrm{MF}_{dg}(I\mathcal{X}, w))[[u]] & \xrightarrow{\mathrm{tr}_{\nabla,I\mathcal{X}}} & \Gamma(\check{\mathrm{C}}(\Omega^\bullet_{I\mathcal{X}}))[[u]] \\ \downarrow & & & & \downarrow \\ \mathbb{R}\Gamma\underline{\overline{C}}^{II}(q\mathrm{MF}_{dg}(\mathcal{X}, w))[[u]] & \xrightarrow[\mathbb{R}\Gamma(\underline{\tau}_m)]{\sim} & \mathbb{R}\Gamma\underline{\overline{C}}^{II}(q\mathrm{MF}_{I\mathcal{X},w})[[u]] & \xrightarrow[\mathbb{R}\Gamma(\mathrm{tr}_\nabla)]{} & \mathbb{R}\Gamma(\check{\mathrm{C}}(\Omega^\bullet_{I\mathcal{X}}))[[u]]. \end{array} \tag{5.1}$$

Note that the diagram is commutative, and all arrows may possibly be quasi-isomorphisms but the two top horizontal arrows are quasi-isomorphisms. Hence, we have the following corollaries.

Corollary 5.1. *The chain map*

$$\mathrm{tr}_{\nabla,I\mathcal{X}} \circ \mathfrak{tw} \circ \rho^*_{\mathcal{X}} : C(q\mathrm{MF}_{dg}(\mathcal{X}, w))[[u]] \to \Gamma(\check{C}(\Omega^\bullet_{I\mathcal{X}}))[[u]]$$

is a quasi-isomorphism.

Corollary 5.2. *Under the isomorphism in equation (4.11), the Chern character* $\mathrm{ch}_{HN}(P)$ *of* $P \in \mathrm{MF}_{dg}(\mathcal{X}, w)$ *is the class represented by Čech cocycle*

$$\mathrm{tr}\Big(\mathfrak{can}_{P|_{I\mathcal{X}}} \circ \exp(-u\nabla^2_{P|_{I\mathcal{X}}} - [\nabla_{P|_{I\mathcal{X}}}, \delta_{P|_{I\mathcal{X}}} + d_{\check{C}ech}])\Big)$$

in $\check{H}(I\mathfrak{U}, (\Omega^\bullet_{I\mathcal{X}}, -dw|_{I\mathcal{X}}))$.

Example 5.3. Consider a DM stack $\mathcal{X}$ of the form $[X/G]$ with X quasi-projective and G a linearly reductive group. Then there is a finite collection $\mathfrak{U} = \{U_i\}_{i\in I}$ of G-invariant affine open subset U_i of X such that $\bigcup_i U_i = X$. On the other hand, there is a finite subset S of G such that

$$I\mathcal{X} = \sqcup_{g\in S}[X^g/\mathrm{C}_G(g)].$$

Note that $I\mathfrak{U} = \{U^g_i : i \in I, g \in S\}$. Instead of affine étale covering, we may use the affine smooth covering $\mathfrak{U}$ for a Čech model of $q\mathrm{MF}_{dg}(\mathcal{X}, w)$ and the chain map $\mathrm{tr}_{\nabla,I\mathcal{X}}$. Since G is linearly reductive, each $P|_{U_i}$ has a G-equivariant connection, which in turn gives a $\mathrm{C}_G(g)$-equivariant connection $\nabla_{i,g}$ on $P|_{U^g_i}$. This is because of the surjection of the canonical map $\mathrm{Hom}^G_{\mathcal{O}_X}(E, J(E)) \to \mathrm{Hom}^G_{\mathcal{O}_X}(E, E)$ induced from the jet sequence

$$0 \to \Omega^1_X \otimes_{\mathcal{O}_X} E \to J(E) \to E \to 0.$$

We have

$$\mathrm{ch}_{HN}(P) = \oplus_{g\in S}\mathrm{tr}\Big(g \circ \exp(-u\Pi_{i\in I}\nabla^2_{i,g} - [\Pi_{i\in I}\nabla_{i,g}, \delta_{P|_{U^g}} + d_{\check{\mathrm{C}}\mathrm{ech}}])\Big).$$

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When X itself is affine, then the formula simplifies to

$$\mathrm{ch}_{HN}(P) = \oplus_{g\in S}\mathrm{tr}\Big(g\circ\exp(-u\nabla_g^2 - [\nabla_g, \delta_{P|_{Ug}}])\Big),$$

taking into account the fact that $[\nabla_g, d_{\check{\mathrm{C}}\mathrm{ech}}] = 0$.

Let $a \in \oplus_i \mathbb{R}^i\mathrm{End}(P)$, then it determines a class in $H^*(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, -dw|_{I\mathcal{X}}))$ under the HKR isomorphism. Denote the class by $\tau(a)$. The assignment $a \mapsto \tau(a)$ is called the boundary-bulk map.

Corollary 5.4 (The boundary-bulk map formula).

$$\tau(a) = \mathrm{tr}\big(a\circ \mathfrak{can}_{P|_{I\mathcal{X}}}\circ\exp(-[\nabla_{P|_{I\mathcal{X}}}, \delta_{P|_{I\mathcal{X}}} + d_{\check{C}ech}])\big).$$

5.2. *A formula via Atiyah class*

Let $P \in \mathrm{MF}_{dg}(\mathcal{X}, w)$, and let

$$\Omega_{\mathcal{X}}^{-dw} := [\ \Omega^1_{\mathcal{X}} \underset{-dw\wedge}{\overset{0}{\rightleftarrows}} \mathcal{O}_{\mathcal{X}}\ ]$$

be the matrix factorization for $(\mathcal{X}, 0)$ located at amplitude $[-1, 0]$. The Atiyah class $\hat{\mathrm{at}}(P)$ defined in [16, Appendix B] is a suitable element of

$$\mathrm{Ext}^1(P, P\otimes\Omega_{\mathcal{X}}^{-dw}).$$

When $w = 0$, we have the decomposition

$$\mathrm{Ext}^1(P, P\otimes\Omega_{\mathcal{X}}^{-dw}) = \mathrm{Ext}^0(P,P)\oplus\mathrm{Ext}^1(P, P\otimes\Omega^1_{\mathcal{X}})$$

and let

$$\mathrm{at}(P) = proj\circ\hat{\mathrm{at}}(P) \in \mathrm{Ext}^1(P, P\otimes\Omega^1_{\mathcal{X}}),$$

where $proj$ is the projection. For example, when P is a vector bundle F and $\mathcal{X}$ is nonstacky, then $\mathrm{at}(F)$ is the usual Atiyah class [2]. In this case $\hat{\mathrm{at}}(F) = 1_F + \mathrm{at}(F)$.

