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ENUMERATION OF A DUAL SET OF STIRLING PERMUTATIONS BY THEIR ALTERNATING RUNS

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Abstract

In this paper, we count a dual set of Stirling permutations by the number of alternating runs and study properties of the generating functions, including recurrence relations, grammatical interpretations and convolution formulas.

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1. Introduction

Denote by $\left\{ {n \atop k} \right\}$ the *Stirling number of the second kind*, which is the number of ways to partition $[n] = \{1, 2, \dots, n\}$ into k blocks. Let D be the differential operator d/dx and let $\vartheta = xD$. It is clear that $Dx = xD + 1$. A classical result in the theory of normal ordering is the following (see [15]):

$$\vartheta^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^k D^k \quad \text{for } n \geq 1.$$

Let

$$r(x) = \frac{\sqrt{1+x}}{\sqrt{1-x}}.$$

By induction, one can easily verify that

$$\vartheta^n(r(x)) = \frac{\sum_{k=1}^{2n-1} T(n,k)x^k}{(1-x)^n(1+x)^{n-1}\sqrt{1-x^2}} \quad \text{for } n \geq 1,$$

where the $T(n, k)$, $k \in [2n - 1]$, are positive integers. It is clear that the numbers $T(n, k)$ satisfy the initial conditions $T(1, 1) = 1$ and $T(1, k) = 0$ for $k \neq 1$. Let

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$T_n(x) = \sum_{k=1}^{2n-1} T(n,k)x^k$. Using $\vartheta^{n+1}(r(x)) = \vartheta(\vartheta^n(r(x)))$, we see that the polynomials $T_n(x)$ satisfy the recurrence relation

$$T_{n+1}(x) = (2nx+1)xT_n(x) + x(1-x^2)T_n'(x) \tag{1.1}$$

for $n \geq 0$, with the initial values $T_0(x) = 1$. In particular, $T_n(1) = -T_{n+1}(-1) = (2n-1)!!$ for $n \geq 1$. The first few $T_n(x)$ are

$$\begin{aligned}
T_1(x) &= x,\\
T_2(x) &= x + x^2 + x^3,\\
T_3(x) &= x + 3x^2 + 7x^3 + 3x^4 + x^5,\\
T_4(x) &= x + 7x^2 + 29x^3 + 31x^4 + 29x^5 + 7x^6 + x^7.
\end{aligned}$$

Equating the coefficients of x^k on both sides of (1.1), we see that the numbers $T(n,k)$ satisfy the recurrence relation

$$T(n+1,k) = kT(n,k) + T(n,k-1) + (2n-k+2)T(n,k-2). \tag{1.2}$$

The motivating goal of this paper is to find a combinatorial interpretation of the numbers $T(n,k)$.

In [5], Carlitz introduced $C_n(x)$ defined by

$$\sum_{n=0}^{\infty} \left\{ {n+k \atop k} \right\} x^n = \frac{C_n(x)}{(1-x)^{2k+1}}$$

and asked for a combinatorial interpretation of $C_n(x)$. Riordan [16] noted that $C_n(x)$ is the enumerator of trapezoidal words with n elements by number of distinct elements, where trapezoidal words are such that the ith element takes the values $1, 2, \ldots, 2i-1$. Gessel and Stanley [7] gave another combinatorial interpretation of $C_n(x)$ in terms of descents of Stirling permutations. A *Stirling permutation* of order n is a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(2n-1)\sigma(2n)$ of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ such that for each i, $1 \leq i \leq n$, all entries between the two occurrences of i are larger than i. Denote by $\mathcal{Q}_n$ the set of Stirling permutations of order n. For $\sigma \in \mathcal{Q}_n$, let $\sigma(0) = \sigma(2n+1) = 0$ and let

$$\begin{aligned}
\operatorname{des}(\sigma) &= \#\{i \mid \sigma(i) > \sigma(i+1)\},\\
\operatorname{asc}(\sigma) &= \#\{i \mid \sigma(i-1) < \sigma(i)\},\\
\operatorname{plat}(\sigma) &= \#\{i \mid \sigma(i) = \sigma(i+1)\}
\end{aligned}$$

denote the number of descents, ascents and plateaux of σ, respectively. Gessel and Stanley [7] proved that

$$C_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\operatorname{des}\sigma}.$$

