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TEMPERATURE-GRADIENT INDUCED MASS-INSTABILITY THEORY OF GLACIER SURGE

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Abstract. A theory is proposed that glacier surges are the result of a time-independent but positionally-dependent temperature distribution in which the mean effective temperature increases down slope. The theory is modeled by a slab analogy in which plane motion on a plane slope consists of uniform shear in a sub-region called the reservoir region. Assuming the usual power function relationship between stress and strain-rate, a thickening of the glacier in excess of its constant-state condition tends to induce instability; i.e. as the reservoir region thickens the oscillation of the region (up and down the slope) becomes unstable. Assuming an accumulation rate which increases linearly with elevation, this oscillation is represented by a non-linear ordinary differential equation. Numerical results are considered and reasonable assumptions, where data does not exist, render surge cycle times in close agreement with observation. The theory does not require basal sliding but this can be included.

Résumé. *Une théorie des crues rapides des glaciers par l'instabilité de masse due au gradient thermique.* La théorie proposée est que les crues rapides des glaciers sont dues à une distribution des températures indépendante du temps mais variable dans l'espace selon laquelle la température constatée croît d'amont en aval. La théorie prend l'exemple d'une plaque dans laquelle un mouvement plan sur une pente plane consiste en un cisaillement uniforme dans une "sous région" appelée zone réservoir. En admettant l'habituelle loi puissance liant les efforts et la vitesse de déformation, un épaississement du glacier en excès par rapport à son état de stabilité tend à induire son instabilité, c'est-à-dire que plus la zone réservoir s'épaissit, plus l'oscillation de la zone (vers l'amont et vers l'aval) devient instable. Dans l'hypothèse d'une vitesse d'accumulation qui augmenterait linéairement avec l'altitude, cette oscillation est représentée par une équation différentielle ordinaire non linéaire. On a examiné les résultats numériques et, lorsqu'il n'existe pas de données, des hypothèses raisonnables conduisent à un cycle des crues dans le temps qui est en bon accord avec l'observation. La théorie ne requiert pas de glissement sur le lit, mais peut le prendre en compte.

Zusammenfassung. *Eine Theorie der Gletscherausbrüche auf der Basis einer vom Temperaturgradienten induzierten Masseninstabilität.* Es wird eine Theorie entwickelt, nach der Gletscherausbrüche die Folge einer zeitlich unabhängigen, aber ortsabhängigen Temperaturverteilung sind, bei der die (wirksame) Temperatur hangabwärts zunimmt. Als theoretisches Modell dient eine Plattenanalogie, worin eine ebene Bewegung auf einer schiefen Ebene durch einheitliche Scherung in einer als Reservoirregion bezeichneten Subregion ausgelöst wird. Unter Annahme der üblichen Potenzbeziehung zwischen den Verformungs- und Spannungsraten wird eine Verdickung des Gletschers über seinen stationären Zustand hinaus eine Instabilität verursachen, d.h., wenn die Reservoirregion dicker wird, wird die Oszillation der Region (hangauf- und hangabwärts) instabil. Bei Annahme einer Akkumulationsrate, die linear mit der Höhe zunimmt, lässt sich die Oszillation durch eine nichtlineare, gewöhnliche Differentialgleichung darstellen. Numerische Lösungen werden betrachtet. Vernünftige Annahmen von Parametern, für die Messwerte fehlen, ergeben für die Ausbrüche Zykluszeiten, die gut mit den Beobachtungen übereinstimmen. Die Theorie benötigt kein Gleiten am Untergrund, doch kann dies miteinbegriffen werden.

Introduction

There are at least three phenomena which are commonly advanced as agents in triggering and/or propagating glacier surges, as reviewed by Robin (1969): stress instability, water-film instability, and temperature instability. Stress instability as treated by Robin (1967, 1969) is proposed primarily as an explanation for the initiation of surges. Water-film instability, proposed by Weertman (1969), and Robin and Weertman (1973), is essentially a phenomenological theory explaining how a "fast-sliding" mechanism could appear periodically in glaciers with temperate bases. Temperature instability, treated by Robin (1955, 1969), predicts a periodic surge-type phenomenon provided there exists a time-dependent temperature oscillation which periodically produces temperate conditions at the base of the glacier. The last two theories are compatible and can be combined to produce a model in which the basal temperature oscillates and fast sliding occurs periodically.

None of these theories is quantitative. The reason for this lies in the prevailing belief that surging is essentially a fast-sliding phenomenon and sliding is an extremely complicated process which involves a description of water movement and heat transfer among other complexities. On the other hand, efforts at obtaining temperature data on surge-type glaciers, while seemingly aimed at verifying the existence of temperate basal conditions, have not conclusively established that the triggering of surges in cold glaciers does not involve a mechanism other than sliding. Clarke and Goodman (1975) and Goodman and others (1975) in their studies of Rusty and Trapridge Glaciers note that temperate basal conditions exist in a central region but the glaciers are frozen to their bases at the upper and lower elevations. This appears to be a common situation in cold surge-type glaciers. It is not clear that one can assume in such a situation that the active region in a surge initiation involves only temperate basal ice. Even in the situation of the surge of glaciers considered as fully temperate there is some evidence of cooling between surges although a surge itself may involve only temperate ice. There is an obvious pressing need for additional temperature data, particularly at the bases, on surge-type glaciers.

The model to be developed here applies to cold glaciers. Conditions at the base, however, may be either cold, temperate, or a combination. It is necessary that the "mean" temperature through the ice thickness increase down-slope. The model may be viewed as presenting another surge phenomenon independent of previous theories or be used in conjunction with previous theories. Our treatment is quantitative in nature. The model may be applied to predict surge cycle times provided only that data exist on annual balance distribution and temperature. Such predictions are attempted later subject to limitations on data.

