

Intermediate-scale statistics for real-valued lacunary sequences

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Abstract

We study intermediate-scale statistics for the fractional parts of the sequence $(\alpha a_n)_{n=1}^\infty$, where $(a_n)_{n=1}^\infty$ is a positive, real-valued lacunary sequence, and $\alpha \in \mathbb{R}$. In particular, we consider the number of elements $S_N(L, \alpha)$ in a random interval of length L/N , where $L = O(N^{1-\epsilon})$, and show that its variance (the number variance) is asymptotic to L with high probability w.r.t. α , which is in agreement with the statistics of uniform i.i.d. random points in the unit interval. In addition, we show that the same asymptotic holds almost surely in $\alpha \in \mathbb{R}$ when $L = O(N^{1/2-\epsilon})$. For slowly growing L , we further prove a central limit theorem for $S_N(L, \alpha)$ which holds for almost all $\alpha \in \mathbb{R}$.

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1. Introduction

A real-valued sequence $(x_n)_{n=1}^\infty$ is said to be uniformly distributed (or equidistributed) modulo one if, for every interval $I \subseteq [0, 1)$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : \{x_n\} \in I\} = |I|,$$

where $\{x\}$ denotes the fractional part of x , and $|I|$ denotes the length of the interval I . There are many examples of sequences which satisfy this property, e.g., the Kronecker sequence $x_n = \alpha n$ where α is irrational, and more generally (as was shown by Weyl in his pioneering 1916 paper [18]) the sequence $x_n = \alpha_d n^d + \dots + \alpha_1 n + \alpha_0$ ($\alpha_i \in \mathbb{R}$), where at least one of the coefficients $\alpha_1, \dots, \alpha_d$ is irrational. In the metric sense, more can be said: Weyl proved [18] that for any sequence $(a_n)_{n=1}^\infty$ of distinct integers, the sequence $x_n = \alpha a_n$ is uniformly distributed modulo one for (Lebesgue) almost all $\alpha \in \mathbb{R}$. This is also true for real-valued sequences whose elements are sufficiently separated from each other (see, e.g., [6, chapter 1, corollary 4.1]): if $(a_n)_{n=1}^\infty$ is a real-valued sequence, and there exists a positive constant $\delta > 0$ such that $|a_n - a_m| \geq \delta$ for each $n \neq m$, then the sequence $x_n = \alpha a_n$ is uniformly distributed modulo one for almost all $\alpha \in \mathbb{R}$. This condition clearly holds for real-valued, positive, lacunary sequences, i.e., sequences such that $a_1 > 0$, and there exists a constant $C > 1$ such that for all $n \geq 1$ we have

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$$a_{n+1} \geq Ca_n.$$

While the classical theory deals with the distribution of sequences modulo one at “large” scales, there has been a growing interest in recent years in the fluctuations of sequences at smaller scales. For many sequences, it is conjectured (backed up by numerical evidence) that the small-scale statistics (at the scale $1/N$ – the mean gap of the first N elements of the sequence modulo one) are in agreement with the statistics of uniform i.i.d. random points in the unit interval (Poissonian statistics), thus demonstrating pseudo-random behaviour for such sequences. The rigorous study of such questions was initiated by Rudnick and Sarnak [13], who considered the pair correlation (see (1.3), with L fixed) of the sequence $x_n = \alpha n^d$ modulo one ($d \geq 2$), and proved Poissonian limiting behaviour for almost all $\alpha \in \mathbb{R}$; while the same limiting behaviour is conjectured to hold for any *specific* irrational α which is badly approximable by rationals, the question remains open, as are most questions about small-scale statistics of specific sequences (i.e., not in the metric sense).

A popular small-scale statistic is the (normalised) gap distribution of the re-ordered first N elements of the sequence modulo one, which for many sequences is expected to converge to the exponential distribution (“Poissonian gap statistics”) – the almost sure limiting distribution of the gaps in the random model. Lacunary sequences are among the rare examples where such behaviour has been rigorously proved to hold (in the metric sense): Rudnick and Zaharescu proved [16] Poissonian gap statistics for almost all $\alpha \in \mathbb{R}$ for the sequence $x_n = \alpha a_n$ where $(a_n)_{n=1}^\infty$ is an integer-valued lacunary sequence; this was recently extended to *real-valued* lacunary sequences by Chaubey and the author [2]. At the other extreme, Lutsko and Technau recently proved [10] Poissonian gap statistics for the *slowly growing* sequence $x_n = \alpha (\log n)^A$ ($A > 1$) – remarkably, this holds for *any* $\alpha > 0$, and not only in the metric sense – see also the closely related results [8, 9] about Poissonian correlations for the sequence $x_n = \alpha n^\theta$ where θ is small.

Statistics in the “mesoscopic” regime, i.e., at the scale L/N , where $L = L(N) \rightarrow \infty$ and $L = o(N)$ as $N \rightarrow \infty$, provide further information which may capture some interesting features of sequences. An example of such a statistic is the *number variance* (the variance of the number of elements in random intervals, see the definition below in our setting), famously studied for the zeros of the Riemann zeta function, for which at small scales the number variance is consistent with that of the eigenvalues of random matrices drawn from the Gaussian unitary ensemble (GUE), whereas “saturation” occurs at larger scales (see [1]). In the context of sequences modulo one, only a few results have been established so far in the mesoscopic regime, mainly concerning the leading order asymptotics of the long-range correlations of the sequence $x_n = \alpha n^2$ (see [4, 5, 7, 12, 17]); nevertheless, important intermediate-scale statistics such as the number variance have largely remained unexplored. The aim of this paper is to study such statistics for real-valued lacunary sequences.

Let $(a_n)_{n=1}^\infty$ be a positive, real-valued lacunary sequence; we are interested in the distribution of the number of elements modulo one of the sequence $(x_n)_{n=1}^\infty = (\alpha a_n)_{n=1}^\infty$ in intervals of length L/N around points $x \in [0, 1)$, which we denote by

$$S_N(L, \alpha) = S_N(L, \alpha)(x) := \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \chi\left(\frac{\alpha a_j - x + n}{L/N}\right),$$

where $\chi = \chi_{[-1/2, 1/2]}$ is the characteristic function of the interval $[-1/2, 1/2]$.