Bóna [3, Theorem 1] introduced the plateau statistic on $\mathcal{Q}_n$, and proved that descents, ascents and plateaux are equidistributed over $\mathcal{Q}_n$. The reader is referred to [8, 9, 13, 14] for recent progress on the study of Stirling permutations.

In the next section, we show that $T_n(x)$ is the enumerator of a dual set of Stirling permutations of order n by the number of alternating runs.

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2. Combinatorial interpretation of $T(n, k)$

Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(2n) \in \mathcal{Q}_n$. Let Φ be an injection which maps each first occurrence of entry j in σ to $2j$ and the second j to $2j-1$, where $j \in [n]$. For example, $\Phi(221331) = 432651$. The *dual set* $\Phi(\mathcal{Q}_n)$ of $\mathcal{Q}_n$ is defined by

$$\Phi(\mathcal{Q}_n) = \{\pi \mid \sigma \in \mathcal{Q}_n, \Phi(\sigma) = \pi\}.$$

Clearly, $\Phi(\mathcal{Q}_n)$ is a subset of $\mathfrak{S}_{2n}$. For $\pi \in \Phi(\mathcal{Q}_n)$, the entry $2j$ is to the left of $2j-1$, and the entries in π between $2j$ and $2j-1$ are all larger than $2j$, where $1 \leq j \leq n$. Let ab be an ascent in σ, that is, $a < b$. Using Φ, we see that ab maps into $(2a-1)(2b-1)$, $(2a-1)(2b)$, $(2a)(2b-1)$ or $(2a)(2b)$ and vice versa. Note that $\mathrm{asc}\,(\sigma) = \mathrm{asc}\,(\Phi(\sigma)) = \mathrm{asc}\,(\pi)$. Therefore,

$$C_n(x) = \sum_{\pi \in \Phi(\mathcal{Q}_n)} x^{\mathrm{asc}\,(\pi)}.$$

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$. We say that $\pi \in \mathfrak{S}_n$ changes direction at position i if either $\pi(i-1) < \pi(i) > \pi(i+1)$ or $\pi(i-1) > \pi(i) < \pi(i+1)$, where $i \in \{2, 3, \ldots, n-1\}$. We say that π has k *alternating runs* if there are $k-1$ indices i such that π changes direction at these positions. Denote by $\mathrm{altrun}\,(\pi)$ the number of alternating runs in π. It should be noted that $\pi \in \Phi(\mathcal{Q}_n)$ always ends with a descending run. We now present the following result.

Theorem 2.1. *We have* $T(n, k) = \#\{\pi \in \Phi(\mathcal{Q}_n) \mid \mathrm{altrun}\,(\pi) = k\}$.

Proof. There are three ways in which a permutation $\pi \in \Phi(\mathcal{Q}_{n+1})$ with $\mathrm{altrun}\,(\pi) = k$ can be obtained from a permutation $\sigma \in \Phi(\mathcal{Q}_n)$ by inserting the pair $(2n+2)(2n+1)$ into consecutive positions.

(a) If $\mathrm{altrun}\,(\sigma) = k$, then we can insert the pair $(2n+2)(2n+1)$ right before the beginning of each descending run, and right after the end of each ascending run. This accounts for $kT(n, k)$ possibilities.

(b) If $\mathrm{altrun}\,(\sigma) = k-1$, then we distinguish two cases: when σ starts in an ascending run, we insert the pair $(2n+2)(2n+1)$ to the front of σ; when σ starts in a descending run, we insert the pair $(2n+2)(2n+1)$ right after the first entry of σ. This gives $T(n, k-1)$ possibilities.