The surge model and linear analysis

The theory presented here is independent of previous theories, but certainly not in conflict with them. The basic premise is that, as has been frequently noted, a surge consists of a process whereby in a large region of the glacier, termed the "reservoir region", mass is periodically accumulated over a relatively long time and discharged over a short time, during which glacier speeds are typically ten to one hundred times the normal speed. This suggests our first primary assumption that the mass of glaciers, at least those which surge, is not constant even though climate is constant. (We shall neglect seasonal weather changes. There is no evidence of periodic climatic conditions in phase with surge phenomena.) It is an implicit part of the assumption that the net mass balance of glaciers oscillates between positive and negative values; obviously, the net mass balance cannot remain either positive or negative indefinitely. This oscillation in mass of glaciers is the physically observed situation although the cause may normally be attributed to climate change.

The first assumption can be realized in our model only if we make a second assumption that the balance distribution increases up-slope. That is, there exists an equilibrium line at which the annual balance vanishes and above (below) this line the balance increases (decreases). This is the usual physically observed situation. It will be shown that if the glacier temperature is uniform a model obeying these first two assumptions exhibits periodic growth and decay of the reservoir region.

An additional assumption is required in order to ensure that this periodic behaviour becomes unstable. What seems to be required is an asymmetry of the flow process relative to the midpoint (or equilibrium line) on the slope. That is, friction should decrease as we move down the slope. Accordingly, we make the assumption that the glacier temperature increases down-slope. Here, there is not much evidence to go on. Near the surface of subpolar glaciers, melted water refreezes in the accumulation zone. This causes these glaciers to be colder in the ablation than in the accumulation zone—at least near the surface. Our assumption, however, need not apply near the surface, but only in the deeper regions where it appears more reasonable. There, water percolation is absent and the down-slope mean air-temperature gradient should contribute to the postulated effect. In addition, the greater part of frictional dissipation takes place at depth and has an accumulative effect which would tend to increase the temperature down-slope. Finally, geothermal input need not be uniform and there is no reason why on some glaciers it should not increase down-slope. At this early stage it is clear that the model is not applicable to strictly temperate glaciers.

In addition to the three fundamental assumptions above, we make certain simplifications and note in so doing that this is a first-order theory intended primarily to produce qualitative agreement with observation. We consider plane flow on a plane slope (Fig. 1).* The reservoir region, which is assumed not to be the entire glacier, is modeled by a slab, a rectangular control volume of fixed length L_0 and height h which in general varies with time. The mechanical behaviour of the slab will be assumed to be independent of the remainder of the glacier and the deformation field will be assumed to consist of a uniform shear flow. To be consistent with the latter assumption, the weight is assumed to act uniformly through the entire thickness, i.e. the weight is applied to the top of the slab. The temperature distribution is assumed to be independent of time at points fixed with respect to bedrock. The temperature distribution at a section x = constant can thus be replaced by a fixed mean effective temperature $\theta(x)$. Sliding of the glacier on bedrock will be neglected merely as a simplification in presenting results; its inclusion would present no mathematical complication. Additional assumptions relating to motion and mass balance will be made within the context of the mathematical formulation.

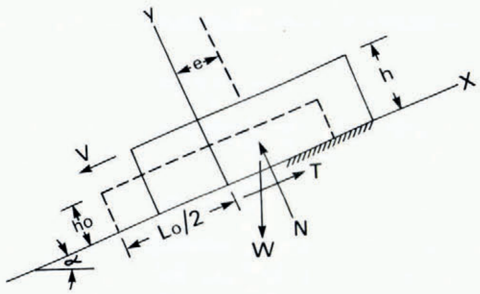

Fig. 1. The reservoir region is represented by a rectangular control volume which we term a slab. In the constant-state condition it is centered at $x = 0$ ($e = 0$) and undergoes a uniform time-independent shear flow. Material flows into the control volume on the right and exists on the left with velocity varying linearly from zero at the base to V_0 at $y = h_0$. Net forces exerted on the control volume by the rest of the glacier vanish. In the general case the control volume moves with its position determined by the position of its upper surface. To the observer it appears as a wave moving "through" the glacier. Although L_0 is fixed, h, e and V vary with time. For $V > V_0$ ($< V_0$) it moves down (up) the slope.

We emphasize that the slab is a control volume which receives mass input from the glacier at its upper boundary (normal to the base) and discharges mass at its lower boundary; the material particles comprising the control volume change with time.† This mass input and output are assumed to balance since we wish to consider the action of the reservoir region independently of the rest of the glacier. The slab is intended to represent the active region of an actual glacier and this is assumed here to be synonymous with the reservoir region. The remainder of the glacier is assumed to be nearly dormant so that any imbalance of mass input or output to the active reservoir region from the remainder of the glacier is likely to be negligible compared to the net balance, accumulation or ablation, on the slab itself.

* Henceforth, all quantities will be expressed per unit distance normal to the x, y plane, Figure 1.

† As one of several alternatives, the entire glacier could be represented by a sliding slab fixed at its upper extremity which undergoes extending flow with shear flow neglected; the location of the terminus as well as the thickness would vary with time. In this model there is no external mass input or output to the slab. The qualitative behaviour of the two models is similar.

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The mass balance of the slab, in accordance with the second fundamental assumption, will depend entirely on the location of the slab relative to the equilibrium line. On the other hand, if the slab thickness is increasing (decreasing) the slab will, as a consequence of equilibrium and flow-law considerations, tend to move down (up) the slope after which motion the net mass balance will tend to decrease (increase). There is thus an analogy to a spring where the slab is "attracted" towards a central position which might be, but is not necessarily, the equilibrium line. If the slab is in a constant-state situation so that its mass and position remain constant, it will appear to the observer to be stationary; that is, the control volume is stationary. Actually the slab is undergoing uniform, time independent, shear flow with material particles flowing into and out of the slab (Fig. 1).