The first statistic that we will study is the number variance

$$\Sigma_N^2(L, \alpha) := \int_0^1 (S_N(L, \alpha)(x) - L)^2 dx,$$

i.e., the variance of $S_N(L, \alpha)$, where we randomise w.r.t. the centre of the interval x . We would like to show that for generic values of $\alpha \in \mathbb{R}$, we have

$$\Sigma_N^2(L, \alpha) = L + o(L), \tag{1.1}$$

which is in agreement with the random model. In our first main result we show that (1.1) holds with high probability in (essentially) the full mesoscopic regime (namely, all the way up to $L = O(N^{1-\epsilon})$ where ϵ is arbitrarily small).

THEOREM 1.1. *Let $\epsilon > 0$, and let I be a bounded interval. Assume that $L = L(N) = O(N^{1-\epsilon})$ as $N \rightarrow \infty$. Then (1.1) holds with high probability w.r.t. α : for any $\delta > 0$, we have*

$$\text{meas} \left\{ \alpha \in I : \left| \Sigma_N^2(L, \alpha) - L \right| > \delta L \right\} = O_{\delta, \epsilon, I} \left(N^{-\epsilon/2} \right)$$

as $N \rightarrow \infty$.

It is desirable to extend this to an almost sure statement, which we are able to establish in a narrower regime $L = O(N^{1/2-\epsilon})$ (along with a technical condition on the oscillations of L , which clearly holds for natural choices of L , e.g., when $L = N^s$ with $s \leq 1/2 - \epsilon$).

THEOREM 1.2. *Let $\epsilon > 0$, and assume that $L = L(N) = O(N^{1/2-\epsilon})$ and that $L(N+1) - L(N) = o(N^{-1/2})$ as $N \rightarrow \infty$. Then for almost all $\alpha \in \mathbb{R}$, we have*

$$\Sigma_N^2(L, \alpha) = L + o(L)$$

as $N \rightarrow \infty$.

For slowly growing L (and under an even milder condition on its oscillations), we will be able to establish a central limit theorem for $S_N(L, \alpha)$. This would hold for example when $L = (\log N)^t$ with $t > 0$.

THEOREM 1.3. *Let $L = L(N) \rightarrow \infty$ as $N \rightarrow \infty$ such that for all $\eta > 0$ we have $L = O(N^\eta)$, and assume that there exists $\epsilon > 0$ such that $L(N+1) - L(N) = O(N^{-\epsilon})$. Then for almost all $\alpha \in \mathbb{R}$, for any $a < b$, we have*

$$\text{meas} \left\{ x \in [0, 1) : a \leq \frac{S_N(L, \alpha)(x) - L}{\sqrt{L}} \leq b \right\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{t^2}{2}} dt$$

as $N \rightarrow \infty$.

We remark that while the condition $L(N) \rightarrow \infty$ in Theorem 1.3 is essential, Theorems 1.1 and 1.2 also hold for fixed L , thus extending the results of [14].

We would like to stress the difference between (1.1) and some weaker notions of long-range Poissonian correlations, as studied, e.g., in [5, 7, 17]. Note that the number variance $\Sigma_N^2(L, \alpha)$ can be expressed in terms of the pair correlation function. Indeed, a direct calculation shows (see, e.g., [11]) that

$$\Sigma_N^2(L, \alpha) = L - L^2 + LR_N^2(L, \alpha, \Delta), \tag{1.2}$$

where

$$R_N^2(L, \alpha, \Delta) = \frac{1}{N} \sum_{i \neq j=1}^N \sum_{n \in \mathbb{Z}} \Delta\left(\frac{\alpha a_i - \alpha a_j + n}{L/N}\right) \tag{1.3}$$

is the (scaled) pair correlation function of $(\alpha a_n)_{n=1}^\infty$ with respect to the test function

$$\Delta = \max\{1 - |x|, 0\}.$$

Hence, (1.1) is equivalent to

$$R_N^2(L, \alpha, \Delta) = L + o(1). \tag{1.4}$$

We thus see that (1.4), and therefore (1.1), is a significantly stronger statement than long-range Poissonian pair correlation in the sense of $R_N^2(L, \alpha, \Delta) = L + o(L)$, where the error term is insufficient for determining the asymptotics of the number variance. Similarly, for $k \geq 2$, consider the k -level correlation function

$$R_N^k(L, \alpha, \Delta) = \frac{1}{N} \sum_{\substack{j_1, \dots, j_k=1 \\ \text{distinct}}}^N \sum_{n_1, \dots, n_{k-1} \in \mathbb{Z}} \Delta\left(\frac{\alpha a_{j_1} - \alpha a_{j_k} + n_1}{L/N}, \dots, \frac{\alpha a_{j_{k-1}} - \alpha a_{j_k} + n_{k-1}}{L/N}\right); \tag{1.5}$$

Proposition 4.1, which is the main ingredient in the proof of Theorem 1.3, is notably stronger than long-range Poissonian higher correlations in the sense of $R_N^k(L, \alpha, \Delta) = L^k + o(L^k)$, which would be insufficient for concluding Theorem 1.3.

2. The number variance

By the Poisson summation formula, we have the following identity for the pair correlation function (1.3)

$$R_N^2(L, \alpha, \Delta) = L - \frac{L}{N} + T_N(L, \alpha), \tag{2.1}$$

where

$$T_N(L, \alpha) = \frac{L}{N^2} \sum_{i \neq j=1}^N \sum_{0 \neq n \in \mathbb{Z}} \widehat{\Delta}\left(\frac{nL}{N}\right) e(n\alpha (a_i - a_j))$$

with

$$\widehat{\Delta}(x) = \frac{\sin^2(\pi x)}{\pi^2 x^2}$$

(we have used the standard notation $e(z) = e^{2\pi iz}$).

We fix a smooth, compactly supported, non-negative weight function $\rho \in C_c^\infty(\mathbb{R})$, $\rho \geq 0$, and denote the weighted L^2 -norm of $T_n(L, \alpha)$ by

$$V_N(L) = \int |T_N(L, \alpha)|^2 \rho(\alpha) d\alpha = \frac{L^2}{N^4} \sum_{\substack{0 \neq n_1 \in \mathbb{Z} \\ 0 \neq n_2 \in \mathbb{Z}}} \widehat{\Delta}\left(\frac{n_1 L}{N}\right) \widehat{\Delta}\left(\frac{n_2 L}{N}\right) w(n_1, n_2, N),$$

where

$$w(n_1, n_2, N) = \sum_{\substack{1 \leq x_1 \neq y_1 \leq N \\ 1 \leq x_2 \neq y_2 \leq N}} \widehat{\rho}(n_1(a_{x_1} - a_{y_1}) - n_2(a_{x_2} - a_{y_2}));$$

the aim of the rest of this section is to give an upper bound for $V_N(L)$.

