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RECIPROCITY RELATIONS FOR A CONDUCTIVE SCATTERER WITH A CHIRAL CORE IN QUASI-STATIC FORM

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Abstract

We analyse a scattering problem of electromagnetic waves by a bounded chiral conductive obstacle, which is surrounded by a dielectric, via the quasi-stationary approximation for the Maxwell equations. We prove the reciprocity relations for incident plane and spherical electric waves upon the scatterer. Mixed reciprocity relations have also been proved for a plane wave and a spherical wave. In the case of spherical waves, the point sources are located either inside or outside the scatterer. These relations are used to study the inverse scattering problems.

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1. Introduction

In this work, we prove reciprocity relations that connect the far-field patterns in the case of two incident plane waves, and the scattered fields in the case of two spherical waves. The two incident plane waves have different directions of propagations and polarizations. The two spherical waves emanate from two different point sources at any locations, outside and inside the scatterer, or a combination of both. We consider a scatterer with a dielectric layer and a core filled with chiral material. Chiral media exhibit optical activity, meaning that chiral materials cannot be brought into congruence under rotation with their mirror images [10].

The conductive transmission problem is a generalization of the classical transmission boundary problem. Angell and Kirsch [1] and Hettlich [8] analysed its well-posedness. Athanasiadis and Stratis [4] studied the solvability of a chiral conductive obstacle in an achiral environment. Athanasiadis et al. [3] proved scattering relations for incident plane waves upon a chiral scatterer in an achiral environment. All the above-mentioned papers discussed this scattering problem in quasi-static form.

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The case of point-source waves in electromagnetism was discussed by Athanasiadis et al. [2]. Athanasiadis and Tsitsas [5, 13] studied reciprocity and scattering relations for a multilayered chiral scatterer with interior dipole excitation and an achiral scatterer with a conductive core, respectively. These relations are used in the study of the inverse scattering problems. In particular, Colton and Kress [6] used the reciprocity relation for plane waves for the definition of the adjoint of the far-field operator, and proved that the far-field operator is normal. Potthast [12] proved the mixed reciprocity relation, which is a combination of a plane wave and a spherical wave, and is the basis for the inverse scattering method of point sources and multipoles. Liu et al. [11] used mixed reciprocity relations to prove the uniqueness of the solution of an inverse scattering problem in acoustics, while Kirsch and Kleefeld [9] studied the solvability of the inverse conductive scattering problem via the factorization method in acoustics. Applications of this kind of scattering problem can be found in radar, antennas and medicine production. In particular, Flapan's work [7] is very representative, considering its applications in chemistry.

In Section 2 we proceed with the formulation of the quasi-stationary conductive transmission problem for a dielectric with a chiral core in its interior. In Section 3 we state and prove reciprocity relations for two incident plane waves, and in Section 4 we establish the corresponding reciprocity relations for spherical waves. Finally, in Section 5 we prove mixed reciprocity relations, while in Section 6 we state some concluding remarks.

2. Formulation

Let D be a bounded three-dimensional obstacle with a C^2-boundary S_0, which will be referred to as the scatterer. The scatterer D consists of two layers D_1, D_c, which are divided by a C^2-surface S_1, that is, $D = \bar{D}_1 \cup D_c$.

The interior layer D_c is filled with a homogeneous isotropic chiral medium with electric permittivity ε, magnetic permeability μ, conductivity σ and chirality measure β, such that $\mu\sigma\beta$ is small. The exterior $D_0 = \mathbb{R}^3 \setminus \bar{D}$ of the scatterer is assumed to be simply connected, and is occupied by an infinite isotropic homogeneous medium with corresponding physical parameters ε_0, μ and σ_0, while the layer D_1 is described by ε_1, μ and σ_1.

We further assume that the surfaces S_0, S_1 of the scatterer D (see Figure 1) are covered by thin layers with very high conductivity, such that the integrated conductivity $\tau_0(\mathbf{r})$, with $\mathbf{r} \in S_0$ and $\tau(\mathbf{r})$, with $\mathbf{r} \in S_1$ remain finite [1]. The problem is formulated in the quasi-stationary form [4]. We refer to [1] and the references therein for details about the physical problem and its derivation. Let $(\mathbf{E}^j, \mathbf{H}^j)$ be the total electromagnetic fields in D_j, $j = 0, 1$, while the corresponding field in D_c is denoted by $(\mathbf{E}, \mathbf{H})$. Chirality measure is introduced via the Drude–Born–Fedorov constitutive relations [10]

$$\mathbf{D} = \varepsilon(\mathbf{E} + \beta\nabla \times \mathbf{E}), \quad \mathbf{B} = \mu(\mathbf{H} + \beta\nabla \times \mathbf{H}),$$