The remainder of this section is devoted to representing mathematically the features of this slab oscillation. In particular, we are interested in investigating whether and under what conditions the oscillation might become unstable.

Neglecting inertia forces, equilibrium of the slab implies

$$T = \rho L_0 h \sin \alpha \tag{1}$$

with ρ the weight density. The speed V of the upper surface of the slab is related to the shear strain rate $\dot{\epsilon}$ by

$$\dot{\epsilon} = V/2h. \tag{2}$$

Assuming the usual power-function relationship between shear stress and strain-rate we can write

$$\tau = k(x)(V/2h)^{1/n} \tag{3}$$

where the temperature $\theta(x)$ and hence the friction coefficient $k(x)$ are functions of x. For lack of data we assume a linear relationship:

$$k(x) = k_0 + k_1 x \sin \alpha / L_0 \tag{4}$$

where $k_1 > 0$ corresponds to temperature increasing down-slope.

The origin $x = 0$ is chosen to coincide with the centre-line of the slab when it is in the constant-state position (to be defined). If e measures displacement of the slab relative to the constant-state position (Fig. 1), integration of Equation (4) on $-L_0/2+e$ to $L_0/2+e$ gives for the average or effective friction coefficient $K(e)$:

$$K(e) = k_0 + k_1 e \sin \alpha / L_0. \tag{5}$$

Integration of Equation (3) then gives for the shear force T

$$T = (V/2h)^{1/n}(k_0 + k_1 e \sin \alpha / L_0). \tag{6}$$

In general, as will be demonstrated later, transitory behaviour is possible whereby the slab position moves and h and e vary. On the other hand, a constant-state condition, defined as a state where all quantities are independent of time, is also possible. It is helpful to investigate this special case first. Assume that a constant state exists and let $h = h_0$, $V = V_0$. Since the net balance must vanish it is also necessary that $e = 0$ in Equation (5) is we assume that the balance gradient is antisymmetric with respect to $x = 0$. Equilibrium considerations through Equations (6) and (1) then give

$$V_0 = 2(\rho \sin \alpha / k_0)^n \, h_0^{n+1}. \tag{7}$$

Another relationship between h_0 and V_0 is obtained by considering the actual constant-state profile of a glacier. For this purpose we resort to a more conventional model of Nye (1959) and assume that plane sections normal to the slope remain plane and normal; i.e. shear deformation vanishes except at the base where there is a velocity discontinuity. If we furthermore assume that the particle velocity is uniform and equal to V_0

$$h_0(x) = -\frac{1}{\rho V_0}\int_{L_1/2}^{x} \delta\dot{w}\,\mathrm{d}x = \frac{k_e}{2\rho V_0}\,[(L_1/2)^2 - x^2]\sin\alpha \tag{8}$$

where for lack of data we have assumed a linear relation for the balance gradient $\delta\dot{w}$ (weight per unit time and length)

$$\delta\dot{w} = k_e x \sin\alpha, \qquad k_e > 0. \tag{9}$$

Therefore, the average thickness of the glacier in the constant state condition is

$$h_0 = k_e L_1^2 \sin\alpha / 12\rho V_0. \tag{10}$$

We take h_0 to be the average thickness of the reservoir region.*

From Equations (7) and (10) we obtain

$$h_0 = \left[\frac{k_e L_1^2 \sin\alpha}{24\rho}\right]^{1/(n+2)} \left(\frac{k_0}{\rho\sin\alpha}\right)^{n/(n+2)}, \tag{11}$$

$$V_0 = \left[\frac{k_e L_1^2 \sin\alpha}{12\rho}\right]\left[\frac{24\rho}{k_e L_1^2\sin\alpha}\right]^{1/(n+2)}\left[\frac{\rho\sin\alpha}{k_0}\right]^{n/(n+2)} \tag{12}$$

A unique constant-state condition for the slab has thus been defined.

When the slab is displaced, $e \neq 0$, its mass rate of change is determined by integration of Equation (9). Thus,

$$\dot{W} = k_e L_0 e \sin\alpha. \tag{13}$$

The final relationship necessary in studying the transitory behaviour is the kinematic relationship (Fig. 1).

$$V_0 - V = \mathrm{d}e/\mathrm{d}t. \tag{14}$$

Substitution of Equations (13), (7), (6) and (1) into Equation (14) gives the basic governing equation for the slab thickness

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = \left[\frac{2(\rho\sin\alpha)^n h_0^{n+1}}{k_0^n} - \frac{2(\rho\sin\alpha)^n h^{n+1}}{(k_0 + k_1 e\sin\alpha/L_0)^n}\right]\frac{k_e\sin\alpha}{\rho}. \tag{15}$$

To achieve a convenient dimensionless form of Equation (15) we define variable u by

$$h = h_0 + h_1, \qquad h_1 = u h_0. \tag{16}$$

In addition, the dimensionless time s is defined by

$$s = [2h_0^n k_e (\sin\alpha)^{n+1}\rho^{n-1}/k_0^n]^{\frac{1}{2}} t \tag{17}$$

so that Equation (15) becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} = \left[1 - (1+u)^{n+1}\bigg/\left(1 + \gamma\frac{\mathrm{d}u}{\mathrm{d}s}\right)^n\right] \tag{18}$$

where the single dimensionless parameter

$$\gamma = \sin\alpha\, L_1 k_1 / 2\sqrt{3} L_0 k_0 \tag{19}$$

is obtained after substituting for h_0 from Equation (11) and for e from Equation (13) where $\dot{W} = \rho L_0 \dot{h}$. Other quantities of interest, such as slab position e/L_1 and the ratio of particle velocities V/V_0 are easily expressed in terms of u. Thus,

$$e/L_1 = (1/2\sqrt{3})\,\mathrm{d}u/\mathrm{d}s, \tag{20}$$

$$V/V_0 = (1+u)^{n+1}/(1+\gamma\,\mathrm{d}u/\mathrm{d}s)^n. \tag{21}$$

Furthermore, substitution from Equation (11) into Equation (17) enables us to compute real time in terms of the physical parameters and s; thus

$$s = \sqrt{2}\,[(\sin\alpha)^{2n+1} L_1^n k_e^{n+1} / 24^{n/2}\rho k_0^n]^{1/(n+2)} t. \tag{22}$$

* For $L_0 \approx L_1$ the approximation to Equation (10) is accurate; for $L_0 \ll L_1$, $h_0 \approx k_e L_1^2 \sin\alpha / 8\rho V_0$.