(c) If $\mathrm{altrun}\,(\sigma) = k-2$, then we can insert the pair $(2n+2)(2n+1)$ into the remaining $(2n+1) - (k-2) - 1 = 2n-k+2$ positions. This gives $(2n-k+2)T(n, k-2)$ possibilities.

Therefore, the numbers $T(n, k)$ satisfy the recurrence relation (1.2), and this completes the proof. □

A polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ is *symmetric* if $a_k = a_{n-k}$ for all $0 \leq k \leq n$, while it is *unimodal* if there exists an index m such that

$$a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_n.$$

Theorem 2.2. *The polynomial $T_n(x)$ is symmetric and unimodal.*

Proof. It is immediate from (3.1) that $T_n(x)$ is a symmetric polynomial. We show the unimodality by induction on n. Note that $T_1(x) = x, T_2(x) = x + x^2 + x^3$ and $T_3(x) = x + 3x^2 + 7x^3 + 3x^4 + x^5$ are all unimodal. Thus, it suffices to consider the case $n \geq 3$. Assume that $T_n(x)$ is symmetric and unimodal. For $1 \leq k \leq n+1$, it follows from (1.2) that

$$\begin{aligned}&T(n+1,k) - T(n+1,k-1)\\&\quad = (k-1)(T(n,k) - T(n,k-1)) + (T(n,k-1) - T(n,k-2))\\&\qquad + (2n-k+2)(T(n,k-2) - T(n,k-3)) + (T(n,k) - T(n,k-3)) \geq 0,\end{aligned}$$

where the inequalities follow from the induction hypothesis. This completes the proof. □

3. Grammatical interpretations

The grammatical method was introduced by Chen [6] in the study of exponential structures in combinatorics. For an alphabet A, let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in A. A context-free grammar over A is a function $G : A \to \mathbb{Q}[[A]]$ that replaces a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. More precisely, the derivative $D = D_G$: $\mathbb{Q}[[A]] \to \mathbb{Q}[[A]]$ is defined as follows: for $x \in A$, we have $D(x) = G(x)$; for a monomial u in $\mathbb{Q}[[A]]$, $D(u)$ is defined so that D is a derivation and, for a general element $q \in \mathbb{Q}[[A]]$, $D(q)$ is defined by linearity. In the rest of this section, we first recall some definitions of permutation statistics and then present grammatical interpretations and convolution formulas related to $T_n(x)$.

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. An *interior peak* in π is an index $i \in \{2, 3, \ldots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$. A *left peak* in π is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. Let $\operatorname{ipk}(\pi)$ (respectively $\operatorname{lpk}(\pi)$) be the number of interior peaks (respectively left peaks) in π. Define

$$M_n(x) = \sum_{\pi \in \Phi(Q_n)} x^{\operatorname{ipk}(\pi)}, \quad N_n(x) = \sum_{\pi \in \Phi(Q_n)} x^{\operatorname{lpk}(\pi)}.$$

It follows from [13, Theorem 4] that $M_n(x) = x^n N_n(1/x)$. Moreover, from [13, Theorem 5],

$$(1+x)T_n(x) = xM_n(x^2) + N_n(x^2).$$

We now recall some properties of $N_n(x)$. Let $N_n(x) = \sum_{k=1}^{n} N(n,k)x^k$. Apart from counting permutations in the set $\Phi(Q_n)$ with k left peaks, the number $N(n,k)$ also has the following combinatorial interpretations.

(m_1) Let $e = (e_1, e_2, \ldots, e_n) \in \mathbb{Z}^n$ and let $I_{n,k} = \{e \in \mathbb{Z}^n \mid 0 \leq e_i \leq (i-1)k\}$, the set of n-dimensional *k-inversion sequences* (see [17]). The number of *ascents* of e is defined by

$$\operatorname{asc}(e) = \#\left\{i : 1 \leq i \leq n-1 \;\middle|\; \frac{e_i}{(i-1)k+1} < \frac{e_{i+1}}{ik+1}\right\}.$$

Savage and Viswanathan [18] found $N(n,k) = \#\{e \in I_{n,2} : \operatorname{asc}(e) = n-k\}$.

