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SINGULAR LIMITS FOR 2-DIMENSIONAL ELLIPTIC PROBLEMS INVOLVING EXPONENTIAL NONLINEARITIES WITH SUB-QUADRATIC CONVECTION TERM

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Abstract. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^2$, $q \in [1, 2)$ and $x_1, x_2, \ldots, x_m \in \Omega$. In this paper we are concerned with the following type of problem:

$$-\Delta u - \lambda|\nabla u|^q = \rho^2 e^u,$$

with $u = 0$ on $\partial\Omega$. We use some nonlinear domain decomposition method to construct a positive weak solution $v_{\rho,\lambda}$ in Ω , which tends to a singular function at each x_i as the parameters ρ and λ tend to 0 independently.

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1. Introduction and statement of the results. In this paper, we study the following problem:

$$\begin{cases} -\Delta u - \lambda|\nabla u|^q = \rho^2 e^u & \text{in } \Omega \\ \qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases}, \tag{1}$$

where ∇ is the gradient symbol and Ω is an open smooth bounded subset of $\mathbb{R}^2$. In the following, we denote by ε the smallest positive parameter satisfying

$$\rho^2 = \frac{8\,\varepsilon^2}{(1+\varepsilon^2)^2}\,. \tag{2}$$

Remark that $\rho \sim \varepsilon$ as $\varepsilon \longrightarrow 0$. We will ask the following question: Does there exist $v_{\varepsilon,\lambda}$ a sequence of solutions of (1) which converges to some nontrivial function as the parameters ε and λ tend to 0? A positive answer to this question has been given by Baraket et al. in [**2**] for problem $(1)_{|q=2}$, under the assumption

(A) : If $0 < \varepsilon < \lambda$, then $\lambda^{1+\delta/2}\varepsilon^{-\delta} \longrightarrow 0$ as $\lambda \longrightarrow 0$, for any $\delta \in (0, 1)$.

In particular, if we take $\lambda = \mathcal{O}(\varepsilon^{2/3})$, then condition (A) is satisfied. With assumption (A), problem $(1)_{|q=2}$, can be treated as a perturbation of the Liouville equation

$$-\Delta u = \rho^2 e^u \quad \text{in } \Omega \subset \mathbb{R}^2.$$

The problem $(1)_{|q=2}$, can be transformed to another one studied by Ren and Wei (see [**17**]). Indeed, if u is a solution of $(1)_{|q=2}$, then the function

$$w = (\lambda\rho^2 e^u)^\lambda,$$

satisfies

$$\begin{cases} -\Delta w = w^{\frac{\lambda+1}{\lambda}} & \text{in } \Omega \\ \quad w = (\lambda\rho^2)^\lambda & \text{on } \partial\Omega \end{cases}. \tag{3}$$

Remark that the exponent $p = \frac{\lambda+1}{\lambda}$ tends to infinity as λ tends to 0, see also [**7**]. We shall therefore mainly consider the case where $q \in [1, 2)$ is a real number. We look for a sequence of solutions $v_{\varepsilon,\lambda}$ of (1) which converges to some nontrivial singular function on some set as the parameters ε and λ tend to 0 without considering any condition like (A), and to see how the presence of the convection term (gradient) can have significant influence on the existence of a solution, as well as on its asymptotic behaviour.

Note that Ghergu and Radulescu in [**8**] have studied more general problem on a domain $\Sigma \subset \mathbb{R}^n$, $n \geq 2$

$$\begin{cases} -\Delta u - \lambda|\nabla u|^a = g(u) + \mu f(x, u) & \text{in } \Sigma \\ \qquad u = 0 & \text{on } \partial\Sigma \end{cases}, \tag{4}$$

with $0 < a \leq 2$, $\lambda, \mu > 0$ and some assumptions on f and g. Problems of the type (4) arise in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrically conducting materials. It also includes some simple prototype models from boundary-layer theory of viscous fluids [**23**]. Problem (1) with the condition $u_{|\partial\Omega} = 0$ replaced by $u_{|\partial\Omega} = +\infty$ arises from many branches of mathematics and applied mathematics, and has been discussed by many authors in many contexts, see, e.g. [**1**, **6**, **9–12**, **14**, **15**, **19**, **24**].