In general, Equation (18) does not possess closed-form solutions. Qualitative behaviour for $|u| \ll 1$ is easily discussed in terms of a linearized approximation. Expansion of $(1+u)^{n+1}/(1+\gamma\, du/ds)^n$ in a Taylor series results in the linearization

$$d^2u/ds^2 - n\gamma\, du/ds + (n+1)u = 0. \quad (23)$$

In case $n^2\gamma^2 < 4(n+1)$ all solutions may be expressed in terms of

$$[\exp(n\gamma s/2)\cos\{1/2(4(n+1)-n^2\gamma^2)^{\frac{1}{2}}s\}, \exp(n\gamma s/2)\sin\{1/2(4(n+1)-n^2\gamma^2)^{\frac{1}{2}}s\}]. \quad (24)$$

Solutions corresponding to $\gamma = 0$ are periodic whereas for $\gamma > 0$ there is an exponential growth superimposed upon the periodic behaviour;* subsequent discussion applies to $\gamma > 0$. Note that any solution must satisfy the inequality $h \geqslant 0$; from (16) this implies that $u \geqslant -1$. The mathematical result is that the condition $u = -1$, corresponding to complete melting, will occur within a finite time. Physically, the slab has moved sufficiently far down the slope that it "melts" before returning. The solution should be interpreted as ceasing to exist from this moment. Once the reservoir region ceases to exist, another active reservoir region comes into existence and the same procedure is repeated. This discussion of course applies only to the linear solution which has limited application; later numerical studies of the non-linear equation demonstrate a different qualitative behaviour whereby in a finite time the speed of the slab may become unbounded (the slab moves off the slope before melting takes place).

If the initial conditions $u(0) = -U$, $U > 0$, $du/ds(0) = 0$ are applied, Equation (24) gives

$$u(s) = -U\exp(n\gamma s/2)\cos[1/2(4n+1)-n^2\gamma^2)^{\frac{1}{2}}s]. \quad (25)$$

(Initial conditions of this type, which correspond to the reservoir region initially in the constant-state position but thinner than h_0, will be applied in the numerical study of Equation (18).) Quantities e and V behave in much the same manner as Equation (25). Negative speeds V can result, but are not possible in the non-linear model since $T \geqslant 0$ and Equations (1) and (3) imply that $V \geqslant 0$.

The point $u = 0$, $e = 0$ (or $u = 0$, $du/ds = 0$), which represents the constant-state condition, is an isolated critical point in the terminology used in the study of non-linear differential equations. The functions $u = 0$, $du/ds = 0$ clearly are a solution of Equation (18)—the constant-state solution. What we have shown in the linearized analysis is that the critical point is unstable. That is, any small change from the constant-state condition results in unstable behaviour—an exponential growth of the solution.

Non-linear behaviour—numerical studies

In the following examples applied to the numerical solution of Equation (18) $n = 3$ is used throughout as a generally accepted value.† Figure 2 illustrates a type of behaviour which culminates in slab melting. The initial conditions $u(0) = -0.15$, $e(0) = 0$ correspond to the reservoir region initially in the constant state position but 15% thinner than the constant-state thickness h_0; consequently, the slab initially moves up the slope. As it does so, the particle speed V (down the slope) increases slowly. Eventually, V increases to the extent that e reaches a maximum; by Equation (14) this occurs when $V/V_0 = 1$. As the slab moves down the slope V increases more rapidly but does not reach a relative maximum until after e returns to zero. The period from $e = 0$ through e_{max} and back to $e = 0$ will be called the accumulation half-cycle.

* The exponential growth also applies to cases where $n^2\gamma^2 \geqslant 4(n+1)$. Later we show that actual glaciers seem to correspond to $n^2\gamma^2 < 4(n+1)$.

† Mellor and Testa (1969[a]) regard $n = 1.8$ as a better value at low stress levels such as exist on a glacier.

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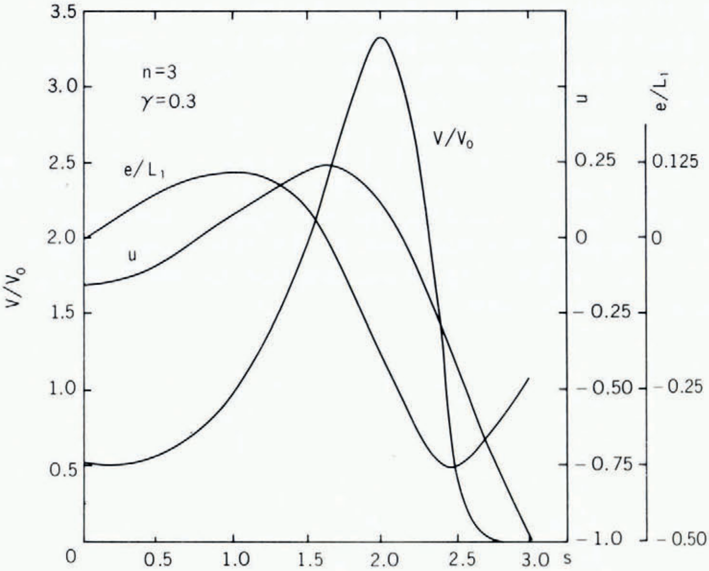

Fig. 2. Typical slab history corresponding to a point $u(0) = -0.15$, $\gamma = 0.3$ in the melt region of Figure 4 which results in complete melting, after being displaced down-slope a distance one-quarter of the glacier length. The maximum speed V/V_0 is only 3.33; the maximum increase in thickness is 25%. The action has a resemblance to a kinematic wave but is not similar to a surge.