Definition 5.5. Taking into account the convention of the exponential $\underline{\exp}$ of $\hat{P}$ as explained in [16, 24], we define a naive Chern character of P by

$$\mathrm{ch}(P) := \mathrm{tr}\big(\underline{\exp}(\hat{\mathrm{at}}(P))\big) \in H^*(\mathcal{X}, (\Omega^\bullet_{\mathcal{X}}, -dw)).$$

For simplicity, we abuse notation writing $\exp(\hat{\mathrm{at}}(P))$ for $\underline{\exp}(\hat{\mathrm{at}}(P))$.

The correct formula for $\mathrm{ch}_{HH}(P)$ in [25] is

$$\begin{aligned}\mathrm{ch}_{HH}(P) &= \mathrm{tr}\big(\mathfrak{can}_{P|_{I\mathcal{X}}}\circ\exp(\hat{\mathrm{at}}(P|_{I\mathcal{X}}))\big) \\ &= \mathrm{ch}(P) + \text{ twisted part }.\end{aligned} \tag{5.2}$$

We note that this formula agrees with the formula in Corollary 5.2 since Atiyah class $\hat{\mathrm{at}}(P|_{I\mathcal{X}})$ is representable as

$$(\mathrm{id}_P, -[\nabla_{P|_{I\mathcal{X}}}, \delta_{P|_{I\mathcal{X}}} + d_{\check{\mathrm{C}}\mathrm{ech}}]) \in \Gamma(I\mathcal{X}, \mathrm{End}(P|_{I\mathcal{X}})\otimes\Omega_{\mathcal{X}}^{-dw}\otimes\check{\mathrm{C}}(\mathcal{O}_{I\mathcal{X}}))$$

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in Čech cohomology $\check{H}(I\mathfrak{U}, \Omega^\bullet_{I\mathcal{X}}, -dw|_{I\mathcal{X}})$ (see [24, Proposition 1.3]) and

$$\underline{\exp}(\mathrm{id}_P, -[\nabla_{P|_{I\mathcal{X}}}, \delta_{P|_{I\mathcal{X}}} + d_{\text{Čech}}]) = \exp(-[\nabla_{P|_{I\mathcal{X}}}, \delta_{P|_{I\mathcal{X}}} + d_{\text{Čech}}]).$$

The boundary bulk map formula can also be written in terms of the Atiyah class:

$$\tau(a) = \mathrm{tr}\big(a \circ \mathfrak{can}_{P|_{I\mathcal{X}}} \circ \exp(\hat{\mathrm{at}}(P|_{I\mathcal{X}}))\big).$$

Definition 5.6. For a vector bundle E on $I\mathcal{X}$, we define

$$\mathrm{ch}_{tw}(E) := \mathrm{tr}(\mathfrak{can}_E \exp(\mathrm{at}(E))$$

and Todd class $\mathrm{td}_{tw}(E)$ of E by the expression of Todd class in terms of the Chern character $\mathrm{ch}_{tw}(E)$. For example, $\mathrm{td}_{tw}(T_{I\mathcal{X}})$ is defined. Since $T_{I\mathcal{X}}$ is fixed under the canonical automorphism, we simply write $\mathrm{td}(T_{I\mathcal{X}})$ for $\mathrm{td}_{tw}(T_{I\mathcal{X}})$.

5.3. Proof of Theorem 1.1

The first statement of Theorem 1.1 is Theorem 4.19. The second statement follows from equation (5.2).

5.4. Compactly supported case

Let Z be a closed substack of $\mathcal{X}$ proper over k. Let P be a matrix factorization for $(\mathcal{X}, w)$ which is coacyclic over $\mathcal{X} - Z$. Note that

$$\hat{\mathrm{at}}(P|_{I\mathcal{X}}) \in \mathrm{Ext}^1(P|_{I\mathcal{X}}, P|_{I\mathcal{X}} \otimes \Omega^{-dw|_{I\mathcal{X}}}_{I\mathcal{X}}) = \mathrm{Ext}^1_{IZ}(P|_{I\mathcal{X}}, P|_{I\mathcal{X}} \otimes \Omega^{-dw|_{I\mathcal{X}}}_{I\mathcal{X}}).$$

To emphasize that $\hat{\mathrm{at}}(P|_{I\mathcal{X}})$ can be considered as an IZ-supported extension class, write $\hat{\mathrm{at}}_Z(P|_{I\mathcal{X}})$ for $\hat{\mathrm{at}}(P|_{I\mathcal{X}})$. Let $\mathrm{MF}_{dg}(\mathcal{X}, w))_Z$ be the full subcategory of $\mathrm{MF}_{dg}(\mathcal{X}, w)$ consisting of all matrix factorization for $(\mathcal{X}, w)$ that are coacyclic over $\mathcal{X} - Z$.

Corollary 5.7. *There is an isomorphism*

$$\mathrm{MC}(\mathrm{MF}_{dg}(\mathcal{X}, w))_Z \cong \mathbb{R}\Gamma_Z(\Omega^\bullet_{I\mathcal{X}}, -dw|_{I\mathcal{X}}, d)$$

in the derived category of mixed complexes. Under the isomorphism ch^Z_{HH}, (P) *is equal to*

$$\mathrm{tr}\big(\mathfrak{can}_{P|_{I\mathcal{X}}} \exp(\hat{\mathrm{at}}_Z(P|_{I\mathcal{X}}))\big)$$

in $\mathbb{H}^*_{IZ}(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, -dw|_{I\mathcal{X}}))$.