We first observe the following identity which will be useful for estimating sums involving $\widehat{\Delta}$.

LEMMA 2.1. *Let $1 \leq L < N$. We have*

$$\sum_{n \in \mathbb{Z}} \widehat{\Delta}\left(\frac{nL}{N}\right) = \frac{N}{L}.$$

Proof. Let $\Delta_{N/L}(x) = \Delta\left(\frac{N}{L}x\right)$. By the Poisson summation formula we have

$$\sum_{n \in \mathbb{Z}} \widehat{\Delta}\left(\frac{nL}{N}\right) = \frac{N}{L} \sum_{n \in \mathbb{Z}} \widehat{\Delta_{N/L}}(n) = \frac{N}{L} \sum_{n \in \mathbb{Z}} \Delta_{N/L}(n) = \frac{N}{L} \Delta(0) = \frac{N}{L}.$$

In the next lemma, we will see that up to an error term of order $O(N^{-1})$, the ranges of the summations defining $V_N(L)$ can be significantly restricted.

LEMMA 2.2. *Let $1 \leq L < N$ and let $\epsilon > 0$. We have*

$$V_N(L) = \frac{L^2}{N^4} \sum_{\substack{0 \neq |n_1| \leq N^4 \\ 0 \neq |n_2| \leq N^4}} \widehat{\Delta}\left(\frac{n_1L}{N}\right) \widehat{\Delta}\left(\frac{n_2L}{N}\right) \tilde{w}(n_1, n_2, N) + O(N^{-1}),$$

where

$$\tilde{w}(n_1, n_2, N) = \sum_{\substack{1 \leq x_1 \neq y_1 \leq N \\ 1 \leq x_2 \neq y_2 \leq N \\ \max\{x_1, x_2, y_1, y_2\} > N^{1/4} \\ |n_1(a_{x_1} - a_{y_1}) - n_2(a_{x_2} - a_{y_2})| \leq N^\epsilon}} \widehat{\rho}(n_1(a_{x_1} - a_{y_1}) - n_2(a_{x_2} - a_{y_2})).$$

Proof. We have

$$\begin{aligned} w(n_1, n_2, N) - \tilde{w}(n_1, n_2, N) &\ll \sum_{\substack{x_1 \neq y_1 \geq 1 \\ x_2 \neq y_2 \geq 1 \\ \max\{x_1, x_2, y_1, y_2\} \leq N^{1/4}}} |\widehat{\rho}(n_1(a_{x_1} - a_{y_1}) - n_2(a_{x_2} - a_{y_2}))| \\ &+ \sum_{\substack{1 \leq x_1 \neq y_1 \leq N \\ 1 \leq x_2 \neq y_2 \leq N \\ |n_1(a_{x_1} - a_{y_1}) - n_2(a_{x_2} - a_{y_2})| > N^\epsilon}} |\widehat{\rho}(n_1(a_{x_1} - a_{y_1}) - n_2(a_{x_2} - a_{y_2}))| \ll N, \end{aligned}$$

where we bounded the first summation using the bound $\widehat{\rho} \ll 1$ and the second summation using $\widehat{\rho}(x) \ll x^{-k}$ for all $k > 0$. Thus,

$$\frac{L^2}{N^4} \sum_{\substack{0 \neq n_1 \in \mathbb{Z} \\ 0 \neq n_2 \in \mathbb{Z}}} \widehat{\Delta}\left(\frac{n_1 L}{N}\right) \widehat{\Delta}\left(\frac{n_2 L}{N}\right) (w(n_1, n_2, N) - \tilde{w}(n_1, n_2, N)) \ll \frac{L^2}{N^3} \left(\sum_{n \in \mathbb{Z}} \widehat{\Delta}\left(\frac{nL}{N}\right) \right)^2 = \frac{1}{N},$$

where in the last equality we used Lemma 2.1. Finally, by bounding \tilde{w} trivially and applying the bound $\widehat{\Delta}(x) \ll x^{-2}$, we have

$$\begin{aligned} \frac{L^2}{N^4} \sum_{\substack{0 \neq n_1 \in \mathbb{Z} \\ 0 \neq n_2 \in \mathbb{Z} \\ \max\{|n_1|, |n_2|\} > N^4}} \widehat{\Delta}\left(\frac{n_1 L}{N}\right) \widehat{\Delta}\left(\frac{n_2 L}{N}\right) \tilde{w}(n_1, n_2, N) &\ll L^2 \sum_{m > N^4} \widehat{\Delta}\left(\frac{mL}{N}\right) \sum_{n \in \mathbb{Z}} \widehat{\Delta}\left(\frac{nL}{N}\right) \\ &= NL \sum_{m > N^4} \widehat{\Delta}\left(\frac{mL}{N}\right) \ll \frac{N^3}{L} \sum_{m > N^4} m^{-2} \ll \frac{1}{LN} \end{aligned}$$

which concludes the proof.

We will now analyse when the summation defining \tilde{w} does not vanish.

PROPOSITION 2.3. *Fix n_1 such that $0 < |n_1| \leq N^4$, and x_1, y_1 such that $1 \leq y_1 < x_1 \leq N$, $x_1 > N^{1/4}$. Then there exist at most $O(N^\epsilon \log N)$ values of n_2, x_2, y_2 such that $0 < |n_2| \leq N^4$, $x_2 \leq x_1$, $1 \leq y_2 < x_2 \leq N$, and*

$$|n_1(a_{x_1} - a_{y_1}) - n_2(a_{x_2} - a_{y_2})| \leq N^\epsilon. \tag{2.2}$$

Proof. We follow the ideas of [14, 15]. We have

$$|n_1| (a_{x_1} - a_{y_1}) \geq a_{x_1} - a_{x_1-1} = a_{x_1} \left(1 - \frac{a_{x_1-1}}{a_{x_1}}\right) \geq \left(1 - \frac{1}{C}\right) a_{x_1}; \tag{2.3}$$

on the other hand,

$$|n_2| (a_{x_2} - a_{y_2}) \leq N^4 a_{x_2} = N^4 a_{x_1} \frac{a_{x_2}}{a_{x_1}} \leq a_{x_1} \frac{N^4}{C^{x_1-x_2}}. \tag{2.4}$$