The ensuing half-cycle, if it is completed (which does not occur in this example) is roughly the reverse of the accumulation half-cycle as far as e and u are concerned and will be called the ablation half-cycle. The speed V does not observe this rough anti-symmetry property and continues to increase rapidly, reaching the maximum value $V/V_0 = 3.33$ when $e/L_1 = -0.225$; after this the speed drops rapidly to zero at which time complete melting takes place. Reversal of the direction of slab motion occurs when $V/V_0 = 1$. The great majority of melting takes place when the slab is positioned in the range $-0.375 < e/L_1 < -0.25$ corresponding to an elapsed time $\Delta s \approx 1.0$ compared to the total elapsed time $\Delta s = 3.0$. Hence, during this last third of the elapsed time, the slab has been significantly displaced from the constant-state position; provided $L_0/L_1 < \frac{1}{2}$ it is entirely removed from the original reservoir region. Therefore, we might consider that for $s > 2.0$ another slab begins life in the original reservoir region and the process is repeated.

This first example is qualitatively similar to the linear results in terms of expression (24). The period $\Delta s = 3.0$ (Fig. 2) compares with $\Delta s = 3.22$ of expression (24). The most noticeable changes are that the elapsed time for the accumulation half-cycle has become twice that of the ablation half-cycle, and that there is evidence of a further increase of slab speed during downward motion.

Henceforth, we shall arbitrarily designate $V/V_0 = 2.0$ as the boundary between low and high speed. The corresponding time duration of low (high) speed will be denoted by $\Delta s_L(\Delta s_H)$. The parameter $\Delta s_H/\Delta s_L$ will, in later examples, be a meaningful parameter for comparison with actual surges.

Figure 3 employs the same initial conditions as Figure 2 with a different parameter value, $\gamma = 0.4$ instead of 0.3. The behaviour of the two examples is virtually identical for $s < 1.5$. For $s > 1.5$ the second example exhibits a rapidly increasing speed culminating in instability. The slab can be regarded as moving off the slope before complete melting can take place. Many of the qualitative features of a surge are met by such behaviour. First and most obvious is the high speed. (We shall later address the problem of rationalizing infinite speeds. At the present time this should be considered as a product of the degree of simplification incorporated in the model, but something which can be remedied by a slightly more sophisticated model.) The second attribute of a real surge is the gradual thickening of the reservoir region during the accumulation half-cycle until it reaches a maximum thickness u_M, 30% greater than that of constant state, before the ablation half-cycle begins. Values of u_M on this order agree with data of Robin and Weertman (1973) for Muldrow Glacier and Finsterwalderbreen. The third comparison is that the material of the reservoir region is discharged to the lower region of the glacier over a relatively short time interval. For this example $\Delta s_H/\Delta s_L = 0.33$. This value should be closer to 0.05 for a surge, illustrating that the model, or at least this example, develops high speeds relatively slowly.

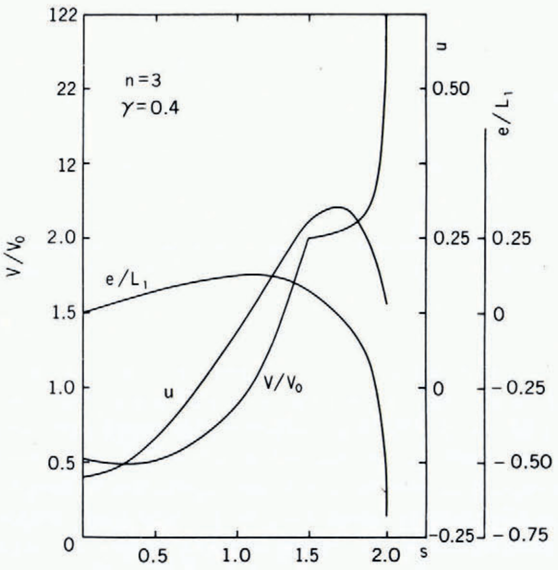

Fig. 3. Typical slab history corresponding to a point $u(0) = -0.15$, $\gamma = 0.4$ in the instability region of Figure 4. Here, the speed of the slab becomes unbounded before complete melting takes place. The slab moves down beyond the original glacier region. The maximum increase in thickness is 30%.

With these two prototype responses in mind, we will now consider the question of whether the type of response can be determined if the initial conditions and parameters γ and n are known—without having to resort to a numerical solution for each particular case. Also important is whether the unstable response corresponds to parameter values associated with actual surge-type glaciers. Extensive numerical studies enable us to answer these questions in the following paragraphs.

For fixed $\gamma > 0.145$ the slab will behave as follows when subject to the initial conditions $u(0) = -U$, $e(0) = 0$ where $0 < U < 1$: if U is sufficiently small the slab will undergo at least one cycle without melting or becoming unstable; at the end of the first cycle, $s = s_1$ and the new initial conditions are $u(s_1) = -U_1 < -U$, $e(s_1) = 0$. At the end of the second cycle either complete melting has occurred or else we arrive at a state $U(s_2) = -U_2 < -U_1$, and so on. Eventually, either complete melting takes place during a cycle or else the slab enters a cycle in which instability occurs.

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The transition between the behaviour illustrated by Figures 2 and 3 is abrupt. Curve ABC in Figure 4 represents the boundary between the two types of behaviour as reflected in the first cycle of response. Although analytically there is a continuous dependence of solutions upon the parameters $u(0)$ and γ, the dependence shows up as an abrupt change when dealing with numerical solutions. The transition appears as a discontinuity to within better than two-figure accuracy in moving across ABC. We shall term this curve the "surge boundary" and the regions which it separates the "melt region" and the "instability region". Note that for $\gamma < 0.145$ all slabs melt eventually after sufficient oscillation. We shall return later to this diagram in relating the model to a surge theory.