Many papers have been devoted to the case $\lambda = 0$, where problem (1) becomes

$$\begin{cases} -\Delta u = \rho^2 e^u & \text{in } \Omega \in \mathbb{R}^2 \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}. \tag{5}$$

The study of this equation goes back to 1853 when Liouville [**13**] derived a representation formula for all solutions of (5) which are defined in $\mathbb{R}^2$. It turns out that, besides the applications in geometry, elliptic equations with exponential nonlinearity also arise in modelling many physical phenomena such as thermionic emission, isothermal gas sphere, gas combustion and gauge theory (see [**20**]). . . .

When ρ tends to 0, the asymptotic behaviour of nontrivial branches of solutions of (5) is well understood, thanks to the work of Suzuki [**18**], which characterizes the possible limit of nontrivial branches of solutions of (5). The existence of nontrivial branches of solutions was first proved by Weston [**22**], and then a general positive result for problem (5) has been obtained by Baraket and Pacard [**4**]. These results were extended, applying to the Chern–Simons vortex theory in mind, by Esposito et al. [**6**] and Del Pino et al. [**5**] to handle equations of the form

$$-\Delta u = \rho^2 \, V \, e^u,$$

where V is a non-constant (positive) function.

We introduce the Green's function $G(x, x')$ defined on $\Omega \times \Omega$, to be the solution of

$$\begin{cases} -\Delta G(x, x') = 8\pi \, \delta_{x=x'} & \text{in } \Omega \\ G(x, x') = 0 & \text{on } \partial\Omega \end{cases}$$

and let its regular part

$$H(x, x') = G(x, x') + 4 \log |x - x'|.$$

Let $m \in \mathbb{N}$, we set

$$\mathcal{F}(x_1, \cdots, x_m) = \sum_{j=1}^{m} H(x_j, x_j) + \sum_{i \neq j} G(x_i, x_j), \tag{6}$$

which is well defined in Ω^m for $x_i \neq x_j$ if $i \neq j$.

Following is our main result:

THEOREM 1. *Given $\alpha \in (0, 1)$ and $q \in [1, 2)$ is a real number. Let Ω be an open smooth bounded set of $\mathbb{R}^2$ and $S = \{x_1, \ldots, x_m\} \subset \Omega$ be a non-empty set. Assume that $(x_1, \ldots, x_m)$ is a* non-degenerate *critical point of the function*

$$\mathcal{F}(x_1, \ldots, x_m) = \sum_{j=1}^{m} H(x_j, x_j) + \sum_{i \neq j} G(x_i, x_j) \quad \text{in } (\Omega)^m,$$

then there exist $\rho_0 > 0$, $\lambda_0 > 0$, and $\{v_{\rho,\lambda}\}_{\substack{0<\rho<\rho_0 \\ 0<\lambda<\lambda_0}}$ a family of solutions of

$$\begin{cases} -\Delta v - \lambda |\nabla v|^q = \rho^2 e^v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

such that

$$\lim_{\substack{\rho \to 0 \\ \lambda \to 0}} v_{\rho,\lambda} = \sum_{j=1}^{m} G(x_j, \cdot)$$

in $C^{2,\alpha}_{loc}(\Omega - \{x_1, \ldots, x_m\})$.

Our result reduces the study of nontrivial branches of solutions of (1) to the search for critical points of the function $\mathcal{F}$ defined in (6). Observe that the assumption on the non-degeneracy of the critical point is a rather mild assumption since it is certainly fulfilled for generic choice of the open domain Ω.

2. Construction of the approximate solution. Let $q \in [1, 2)$. We first describe the rotationally symmetric approximate solutions of

$$-\Delta u - \lambda |\nabla u|^q = \rho^2 e^u \tag{7}$$

in $\mathbb{R}^2$ which will play a central role in our analysis. Given $\varepsilon > 0$, we define

$$u_\varepsilon(x) := 2\log(1+\varepsilon^2) - 2\log(\varepsilon^2 + |x|^2), \tag{8}$$

which is clearly a solution of

$$-\Delta u = \rho^2 e^u \tag{9}$$

in $\mathbb{R}^2$.