Applying the reverse triangle inequality to (2.2), we have

$$|n_1| (a_{x_1} - a_{y_1}) - |n_2| (a_{x_2} - a_{y_2}) \leq |n_1(a_{x_1} - a_{y_1}) - n_2(a_{x_2} - a_{y_2})| \leq N^\epsilon;$$

substituting the estimates (2.3) and (2.4), we obtain

$$1 - \frac{1}{C} - \frac{N^4}{C^{x_1-x_2}} \leq N^\epsilon a_{x_1}^{-1}.$$

Since $x_1 > N^{1/4}$, we have $a_{x_1} \geq a_1 C^{x_1-1} > a_1 C^{N^{1/4}-1}$, and hence

$$1 - \frac{1}{C} - \frac{N^4}{C^{x_1-x_2}} \leq N^\epsilon a_1^{-1} C^{-(N^{1/4}-1)},$$

and therefore for sufficiently large N we have

$$C^{x_1-x_2} \leq \frac{N^4}{1 - \frac{1}{C} - N^\epsilon a_1^{-1} C^{-(N^{1/4}-1)}} \ll N^4$$

so that $x_1 - x_2 \ll \log N$. Thus, there are at most $O(\log N)$ possible values for x_2 , and moreover we conclude that $x_2 \gg N^{1/4}$, and hence $a_{x_2} \gg C^{N^{1/4}}$.

We now fix x_2 . Since

$$\left| n_2 - n_1 \frac{a_{x_1} - a_{y_1}}{a_{x_2} - a_{y_2}} \right| \leq \frac{N^\epsilon}{a_{x_2} - a_{y_2}} \leq \frac{N^\epsilon}{\left(1 - \frac{1}{C}\right) a_{x_2}} \ll \frac{N^\epsilon}{C^{N^{1/4}}},$$

we see that for sufficiently large N , the integer n_2 is uniquely determined by the values of x_1, x_2, y_1, y_2, n_1 . It is therefore sufficient to bound the number of possible values of y_2 . There are $O(\log N)$ values of y_2 such that $x_2 - y_2 \leq 5 \log_C N$. We will therefore count the number of possible values of y_2 such that $x_2 - y_2 > 5 \log_C N$. For such y_2 we have

$$a_{y_2} = a_{x_2} \frac{a_{y_2}}{a_{x_2}} \leq \frac{a_{x_2}}{C^{x_2-y_2}} < \frac{a_{x_2}}{N^5}$$

and therefore

$$\begin{aligned} n_1(a_{x_1} - a_{y_1}) &= n_2(a_{x_2} - a_{y_2}) + O(N^\epsilon) = n_2 a_{x_2} \left(1 - \frac{a_{y_2}}{a_{x_2}} + O\left(\frac{N^\epsilon}{C^{N^{1/4}}}\right)\right) \\ &= n_2 a_{x_2} \left(1 + O\left(N^{-5}\right)\right). \end{aligned}$$

Hence, given (y_2, n_2) and (y'_2, n'_2) such that $x_2 - y_2 > 5 \log_C N$ and $x_2 - y'_2 > 5 \log_C N$, we have

$$n_2 a_{x_2} \left(1 + O\left(N^{-5}\right)\right) = n'_2 a_{x_2} \left(1 + O\left(N^{-5}\right)\right)$$

and since $|n_2| \leq N^4$ we conclude that

$$n'_2 = n_2 + O\left(N^{-1}\right)$$

so in fact $n'_2 = n_2$. We therefore see that the value of n_2 is identical for each y_2 such that $x_2 - y_2 > 5 \log_C N$. But, for such y_2 , (2.2) gives

$$a_{y_2} \in \left[a_{x_2} - \frac{n_1(a_{x_1} - a_{y_1})}{n_2} - \frac{N^\epsilon}{n_2}, a_{x_2} - \frac{n_1(a_{x_1} - a_{y_1})}{n_2} + \frac{N^\epsilon}{n_2} \right]$$

so that a_{y_2} lies in an interval of length $O(N^\epsilon)$, and since

$$a_{n+1} - a_n = a_{n+1} \left(1 - \frac{a_n}{a_{n+1}}\right) \geq a_{n+1} \left(1 - \frac{1}{C}\right) \gg 1$$

there could be at most $O(N^\epsilon)$ values of y_2 in this interval.

As an immediate corollary of Lemma 2.2 and Proposition 2.3, we obtain an upper bound for $V_N(L)$.

COROLLARY 2.4. *Let $1 \leq L < N$ and let $\epsilon < 0$. We have*

$$V_N(L) = O\left(LN^{-1+\epsilon}\right). \tag{2.5}$$

Proof. We use the bound $\widehat{\rho} \ll 1$ and Lemma 2.2 to conclude that

$$V_N(L) \ll \frac{L^2}{N^4} \sum_{\substack{0 \neq |n_1| \leq N^4 \\ 1 \leq x_1 \neq y_1 \leq N}} \widehat{\Delta}\left(\frac{n_1 L}{N}\right) \sum_{\substack{0 \neq |n_2| \leq N^4 \\ 1 \leq x_2 \neq y_2 \leq N \\ \max\{x_1, x_2, y_1, y_2\} > N^{1/4} \\ |n_1(a_{x_1} - a_{y_1}) - n_2(a_{x_2} - a_{y_2})| \leq N^{\epsilon/2}}} \widehat{\Delta}\left(\frac{n_2 L}{N}\right) + N^{-1}.$$

By symmetry we can assume that $y_1 < x_1$, $x_2 \leq x_1$ and $y_2 < x_2$, so that by the bound $\widehat{\Delta} \ll 1$ and by Proposition 2.3, the inner summation is $O(N^{\epsilon/2} \log N)$. Hence, Lemma 2.1 gives the required bound (2.5).

3. Proofs of Theorems 1.1 and 1.2

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By (1.2) and (2.1) we have

$$\frac{\Sigma_N^2(L, \alpha) - L}{L} = T_N(L, \alpha) - \frac{L}{N}.$$

Hence, for sufficiently large N we have

$$\begin{aligned} \text{meas}\left\{\alpha \in I : \left|\Sigma_N^2(L, \alpha) - L\right| > \delta L\right\} &= \text{meas}\left\{\alpha \in I : \left|T_N(L, \alpha) - \frac{L}{N}\right| > \delta\right\} \\ &\leq \text{meas}\{\alpha \in I : |T_N(L, \alpha)| > \delta/2\}. \end{aligned}$$

Denote by χ_I the characteristic function of the interval I , and fix a smooth, compactly supported weight function $\rho \in C_c^\infty(\mathbb{R})$ such that $\chi_I(x) \leq \rho(x)$ for all $x \in \mathbb{R}$. By Chebyshev’s inequality we conclude that for sufficiently large N we have

$$\begin{aligned} \text{meas}\left\{\alpha \in I : \left|\Sigma_N^2(L, \alpha) - L\right| > \delta L\right\} &\leq \frac{4 \int_I |T_N(L, \alpha)|^2 d\alpha}{\delta^2} \leq \frac{4V_N(L)}{\delta^2} \\ &\ll_{\delta, \epsilon, I} LN^{-1+\epsilon/2} \ll N^{-\epsilon/2} \end{aligned} \tag{3.1}$$

where we used (2.5) with $\epsilon/2$.