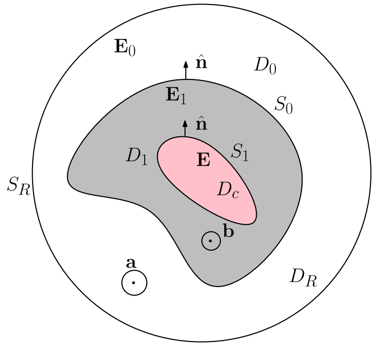

FIGURE 1. Scatterer D with layer D_1 and core D_c.

where $\mathbf{D}$ and $\mathbf{B}$ are the electric and the magnetic flux density vectors, respectively. Considering the quasi-stationary approximation for the Maxwell equations in D_j and D_c, we get the symmetrized system [4]

$$\nabla \times \mathbf{E}^j - ik_j\mathbf{H}^j = \mathbf{0}, \quad \nabla \times \mathbf{H}^j + ik_j\mathbf{E}^j = \mathbf{0} \quad \text{in } D_j, \tag{2.1}$$

$$\nabla \times \mathbf{E} - ik\mathbf{H} = \lambda\mathbf{E}, \quad \nabla \times \mathbf{H} + ik\mathbf{E} = \lambda\mathbf{H} \quad \text{in } D_c, \tag{2.2}$$

where $k_j^2 = i\omega\mu\sigma_j$, with k_j being the wave number in D_j, and

$$k^2 = \frac{i\omega\mu\sigma}{(1-\varepsilon\mu\omega^2\beta^2)^2}, \quad \lambda = \frac{\varepsilon\mu\omega^2\beta}{1-\varepsilon\mu\omega^2\beta^2}.$$

Note that k is not a wave number in D_c but a shorthand notation. The total exterior field $(\mathbf{E}^0, \mathbf{H}^0)$ is the superposition of the incident field $(\mathbf{E}^i, \mathbf{H}^i)$, which is either a plane wave or a spherical wave, and the scattered field $(\mathbf{E}^s, \mathbf{H}^s)$ with

$$\mathbf{E}^0 = \mathbf{E}^i + \mathbf{E}^s, \quad \mathbf{H}^0 = \mathbf{H}^i + \mathbf{H}^s. \tag{2.3}$$

The equations (2.3) are valid in $\mathbb{R}^3$ for plane waves and in $\mathbb{R}^3$, apart from the locations of the point sources, for spherical waves. The scattered field should attenuate away from the scatterer satisfying the Silver–Müller radiation condition

$$\lim_{r\to\infty}(\mathbf{r} \times \nabla \times \mathbf{E}^s + ik_0 r\mathbf{E}^s) = \mathbf{0} \tag{2.4}$$

uniformly in all directions $\hat{\mathbf{r}} = \mathbf{r}/r \in S^2$, where S^2 is the unit sphere and $r = |\mathbf{r}|$. In what follows, for a vector $\mathbf{u}$ we will denote by $u = |\mathbf{u}|$ the measure of $\mathbf{u}$ and $\hat{\mathbf{u}} = \mathbf{u}/u$ the corresponding vector unit.

In order to prove various reciprocity relations for electromagnetic scattering by a conductive obstacle, we consider the decoupling of electric and magnetic fields in the system (2.1)–(2.2) by eliminating the magnetic field. Thus, the equation system turns into

$$\nabla \times \nabla \times \mathbf{E}^j - k_j^2\mathbf{E}^j = \mathbf{0} \quad \text{in } D_j, \tag{2.5}$$

$$\nabla \times \nabla \times \mathbf{E} - 2\lambda\nabla \times \mathbf{E} - (k^2 - \lambda^2)\mathbf{E} = \mathbf{0} \quad \text{in } D_c \tag{2.6}$$

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for $j = 0, 1$. The conductive transmission conditions which were formulated by Athanasiadis and Stratis [4] and Angell and Kirsch [1] take the form

$$\hat{\mathbf{n}} \times \mathbf{E}^0 = \hat{\mathbf{n}} \times \mathbf{E}^1 \quad \text{on } S_0, \tag{2.7}$$

$$\hat{\mathbf{n}} \times \nabla \times \mathbf{E}^0 = \hat{\mathbf{n}} \times \nabla \times \mathbf{E}^1 + i\mu\tau_0\omega\hat{\mathbf{n}} \times \mathbf{E}^1 \quad \text{on } S_0, \tag{2.8}$$

$$\hat{\mathbf{n}} \times \mathbf{E}^1 = \hat{\mathbf{n}} \times \mathbf{E} \quad \text{on } S_1, \tag{2.9}$$

$$\hat{\mathbf{n}} \times \nabla \times \mathbf{E}^1 = \hat{\mathbf{n}} \times \nabla \times \mathbf{E} - \lambda\hat{\mathbf{n}} \times \mathbf{E} + i\frac{\lambda\tau}{\omega\varepsilon\beta}(\hat{\mathbf{n}} \times \mathbf{E}) \times \hat{\mathbf{n}} \quad \text{on } S_1, \tag{2.10}$$

where $\hat{\mathbf{n}}$ is the unit outward normal vector on each surface. The electric far-field pattern $\mathbf{E}^\infty$ is closely related to the asymptotic behaviour of the scattered electric field $\mathbf{E}^s$, as it is the coefficient of the zeroth-order spherical Hankel function of the first kind, $h(x) = e^{ix}/ix$, and

$$\mathbf{E}^s(\mathbf{r}) = h(kr)\mathbf{E}^\infty(\hat{\mathbf{r}}) + O\left(\frac{1}{r^2}\right), \quad r \to \infty. \tag{2.11}$$