Let us note that (9) is invariant under dilation in the following sense : If v is a solution of (9) and $\tau > 0$, then $v(\tau\,\cdot) + 2\log\tau$ is also a solution of (9). With this observation in mind, we define for all $\tau > 0$

$$u_{\varepsilon,\tau}(x) := 2\log(1+\varepsilon^2) + 2\log\tau - 2\log(\varepsilon^2 + |\tau x|^2). \tag{10}$$

2.1. A linearized operator on $\mathbb{R}^2$. For all $\varepsilon, \tau > 0$, we define

$$R_{\varepsilon,\lambda} := \tau r_{\varepsilon,\lambda} / \varepsilon,$$

where

$$r_{\varepsilon,\lambda} := \max(\sqrt{\varepsilon}, \sqrt{\lambda}). \tag{11}$$

We define the linear second-order elliptic operator

$$\mathbb{L} := -\Delta - \frac{8}{(1+|x|^2)^2}, \tag{12}$$

which corresponds to the linearization of (9) about the solution $u_1 (= u_{\varepsilon=\tau=1})$ which has been defined in the previous section.

We are interested in the classification of bounded solutions of $\mathbb{L}\, w = 0$ in $\mathbb{R}^2$. Some solutions are easy to find. For example, we can define

$$\phi_0(x) := \frac{r}{2}\,\partial_r u_1(x) + 1 = 2\,\frac{1-r^2}{1+r^2},$$

where $r = |x|$. Clearly, $\mathbb{L}\,\phi_0 = 0$ and this reflects the fact that (9) is invariant under the group of dilations $\tau \longrightarrow u(\tau\,\cdot) + 2\,\log\tau$. We also define, for $i = 1, 2$,

$$\phi_i(x) := -\partial_{x_i} u_1(x) = \frac{2\,x_i}{1+|x|^2},$$

which are also solutions of $\mathbb{L}\,\phi_i = 0$, since these solutions correspond to the invariance of the equation under the group of translations $a \longrightarrow u(\cdot + a)$.

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We recall the following result which classifies all bounded solutions of $\mathbb{L}\, w = 0$ which are defined in $\mathbb{R}^2$.

LEMMA 1 [**4**]. *Any bounded solution of* $\mathbb{L}\, w = 0$ *defined in* $\mathbb{R}^2$ *is a linear combination of* ϕ_i *for* $i = 0, 1, 2$.

Let B_r denotes the ball of radius r centred at the origin in $\mathbb{R}^2$.

DEFINITION 1. Given $k \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\mu \in \mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}^{k,\alpha}_{\mu}(\mathbb{R}^2)$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{loc}(\mathbb{R}^2)$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_{\mu}(\mathbb{R}^2)} := \|w\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1)} + \sup_{r\geq 1} \left((1 + r^2)^{-\mu/2} \, \|w(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1 - B_{1/2})} \right)$$

is finite.

We also define

$$\mathcal{C}^{k,\alpha}_{\mathrm{rad},\mu}(\mathbb{R}^2) = \{f \in \mathcal{C}^{k,\alpha}_{\mu}(\mathbb{R}^2);\ f(x) = f(|x|), \forall\, x \in \mathbb{R}^2\}.$$

As a consequence of the result of Lemma 1, we recall the surjectivity result of $\mathbb{L}$ given in [**4**].

PROPOSITION 1 [**4**].
(i) Assume that $\mu > 1$ *and* $\mu \notin \mathbb{N}$, *then*

$$\begin{array}{rccc} L_\mu : & \mathcal{C}^{2,\alpha}_{\mu}(\mathbb{R}^2) & \longrightarrow & \mathcal{C}^{0,\alpha}_{\mu-2}(\mathbb{R}^2) \\ & w & \longmapsto & \mathbb{L}\, w \end{array}$$

is surjective.
(ii) Assume that $\delta > 0$ *and* $\delta \notin \mathbb{N}$, *then*

$$\begin{array}{rccc} L_\delta : & \mathcal{C}^{2,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^2) & \longrightarrow & \mathcal{C}^{0,\alpha}_{\mathrm{rad},\delta-2}(\mathbb{R}^2) \\ & w & \longmapsto & \mathbb{L}\, w \end{array}$$

is surjective.

We set $\bar{B}^*_1 = \bar{B}_1 - \{0\}$.

DEFINITION 2. Given $k \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\mu \in \mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}^{k,\alpha}_{\mu}(\bar{B}^*_1)$ as the space of functions in $\mathcal{C}^{k,\alpha}_{loc}(\bar{B}^*_1)$ for which the following norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_{\mu}(\bar{B}^*_1)} = \sup_{r\leq 1/2} \left(r^{-\mu} \, \|u(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_2 - B_1)} \right)$$

is finite.