Proof. The first statement immediately follows from Theorem 1.1. The second statement follows from a concrete chain map for the Hochschild-type chain complexes; see [10, §6.2] for some details. □

6. Riemann–Roch for stacky matrix factorizations

6.1. The categorical HRR

The results in this subsection are taken from [35, 41] with a weaker condition on a dg category $\mathcal{A}$. Instead of assuming that $\mathcal{A}$ is saturated, we assume that $\mathcal{A}$ is locally proper and smooth.

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Definition 6.1. Let $\mathcal{A}$ be a *locally proper* dg category: That is, for every $x, y \in \mathcal{A}$, the dimension $\sum_{i\in\mathbb{G}} \dim H^i\mathrm{Hom}_{\mathcal{A}}(x, y)$ of total cohomology of $\mathrm{Hom}_{\mathcal{A}}(x, y)$ is finite. Let

$$\langle , \rangle_{can} : HH_*(\mathcal{A}) \times HH_*(\mathcal{A}^{op}) \to k$$

be the *canonical pairing* of $\mathcal{A}$ defined by Shklyarov. It is a k-linear pairing.

6.1.1. Transformations by bimodules

Definition 6.2. For a dg category $\mathcal{C}$, we take the projective model structure on the category $\mathrm{Mod}(\mathcal{C})$ of right $\mathcal{C}$-modules. The cofibrant objects are exactly the summands of semifree dg-modules. A right $\mathcal{C}$-module N is called *perfect* if N is a cofibrant object which is compact in the derived category $D(\mathcal{C})$ of right $\mathcal{C}$-modules. Let $\mathrm{Mod}_{dg}(\mathcal{C})$ be the dg category of right $\mathcal{C}$-modules. Let $\mathrm{Perf}(\mathcal{C})$ be the full subcategory of $\mathrm{Mod}_{dg}(\mathcal{C})$ consisting of all perfect $\mathcal{C}$-modules. We call a dg category $\mathcal{C}$ is *smooth* if the diagonal bimodule $\Delta_{\mathcal{A}}$ is a perfect bimodule.

From now on, let $\mathcal{A}$ and $\mathcal{B}$ be locally perfect and smooth dg categories unless otherwise stated.

Lemma 6.3. *The total dimension of Hochschild homology $HH_*(\mathcal{A})$ of $\mathcal{A}$ is finite. The dg category* $\mathrm{Perf}(\mathcal{A} \otimes \mathcal{B})$ *is locally perfect and smooth.*

Proof. The first claim amounts $\Delta_{\mathcal{A}} \otimes^{\mathbb{L}}_{\mathcal{A}^{op}\otimes\mathcal{A}} \Delta_{\mathcal{A}}$ is a perfect dg k-module, which follows from tensor-hom adjunction and the conditions on $\mathcal{A}$. The second claim follows from [30, Lemma 2.13, 2.14, 2.15] since k is a field. □

For a right $\mathcal{A}^{op} \otimes \mathcal{B}$-module M, there is a dg functor $T_M : \mathrm{Perf}(\mathcal{A}) \to \mathrm{Mod}_{dg}(\mathcal{B})$ sending N to $N \otimes_{\mathcal{A}} M$. If M is representable, then T_M factors though $\mathrm{Perf}(\mathcal{B})$ since $\mathcal{A}$ is locally proper. Hence, this is the case for every perfect $\mathcal{A}^{op} \otimes \mathcal{B}$-module M.

Let $M \in \mathrm{Perf}(\mathcal{A}^{op} \otimes \mathcal{B})$, and let $\mathrm{Ch}(M) = \sum_i \gamma_i \otimes \gamma^i$ under the Künneth isomorphism $HH_*(\mathrm{Perf}(\mathcal{A}^{op} \otimes \mathcal{B})) \cong HH_*(\mathrm{Perf}(\mathcal{A}^{op})) \otimes HH_*(\mathrm{Perf}(\mathcal{B}))$. Let $HH(T_M) : HH_*(\mathrm{Perf}(\mathcal{A})) \to HH_*(\mathrm{Perf}(\mathcal{B}))$ be the homomorphism induced by $T_M : \mathrm{Perf}(\mathcal{A}) \to \mathrm{Perf}(\mathcal{B})$.

Proposition 6.4. *If $\langle , \rangle_{can}$ denotes the canonical pairing of* $\mathrm{Perf}(\mathcal{A}^{op} \otimes \mathcal{B})$*, then for every $\sigma \in HH_*(\mathrm{Perf}(\mathcal{A}))$ we have*

$$HH(T_M)(\sigma) = \sum_i \langle \sigma, \gamma_i \rangle_{can} \gamma^i.$$

6.1.2. The characteristic property

There are natural isomorphisms

$$HH_*(\mathrm{Perf}(\mathcal{A}^{op} \otimes \mathcal{A})) \cong HH_*(\mathcal{A}^{op} \otimes \mathcal{A}) \cong HH_*(\mathcal{A}^{op}) \otimes HH_*(\mathcal{A})$$

by the Morita invariance and the Künneth isomorphism. Write

$$\mathrm{Ch}_{HH}(\Delta_{\mathcal{A}}) = \sum_i T^i \otimes T_i \in HH_*(\mathcal{A}^{op}) \otimes HH_*(\mathcal{A}).$$

Then by Proposition 6.4 we obtain this.

Corollary 6.5. *The canonical pairing $\langle , \rangle_{can}$ is characterized by two conditions: (1) it is nondegenerate and (2) it satisfies the 'diagonal decomposition' property:*

$$\sum_i \langle \gamma, T^i \rangle \langle T_i, \gamma' \rangle = \langle \gamma, \gamma' \rangle$$

for every $\gamma \in HH_*(\mathcal{A}), \gamma' \in HH_*(\mathcal{A}^{op})$.