We now turn to the proof of Theorem 1.2, that is, we will show that (1.1) (or equivalently (1.4)) holds for almost all $\alpha \in \mathbb{R}$. It is sufficient to prove this for $\alpha \in I$ where I is a bounded interval. We first show that almost sure convergence of the pair correlation holds along a subsequence.

LEMMA 3.1. *Let I be a bounded interval, and let $\epsilon > 0$. Assume that $L = L(N) = O(N^{1/2-\epsilon})$ as $N \rightarrow \infty$. Let $N_m = m^2$, and denote $L_m = L(N_m)$. Then for almost all $\alpha \in I$, we have*

$$R_{N_m}^2(L_m, \alpha, \Delta) = L_m + o(1) \tag{3.2}$$

as $m \rightarrow \infty$.

Proof. Applying (2.5) as in (3.1), for every $\delta > 0$ and N sufficiently large we have

$$\text{meas}\left\{\alpha \in I : \left|R_N^2(L, \alpha, \Delta) - L\right| > \delta\right\} \leq \text{meas}\{\alpha \in I : |T_N(L, \alpha)| > \delta/2\} \\ \ll_{\delta, \epsilon, I} LN^{-1+\epsilon/2} \ll N^{-1/2-\epsilon/2}.$$

so that

$$\text{meas}\left\{\alpha \in I : \left|R_{N_m}^2(L_m, \alpha, \Delta) - L_m\right| > \delta\right\} \ll_{\delta, \epsilon, I} m^{-1-\epsilon}; \tag{3.3}$$

the asymptotic (3.2) thus holds for almost all $\alpha \in I$ by the Borel–Cantelli lemma.

We now have all that is needed to prove Theorem 1.2.

Proof of Theorem 1.2. It is sufficient to show that

$$R_N^2(L, \alpha, \Delta) = L + o(1) \tag{3.4}$$

as $N \rightarrow \infty$ for almost all $\alpha \in I$. Let $N_m = m^2$. For any N there exists m such that $N_{m-1} \leq N < N_m$. Moreover, $N_m/N = 1 + O(m^{-1})$, and by the assumption

$$L(N+1) - L(N) = o(N^{-1/2})$$

we have

$$L(N) = L_m + o\left(\frac{N - N_m}{\sqrt{N}}\right) = L_m + o(1);$$

thus, there exists a constant $C > 0$ such that for any $\delta > 0$, for sufficiently large N we have

$$R_N^2(L, \alpha, \Delta) \leq \frac{N_m}{N} R_{N_m}^2\left(L \cdot \frac{N_m}{N}, \alpha, \Delta\right) \\ \leq \left(1 + Cm^{-1}\right) R_{N_m}^2\left((L_m + \delta) \cdot \left(1 + Cm^{-1}\right), \alpha, \Delta\right);$$

by applying Lemma 3.1 with $(L + \delta) \cdot (1 + CN^{-1/2})$ instead of L (and therefore with $(L_m + \delta) \cdot (1 + Cm^{-1})$ instead of L_m), as $m \rightarrow \infty$ we have

$$R_{N_m}^2\left((L_m + \delta) \cdot \left(1 + Cm^{-1}\right), \alpha, \Delta\right) = (L_m + \delta) \cdot \left(1 + Cm^{-1}\right) + o(1) \\ = L_m + \delta + o(1)$$

for all $\alpha \in I_\delta$, where I_δ is a full measure set in I (note that $L_m = o(m)$ by the assumption $L = O(N^{1/2-\epsilon})$). Hence, for sufficiently large N we have

$$R_N^2(L, \alpha, \Delta) \leq \left(1 + Cm^{-1}\right) (L_m + \delta + o(1)) = L_m + \delta + o(1) = L + \delta + o(1) \tag{3.5}$$

for all $\alpha \in I_\delta$. Symmetrically,

$$R_N^2(L, \alpha, \Delta) \geq L - \delta - o(1) \tag{3.6}$$

for α in a full measure set in I (depending on δ); since $\delta > 0$ can be taken arbitrarily small along a countable sequence of values, and a countable intersection of full measure sets is still of full measure, the bounds (3.5) and (3.6) imply (3.4) for almost all $\alpha \in I$.

Remark. The faster L grows, the sparser the subsequence N_m one has to take in order to apply the Borel–Cantelli lemma in the proof of Lemma 3.1. On the other hand, since we require the condition $L_m = o(m)$, the subsequence N_m cannot be too sparse. For example, if $L = N^s$, and $N_m = \lfloor m^t \rfloor$, one needs $t > 1/(1 - s)$ for (3.3) to hold, but also $t < 1/s$, so that $s < 1/2$. This explains why the above argument only works for L growing slower than $N^{1/2}$.

4. Higher order correlations – proof of Theorem 1.3

Taking expectations w.r.t x , for all $k \geq 2$, we have

$$\begin{aligned} \mathbb{E} \left[(S_N(L, \alpha))^k \right] &= \sum_{j_1, \dots, j_k=1}^N \sum_{n_1, \dots, n_k \in \mathbb{Z}} \int_0^1 \prod_{i=1}^k \chi \left(\frac{\alpha a_{j_i} - x + n_i}{L/N} \right) dx \\ &= \sum_{j_1, \dots, j_k=1}^N \sum_{n_1, \dots, n_{k-1} \in \mathbb{Z}} \int_{\mathbb{R}} \prod_{i=1}^{k-1} \chi \left(\frac{\alpha a_{j_i} - x + n_i}{L/N} \right) \chi \left(\frac{\alpha a_{j_k} - x}{L/N} \right) dx \\ &= \frac{L}{N} \sum_{j_1, \dots, j_k=1}^N \sum_{n_1, \dots, n_{k-1} \in \mathbb{Z}} \Delta \left(\frac{\alpha a_{j_1} - \alpha a_{j_k} + n_1}{L/N}, \dots, \frac{\alpha a_{j_{k-1}} - \alpha a_{j_k} + n_{k-1}}{L/N} \right), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \Delta(t_1, \dots, t_{k-1}) &= \int_{\mathbb{R}} \prod_{i=1}^{k-1} \chi(t_i - x) \chi(x) dx \\ &= \max\{1 - \max\{0, t_1, \dots, t_{k-1}\} + \min\{0, t_1, \dots, t_{k-1}\}, 0\}; \end{aligned}$$

we have (see [3, lemma 13])

$$\int_{\mathbb{R}^{k-1}} \Delta(t_1, \dots, t_{k-1}) dt_1 \dots dt_{k-1} = 1. \tag{4.2}$$