The electric far-field pattern $\mathbf{E}^\infty$ is given by

$$\mathbf{E}^\infty(\hat{\mathbf{r}}) = \frac{ik_0}{4\pi}(\tilde{\mathbf{I}} - \hat{\mathbf{r}}\hat{\mathbf{r}}) \cdot \int_S [\hat{\mathbf{n}}' \times (\nabla \times \mathbf{E}^s(\mathbf{r}')) + ik_0\hat{\mathbf{r}} \times (\hat{\mathbf{n}}' \times \mathbf{E}^s(\mathbf{r}'))]e^{-ik_0\hat{\mathbf{r}}\cdot\mathbf{r}'}\,ds(\mathbf{r}'), \tag{2.12}$$

where $\tilde{\mathbf{I}} = \hat{\mathbf{x}}_1\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2\hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_3\hat{\mathbf{x}}_3$ is the identity dyadic and $\hat{\mathbf{n}}' = \hat{\mathbf{n}}(\mathbf{r}')$. Equation (2.12) is also valid for the scattered electric field replaced by the total electric field. Multiplying with a constant vector $\hat{\mathbf{q}}$ for which $\hat{\mathbf{q}} \cdot \hat{\mathbf{r}} = 0$,

$$\begin{aligned} \hat{\mathbf{q}} \cdot \mathbf{E}^\infty(\hat{\mathbf{r}}) = \frac{ik_0}{4\pi} \int_S &[-(\nabla \times \mathbf{E}^s(\mathbf{r}')) \cdot (\hat{\mathbf{n}}' \times \hat{\mathbf{q}}e^{-ik_0\hat{\mathbf{r}}\cdot\mathbf{r}'}) \\ &+ (\hat{\mathbf{n}}' \times \mathbf{E}^s(\mathbf{r}')) \cdot (\nabla \times (\hat{\mathbf{q}}e^{-ik_0\hat{\mathbf{r}}\cdot\mathbf{r}'})]\,ds(\mathbf{r}'). \end{aligned} \tag{2.13}$$

In what follows, we will employ the Twersky notation [2] for two vector functions

$$\{\mathbf{u}, \mathbf{v}\}_S = \int_S [(\hat{\mathbf{n}} \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) - (\hat{\mathbf{n}} \times \mathbf{v}) \cdot (\nabla \times \mathbf{u})]\,ds. \tag{2.14}$$

3. Plane waves

Let

$$\mathbf{E}^i(\mathbf{r}; \hat{\mathbf{d}}, \hat{\mathbf{p}}) = \hat{\mathbf{p}}e^{ik_0\hat{\mathbf{d}}\cdot\mathbf{r}}, \quad \mathbf{H}^i(\mathbf{r}; \hat{\mathbf{d}}, \hat{\mathbf{p}}) = (\hat{\mathbf{d}} \times \hat{\mathbf{p}})e^{ik_0\hat{\mathbf{d}}\cdot\mathbf{r}} \tag{3.1}$$

be an incident time-harmonic electromagnetic plane wave. The real unit vectors $\hat{\mathbf{d}}$ and $\hat{\mathbf{p}}$ describe the directions of propagation and polarization, respectively, and are connected: $\hat{\mathbf{d}} \cdot \hat{\mathbf{p}} = \mathbf{0}$. In the sequel, we will denote the electric scattered field, the total field and the far-field pattern by writing $\mathbf{E}^s(\mathbf{r}; \hat{\mathbf{d}}, \hat{\mathbf{p}})$, $\mathbf{E}^t(\mathbf{r}; \hat{\mathbf{d}}, \hat{\mathbf{p}})$ and $\mathbf{E}^\infty(\mathbf{r}; \hat{\mathbf{d}}, \hat{\mathbf{p}})$, respectively, indicating the dependence on the incident direction $\hat{\mathbf{d}}$ and the polarization $\hat{\mathbf{p}}$.