6.1.3. The Cardy condition

Consider objects $x, y \in \mathcal{A}$. Let a and b be closed endomorphisms of x and y, respectively. Let

$$L_b \circ R_a : \mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{A}}(x, y),\ (-1)^{|a||c|} \mapsto b \circ c \circ a.$$

Theorem 6.6. *We have*

$$\mathrm{tr}(L_b \circ R_a) = \langle [b], [a] \rangle_{can}.$$

For the identities $a = 1_x$, $b = 1_y$, it is specialized to

$$\chi(x, y) = \langle \mathrm{Ch}_{HH}(y), \mathrm{Ch}_{HH}(x) \rangle_{can}.$$

6.2. *On* $\mathrm{MF}_{dg}(\mathcal{X}, w)$

Consider two proper LG models $(\mathcal{X}, w)$ and $(\mathcal{Y}, v)$. We want to show that $\mathrm{MF}_{dg}(\mathcal{X}, w)$ is locally proper, smooth; and the following (†) and (★) hold:

(†) There is a natural dg functor

$$\mathrm{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, w \boxplus v) \to \mathrm{Perf}(\mathrm{MF}_{dg}(\mathcal{X}, w) \otimes \mathrm{MF}_{dg}(\mathcal{Y}, v)) \quad \text{defined by}$$
$$E \mapsto \Psi(E) : x \otimes y \mapsto \mathrm{Hom}_{\mathrm{MF}_{dg}(\mathcal{X}\times\mathcal{Y}, w\boxplus v)}(x \boxtimes y, E).$$

Here, $w \boxplus v$ denotes $w \otimes 1 + 1 \otimes v$.

(★) The triangulated category $[\mathrm{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, w \boxplus v)]$ is the smallest full triangulated subcategory containing all exterior products closed under finite coproducts and summands. Here, an object of $[\mathrm{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, w \boxplus v)]$ is called an exterior product if it is isomorphic to $x \boxtimes y$ for some $x \in \mathrm{MF}_{dg}(\mathcal{X}, w)$, $y \in \mathrm{MF}_{dg}(\mathcal{Y}, v)$.

Lemma 6.7.

1. (★) *implies* (†).
2. (†) *implies that the smoothness of* $\mathrm{MF}_{dg}(\mathcal{X}, w)$.

Proof. (1) is clear. Let $\Delta : \mathcal{X} \to \mathcal{X}^2$ be the diagonal morphism. Then (2) follows from the fact that $\Psi(\Delta \mathcal{O}_{\mathcal{X}})$ is quasi-isomorphic to the diagonal bimodule. □

Since $\mathrm{MF}_{dg}(\mathcal{X}, w)$ is clearly locally proper, it is enough to show (★). We check this when $\mathcal{X}$ is a stack quotient $[X/G]$ of a smooth variety by an action of an affine algebraic group G. When $\mathbb{G} = \mathbb{Z}$, then $w = 0$. Note that (★) holds by Theorem 2.29 and Corollary 4.21 of [3]. When $\mathbb{G} = \mathbb{Z}/2$, then w is a G-invariant function on X, not identically zero on any component of X. Note that (★) holds by Theorem 2.29 and Lemma 4.23 of [3].

6.3. *A geometric realization of the diagonal module*

Consider two proper LG models $(\mathcal{X}, w), (\mathcal{Y}, v)$. Suppose that $\mathcal{X}$, $\mathcal{Y}$ are stack quotients of smooth varieties by actions of affine algebraic groups. Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism with $f^* v = w$. We call $f : (\mathcal{X}, w) \to (\mathcal{Y}, v)$ an *proper LG morphism*. Choose an affine étale cover $\mathfrak{U} \to \mathcal{X}$ and $\mathfrak{U}' \to \mathcal{Y}$. Denote

$$\mathcal{A} := \mathrm{MF}_{dg}(\mathcal{X}, w),\ \mathcal{B} := \mathrm{MF}_{dg}(\mathcal{Y}, v).$$

They are locally proper and smooth as seen in §6.2.

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Let $-w \boxplus v := -w \otimes 1 + 1 \otimes v$, and let $\mathrm{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, -w \boxplus v)$ be the Čech dg model of the matrix factorizations for $(\mathcal{X} \times \mathcal{Y}, -w \boxplus v)$ with respect to the affine cover $\mathfrak{U} \times \mathfrak{U}' \to \mathcal{X} \times \mathcal{Y}$. Then by (†) we have a natural dg functor

$$\Psi : \mathrm{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, -w \boxplus v) \to \mathrm{Perf}(\mathcal{A}^{op} \otimes \mathcal{B}).$$

Let $D : \mathcal{A}^{op} \to \mathrm{MF}_{dg}(\mathcal{X}, -w)$ be the duality functor. Then we have a commuting diagram of isomorphisms

$$\begin{array}{ccc} HH_*(\mathrm{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, -w \boxplus v)) & \xrightarrow{HH(\Psi)} & HH_*(\mathrm{Perf}(\mathcal{A}^{op} \otimes \mathcal{B})) \\ \Big\downarrow{\scriptstyle \mathrm{HKR}} & & \Big\downarrow{\scriptstyle \mathrm{HKR} \circ HH(D) \otimes \mathrm{id} \circ \text{Künneth}} \\ \mathbb{H}^{-*}(\Omega^\bullet_{I\mathcal{X} \times I\mathcal{Y}}, d(w \boxplus -v)) & \xrightarrow[\text{Künneth}]{} & \mathbb{H}^{-*}(\Omega^\bullet_{I\mathcal{X}}, dw) \otimes \mathbb{H}^{-*}(\Omega^\bullet_{I\mathcal{Y}}, -dv), \end{array} \tag{6.1}$$

where HKR and Künneth are the HKR-type isomorphisms in §4.8 and the Künneth isomorphisms, respectively.