For $0 \leq j \leq k$, denote by $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}$ the Stirling number of the second kind, i.e., the number of ways to partition a set of k elements into j non-empty subsets. We partition the sum over j_1, \dots, j_k on the right-hand side of (4.1) into sums with j distinct indices. The term corresponding to $j = 1$ is clearly equal to L . Recalling the definition (1.5) of the j -level correlation functions $R_N^j(L, \alpha, \Delta)$, we then have

$$\mathbb{E} \left[(S_N(L, \alpha))^k \right] = L + L \sum_{j=2}^k \left\{ \begin{matrix} k \\ j \end{matrix} \right\} R_N^j(L, \alpha, \Delta).$$

In view of Lemma A.1, Theorem 1.3 will be a direct consequence of the following proposition.

PROPOSITION 4.1. Let $L = L(N)$ such that for all $\eta > 0$ we have $L = O(N^\eta)$, and assume that there exists $\epsilon > 0$ such that $L(N + 1) - L(N) = O(N^{-\epsilon})$. Then for almost all $\alpha \in \mathbb{R}$, we have

$$R_N^j(L, \alpha, \Delta) = L^{j-1} + O(L^{-s}) \tag{4.3}$$

for all $j \geq 2$ and all $s > 0$.

We apply the following strategy for proving Proposition 4.1. We first prove an analogous result with a smooth test function along a subsequence. We then unsmooth along the subsequence, and finally deduce the result along the full sequence. We would like to use the results of [2], and for that it would be more convenient to work with a “transformed” correlation function: for $k \geq 2$ and for a smooth, compactly supported function $\psi : \mathbb{R}^{k-1} \rightarrow \mathbb{R}$ (which may depend on N), we denote the smoothed k -level correlation function

$$R_N^k(L, \alpha, \psi) = \frac{1}{N} \sum_{\substack{j_1, \dots, j_k=1 \\ \text{distinct}}}^N \sum_{n_1, \dots, n_{k-1} \in \mathbb{Z}} \psi \left(\frac{\alpha a_{j_1} - \alpha a_{j_k} + n_1}{L/N}, \dots, \frac{\alpha a_{j_{k-1}} - \alpha a_{j_k} + n_{k-1}}{L/N} \right)$$

and the transformed smoothed k -level correlation function

$$\tilde{R}_N^k(\alpha, \psi) = \frac{1}{N} \sum_{\substack{j_1, \dots, j_k=1 \\ \text{distinct}}}^N \sum_{n_1, \dots, n_{k-1} \in \mathbb{Z}} \psi(N(\alpha a_{j_1} - \alpha a_{j_2} + n_1), \dots, N(\alpha a_{j_{k-1}} - \alpha a_{j_k} + n_{k-1})).$$

Then

$$R_N^k(L, \alpha, \psi) = \tilde{R}_N^k(\alpha, \tilde{\psi}_L), \tag{4.4}$$

where

$$\tilde{\psi}_L(t_1, \dots, t_{k-1}) = \psi \left(\frac{t_1 + \dots + t_{k-1}}{L}, \frac{t_2 + \dots + t_{k-1}}{L}, \dots, \frac{t_{k-1}}{L} \right).$$

For the transformed correlation function we have the following L^2 -norm estimate: let I be a bounded interval, and let

$$V(\tilde{R}_N^k(\psi)) = \int_I (\tilde{R}_N^k(\alpha, \psi) - C_k(N) \widehat{\psi}(0))^2 d\alpha,$$

where

$$C_k(N) = \left(1 - \frac{1}{N}\right) \dots \left(1 - \frac{k-1}{N}\right).$$

LEMMA 4.2. Let $k \geq 2$. For each $\eta > 0$ there exists $r = r(\eta)$ such that

$$V(\tilde{R}_N^k(\psi)) = O(\|\psi\|_{r,1}^2 N^{-1+\eta}) \tag{4.5}$$

where $\|\psi\|_{r,1} = \sum_{|\alpha| \leq r} \|\partial^\alpha \psi\|_1$.

Proof. For a smooth, compactly supported, non-negative weight function $\rho \in C_c^\infty(\mathbb{R})$, $\rho \geq 0$, denote

$$V(\tilde{R}_N^k(\psi), \rho) = \int_{\mathbb{R}} \left(\tilde{R}_N^k(\alpha, \psi) - C_k(N) \widehat{\psi}(0) \right)^2 \rho(\alpha) d\alpha.$$

Then proposition 7 in [2] implies that for each $\eta > 0$ there exists $r = r(\eta)$ such that

$$V(\tilde{R}_N^k(\psi), \rho) = O\left(\|\psi\|_{r,1}^2 N^{-1+\eta}\right); \tag{4.6}$$

while the term $\|\psi\|_{r,1}^2$ is not explicitly stated there, it follows from the proof, which we now sketch (for the full details we refer the reader to [2]): for $x = (x_1, \dots, x_k)$, denote

$$\Delta_{(a_n)}(x) = (a_{x_1} - a_{x_2}, \dots, a_{x_{k-1}} - a_{x_k});$$

by the Poisson summation formula, we have

$$\tilde{R}_N^k(\alpha, \psi) = C_k(N) \widehat{\psi}(0) + \frac{1}{N^k} \sum_{0 \neq n \in \mathbb{Z}^{k-1}} \widehat{\psi}\left(\frac{n}{N}\right) \sum_{\substack{x=(x_1, \dots, x_k) \\ 1 \leq x_1, \dots, x_k \leq N \text{ distinct}}} e(\alpha n \cdot \Delta_{(a_n)}(x)),$$

and hence

$$V(\tilde{R}_N^k(\psi), \rho) = \frac{1}{N^{2k}} \sum_{0 \neq n, m \in \mathbb{Z}^{k-1}} \widehat{\psi}\left(\frac{n}{N}\right) \overline{\widehat{\psi}\left(\frac{m}{N}\right)} \sum^* \widehat{\rho}(n \cdot \Delta_{(a_n)}(x) - m \cdot \Delta_{(a_n)}(y)),$$