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THEOREM 3.1. *Let $\mathbf{E}_1^i = \mathbf{E}^i(\mathbf{r}'; \hat{\mathbf{d}}, \hat{\mathbf{p}})$, $\mathbf{E}_2^i = \mathbf{E}^i(\mathbf{r}'; \hat{\mathbf{r}}, \hat{\mathbf{q}})$ be two plane electric waves incident upon the scatterer D. For the corresponding far-field patterns, the following reciprocity relation holds:*

$$\hat{\mathbf{p}} \cdot \mathbf{E}^\infty(\hat{\mathbf{d}}; \hat{\mathbf{r}}, \hat{\mathbf{q}}) = \hat{\mathbf{q}} \cdot \mathbf{E}^\infty(-\hat{\mathbf{r}}; -\hat{\mathbf{d}}, \hat{\mathbf{p}}). \tag{3.2}$$

PROOF. Let $\mathbf{E}_1^s$, $\mathbf{E}_2^s$ and $\mathbf{E}_1^0$, $\mathbf{E}_2^0$ be the corresponding scattered and total exterior fields. In view of bilinearity of the form (2.14) and relation (2.3), we take

$$\{\mathbf{E}_1^0, \mathbf{E}_2^0\}_{S_0} = \{\mathbf{E}_1^i, \mathbf{E}_2^i\}_{S_0} + \{\mathbf{E}_1^i, \mathbf{E}_2^s\}_{S_0} + \{\mathbf{E}_1^s, \mathbf{E}_2^i\}_{S_0} + \{\mathbf{E}_1^s, \mathbf{E}_2^s\}_{S_0}.$$

From equations (2.5) and (2.6), the conductive transmission conditions (2.7)–(2.10), the relation $\{\mathbf{E}_1^0, \mathbf{E}_2^0\}_{S_0} = \{\mathbf{E}_1^1, \mathbf{E}_2^1\}_{S_0}$ and using Green's second vector theorem, we take $\{\mathbf{E}_1^0, \mathbf{E}_2^0\}_{S_0} = 0$. The incident plane waves are regular solutions of (2.5) for $j = 0$; thus, $\{\mathbf{E}_1^i, \mathbf{E}_2^i\}_{S_0} = 0$. For the scattered fields, we consider a sphere S_R centred at the origin with radius R large enough to include the scatterer D in its interior. The fields $\mathbf{E}_1^s$, $\mathbf{E}_2^s$ are regular solutions of (2.5) for $j = 0$ in the region D_R between the sphere and the scatterer. We apply the divergence theorem over the surface S_0 to get

$$\{\mathbf{E}_1^s, \mathbf{E}_2^s\}_{S_0} = \{\mathbf{E}_1^s, \mathbf{E}_2^s\}_{S_R}.$$

Letting $R \to \infty$, we pass to the radiation zone; therefore, we can use the asymptotic form (2.11) to get $\{\mathbf{E}_1^s, \mathbf{E}_2^s\}_{S_R} = 0$. Finally, from (2.13) and (2.14), we have for the remaining terms

$$\{\mathbf{E}_1^i, \mathbf{E}_2^s\}_{S_0} = -\frac{4\pi}{ik_0}\hat{\mathbf{p}} \cdot \mathbf{E}^\infty(\hat{\mathbf{d}}; \hat{\mathbf{r}}, \hat{\mathbf{q}}), \quad \{\mathbf{E}_1^s, \mathbf{E}_2^i\}_{S_0} = \frac{4\pi}{ik_0}\hat{\mathbf{q}} \cdot \mathbf{E}^\infty(-\hat{\mathbf{r}}; -\hat{\mathbf{d}}, \hat{\mathbf{p}}).$$

Combining all the above, we deduce (3.2). □

4. Spherical waves

Next we consider an incident spherical electromagnetic wave due to a source located at a point with vector position **a** with respect to the origin. This incident field was given by Athanasiadis et al. [2] as

$$\begin{aligned} \mathbf{E}_{\mathbf{a}}^i(\mathbf{r}; \hat{\mathbf{p}}) &= \frac{ae^{-ik_j a}}{ik_j} \nabla \times \left(\frac{e^{ik_j|\mathbf{r}-\mathbf{a}|}}{|\mathbf{r}-\mathbf{a}|} \hat{\mathbf{a}} \times \hat{\mathbf{p}} \right), \\ \mathbf{H}_{\mathbf{a}}^i(\mathbf{r}; \hat{\mathbf{p}}) &= \frac{1}{ik_j}\left(\frac{\varepsilon_j}{\mu}\right)^{1/2} \nabla \times \mathbf{E}_{\mathbf{a}}^i(\mathbf{r}; \hat{\mathbf{p}}), \end{aligned} \tag{4.1}$$

where $\hat{\mathbf{p}}$ is a constant vector for which $\hat{\mathbf{p}} \cdot \hat{\mathbf{a}} = 0$ and k_j the wave number of the region where the point source is located $\mathbf{a} \in D_j$, $j = 0, 1$. This incident field is generated by a magnetic dipole with dipole moment $\hat{\mathbf{a}} \times \hat{\mathbf{p}}$. The total spherical electric wave again is the superposition of incident and scattered fields that satisfies

$$\mathbf{E}_{\mathbf{a}}^0(\mathbf{r}; \hat{\mathbf{p}}) = \mathbf{E}_{\mathbf{a}}^i(\mathbf{r}; \hat{\mathbf{p}}) + \mathbf{E}_{\mathbf{a}}^s(\mathbf{r}; \hat{\mathbf{p}}), \quad \mathbf{r} \in D_0 \setminus \{\mathbf{a}\}, \tag{4.2}$$