Consider a matrix factorization K for $(\mathcal{X} \times \mathcal{Y}, -w \boxplus v)$. For example, we have a coherent factorization

$$\mathcal{O}_{\Gamma_f} := (\Gamma_f)_* \mathcal{O}_{\mathcal{X}} \text{ for } (\mathcal{X} \times \mathcal{Y}, -w \boxplus v).$$

Since $\mathcal{X} \times \mathcal{Y}$ satisfies the resolution property by [3, Theorem 2.29], $\mathcal{O}_{\Gamma_f}$ is quasi-isomorphic to a matrix factorization.

For all $x \in \mathcal{A}, y \in \mathcal{B}$, there is a natural quasi-isomorphism

$$\mathbb{R}\mathrm{Hom}(y, q_*(K \otimes p^* x)) \sim_{qiso} \mathrm{Hom}_{\mathrm{MF}_{dg}(\mathcal{X} \times \mathcal{Y}, -w \boxplus v)}(x^\vee \boxtimes y, K)$$

functorial under the morphisms in the categories $\mathcal{B}$ and $\mathcal{A}$. This shows the following, which will be used later.

Lemma 6.8. *For easy notation, write T_K for $T_{\Psi(K)}$. Then:*

1. *The transformation $T_K : \mathrm{Perf}(\mathcal{A}) \to \mathrm{Perf}(\mathcal{B})$ is a dg enhancement of the Fourier–Mukai transform $[\mathcal{A}] \to [\mathcal{B}]$ attached to the kernel K. In particular, $T_{\mathcal{O}_{\Gamma_f}}$ represents $\mathbb{R}f_* : [\mathcal{A}] \to [\mathcal{B}]$.*
2. *The bimodule $\Psi(\mathcal{O}_{\Gamma_{\mathrm{id}}})$ and the diagonal bimodule $\Delta_{\mathcal{A}}$ are quasi-isomorphic.*

The second statement in the above lemma is also in Lemma 5.24 of [3].

6.4. An explicit realization of the canonical pairing

Theorem 6.9. *Let $(\mathcal{X}, w)$ be a proper LG model. Assume that $\mathcal{X}$ is a smooth quotient DM stack which satisfies the resolution property. Then the canonical pairing coincides with the pairing defined by*

$$\int_{I\mathcal{X}} (-1)^{\binom{\dim_{I\mathcal{X}}+1}{2}} \cdot \wedge \cdot \wedge \frac{\mathrm{td}(T_{I\mathcal{X}})}{\mathrm{ch}_{tw}(\lambda_{-1}(N^\vee_{I\mathcal{X}/\mathcal{X}}))}, \tag{6.2}$$

where $\dim_{I\mathcal{X}}$ is the locally constant dimension function of $I\mathcal{X}$.

Proof. We prove the characteristic property in Corollary 6.5 for the pair (6.2). The nondegeneracy follows from Serre duality [32] as argued in [16, §4.1]. By Lemma 6.8 the 'diagonal decomposition' is

$$\sum_i \int_{I\mathcal{X}} \gamma \cdot t^i \cdot \widetilde{\mathrm{td}}(T_{I\mathcal{X}}) \int_{I\mathcal{X}} t^i \cdot \gamma' \cdot \widetilde{\mathrm{td}}(T_{I\mathcal{X}}) = \int_{I\mathcal{X}} (-1)^{\binom{\dim_{I\mathcal{X}}+1}{2}} \gamma'' \cdot \widetilde{\mathrm{td}}(T_{I\mathcal{X}}), \tag{6.3}$$

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where

$$\sum_i t^i \otimes t_i = \mathrm{ch}_{HH}(\Delta_* \mathcal{O}_{\mathcal{X}}) \in \mathbb{H}^*(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, -dw|_{I\mathcal{X}})) \otimes \mathbb{H}^*(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, dw|_{I\mathcal{X}}))$$

$$\gamma \in \mathbb{H}^*(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, dw|_{I\mathcal{X}})), \gamma' \in \mathbb{H}^*(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, -dw|_{I\mathcal{X}}))$$

$$\widetilde{\mathrm{td}}(T_{I\mathcal{X}}) := \frac{\mathrm{td}(T_{I\mathcal{X}})}{\mathrm{ch}_{tw}(\lambda_{-1}(N^\vee_{I\mathcal{X}/\mathcal{X}}))}. \tag{6.4}$$

To show equation (6.3), we use the deformation to the normal cone.

Over $\mathbb{P}^1$, there is a deformation stack $\mathcal{M}^\circ$ to the normal cone $N_{\mathcal{X}/\mathcal{X}^2}$: The general fiber is $\mathcal{X}^2$ and the special fiber, say over ∞, is the vector bundle stack $N_{\mathcal{X}/\mathcal{X}^2} \cong T_{\mathcal{X}}$. It comes with a natural morphism $h : \mathcal{M}^\circ \to \mathcal{X}^2$, a flat morphism $pr : \mathcal{M}^\circ \to \mathbb{P}^1$, and a morphism $\widetilde{\Delta} : \mathcal{X} \times \mathbb{P}^1 \to \mathcal{M}^\circ$ such that $(h, pr) \circ \widetilde{\Delta} = \Delta \times \mathrm{id}_{\mathbb{P}^1}$. Consider the fiber square diagram

$$\begin{array}{ccccc}
\mathcal{X} \times 0 & \longrightarrow & \mathcal{X} \times \mathbb{P}^1 & \longleftarrow & \mathcal{X} \times \infty \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \widetilde{\Delta}} & & \downarrow{\scriptstyle \delta} \\
\mathcal{X}^2 & \longrightarrow & \mathcal{M}^\circ & \longleftarrow & T_{\mathcal{X}} \\
\uparrow{\scriptstyle \rho_{\mathcal{X}^2}} & & \uparrow{\scriptstyle \rho_{\mathcal{M}^\circ}} & & \uparrow{\scriptstyle \rho_{T_{\mathcal{X}}}} \\
I\mathcal{X}^2 & \longrightarrow & I\mathcal{M}^\circ & \longleftarrow & IT_{\mathcal{X}}.
\end{array}$$

Here, we use facts that $I\mathcal{X}^2 \cong I\mathcal{X} \times I\mathcal{X}$ and $T_{I\mathcal{X}} \cong IT_{\mathcal{X}}$ by Lemma 3.1.