Then we define the subspace of radial functions in $\mathcal{C}^{k,\alpha}_{\mathrm{rad},\delta}(\bar{B}^*_1)$ by

$$\mathcal{C}^{k,\alpha}_{\mathrm{rad},\delta}(\bar{B}^*_1) = \{f \in \mathcal{C}^{k,\alpha}_{\delta}(\bar{B}^*_1);\ f(x) = f(|x|), \forall\, x \in \bar{B}^*_1\}.$$

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We would like to find a solution u of

$$\Delta u + \lambda |\nabla u|^q + \rho^2 e^u = 0 \tag{13}$$

in $\bar{B}_{r_{\varepsilon,\lambda}}$. Using the transformation

$$v(x) = u\left(\frac{\varepsilon}{\tau} x\right) + 4 \log \varepsilon - 2 \log(\tau(1+\varepsilon^2)/2),$$

then equation (13) is equivalent to

$$\Delta v + \lambda \left(\frac{\varepsilon}{\tau}\right)^{2-q} |\nabla v|^q + 2e^v = 0 \tag{14}$$

in $\bar{B}_{R_{\varepsilon,\lambda}}$. Now we look for a solution of (14) of the form

$$v(x) = u_1(x) + h(x),$$

this amounts to solve

$$\mathbb{L}\, h = \frac{8}{(1+|x|^2)^2}(e^h - h - 1) + \lambda \left(\frac{\varepsilon}{\tau}\right)^{2-q} |\nabla(u_1 + h)|^q \tag{15}$$

in $\bar{B}_{R_{\varepsilon,\lambda}}$. We will need the following definition.

DEFINITION 3. Given $\bar{r} \geq 1$, $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, the weighted space $\mathcal{C}^{k,\alpha}_{\mu}(B_{\bar{r}})$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}(B_{\bar{r}})$ endowed with the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_{\mu}(\bar{B}_{\bar{r}})} := \|w\|_{\mathcal{C}^{k,\alpha}(B_1)} + \sup_{1 \leq r \leq \bar{r}} \left(r^{-\mu} \, \|w(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1 - B_{1/2})} \right).$$

For all $\sigma \geq 1$, we denote by

$$\mathcal{E}_\sigma : \mathcal{C}^{0,\alpha}_{\mu}(\bar{B}_\sigma) \longrightarrow \mathcal{C}^{0,\alpha}_{\mu}(\mathbb{R}^2)$$

the extension operator defined by

$$\mathcal{E}_\sigma(f)(x) = \begin{cases} f(x) & \text{for } |x| \leq \sigma \\ \chi\left(\frac{|x|}{\sigma}\right) f\left(\sigma \frac{x}{|x|}\right) & \text{for } |x| \geq \sigma \end{cases}, \tag{16}$$

where $t \longmapsto \chi(t)$ is a smooth non-negative cutoff function identically equal to 1 for $t \leq 1$ and identically equal to 0 for $t \geq 2$. It is easy to check that there exists a constant $c = c(\mu) > 0$ independent of $\sigma \geq 1$ such that

$$\|\mathcal{E}_\sigma(w)\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathbb{R}^2)} \leq c\, \|w\|_{\mathcal{C}^{0,\alpha}_{\mu}(\bar{B}_\sigma)}. \tag{17}$$

We fix

$$\delta \in (0, 2-q)$$

and denote by $\mathcal{G}_\delta$ to be a right inverse of $\mathbb{L}_\delta$ provided by Proposition 1. To find a solution of (15) it is enough to find a fixed point h, in a small ball of $\mathcal{C}^{2,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^2)$,

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solution of

$$h = \aleph(h), \tag{18}$$

where

$$\aleph(h) := \mathcal{G}_\delta \circ \mathcal{E}_{R_{\varepsilon,\lambda}} \circ \mathfrak{R}(h)$$

with

$$\mathfrak{R}(h) = \frac{8}{(1+|x|^2)^2}(e^h - h - 1) + \lambda\Big(\frac{\varepsilon}{\tau}\Big)^{2-q}|\nabla(u_1 + h)|^q.$$

We have

$$|\mathfrak{R}(0)| = \lambda\Big(\frac{\varepsilon}{\tau}\Big)^{2-q}|\nabla u_1|^q.$$