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where $\mathbf{E}^s_{\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}})$ is the scattered field satisfying (2.4). Note that in the case of spherical waves, the radiation condition (2.4), apart from the scattered field, is satisfied by the incident and total fields as well. The asymptotic behaviour of the incident and scattered fields is given similarly to (2.11) by

$$\mathbf{E}^i_{\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}}) = h(kr)\mathbf{E}^{i,\infty}_{\mathbf{a}}(\hat{\mathbf{r}};\hat{\mathbf{p}}) + O\Big(\frac{1}{r^2}\Big)\, r\to\infty, \tag{4.3}$$

$$\mathbf{E}^s_{\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}}) = h(kr)\mathbf{E}^{\infty}_{\mathbf{a}}(\hat{\mathbf{r}};\hat{\mathbf{p}}) + O\Big(\frac{1}{r^2}\Big)\, r\to\infty, \tag{4.4}$$

$$\mathbf{E}^0_{\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}}) = h(kr)\mathbf{E}^{0,\infty}_{\mathbf{a}}(\hat{\mathbf{r}};\hat{\mathbf{p}}) + O\Big(\frac{1}{r^2}\Big)\, r\to\infty,$$

where $\mathbf{E}^{i,\infty}_{\mathbf{a}}(\hat{\mathbf{r}};\hat{\mathbf{p}}) = ikae^{-ika(1+\hat{\mathbf{r}}\cdot\hat{\mathbf{a}})}(\hat{\mathbf{r}}\times(\hat{\mathbf{a}}\times\hat{\mathbf{p}}))$ is the far-field pattern of the point-source incident wave and $\mathbf{E}^{\infty}_{\mathbf{a}}(\hat{\mathbf{r}})$ and $\mathbf{E}^{0,\infty}_{\mathbf{a}}(\hat{\mathbf{r}})$ are the far-field patterns of the corresponding scattered and total fields, respectively. We proceed with a reciprocity theorem for the total spherical waves similar to Theorem 3.1.

THEOREM 4.1. *Let* $\mathbf{E}^i_{\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}})$, $\mathbf{E}^i_{\mathbf{b}}(\mathbf{r};\hat{\mathbf{q}})$ *be two spherical waves due to two point sources* $\mathbf{a}$ *and* $\mathbf{b}$ *with polarization constant unit vectors* $\hat{\mathbf{p}}$ *and* $\hat{\mathbf{q}}$, *respectively. Then, for the scattering problem* (2.3)–(2.10), *we have the following result.*

(i) *If* $\mathbf{a},\mathbf{b}\in D_j$ *for* $j=0,1$, *then*

$$h(k_j a)(\hat{\mathbf{b}}\times\hat{\mathbf{q}})\cdot(\nabla\times\mathbf{E}^j_{\mathbf{a}}(\mathbf{b};\hat{\mathbf{p}})) = h(k_j b)(\hat{\mathbf{a}}\times\hat{\mathbf{p}})\cdot(\nabla\times\mathbf{E}^j_{\mathbf{b}}(\mathbf{a};\hat{\mathbf{q}})). \tag{4.5}$$

(ii) *If* $\mathbf{a}\in D_0$ *and* $\mathbf{b}\in D_1$, *then*

$$k_0^2 h(k_0 a)(\hat{\mathbf{b}}\times\hat{\mathbf{q}})\cdot(\nabla\times\mathbf{E}^1_{\mathbf{a}}(\mathbf{b};\hat{\mathbf{p}})) = k_1^2 h(k_1 b)(\hat{\mathbf{a}}\times\hat{\mathbf{p}})\cdot(\nabla\times\mathbf{E}^0_{\mathbf{b}}(\mathbf{a};\hat{\mathbf{q}})). \tag{4.6}$$

PROOF. (i) We consider two small spheres in D_0, $S_{R_{\mathbf{a}}} = \{\mathbf{r}\in\mathbb{R}^3 : |\mathbf{a}-\mathbf{r}| = R_{\mathbf{a}}\}$, $S_{R_{\mathbf{b}}} = \{\mathbf{r}\in\mathbb{R}^3 \mid |\mathbf{b}-\mathbf{r}| = R_{\mathbf{b}}\}$ centred at $\mathbf{a}$ and $\mathbf{b}$ with radii $R_{\mathbf{a}}$ and $R_{\mathbf{b}}$, respectively. A large sphere S_R of radius R centred at the origin contains the scatterer and the two small spheres. Let D_R be the space which is surrounded by the surfaces of the spheres S_R, $S_{R_{\mathbf{a}}}$, $S_{R_{\mathbf{b}}}$ and the surface S_0. We apply the vector version of the second Green's theorem in D_R for the functions $\mathbf{E}^0_{\mathbf{a}}$, $\mathbf{E}^0_{\mathbf{b}}$, which are regular solutions of (2.5),

$$\{\mathbf{E}^0_{\mathbf{a}},\mathbf{E}^0_{\mathbf{b}}\}_{S_R} = \{\mathbf{E}^0_{\mathbf{a}},\mathbf{E}^0_{\mathbf{b}}\}_{S_0} + \{\mathbf{E}^0_{\mathbf{a}},\mathbf{E}^0_{\mathbf{b}}\}_{S_{R_{\mathbf{a}}}} + \{\mathbf{E}^0_{\mathbf{a}},\mathbf{E}^0_{\mathbf{b}}\}_{S_{R_{\mathbf{b}}}}. \tag{4.7}$$