Let

$$\pi_{\mathcal{X}} : T_{\mathcal{X}} \to \mathcal{X} \text{ and } \pi_{I\mathcal{X}} : T_{I\mathcal{X}} \to I\mathcal{X}$$

be the projections from vector bundles. Then left-hand side of equation (6.3) becomes

$$\int_{N_{I\mathcal{X}/I\mathcal{X}^2}} \pi^*_{I\mathcal{X}}(\gamma'')(\mathrm{ch}_{tw}(\mathbb{L}\rho^*_{T_{\mathcal{X}}}\delta_* \mathcal{O}_{\mathcal{X}})) \cdot \pi^*_{I\mathcal{X}}(\widetilde{\mathrm{td}}_{I\mathcal{X}})^2 \tag{6.5}$$

by the Tor independence of the pair $(\mathcal{X} \times \mathbb{P}^1, \mathcal{M}^\circ \times_{\mathbb{P}^1} p)$ over $\mathcal{M}^\circ$ for $p = 0, \infty$ and the base change II in §7.0.1; for details, see the proof of [23, §3.3]. Let σ be the diagonal section of the vector bundle $\pi^*_{\mathcal{X}} T_{\mathcal{X}}$ on $T_{\mathcal{X}}$, and let $\mathrm{Kos}(\sigma)$ denote the Koszul complex associated to the section σ. Then equation (6.5) becomes

$$\int_{N_{I\mathcal{X}/I\mathcal{X}^2}} \pi^*_{I\mathcal{X}}(\gamma'')(\mathrm{ch}_{tw}(\rho^*_{T_{\mathcal{X}}}\mathrm{Kos}(\sigma))) \cdot \pi^*_{I\mathcal{X}}(\widetilde{\mathrm{td}}_{I\mathcal{X}})^2,$$

which equals, by the functoriality and the projection formula §7.0.1,

$$\int_{I\mathcal{X}} \left((\gamma'' \cdot \widetilde{\mathrm{td}}(T_{I\mathcal{X}}) \int_{\pi_{I\mathcal{X}}} (\mathrm{ch}_{tw}(\rho^*_{T_{\mathcal{X}}}\mathrm{Kos}(\sigma))) \cdot \pi^*_{I\mathcal{X}}(\widetilde{\mathrm{td}}_{I\mathcal{X}}) \right). \tag{6.6}$$

Let $I\sigma$ be the diagonal section of the vector bundle $\pi^*_{I\mathcal{X}} T_{I\mathcal{X}}$ on $T_{I\mathcal{X}}$. From the short exact sequence in §3.4.2, we have a short exact sequence

$$0 \to \pi^*_{I\mathcal{X}} T_{I\mathcal{X}} \xrightarrow{\iota} \pi^*_{I\mathcal{X}}(T_{\mathcal{X}}|_{I\mathcal{X}}) \to \pi^*_{I\mathcal{X}} N_{I\mathcal{X}/\mathcal{X}} \to 0; \tag{6.7}$$
$$\text{with } \iota(I\sigma) = \pi^*_{I\mathcal{X}}\sigma$$

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and an equality

$$T_{\mathcal{X}}|_{I\mathcal{X}}^{\text{fixed}} = T_{I\mathcal{X}}. \tag{6.8}$$

Then equation (6.6) becomes, by equations (6.7) and (6.8),

$$\int_{I\mathcal{X}} \left((\gamma'' \cdot \widetilde{\text{td}}(T_{I\mathcal{X}})) \int_{\pi_{I\mathcal{X}}} \text{ch}(\text{Kos}(I\sigma))\pi_{I\mathcal{X}}^{*}\text{td}(T_{I\mathcal{X}}) \right)$$

which becomes, by §7.0.2,

$$\int_{I\mathcal{X}} (-1)^{\binom{\dim_{I\mathcal{X}}+1}{2}} \gamma'' \cdot \widetilde{\text{td}}(T_{I\mathcal{X}}).$$

This completes the proof. □

6.5. *Proof of Theorem 1.2*

This follows from Theorems 6.6 and 6.9.

6.6. *GRR*

Consider a proper morphism $f : \mathcal{X} \to \mathcal{Y}$ with $f^{*}v = w$ as in §6.3. Let $K_0(\mathcal{A})$ be the Grothendieck group of the homotopy category of a pretriangulated dg category $\mathcal{A}$. Denote $f_! : K_0(\text{MF}_{dg}(\mathcal{X}, w)) \to K_0(\text{MF}_{dg}(\mathcal{Y}, v))$ be the homomorphism in the Grothendieck groups induced from $\mathbb{R}f_*$.

Theorem 6.10 (=Theorem 1.3). *The diagram*

$$\begin{array}{ccc}
K_0(\text{MF}_{dg}(\mathcal{X}, w)) & \xrightarrow{\quad f_! \quad} & K_0(\text{MF}_{dg}(\mathcal{Y}, v)) \\
\downarrow{\scriptstyle \text{Ch}_{HH}} & & \downarrow{\scriptstyle \text{Ch}_{HH}} \\
HH_*(\text{MF}_{dg}(\mathcal{X}, w)) & \xrightarrow{\quad HH(\mathbb{R}f_*) \quad} & HH_*(\text{MF}_{dg}(\mathcal{Y}, v)) \\
\downarrow{\scriptstyle I_{HKR}} & & \downarrow{\scriptstyle I_{HKR}} \\
\mathbb{H}^{-*}(I\mathcal{X}, (\Omega^{\bullet}_{I\mathcal{X}}, -dw|_{I\mathcal{X}})) & \xrightarrow[\int_{If}(-1)^{\dim_{If}} \cdot \wedge \widetilde{\text{td}}(T_{If})]{} & \mathbb{H}^{-*}(I\mathcal{Y}, (\Omega^{\bullet}_{I\mathcal{Y}}, -dv|_{I\mathcal{Y}}))
\end{array}$$

is commutative. Here, $\widetilde{\text{td}}(T_{If}) := \widetilde{\text{td}}(T_{I\mathcal{X}})/If^{*}\widetilde{\text{td}}(T_{I\mathcal{Y}})$ *and* $\dim_{If} = \dim_{I\mathcal{X}} - \dim_{I\mathcal{Y}}$*, where* $\widetilde{\text{td}}(T_?)$ *is* $\widetilde{\text{td}}$ *for* $T_?$ *in equation (6.4).*