where the range of the summation \sum^* is over $x = (x_1, \dots, x_k)$ where $1 \leq x_1, \dots, x_k \leq N$ are distinct, and $y = (y_1, \dots, y_k)$ where $1 \leq y_1, \dots, y_k \leq N$ are distinct. Fix $\epsilon > 0$; by splitting the summation over n, m into different ranges and using the bounds $\widehat{\rho} \ll 1$, $|\widehat{\psi}| \leq \|\psi\|_1 \leq \|\psi\|_{r,1}$ and $|\widehat{\psi}| \ll \|\psi\|_{r,1} \|x\|_\infty^{-r}$ (for arbitrarily large r), we obtain

$$\begin{aligned} V(\tilde{R}_N^k(\psi), \rho) &\ll \|\psi\|_{r,1}^2 \left(\frac{1}{N^{2k}} \sum_{\|n\|_\infty, \|m\|_\infty > N^{1+\epsilon}} \left\| \frac{n}{N} \right\|_\infty^{-r} \left\| \frac{m}{N} \right\|_\infty^{-r} \sum^* 1 \right. \\ &\quad + \frac{1}{N^{2k}} \sum_{\|n\|_\infty > N^{1+\epsilon}, 0 < \|m\|_\infty \leq N^{1+\epsilon}} \left\| \frac{n}{N} \right\|_\infty^{-r} \sum^* 1 \\ &\quad \left. + \frac{1}{N^{2k}} \sum_{0 < \|n\|_\infty, \|m\|_\infty \leq N^{1+\epsilon}} \sum^* |\widehat{\rho}(n \cdot \Delta_{(a_n)}(x) - m \cdot \Delta_{(a_n)}(y))| \right). \end{aligned}$$

The contribution of the first two terms is negligible by a trivial estimate, and so is the contribution of the third term restricted to the range $|n \cdot \Delta_{(a_n)}(x) - m \cdot \Delta_{(a_n)}(y)| > N^\epsilon$ (choosing r sufficiently large depending on ϵ). The rest of the contribution from the third term is then bounded by [2, proposition 2] which states that there are at most $O(N^{2k-1+4k\epsilon})$ values of n, m, x, y in the above ranges such that $|n \cdot \Delta_{(a_n)}(x) - m \cdot \Delta_{(a_n)}(y)| \leq N^\epsilon$, which gives (4.6).

Finally, if we choose ρ such that $\rho \geq \chi_I$, then

$$V(\tilde{R}_N^k(\psi)) \leq V(\tilde{R}_N^k(\psi), \rho),$$

and (4.5) follows.

Fix $\eta > 0$, and let $r = r(\eta) > 1$ be as in Lemma 4.2; let $0 < \epsilon < 1$, $\delta = N^{-\frac{\epsilon}{8r}}$, and assume that $\psi \in C_c^\infty(\mathbb{R}^{k-1})$ is a smooth approximation to Δ , such that $\|\Delta - \psi\|_\infty \ll \delta$ and such that $\|\psi\|_{r,1} \ll \delta^{-r}$. By (4.2), if L grows slower than any power of N , then

$$\widehat{\psi}_L(0) = L^{k-1} \widehat{\psi}(0) = L^{k-1} + O(\delta L^{k-1}) = L^{k-1} + O(N^{-\frac{\epsilon}{8r} + \eta}).$$

Moreover, we have

$$\|\tilde{\psi}_L\|_{r,1}^2 \ll L^{2(k-1)} \|\psi\|_{r,1}^2 \ll L^{2(k-1)} \delta^{-2r} \ll N^{\frac{\epsilon}{4} + \eta}.$$

We deduce almost sure convergence along a subsequence.

LEMMA 4.3. *Let $L = L(N)$ be such that for all $\eta > 0$ we have $L = O(N^\eta)$ and let $k \geq 2$. Let $0 < \epsilon < 1$, $N_m = \lfloor m^{1+\epsilon} \rfloor$, and denote $L_m = L(N_m)$. Then for almost all $\alpha \in I$, we have*

$$R_{N_m}^k(L_m, \alpha, \Delta) = L_m^{k-1} + O(L_m^{-s}) \tag{4.7}$$

for all $s > 0$, as $m \rightarrow \infty$,

Proof. It is sufficient to show that for any fixed $s > 0$, (4.7) holds for almost all $\alpha \in I$. By identity (4.4), Lemma 4.2, and the upper bound on L , for each $\eta > 0$ there exists $r = r(\eta)$ such that

$$\begin{aligned} & \int_I \left(L_m^s \left(R_{N_m}^k(L_m, \alpha, \psi) - C_k(N_m) \widehat{\psi}_{L_m}(0) \right) \right)^2 d\alpha = L_m^{2s} V \left(\tilde{R}_{N_m}^k(\tilde{\psi}_{L_m}) \right) \\ & = O \left(\|\tilde{\psi}_{L_m}\|_{r,1}^2 L_m^{2s} N_m^{-1+\eta} \right) = O \left(N_m^{-1+\frac{\epsilon}{4}+3\eta} \right) = O \left(m^{(1+\epsilon)(-1+\frac{\epsilon}{4}+3\eta)} \right). \end{aligned}$$

Hence, by the Borel–Cantelli lemma, for η sufficiently small we have

$$L_m^s \left(R_{N_m}^k(L_m, \alpha, \psi) - C_k(N_m) \widehat{\psi}_{L_m}(0) \right) = o(1)$$

for almost all $\alpha \in I$, and in particular

$$R_{N_m}^k(L_m, \alpha, \psi) = C_k(N_m) \widehat{\psi}_{L_m}(0) + O(L_m^{-d}) = L_m^{k-1} + O(L_m^{-s}), \tag{4.8}$$

where we used again the upper bound on L .

Let $\psi = \psi_\pm$ be approximations to Δ satisfying the above assumptions such that $\psi_- \leq \Delta \leq \psi_+$; a simple way to construct such approximations is to convolve the functions

$$\Delta_\delta^\pm(t_1, \dots, t_{k-1}) := \max\{1 \pm \delta - \max\{0, t_1, \dots, t_{k-1}\} + \min\{0, t_1, \dots, t_{k-1}\}, 0\}$$

with $\varphi_{\delta/10}(t)$, where $\varphi_\epsilon(t) = \epsilon^{-(k-1)} \varphi(t/\epsilon)$, and $\varphi \in C_c^\infty(\mathbb{R}^{k-1})$ is the standard mollifier. We then have

$$R_{N_m}^k(L_m, \alpha, \psi^-) \leq R_{N_m}^k(L_m, \alpha, \Delta) \leq R_{N_m}^k(L_m, \alpha, \psi^+);$$

substituting the asymptotics (4.8), we conclude that (4.7) holds for almost all $\alpha \in I$.