For the left-hand-side term, we have $\{\mathbf{E}^0_{\mathbf{a}},\mathbf{E}^0_{\mathbf{b}}\}_{S_R}=0$ due to relation (4.2) and due to the asymptotic forms (4.3), (4.4). For the first term in the right-hand side, we have $\{\mathbf{E}^0_{\mathbf{a}},\mathbf{E}^0_{\mathbf{b}}\}_{S_0} = \{\mathbf{E}^1_{\mathbf{a}},\mathbf{E}^1_{\mathbf{b}}\}_{S_0}$ due to (2.7), (2.8). Then we apply the second Green's theorem [2] in D_1 and D_c, successively taking into account that these fields are regular solutions of (2.5), (2.6) and due to the boundary conditions (2.9), (2.10), we have $\{\mathbf{E}^1_{\mathbf{a}},\mathbf{E}^1_{\mathbf{b}}\}_{S_0} = \{\mathbf{E}^1_{\mathbf{a}},\mathbf{E}^1_{\mathbf{b}}\}_{S_1}=0$. For the term $\{\mathbf{E}^0_{\mathbf{a}},\mathbf{E}^0_{\mathbf{b}}\}_{S_{R_{\mathbf{a}}}}$, we have the following analysis:

$$\{\mathbf{E}^0_{\mathbf{a}},\mathbf{E}^0_{\mathbf{b}}\}_{S_{R_{\mathbf{a}}}} = \{\mathbf{E}^i_{\mathbf{a}},\mathbf{E}^0_{\mathbf{b}}\}_{S_{R_{\mathbf{a}}}} + \{\mathbf{E}^s_{\mathbf{a}},\mathbf{E}^0_{\mathbf{b}}\}_{S_{R_{\mathbf{a}}}}. \tag{4.8}$$

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The functions of the integral $\{\mathbf{E}^s_{\mathbf{a}}, \mathbf{E}^0_{\mathbf{b}}\}_{S_{R_\mathbf{a}}}$ are regular solutions of (2.5) in the domain of the sphere $S_{R_\mathbf{a}}$ and by applying the second Green's theorem the term equals zero. For $\{\mathbf{E}^i_{\mathbf{a}}, \mathbf{E}^0_{\mathbf{b}}\}_{S_{R_\mathbf{a}}}$,

$$\begin{aligned}\{\mathbf{E}^i_{\mathbf{a}}, \mathbf{E}^0_{\mathbf{b}}\}_{S_{R_\mathbf{a}}} = ae^{-ik_0a}\int_{S_{R_\mathbf{a}}} \hat{\mathbf{n}}\cdot\nabla\times[(\hat{\mathbf{a}}\times\hat{\mathbf{p}})\cdot(\nabla h(k_0R_a))\mathbf{E}^0_{\mathbf{b}}(\mathbf{r};\hat{\mathbf{q}})]\,ds \\ - ae^{-ik_0a}\int_{S_{R_\mathbf{a}}} \hat{\mathbf{n}}\cdot\nabla h(k_0R_a))[(\nabla\times\mathbf{E}^0_{\mathbf{b}}(\mathbf{r};\hat{\mathbf{q}}))\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{p}})]\,ds \\ + k^2ae^{-ik_0a}\int_{S_{R_\mathbf{a}}} \hat{\mathbf{n}}\cdot[\mathbf{E}^0_{\mathbf{b}}(\mathbf{r};\hat{\mathbf{q}})\times h(k_0R_a)(\hat{\mathbf{a}}\times\hat{\mathbf{p}})]\,ds.\end{aligned} \tag{4.9}$$

We apply Stokes' theorem on the first integral on the right-hand side of equation (4.9), and therefore it vanishes. Then we apply the mean value theorem on the other two integrals in (4.9) and, letting $R_a \to 0$,

$$\lim_{R_a\to 0}\{\mathbf{E}^i_{\mathbf{a}}, \mathbf{E}^0_{\mathbf{b}}\}_{S_{R_\mathbf{a}}} = -h(k_0a)(\hat{\mathbf{a}}\times\hat{\mathbf{p}})\cdot(\nabla\times\mathbf{E}^0_{\mathbf{b}}(\mathbf{a};\hat{\mathbf{q}})) \tag{4.10}$$