Proof. The proof is parallel to that of Theorem 3.6 of [23]. The upper rectangle is clearly commutative. We show the commutativity of the lower rectangle. For $\gamma \in HH_*(\text{MF}_{dg}(\mathcal{X}, w))$, let $\alpha := I_{HKR}(\gamma)$, $\alpha' := I_{HKR}(HH(\mathbb{R}f_*)(\gamma))$, and let

$$\text{ch}(\Psi(\mathcal{O}_{\Gamma_f})) = \sum_i T^i \otimes T_i \in \mathbb{H}^*(I\mathcal{X}, (\Omega^{\bullet}_{I\mathcal{X}}, dw|_{I\mathcal{X}})) \otimes \mathbb{H}^*(I\mathcal{Y}, (\Omega^{\bullet}_{I\mathcal{Y}}, -dv|_{I\mathcal{Y}})),$$

then by Proposition 6.4 and Theorem 6.9 we have for $\beta \in \mathbb{H}^{-*}(I\mathcal{Y}, (\Omega^{\bullet}_{I\mathcal{Y}}, dv|_{I\mathcal{Y}}))$

$$\int_{I\mathcal{Y}} \alpha' \wedge \beta \wedge \widetilde{\text{td}}(T_{I\mathcal{Y}}) = \sum_i \int_{I\mathcal{X}} (-1)^{\binom{\dim_{I\mathcal{X}}+1}{2}} \alpha \wedge T^i \wedge \widetilde{\text{td}}(T_{I\mathcal{X}}) \int_{I\mathcal{Y}} T_i \wedge \beta \wedge \widetilde{\text{td}}(T_{I\mathcal{Y}}). \tag{6.9}$$

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Let π denote the projection $If^*T_{I\mathcal{Y}} \to I\mathcal{X}$, and let s be the diagonal section of $\pi^*If^*T_{I\mathcal{Y}}$ on $If^*T_{I\mathcal{Y}}$. Then the deformation argument for $\Gamma_f : \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ as in the proof of Theorem 6.9 shows that

$$\begin{aligned}
&\text{right-hand side of equation (6.9)}\\
&= \int_{I\mathcal{X}\times I\mathcal{Y}} (-1)^{\binom{\dim_{I\mathcal{X}}+1}{2}} (\alpha \otimes \beta) \wedge \mathrm{ch}(\mathcal{O}_{\Gamma_f}^{w\boxminus v}) \wedge (\widetilde{\mathrm{td}}(T_{I\mathcal{X}}) \otimes \widetilde{\mathrm{td}}(T_{I\mathcal{Y}}))\\
&= \int_{f^*T_{I\mathcal{Y}}} (-1)^{\binom{\dim_{I\mathcal{X}}+1}{2}} \pi^*(\alpha \wedge If^*\beta \wedge \widetilde{\mathrm{td}}(If^*T_{I\mathcal{Y}}) \wedge \widetilde{\mathrm{td}}(T_{I\mathcal{X}})) \wedge \mathrm{ch}(\mathrm{Kos}(s))\\
&= \int_{I\mathcal{X}} (-1)^{\binom{\dim_{If}+1}{2}} \alpha \wedge f^*\beta \wedge \widetilde{\mathrm{td}}(T_{I\mathcal{X}}) = \int_{I\mathcal{Y}} (\int_{If} (-1)^{\binom{\dim_{If}+1}{2}} \alpha \wedge \widetilde{\mathrm{td}}(T_{I\mathcal{X}})) \wedge \beta.
\end{aligned}$$

This completes the proof. □

Remark 6.11. We briefly discuss how the GRR for Δ would compute the canonical pairing, which shows some relationship between GRR and the canonical pairing.

Consider the Riemann–Roch map

$$\begin{aligned}
\mathrm{ch}^\tau : K_0(\mathcal{X}, w) &\to \mathbb{H}^*(I\mathcal{X}, (\Omega^\bullet_{I\mathcal{X}}, -dw))\\
E &\mapsto \mathrm{ch}_{HH}(E)\widetilde{\mathrm{td}}(T_{I\mathcal{X}}).
\end{aligned}$$

Suppose that we have a GRR type theorem for the diagonal map $\Delta : \mathcal{X} \to \mathcal{X}^2$:

$$\Delta_*\mathrm{ch}^\tau(\mathcal{O}_X) = \mathrm{ch}^\tau(\Delta_*\mathcal{O}_\mathcal{X}) = \frac{\mathrm{td}(I\mathcal{X}^2)\mathrm{ch}_{HH}(\Delta_*\mathcal{O}_\mathcal{X})}{\mathrm{ch}_{tw}(N_{I\mathcal{X}^2/\mathcal{X}^2})}. \tag{6.10}$$