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. The argument is similar to that of the proof of Theorem 1.2. Let $k \geq 2$; it is enough to show that for almost all $\alpha \in I$, we have

$$R_N^k(L, \alpha, \Delta) = L^{k-1} + O(L^{-s})$$

for all $s > 0$. Let $N_m = \lfloor m^{1+\epsilon/2} \rfloor$, so that for any N there exists m such that $N_{m-1} \leq N < N_m$. Moreover, $N_m/N = 1 + O(m^{-1})$, and by the assumption

$$L(N + 1) - L(N) = O(N^{-\epsilon})$$

we have

$$L = L_m + O\left(\frac{N - N_m}{N^\epsilon}\right) = L_m + O(N_m^{-\epsilon/2});$$

hence, there exists a constant $C > 0$ such that for sufficiently large N we have

$$\begin{aligned} R_N^k(L, \alpha, \Delta) &\leq \frac{N_m}{N} R_{N_m}^k\left(L \cdot \frac{N_m}{N}, \alpha, \Delta\right) \\ &\leq \left(1 + Cm^{-1}\right) R_{N_m}^k\left(\left(L_m + CN_m^{-\epsilon/2}\right) \cdot \left(1 + CN_m^{-\frac{1}{1+\epsilon/2}}\right), \alpha, \Delta\right); \end{aligned}$$

by the upper bound on L and Lemma 4.3 with $(L + CN^{-\epsilon/2}) \cdot (1 + CN^{-\frac{1}{1+\epsilon/2}})$ instead of L , for almost all $\alpha \in I$ we have

$$\begin{aligned} &R_{N_m}^k\left(\left(L_m + CN_m^{-\epsilon/2}\right) \cdot \left(1 + CN_m^{-\frac{1}{1+\epsilon/2}}\right), \alpha, \Delta\right) \\ &= \left(L_m + CN_m^{-\epsilon/2}\right)^{k-1} \cdot \left(1 + CN_m^{-\frac{1}{1+\epsilon/2}}\right)^{k-1} + O(L_m^{-s}) = L_m^{k-1} + O(L_m^{-s}) \end{aligned}$$

for all $s > 0$. Thus, for sufficiently large N we have (using again the upper bound on L), for almost all $\alpha \in I$ we have

$$R_N^k(L, \alpha, \Delta) \leq \left(1 + Cm^{-1}\right) \left(L_m^{k-1} + O(L_m^{-s})\right) = L_m^{k-1} + O(L_m^{-s}) = L^{k-1} + O(L^{-s}) \tag{4.9}$$

for all $s > 0$. Similarly, for almost all $\alpha \in I$ we have

$$R_N^k(L, \alpha, \Delta) \geq L^{k-1} - O(L^{-s}) \tag{4.10}$$

for all $s > 0$; the bounds (4.9) and (4.10) give (4.3).

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Appendix A. Normal approximation to the Poisson distribution

We require a normal approximation to a random variable whose moments are close to the Poisson moments. Denote

$$\mu_k^{Poisson}(L) = \sum_{j=0}^k \binom{k}{j} L^j$$

the k th moment of a Poisson-distributed random variable with parameter L , and

$$\mu_k^{normal} = \begin{cases} 0 & k \text{ odd} \\ (k-1)!! & k \text{ even} \end{cases}$$

the k th moment of a standard Gaussian random variable.

LEMMA A.1. Let $L = L(N) \rightarrow \infty$ as $N \rightarrow \infty$, and let $(X_N)_{N=1}^\infty$ be a sequence of random variables such that for all $j \geq 1$ and for all $s > 0$ we have

$$\mathbb{E} \left[X_N^j \right] = \mu_j^{Poisson}(L) + O(L^{-s}) \tag{A.1}$$

as $N \rightarrow \infty$. Then

$$\frac{X_N - L}{\sqrt{L}} \xrightarrow{d} \mathcal{N}(0, 1)$$

as $N \rightarrow \infty$, where $\mathcal{N}(0, 1)$ is the standard Gaussian distribution.

Proof. Let $\widehat{X}_N = (X_N - L)/\sqrt{L}$. It is sufficient to prove that for all $k \geq 1$ we have $\lim_{N \rightarrow \infty} \mathbb{E} \left[\widehat{X}_N^k \right] = \mu_k^{normal}$. By (A.1), we have

$$\mathbb{E} \left[\widehat{X}_N^k \right] = L^{-k/2} \sum_{j=0}^k \binom{k}{j} \left(\mu_j^{Poisson}(L) \right)^j (-L)^{k-j} + o(1),$$

so we have to show that for all $k \geq 1$ we have

$$L^{-k/2} \sum_{j=0}^k \binom{k}{j} \left(\mu_j^{Poisson}(L) \right)^j (-L)^{k-j} = \mu_k^{normal} + o(1). \tag{A.2}$$

Let Y_L be a Poisson-distributed random variable with parameter L and $\widehat{Y}_L = (Y_L - L)/\sqrt{L}$; we have to show that for all $k \geq 1$ we have $\lim_{N \rightarrow \infty} \mathbb{E} \left[\widehat{Y}_L^k \right] = \mu_k^{normal}$. Let $M_{\widehat{Y}_L}(t)$ be the moment-generating function of \widehat{Y}_L . Then for any t we have

$$\begin{aligned} M_{\widehat{Y}_L}(t) &= e^{-t\sqrt{L}+L(e^{t/\sqrt{L}}-1)} = e^{-t\sqrt{L}+L(t/\sqrt{L}+t^2/(2L)+O(L^{-3/2}))} \\ &= e^{\frac{t^2}{2}+O(L^{-1/2})} \xrightarrow{N \rightarrow \infty} e^{t^2/2} \end{aligned} \tag{A.3}$$

so that the limit is the moment-generating function of a standard Gaussian random variable. Since the convergence in (A.3) is uniform in a complex neighbourhood of $t = 0$ and all the functions involved are (complex) analytic, convergence of the moments (which can be expressed as the derivatives of the moment-generating function evaluated at zero) easily follows from Cauchy’s integral formula.