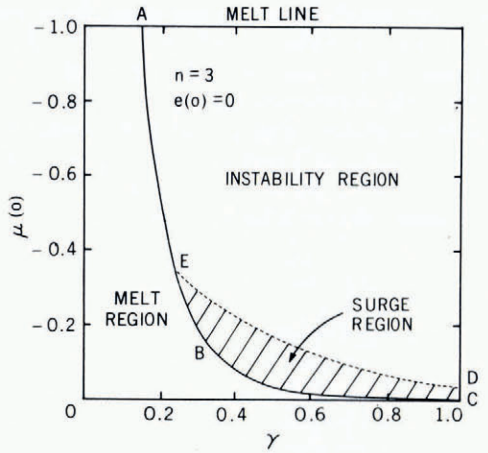

Fig. 4. Corresponding to the initial condition $e(0) = 0$, the qualitative behaviour of a single cycle of slab motion is conveniently represented in the $u(0)$, γ plane. Below ABC, the "surge boundary", speeds are bounded and melting may occur. Above the surge boundary speeds become unbounded. Conditions interior to CBED result in a maximum slab thickening less than 40% in excess of h_0.

Table I lists values of various parameters for a variety of solutions in the instability region. Values of $\Delta s_H/\Delta s_L$ are generally high compared to actual surges but approach reasonable levels for γ on the order unity. The elapsed time Δs is fairly constant; hereafter we shall adopt the value $\Delta s = 2.0$ as a uniform measure for all $\gamma < 1.0$. (Note that the linear solution in Equation (25) for $n = 3$ indicates a dependence of Δs upon γ where $\Delta s \rightarrow \infty$ as $\gamma \rightarrow \frac{4}{3}$.)

TABLE I. CRITICAL PARAMETERS FOR SOLUTIONS IN THE INSTABILITY REGION OF FIGURE 4

Δs is total elapsed dimensionless time; Δs_H, (Δs_L) is the elapsed time during which $V/V_0 > 2$, (< 2); $(e/L_1)_M$, $(e/L_1)_m$ are the maximum and minimum slab displacements ratios, respectively; $100u_M$ is the maximum percentage increase in glacier thickness compared with h_0.

	$u(0) = -0.025$				$u(0) = -0.05$				
γ	0.6	0.8	1.0	1.1	0.5	0.6	0.7	0.8	0.9
$\Delta s_H/\Delta s_L$	0.18	0.12	0.11	0.097	0.25	0.20	0.16	0.14	0.13
Δs	2.37	2.13	2.20	2.27	2.27	2.07	2.01	2.01	2.05
$(e/L_1)_M$	0.036	0.057	0.096	0.123	0.056	0.067	0.079	0.096	0.120
$(e/L_1)_m$	−0.45	−0.33	−0.27	−0.23	−0.54	−0.41	−0.38	−0.34	−0.29
u_M	0.11	0.20	0.37	0.50	0.15	0.20	0.25	0.33	0.42

	$u(0) = -0.10$				$u(0) = -0.15$				$u(0) = -0.30$			
γ	0.4	0.5	0.6	0.7	0.4	0.5	0.6	0.7	0.25	0.3	0.4	0.5
$\Delta s_H/\Delta s_L$	0.38	0.26	0.21	0.18	0.34	0.29	0.23	0.17	0.57	0.41	0.36	0.29
Δs	2.19	1.98	1.92	1.92	2.00	1.94	1.92	1.89	2.21	2.00	1.92	1.91
$(e/L_1)_M$	0.088	0.10	0.11	0.13	0.12	0.14	0.16	0.17	0.18	0.19	0.21	0.23
$(e/L_1)_m$	−0.68	−0.51	−0.44	−0.40	−0.67	−0.56	−0.46	−0.37	−1.1	−0.90	−0.60	−0.39
u_M	0.22	0.27	0.34	0.42	0.30	0.32	0.45	0.54	0.38	0.42	0.50	0.59

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For $\gamma > 0.145$ we have noted qualitative agreement between model behaviour during the final unstable cycle and surge behaviour. It seems reasonable that we should insist that if the model is to represent a surge, it must undergo instability during the initial cycle. This is because we have assumed that the slab motion is independent of constraints imposed by the remainder of the glacier. This assumption is certainly questionable for a multi-cycled situation where the quantitative accuracy in rendering initial conditions for ensuing cycles is important; on the other hand, quantitative accuracy should not be so important during a single instability cycle. Thus, the model might adequately describe the thickening of the reservoir during an initial slow upward movement of the slab followed by the unstable downward motion. A continued quasi-cyclic behaviour on the other hand, would seem to require a refined analysis outside the scope of the model. We therefore restrict the labelling of instability in the response of the model to the instability region of Figure 4, where it occurs during the initial cycle.

In order to conform to a surge there must be limits imposed on the maximum thickness u_M of the reservoir.* If we arbitrarily impose an upper limit of $u_M = 1.4$ the unstable response region is further restricted to lie in EBCDE (Fig. 4) which we term the surge region. Curve ED is obtained from data such as shown in Table I. Figure 4 indicates that, for large values of γ close to unity, the initial conditions $u(0)$, $e(0)$ must tend to a small perturbation from the constant-state condition. The surge region is bounded at the left point E by $\gamma = 0.25$.

Quantitative model predictions

We shall attempt here to evaluate the ability of the model to represent actual surges. At this time any such study is subject to a deficiency of data. There is insufficient temperature data on surging glaciers with which to test the third fundamental assumption. Most data is at the ten-meter depth and the model requires the distribution of temperature through the thickness. Balance data is available for determining k_e but the author knows of none for surging glaciers.† Moreover, balance data must be collected over many years to be useful. Reliable data on bedrock profiles is also scarce.