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(m_2) We say that an index $i \in [2n-1]$ is an *ascent plateau* of $\pi \in \mathcal{Q}_n$ if $\pi(i-1) < \pi(i) = \pi(i+1)$. The number $N(n,k)$ counts Stirling permutations in $\mathcal{Q}_n$ with k ascent plateaux (see [13, Theorem 3]).

(m_3) The number $N(n,k)$ counts perfect matching on $[2n]$ with the restriction that there are only k matching pairs with even maximal elements (see [14]).

The polynomials $N_n(x)$ satisfy the recurrence relation

$$N_{n+1}(x) = (2n+1)xN_n(x) + 2x(1-x)N_n'(x)$$

with initial value $N_0(x) = 1$. The first few of the $N_n(x)$ are

$$N_1(x) = x, \quad N_2(x) = 2x + x^2,$$
$$N_3(x) = 4x + 10x^2 + x^3, \quad N_4(x) = 8x + 60x^2 + 36x^3 + x^4.$$

The exponential generating function for $N_n(x)$ is given by (see [10, Section 5])

$$N(x,z) = \sum_{n\geq 0} N_n(x)\frac{z^n}{n!} = \sqrt{\frac{1-x}{1-xe^{2z(1-x)}}}. \tag{3.1}$$

Let

$$R_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\text{altrun}\,(\pi)} = \sum_{k=1}^{n-1} R(n,k)x^k.$$

The study of alternating runs of permutations was initiated by André [2] and he proved that the numbers $R(n,k)$ satisfy the recurrence relation

$$R(n,k) = kR(n-1,k) + 2R(n-1,k-1) + (n-k)R(n-1,k-2)$$

for $n,k \geq 1$, where $R(1,0) = 1$ and $R(1,k) = 0$ for $k \geq 1$. There is a large literature devoted to the numbers $R(n,k)$ (see [19, A059427]). The reader is referred to [4, 11] for recent results on this subject.

Recall that a *descent* of a permutation $\pi \in \mathfrak{S}_n$ is a position i such that $\pi(i) > \pi(i+1)$. Denote by des (π) the number of descents of π. Then the equations

$$A_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\text{des}\,(\pi)+1} = \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle x^k$$

define the *Eulerian polynomial* $A_n(x)$ and the *Eulerian number* $\left\langle {n \atop k} \right\rangle$. Denote by B_n the hyperoctahedral group which is the group of signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i, where $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. For each $\pi \in B_n$, we define

$$\text{des}_A(\pi) := \#\{i \in \{1,2,\ldots,n-1\} \mid \pi(i) > \pi(i+1)\},$$
$$\text{des}_B(\pi) := \#\{i \in \{0,1,2,\ldots,n-1\} \mid \pi(i) > \pi(i+1)\},$$

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where $\pi(0) = 0$. Following [1], the *flag descent number* of π is defined by

$$\operatorname{fdes}(\pi) := \begin{cases} 2\operatorname{des}_A(\pi) + 1 & \text{if } \pi(1) < 0, \\ 2\operatorname{des}_A(\pi) & \text{otherwise.} \end{cases}$$

Let

$$B_n(x) = \sum_{\pi \in B_n} x^{\operatorname{des}_B(\pi)} = \sum_{k=0}^{n} B(n,k)x^k,$$

$$S_n(x) = \sum_{\pi \in B_n} x^{\operatorname{fdes}(\pi)} = \sum_{k=1}^{2n} S(n,k)x^{k-1}.$$

The polynomial $B_n(x)$ is called an *Eulerian polynomial of type B*, while $B(n,k)$ is called an *Eulerian number of type B* (see [19, A060187]). It follows from [1, Theorem 4.3] that the numbers $S(n,k)$ satisfy the recurrence relation

$$S(n,k) = kS(n-1,k) + S(n-1,k-1) + (2n-k+1)S(n-1,k-2)$$

for $n, k \geq 1$, where $S(1,1) = S(1,2) = 1$ and $S(1,k) = 0$ for $k \geq 3$. The polynomial $S_n(x)$ is closely related to the Eulerian polynomial $A_n(x)$:

$$S_n(x) = \frac{1}{x}(1+x)^n A_n(x) \quad \text{for } n \geq 1,$$

which was established by Adin *et al.* [1].