This implies that given $\kappa > 1$, there exists $c_\kappa > 0$ (which can depend only on κ) such that for $|x| = r$, we have

$$\begin{aligned}\sup_{r\le R_{\varepsilon,\lambda}} r^{2-\delta}\ |\mathfrak{R}(0)| &\le \sup_{r\le R_{\varepsilon,\lambda}} r^{2-\delta}\lambda\Big(\frac{\varepsilon}{\tau}\Big)^{2-q}|\nabla u_1|^q,\\ &\le c_\kappa \lambda \varepsilon^{2-q} R_{\varepsilon,\lambda}^{2-q-\delta} \le c_\kappa \lambda \varepsilon^{\delta} r_{\varepsilon,\lambda}^{2-q-\delta}.\end{aligned}$$

Making use of Proposition 1 together with (17), we conclude that

$$\|\aleph(0)\|_{C^{2,\alpha}_{\mathrm{rad},\delta}} \le c_\kappa \varepsilon^\delta r^2_{\varepsilon,\lambda}. \tag{19}$$

Now, we recall an important result that plays a centre role in our estimates. See for example [**16**] and some references therein.

LEMMA 2. *Given x and y two real numbers, $x > 0$, $q \ge 1$ and for all small $\eta > 0$ there exists a positive constant C_η such that*

$$\Big||x+y|^q - x^q\Big| \le (1+\eta)qx^{q-1}|y|\ + C_\eta|y|^q.$$

Now, let h_1, h_2 in $B(0, 2c_\kappa\varepsilon^\delta r^2_{\varepsilon,\lambda})$ of $C^{2,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^2)$, there exist $c_\kappa^{(i)} > 0$, $1 \le i \le 4$ (only depend on κ) such that

$$\begin{aligned}&\sup_{r\le R_{\varepsilon,\lambda}} r^{2-\delta}\ |\mathfrak{R}(h_2) - \mathfrak{R}(h_1)|\\ &\quad\le c_\kappa^{(1)} \sup_{r\le R_{\varepsilon,\lambda}} r^{2-\delta}(1+|x|^2)^{-2}\left|e^{h_2} - e^{h_1} + h_1 - h_2\right|\\ &\qquad + c_\kappa^{(2)}\Big(\frac{\varepsilon}{\tau}\Big)^{2-q} \sup_{r\le R_{\varepsilon,\lambda}} \lambda r^{2-\delta}(|\nabla(u_1+h_2)|^q - |\nabla(u_1+h_1)|^q)\\ &\quad\le c_\kappa^{(3)} \sup_{r\le R_{\varepsilon,\lambda}} r^{-2-\delta}\left|e^{h_2} - e^{h_1} + h_1 - h_2\right|\\ &\qquad + c_\kappa^{(4)}\lambda\Big(\frac{\varepsilon}{\tau}\Big)^{2-q} \sup_{r\le R_{\varepsilon,\lambda}} r^{2-\delta}(|\nabla(u_1+h_1) + \nabla(h_2-h_1)|^q - |\nabla(u_1+h_1)|^q).\end{aligned}$$

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Making use of Lemma 2, there exist $c_\kappa^{(i)} > 0,\ 5 \le i \le 10$ and $\bar{c}_\kappa > 0$ (only depend on κ) such that

$$\begin{aligned}
&\sup_{r \le R_{\varepsilon,\lambda}} r^{2-\delta}\ |\mathfrak{R}(h_2) - \mathfrak{R}(h_1)| \\
&\quad \le c_\kappa^{(5)} \sup_{r \le R_{\varepsilon,\lambda}} r^{-2-\delta} |h_2 - h_1||h_2 + h_1| \\
&\qquad + c_\kappa^{(6)} \lambda \left(\frac{\varepsilon}{\tau}\right)^{2-q} \sup_{r \le R_{\varepsilon,\lambda}} r^{2-\delta} \Big[|\nabla(u_1 + h_1)|^{q-1} + |\nabla(h_2 - h_1)|^{q-1} \Big] |\nabla(h_2 - h_1)| \\
&\quad \le c_\kappa^{(7)} \sup_{r \le R_{\varepsilon,\lambda}} r^{-2-\delta} |h_2 - h_1||h_2 + h_1| \\
&\qquad + c_\kappa^{(8)} \left(\frac{\varepsilon}{\tau}\right)^{2-q} \sup_{r \le R_{\varepsilon,\lambda}} \lambda r^{2-\delta} \Big[|\nabla u_1|^{q-1} + |\nabla h_1|^{q-1} + |\nabla h_2|^{q-1} \Big] |\nabla(h_2 - h_1)| \\
&\quad \le 2c_\kappa^{(9)} \|h_i\|_{\mathcal{C}^{2,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^2)} \|h_2 - h_1\|_{\mathcal{C}^{2,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^2)} + c_\kappa^{(10)} \lambda r_{\varepsilon,\lambda}^{2-q} \Big(1 + r_{\varepsilon,\lambda}^{(2+\delta)(q-1)}\Big) \|h_2 - h_1\|_{\mathcal{C}^{2,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^2)} \\
&\quad \le \bar{c}_\kappa r_{\varepsilon,\lambda}^2 \|h_2 - h_1\|_{\mathcal{C}^{2,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^2)},
\end{aligned}$$