Quantitative information on model behaviour is almost entirely described by the values of γ from Equation (19) and Δt from Equation (22), where Δt is real time corresponding to Δs. In order for the model to exhibit instability, it is necessary that values of γ greater than 0.25 be exhibited, in accordance with Figure 4. Values of Δt corresponding to surge glaciers have been tabulated by Meier and Post (1969) and are typically on the order of 20 to 40 years with the active phase, corresponding roughly to our Δs_H, being on the order of two to three years.

Tables II and III evaluate γ and Δt for ranges of the parameters which include all types of surging glaciers: first, large glaciers of very small slope such as Bering Glacier, Chugach Range where $L_1 \approx 200$ km, $\alpha \approx 1°$; second, large glaciers of moderate slope such as Muldrow Glacier, Alaska Range where $L_1 \approx 63$ km, $\alpha \approx 3.5°$; third, moderately large glaciers of increased slope such as Middle Fork Glacier, Wrangell Range where $L_1 \approx 14$ km, $\alpha \approx 5°$; fourth, a glacier such as Tyeen Glacier, Fairweather Range where $L_1 \approx 7$ km, $\alpha \approx 18°$; and finally, extremely small glaciers of steep slope such as the outlet glacier on the south side of Tenas Tikke Glacier, Alsek Range where $L_1 \approx 2$ km, $\alpha \approx 25°$.‡ Various ranges of k_0, $k_1' = k_1/L_0$ and k_e have been chosen which will, hopefully, bracket the correct values.

* Values of u_M increase rapidly with γ. For example, the case $u(0) = -0.025$, $e(0) = 0$, $\gamma = 3$ results in a maximum value $u_M = 43$ before sudden instability sets in. Lacking accurate data it is difficult to estimate the value h_0 for a given glacier. It could be, for example, that large values u_M could occur, i.e. perhaps surging glaciers satisfy the condition $h \gg h_0$ except during an active surge phase. We are tacitly assuming the opposite and that values of u_M should be below 1.4.

† Temperature and balance data for Rusty and Trapridge Glaciers is currently insufficient for our purposes. More data for these glaciers are expected.

‡ In this model the length L_1 of outlet glaciers should be increased beyond the actual length in order to correct for the influence of another glacier.

Table II. Values of γ for five different glacier types

k_0 values of [14, 19, 24, 29] Mgf $a^{\frac{1}{3}}$/m correspond, respectively, to mean temperatures θ_M of [−1, −3, −6, −9] °C. k_1' (kgf $a^{\frac{1}{3}}/m^2$) values correspond to moderate temperature gradients as exemplified in the text and are consistent with θ_M such that glacier temperature does not exceed 0°C. Where values of γ are not supplied the temperature would exceed 0°C. Values of $\gamma > 0.25$ have corresponding states in surge region (Fig. 4).

	L_1 = 2 km, α = 25°				L_1 = 7 km, α = 18°				L_1 = 14 km, α = 5°			
k_0 \ k_1'	17.5	24	30.5	37	17.5	24	30.5	37	17.5	24	30.5	37
14	0.31	0.42	—	—	0.78	—	—	—	0.44	—	—	—
19	0.23	0.31	0.39	—	0.58	0.79	—	—	0.32	0.45	—	—
24	0.18	0.24	0.31	0.38	0.46	0.63	0.80	—	0.26	0.35	0.45	—
29	0.15	0.20	0.26	0.31	0.38	0.52	0.66	0.80	0.21	0.29	0.37	0.45

	L_1 = 63 km, α = 3.5°				L_1 = 200 km, α = 1°		
k_0 \ k_1'	11	17.5	24	30.5	11	17.5	24
14	0.87	—	—	—	—	—	—
19	0.64	1.0	—	—	0.58	—	—
24	0.51	0.81	1.1	—	0.46	0.74	—
29	0.42	0.67	0.92	1.2	0.38	0.61	0.84

Table III. Surge cycle time in years for five different glacier types

Range of k_e (kgf/a m²) corresponds to high accumulation-rate gradients. As reported by Meier and Post (1969) times for prototype glaciers are: Tenas Tikke Glacier *c.* 20±1, Tyeen Glacier *c.* 20±1, Middle Fork Glacier ?, Muldrow Glacier *c.* 50±10, Bering Glacier *c.* 30±15. Slope α of bedrock in accumulation region is given by Post (1960) for Muldrow Glacier and is estimated for other glaciers.

	Tenas Tikke Glacier L_1 = 2 km, α = 25°					Tyeen Glacier L_1 = 7 km, α = 18°					Middle Fork Glacier L_1 = 14 km, α = 5°				
k_0 \ k_e	5	10	20	30	40	5	10	20	30	40	5	10	20	30	40
14	41	23	13	10	8	30	17	10	7	6	116	66	38	28	22
19	49	28	16	12	9	36	21	12	9	7	139	80	46	33	26
24	56	32	19	13	11	41	24	14	10	8	160	92	53	38	30
29	63	36	21	15	12	46	26	15	11	9	179	103	59	43	34

	Muldrow Glacier L_1 = 63 km, α = 3.5°					Bering Glacier L_1 = 200 km, α = 1°				
k_0 \ k_e	5	10	20	30	40	5	10	20	30	40
14	77	44	25	18	15	—	—	—	—	—
19	93	53	31	22	18	267	154	88	64	51
24	107	61	35	25	20	307	177	102	73	58
29	119	69	39	28	23	345	198	114	82	65

The manner of definition of k_e in Equation (9) implies that it is independent of α and should depend only upon climate. Accumulation data computed for four glaciers in widely different locations give:

k_e = 9 kgf/m² a = 88 N/m² a for Blue Glacier, Washington, U.S.A. (LaChapelle, 1965);
k_e = 9 kgf/m² a = 88 N/m² a for Nigardsbreen, Norway (Paterson, 1969, p. 36);
k_e = 20 kgf/m² a = 196 N/m² a for South Cascade Glacier, Washington, U.S.A. (Meier and Tangborn, 1965);
k_e = 4 kgf/m² a = 39 N/m² a for Hintereisferner, Ötztal Alps (Hoinkes and Rudolph, 1962).