Consider the context-free grammar

$$A = \{x, y, z\}, \quad G = \{x \to p(x,y,z), y \to q(x,y,z), z \to r(x,y,z)\},$$

where $p(x,y,z), q(x,y,z)$ and $r(x,y,z)$ are polynomials in x, y and z. The *diamond product* of z with the grammar G is defined by

$$G \diamond z = \{x \to p(x,y,z)z, y \to q(x,y,z)z, z \to r(x,y,z)z\}.$$

We now recall two results on context-free grammars.

Proposition 3.1 [11, Theorem 6]. *If*

$$G = \{x \to xy, y \to yz, z \to y^2\}, \tag{3.2}$$

then

$$D^n(x^2) = x^2 \sum_{k=0}^{n} R(n+1,k) y^k z^{n-k}.$$

Setting $x = z = 1$, we have $D^n(x^2)|_{x=z=1} = R_{n+1}(y)$.

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Proposition 3.2 [12, Theorem 10]. *Consider the context-free grammar*

$$G' = \{x \to xyz, y \to yz^2, z \to y^2z\}, \tag{3.3}$$

which is the diamond product of z with the grammar G defined by (3.2). For $n \geq 1$,

$$D^n(xy) = x\sum_{k=1}^{2n} S(n,k)y^{2n-k+1}z^k,$$

$$D^n(yz) = \sum_{k=0}^{n} B(n,k)y^{2n-2k+1}z^{2k+1},$$

$$D^n(y) = \sum_{k=1}^{n} N(n,k)y^{2n-2k+1}z^{2k},$$

$$D^n(z) = \sum_{k=1}^{n} N(n,n-k+1)y^{2n-2k+2}z^{2k-1},$$

$$D^n(y^2) = 2^n\sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle y^{2n-2k+2}z^{2k}.$$

We can now deduce the following result.

Theorem 3.3. *Let G' be the context-free grammar given by (3.3). Then, for $n \geq 1$,*

$$D^n(x) = x\sum_{k=1}^{2n-1} T(n,k)y^kz^{2n-k},$$

$$D^n(x^2) = 2x^2(y+z)^{n-1}\sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle y^kz^{n-k+1}.$$

Setting $x = z = 1$, we have $D^n(x)|_{x=z=1} = T_n(y)$ and $D^n(x^2)|_{x=z=1} = 2(1+y)^{n-1}A_n(y)$.

Proof. Note that $D(x) = xyz$ and $D^2(x) = xyz^3 + xy^2z^2 + xy^3z$. For $n \geq 1$, we define $t(n,k)$ by

$$D^n(x) = x\sum_{k\geq 1} t(n,k)y^kz^{2n-k}.$$

Then

$$D^{n+1}(x) = D(D^n(x)) = x\sum_{k\geq 1} t(n,k)y^{k+1}z^{2n-k+1} + x\sum_{k\geq 1} kt(n,k)y^kz^{2n-k+2} + x\sum_{k\geq 1}(2n-k)t(n,k)y^{k+2}z^{2n-k}.$$

Hence,

$$t(n+1,k) = kt(n,k) + t(n,k-1) + (2n-k+2)t(n,k-2). \tag{3.4}$$

By comparing (3.4) with (1.2), we see that the numbers $t(n,k)$ satisfy the same recurrence relation and initial conditions as $T(n,k)$, so they agree. The assertion for $D^n(x^2)$ can be proved in a similar way. □

It follows from Leibniz's formula that

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u) D^{n-k}(v).$$

Hence,

$$D^n(x^2) = \sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(x),$$

$$D^{n+1}(x) = D^n(xyz) = \sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(yz) = \sum_{k=0}^{n} \binom{n}{k} D^k(xy) D^{n-k}(z).$$

Therefore, we can use Proposition 3.2 and Theorem 3.3 to get several convolution identities.