This yields a formula

$$\mathrm{ch}_{HH}(\Delta_*(\mathcal{O}_\mathcal{X})) = \Delta_*\left(\frac{\mathrm{ch}_w(N_{I\mathcal{X}/\mathcal{X}})}{\mathrm{td}(I\mathcal{X})}\mathrm{ch}_{HH}(\mathcal{O}_\mathcal{X})\right) = \Delta_*\frac{\mathrm{ch}_w(N_{I\mathcal{X}/\mathcal{X}})}{\mathrm{td}(I\mathcal{X})}$$

since $\mathrm{ch}_{HH}(\mathcal{O}_\mathcal{X}) = 1$. Denote $\widetilde{\mathrm{td}} = \widetilde{\mathrm{td}}T_{I\mathcal{X}}$. Then

$$\begin{aligned}
&\int_{I\mathcal{X}} (-1)^{\binom{\dim_{I\mathcal{X}}+1}{2}} \gamma \cdot t^i \cdot \widetilde{\mathrm{td}} \int_{I\mathcal{X}} (-1)^{\binom{\dim_{I\mathcal{X}}+1}{2}} t_i \cdot \gamma' \cdot \widetilde{\mathrm{td}}\\
&\qquad = \int_{I\mathcal{X}^2} \gamma \otimes \gamma' \cdot \Delta_* \frac{\mathrm{ch}_w(N_{I\mathcal{X}/\mathcal{X}})}{\mathrm{td}(I\mathcal{X})} \cdot \widetilde{\mathrm{td}} \otimes \widetilde{\mathrm{td}} = \int_{I\mathcal{X}} (-1)^{\binom{\dim_{I\mathcal{X}}+1}{2}} \gamma \cdot \gamma' \cdot \widetilde{\mathrm{td}},
\end{aligned}$$

which is the characteristic property of the canonical pairing. Thus, equation (6.10) implies that the canonical pairing is $\int_{I\mathcal{X}} (-1)^{\binom{\dim_{I\mathcal{X}}+1}{2}} \cdot \wedge \cdot \wedge \widetilde{\mathrm{td}}(T_{I\mathcal{X}})$.

7. Pushforward in Hodge cohomology

We introduced an operation $\int_{I\mathcal{X}}$ to formulate Theorem 6.9. In this subsection, we recall its definition and the basic properties we used.

7.0.1. Basic properties

In this section, all stacks are assumed to be smooth separated DM stacks of finite type over k. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism. Assume that they are pure dimensional, and let d be $\dim \mathcal{X} - \dim \mathcal{Y}$.

Definition 7.1. Once we have the right adjoint functor $f^!$ of $\mathbb{R}f_*$, as in [23] we can define

$$\int_f : H^q_{\sharp_1}(\mathcal{X}, \Omega^p_{\mathcal{X}}) \to H^{q-d}_{\sharp_2}(\mathcal{Y}, \Omega^{p-d}_{\mathcal{Y}}),$$

where $(\sharp_1, \sharp_2)$ is either (c, c) or $(cf, \emptyset)$. When $\mathcal{Y}$ is $\operatorname{Spec} k$, write $\int_{\mathcal{X}}$ for $\int_f$.

The following can be straightforwardly proven as in [23, §3.6].

1. (Base change I) Consider a Cartesian diagram (7.1) below. Assume that f is a flat, proper and locally complete intersection morphism. Then

$$\int_{\mathcal{X}'} v^*(\gamma) = u^*\left(\int_f \gamma\right).$$

2. (Base change II) Consider a Cartesian diagram (7.1) below. Assume that f is a flat morphism, $\mathcal{Y}$ is a connected one-dimensional smooth scheme, and u is the embedding of a closed point $\mathcal{Y}'$ of $\mathcal{Y}$. Then

$$\int_{\mathcal{X}'} v^*(\gamma) = u^*\left(\int_f \gamma\right) \in k.$$

3. (Functoriality) Let $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$ be morphisms. Then

$$\int_{g\circ f} = \int_g \circ \int_f .$$

4. (Projection formula) Let $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ be a morphisms. Then

$$\int_f (f^*\sigma \wedge \gamma) = \sigma \wedge \int_f \gamma$$

for $\gamma \in H^d_{cf}(\mathcal{X}, \Omega^d_{\mathcal{X}})$ and $\sigma \in H^q(\mathcal{Y}, \Omega^p_{\mathcal{Y}})$.

Remark 7.2. In the construction of $\int_f$, Nagata's compactification and the resolution of singularities were used. In our separated DM stack setup, both are known by [39] and [42], respectively. Also, the existence of $f^!$ is also proven for proper morphism $f : \mathcal{X} \to \mathcal{Y}$ between Deligne–Mumford stacks in [32].

Remark 7.3. The base change formula I and II relies on the following form of a base change formula; suppose that we have a tor-independent Cartesian diagram of DM stack

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{v} & \mathcal{X} \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{u} & \mathcal{Y} \end{array} \tag{7.1}$$

such that f is proper. One can ask whether the canonical base change map

$$\beta : v^* f^! \to g^! u^*.$$

is an isomorphism. It is known to be true in scheme cases and generalized to stacks when u is representable affine. (See [19, §2, §3] [28]).

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7.0.2. Computation

Let E be a vector bundle on $\mathcal{X}$ of rank n, let $\pi : E \to \mathcal{X}$ be the projection and let s be the diagonal section of π^*E. Since π is representable, we have

$$\int_\pi \mathrm{ch}(\mathrm{Kos}(s))\mathrm{td}(\pi^*E) = (-1)^{\binom{n+1}{2}}$$

by the base change I in §7.0.1 and the computation of [23, §3.6.6].

Acknowledgements. We thank David Favero and Daniel Pomerleano for answering some questions that we had during the preparation of this manuscript. We thank Jeongseok Oh for comments on an initial draft. The third author thanks Charanya Ravi for discussions. We thank the anonymous reviewer for comments and suggestions that helped improve the paper.

Conflict of Interest. The authors have no conflict of interest to declare.

Financial support. B. Kim is supported by KIAS individual grant MG016404 at Korea Institute for Advanced Study. D. Choa is supported by KIAS individual grant MG079401 at Korea Institute for Advanced Study. B. Sreedhar was supported by the Institute for Basic Science (IBS-R003-D1).

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https://doi.org/10.1017/fms.2022.99 Published online by Cambridge University Press