Following the same procedure for $\{\mathbf{E}^0_{\mathbf{a}}, \mathbf{E}^0_{\mathbf{b}}\}_{S_{R_\mathbf{b}}}$,

$$\lim_{R_b\to 0}\{\mathbf{E}^0_{\mathbf{a}}, \mathbf{E}^0_{\mathbf{b}}\}_{S_{R_\mathbf{b}}} = \{\mathbf{E}^0_{\mathbf{a}}, \mathbf{E}^i_{\mathbf{b}}\}_{S_{R_\mathbf{b}}} = h(k_0b)(\hat{\mathbf{b}}\times\hat{\mathbf{q}})\cdot(\nabla\times\mathbf{E}^0_{\mathbf{a}}(\mathbf{b};\hat{\mathbf{p}})). \tag{4.11}$$

Combining (4.7)–(4.11) yields (4.5). For $\mathbf{a}$, $\mathbf{b} \in D_1$, we follow a similar procedure.

(ii) First we apply the second Green's theorem in D_1 excluding the domain of the sphere $S_{R_\mathbf{b}}$,

$$\{\mathbf{E}^1_{\mathbf{a}}, \mathbf{E}^1_{\mathbf{b}}\}_{\partial D_1} = \{\mathbf{E}^1_{\mathbf{a}}, \mathbf{E}^1_{\mathbf{b}}\}_{S_0} - \{\mathbf{E}^1_{\mathbf{a}}, \mathbf{E}^1_{\mathbf{b}}\}_{S_1} = \{\mathbf{E}^1_{\mathbf{a}}, \mathbf{E}^1_{\mathbf{b}}\}_{S_{R_\mathbf{b}}}. \tag{4.12}$$

We analyse the last term on the right-hand side of (4.12), taking into account that $\mathbf{E}^1_{\mathbf{b}} = \mathbf{E}^i_{\mathbf{b}} + \mathbf{E}^s_{\mathbf{b}}$ and

$$\{\mathbf{E}^1_{\mathbf{a}}, \mathbf{E}^1_{\mathbf{b}}\}_{S_{R_\mathbf{b}}} = \{\mathbf{E}^1_{\mathbf{a}}, \mathbf{E}^i_{\mathbf{b}}\}_{S_{R_\mathbf{b}}} + \{\mathbf{E}^1_{\mathbf{a}}, \mathbf{E}^s_{\mathbf{b}}\}_{S_{R_\mathbf{b}}}.$$

The fields $\mathbf{E}^1_{\mathbf{a}}$, $\mathbf{E}^s_{\mathbf{b}}$ are regular solutions in the interior of $S_{R_\mathbf{b}}$, $\{\mathbf{E}^1_{\mathbf{a}}, \mathbf{E}^s_{\mathbf{b}}\}_{S_{R_\mathbf{b}}} = 0$, and we conclude that

$$\{\mathbf{E}^1_{\mathbf{a}}, \mathbf{E}^1_{\mathbf{b}}\}_{S_{R_\mathbf{b}}} = \{\mathbf{E}^1_{\mathbf{a}}, \mathbf{E}^i_{\mathbf{b}}\}_{S_{R_\mathbf{b}}} = \frac{4i\pi b}{k_1}e^{-ik_1b}(\nabla\times\mathbf{E}^1_{\mathbf{a}}(\mathbf{b};\hat{\mathbf{p}}))\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{q}}). \tag{4.13}$$

We also obtain the following analysis for (4.12) by applying the transmission conditions (2.5) and considering that these functions are regular solutions in D_0 and D_1,

$$\{\mathbf{E}^1_{\mathbf{a}}, \mathbf{E}^1_{\mathbf{b}}\}_{S_0} = \{\mathbf{E}^0_{\mathbf{a}}, \mathbf{E}^0_{\mathbf{b}}\}_{S_0} = \{\mathbf{E}^i_{\mathbf{a}}, \mathbf{E}^0_{\mathbf{b}}\}_{S_0} + \{\mathbf{E}^s_{\mathbf{a}}, \mathbf{E}^0_{\mathbf{b}}\}_{S_0},$$

where

$$\{\mathbf{E}^i_{\mathbf{a}}, \mathbf{E}^0_{\mathbf{b}}\}_{S_0} = -\{\mathbf{E}^i_{\mathbf{a}}, \mathbf{E}^0_{\mathbf{b}}\}_{S_{R_\mathbf{a}}} = \frac{4i\pi a}{k_0}e^{-ik_0\mathbf{a}}(\nabla\times\mathbf{E}^0_{\mathbf{b}}(\mathbf{a};\hat{\mathbf{q}}))\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{p}}). \tag{4.14}$$

Applying Green's vector theorem [2] in D_R and passing to the radiation zone, we get $\{\mathbf{E}^s_{\mathbf{a}}, \mathbf{E}^0_{\mathbf{b}}\}_{S_0} = 0$. Hence, equations (4.13) and (4.14) lead to (4.6). □

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5. Plane and spherical waves

Next we consider an incident time-harmonic electric plane wave of the form (3.1) and a spherical incident electric wave of the form (4.1) with polarization $\hat{\mathbf{q}}$. We note that the spherical wave reduces to a plane wave with opposite direction of propagation once the point source goes to infinity, that is,

$$\lim_{a\to\infty} \mathbf{E}^i_{\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}}) = \mathbf{E}^i(\mathbf{r};-\hat{\mathbf{a}},\hat{\mathbf{p}}),$$

and this holds for the corresponding total, scattering and far-field pattern fields as well. The following mixed reciprocity relation is valid.