The five prototype glaciers cited earlier appear to be situated in areas of high precipitation (Post, 1969). This should be indicative of high accumulation gradients. In terms of yearly water equivalent the data are: Chugach Range (Bering), 2–4 m/a; Alaska Range (Muldrow), 1–4 m/a; Wrangell Range (Middle Fork), 0.5–1 m/a; Fairweather Range (Tyeen), 2–4 m/a; and the Alsek Range (Tenas Tikke), 1–2 m/a. These figures would support values in the range $10 < k_e < 40$ kgf/m² a (= 392 N/m² a).

The coefficient k_0 in Equation (4) is determined by the effective or mean temperature $\theta(0)$ at the section $x = 0$, the equilibrium line of the model. Using data of Mellor and Testa (1969[b])* we have obtained:

$$k_0 = [14\,000, 19\,000, 24\,000, 29\,000] \text{ kgf a}^{\frac{1}{3}}\text{/m}$$
$$= [0.14, 0.19, 0.24, 0.28 \text{ MN a}^{\frac{1}{3}}\text{/m}]$$

corresponding to $[-1, -3, -6, -9]$°C, respectively.

The coefficient $k_1' = k_1/L_0$ in Equation (4) is independent of slope and dependent only upon climate, i.e. the temperature gradient. We present various examples in Table IV where Δz represents the change in elevation of the glacier, θ_T, (θ_B) the effective temperature at the top (and bottom).

TABLE IV. VALUES OF THE COEFFICIENT $k_1' = k_1/L_0$ IN EQUATION (4) FOR VARIOUS ELEVATION CHANGES AND TEMPERATURES

Δz	θ_T	θ_B	k_1'	
km	°C	°C	kgf a$^{\frac{1}{3}}$/m²	N a$^{\frac{1}{3}}$/m²
1	−5	−1	16	157
0.5	−2	0	13	127
0.5	−5	−1	32	314
1.5	−18	−2	37.2	365
1	−6	0	28	275

Table II presents values of γ calculated for a range of k_0 corresponding to the mid-slope temperature range $-1°\text{C} > \theta_M > -9°\text{C}$ and for the range

$$11 \text{ kgf a}^{\frac{1}{3}}\text{/m}^2 = 108 \text{ N a}^{\frac{1}{3}}\text{/m}^2 < k_1' < 37 \text{ kgf a}^{\frac{1}{3}}\text{/m}^2 = 363 \text{ N a}^{\frac{1}{3}}\text{/m}^2$$

where the values are compatible with temperature constraints, i.e. such that θ_B is not greater than 0°C,† and with temperature gradients similar to the above examples. The restriction that $\gamma > 0.25$ is satisfied for reasonable temperature gradients, and we conclude that the surge region of Figure 4 is attainable.

Table III presents the period in years between surges as predicted by the model. With the possible exception of the case $L = 200$ km, $\alpha = 1°$ the cycle times are very much in the range of the observed times.

CONCLUSIONS

It has been commonly assumed that a surge is possible only if most of the base is temperate. This is because, in a glacier where sliding does occur, the contribution to velocity due to sliding overwhelms the contribution due to shear flow. This development shows that even if sliding is not present the model exhibits the characteristics of a surge, albeit the model goes too far and exhibits instability. This is an exaggeration which a second-order model would hopefully correct. In our opinion a more refined model would exhibit bounded velocities while retaining the surge characteristics of the present model.

There are several factors which would tend to limit slab speeds to bounded values without unduly increasing the complexity of the model. First, as speeds increase the down-slope temperature gradient will be reduced as colder material is carried downward. Second, the

* We have attempted to convert their compression data to pure shear.
† Temperatures greater than 0°C will occur in the model if the slab moves below the original glacier region.

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effective temperature $\theta(x)$ cannot increase above 0°C (see footnote on p. 444); the assumed linear variation in $k(x)$ given by Equation (4) cannot continue indefinitely down-slope and must have a lower bound. Third, it is usual for α to decrease down-slope. (An analysis based upon a circular profile base would be appropriate and possible.)

There are additional factors tending to limit slab speeds which would add to the complexities of the model. The inclusion of inertia forces is an obvious example. The inclusion of constraints exerted upon the slab by the remainder of the glacier is another.

Equation (22) indicates the dependence of cycle period Δt upon the parameters k_0, k_e, L_1 and α. However, these parameters are not independent. For a given climate and geothermal conditions k_0, k_e and L_1 should be determined by α. It is curious that all the values of the cycle period tabulated by Meier and Post (1969) fall in the range 20 to 60 years, indicating a very weak dependence upon α. This seems unlikely. There must exist glaciers with very long cycle periods which have not been noted. These could, for example, be cold glaciers with large k_0.*

As noted previously, the model does not apply to strictly temperate glaciers. The basic idea of the model, however, could be applied to what we shall term a cold–hot–cold, c.h.c., glacier such as Trapridge Glacier, which is temperate in a central region and cold at its upper and lower elevations. In a c.h.c. glacier the upper cold region could act as a reservoir region during the accumulation half-cycle. During the ablation half-cycle the mass of the reservoir region would be dumped onto the temperate middle region and instability could result. In this connection we repeat that sliding is not excluded by the model. Equation (3) could be replaced or modified by a basal sliding law which is, mathematically, very similar.

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* Normally, in cold regions k_e is small and thus Δt tends to increase.