COROLLARY 3.4. *For $n \geq 1$,*

$$2(1+x)^{n-1} A_n(x) = \textstyle\sum_{k=0}^{n} \binom{n}{k} T_k(x) T_{n-k}(x), \tag{3.5}$$
$$T_{n+1}(x) = x \textstyle\sum_{k=0}^{n} \binom{n}{k} T_k(x) B_{n-k}(x^2),$$
$$T_{n+1}(x) = x \textstyle\sum_{k=0}^{n} \binom{n}{k} S_k(x) N_{n-k}(x^2).$$

Let $T(x,z) = \sum_{n=0}^{\infty} T_n(x)(z^n/n!)$. Recall that the exponential generating function for $A_n(x)$ is given as follows (see [19, A008292]):

$$A(x,t) = \sum_{n\geq 0} A_n(x) \frac{t^n}{n!} = \frac{1-x}{1-xe^{t(1-x)}}. \tag{3.6}$$

Combining (3.5) and (3.6),

$$T(x,z) = \frac{e^{z(x-1)(x+1)} + x}{1+x} \sqrt{\frac{1-x^2}{e^{2z(x-1)(x+1)} - x^2}}. \tag{3.7}$$

From (3.1),

$$\sum_{n\geq 0} M_n(x^2) \frac{z^n}{n!} = \sum_{n\geq 0} x^{2n} N_n\Big(\frac{1}{x^2}\Big) \frac{z^n}{n!} = \sqrt{\frac{1-x^2}{e^{2z(x-1)(x+1)} - x^2}}.$$

Note that

$$\frac{e^{z(x-1)(x+1)} + x}{1+x} = 1 + \sum_{n\geq 1} (x-1)^n (x+1)^{n-1} \frac{z^n}{n!}.$$

Therefore, from (3.7),

$$T_n(x) = M_n(x^2) + \sum_{k=0}^{n-1} \binom{n}{k} M_k(x^2)(x-1)^{n-k}(x+1)^{n-k-1} \quad \text{for } n \geq 1.$$

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4. Concluding remarks

Let $f(x)$ and $F(x)$ be two polynomials with real coefficients. We say that $f(x)$ *separates* $F(x)$ if $\deg F = \deg f + 2$ and the sequences of real and imaginary parts of the zeros of $f(x)$ respectively separate those of $F(x)$. In other words, if we set $f(x) = a\prod_{j=1}^{n-1}(x + p_j + q_j\mathrm{i})(x + p_j - q_j\mathrm{i})$ and $F(x) = b\prod_{j=1}^{n}(x + s_j + t_j\mathrm{i})(x + s_j - t_j\mathrm{i})$, where a, b are respectively leading coefficients of $f(x)$, $F(x)$, $p_1 \geq p_2 \geq \cdots \geq p_{n-1}$, $q_1 \geq q_2 \geq \cdots \geq q_{n-1}$, $s_1 \geq s_2 \geq \cdots \geq s_n$ and $t_1 \geq t_2 \geq \cdots \geq t_n$, then

$$s_1 \geq p_1 \geq s_2 \geq p_2 \geq \cdots \geq s_{n-1} \geq p_{n-1} \geq s_n,$$
$$t_1 \geq q_1 \geq t_2 \geq q_2 \geq \cdots \geq t_{n-1} \geq q_{n-1} \geq t_n.$$

Based on empirical evidence, we propose the following conjecture.

CONJECTURE 4.1. For $n \geq 2$, all zeros of $T_n(x)/x$ are imaginary and $T_n(x)/x$ separates $T_{n+1}(x)/x$.

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