Theorem 5.1. *Let* $\mathbf{E}^i_{\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}})$, $\hat{\mathbf{a}}\in D_0$ *be an incident spherical electric wave and* $\mathbf{E}^i(\mathbf{r};-\hat{\mathbf{b}},\hat{\mathbf{q}})$ *be an incident plane electric wave. Then*

$$\hat{\mathbf{p}}\cdot(\mathbf{E}^{0,\infty}_{\mathbf{b}}(\hat{\mathbf{a}};\hat{\mathbf{q}})) = \frac{k_0}{k_1} b e^{-ik_1 b}(\hat{\mathbf{b}}\times\hat{\mathbf{q}})\cdot(\nabla\times\mathbf{E}^1(\mathbf{b};-\hat{\mathbf{a}},\hat{\mathbf{p}})). \tag{5.1}$$

Proof. Working as in the proof of Theorem 4.1,

$$\begin{aligned}\{\mathbf{E}^1(\cdot;\,-\hat{\mathbf{a}},\,\hat{\mathbf{p}}),\,\mathbf{E}^1_{\mathbf{b}}(\cdot;\,\hat{\mathbf{q}})\}_{\partial D_1} &= \{\mathbf{E}^1(\cdot;\,-\hat{\mathbf{a}},\,\hat{\mathbf{p}}),\,\mathbf{E}^i_{\mathbf{b}}(\cdot;\,\hat{\mathbf{q}})\}_{\partial D_1} + \{\mathbf{E}^1(\cdot;\,-\hat{\mathbf{a}},\,\hat{\mathbf{p}}),\,\mathbf{E}^s_{\mathbf{b}}(\cdot;\,\hat{\mathbf{q}})\}_{\partial D_1}\\ &= \{\mathbf{E}^1(\cdot;\,-\hat{\mathbf{a}},\,\hat{\mathbf{p}}),\,\mathbf{E}^i_{\mathbf{b}}(\cdot;\,\hat{\mathbf{q}})\}_{S_{R_{\mathbf{b}}}}\\ &= \frac{4\pi b i}{k_1} e^{-ik_1\mathbf{b}}(\nabla\times\mathbf{E}^1(\mathbf{b};\,-\hat{\mathbf{a}},\,\hat{\mathbf{p}}))\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{q}}).\end{aligned} \tag{5.2}$$

Moreover, taking into account equation (2.13),

$$\begin{aligned}\{\mathbf{E}^1(\cdot;\,-\hat{\mathbf{a}},\,\hat{\mathbf{p}}),\,\mathbf{E}^1_{\mathbf{b}}(\cdot;\,\hat{\mathbf{q}})\}_{\partial D_1} &= \{\mathbf{E}^1(\cdot;\,-\hat{\mathbf{a}},\,\hat{\mathbf{p}}),\,\mathbf{E}^1_{\mathbf{b}}(\cdot;\,\hat{\mathbf{q}})\}_{S_0}\\ &= \{\mathbf{E}^0(\cdot;\,-\hat{\mathbf{a}},\,\hat{\mathbf{p}}),\,\mathbf{E}^0_{\mathbf{b}}(\cdot;\,\hat{\mathbf{q}})\}_{S_0}\\ &= \{\mathbf{E}^i(\cdot;\,-\hat{\mathbf{a}},\,\hat{\mathbf{p}}),\,\mathbf{E}^0_{\mathbf{b}}(\cdot;\,\hat{\mathbf{q}})\}_{S_0}\\ &= \frac{4\pi i}{k_0}\hat{\mathbf{p}}\cdot(\mathbf{E}^{0,\infty}_{\mathbf{b}}(\hat{\mathbf{a}};\hat{\mathbf{q}})).\end{aligned} \tag{5.3}$$

Combining equations (5.2) and (5.3), we obtain (5.1). □

6. Conclusions

We have considered a scattering problem of electromagnetic waves by a chiral conductive obstacle, which is surrounded by a dielectric, in the quasi-static form. If $\lambda = 0$, equation (2.2) is reduced to achiral form, and our study covers corresponding results for the achiral case [2, 13]. We note that reciprocity relations are independent of the integrated conductivities, τ_0 and τ [2]. The mixed reciprocity relation in Theorem 5.1 can be used in the study of uniqueness of the inverse scattering problem, as it was applied by Liu et al. [11] for acoustic waves. Finally, our present results can be extended to a more complex scattering problem described by a multilayer scatterer with n layers and a chiral core [5].

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