Similarly, making use of Proposition 1 together with (17), we conclude that given $\kappa > 1$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$ and $\bar{c}_\kappa > 0$ (only depend on κ) such that

$$\|\aleph(h_2) - \aleph(h_1)\|_{\mathcal{C}^{2,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^2)} \le \bar{c}_\kappa r_{\varepsilon,\lambda}^2 \|h_2 - h_1\|_{\mathcal{C}^{2,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^2)}. \tag{20}$$

Reducing $\varepsilon_\kappa > 0$ and $\lambda_\kappa > 0$ if necessary, we can assume that,

$$\bar{c}_\kappa\, r_{\varepsilon,\lambda}^2 \le \frac{1}{2}$$

for all $\varepsilon \in (0, \varepsilon_\kappa)$ and $\lambda \in (0, \lambda_\kappa)$. Then (19) and (20) are enough to show that

$$h \longmapsto \aleph(h)$$

is a contraction from the ball

$$\{h \in \mathcal{C}^{2,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^2) : \|h\|_{\mathcal{C}^{2,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^2)} \le 2c_\kappa \varepsilon^\delta r_{\varepsilon,\lambda}^2\}$$

into itself and hence has a unique fixed point h in this set. This fixed point is a solution of (18) in $\bar{B}_{R_{\varepsilon,\lambda}}$. We summarize this in the following proposition.

PROPOSITION 2. *Given $\kappa > 1$. There exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$ and $c_\kappa > 0$ (which can depend only on κ) such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, for all $\lambda \in (0, \lambda_\kappa)$ and for any $\delta \in (0, 2-q)$, there exists a unique solution $h \in \mathcal{C}^{2,\alpha}_{rad,\delta}(\mathbb{R}^2)$ of (18) such that*

$$v(x) = u_1(x) + h(x)$$

solves (14) in $\bar{B}_{R_{\varepsilon,\lambda}}$. In addition,

$$\|h\|_{\mathcal{C}^{2,\alpha}_{rad,\delta}(\mathbb{R}^2)} \le 2c_\kappa \varepsilon^\delta r_{\varepsilon,\lambda}^2. \tag{21}$$

2.2. Analysis of Laplace operator in weighted spaces. In this section we study the mapping properties of the Laplace operator in weighted Hölder spaces. Given $x_1, \dots, x_m \in \Omega$, we define $\mathbf{x} := (x_1, \dots, x_m)$ and

$$\bar{\Omega}^*(\mathbf{x}) := \bar{\Omega} - \{x_1, \dots x_m\},$$

and choose $r_0 > 0$ so that the balls $B_{r_0}(x_i)$ of centre x_i and radius r_0 are mutually disjoint and included in Ω. For all $r \in (0, r_0)$, we define

$$\bar{\Omega}_r(\mathbf{x}) := \bar{\Omega} - \cup_{i=1}^m B_r(x_i).$$

DEFINITION 4. Given $k \in \mathbb{R}$, $\alpha \in (0, 1)$ and $\nu \in \mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}_\nu^{k,\alpha}(\bar{\Omega}^*(\mathbf{x}))$ as the space of functions $w \in \mathcal{C}_{\mathrm{loc}}^{k,\alpha}(\bar{\Omega}^*(\mathbf{x}))$ for which the following norm

$$\|w\|_{\mathcal{C}_\nu^{k,\alpha}(\bar{\Omega}^*(\mathbf{x}))} := \|w\|_{\mathcal{C}^{k,\alpha}(\bar{\Omega}_{r_0/2})} + \sum_{i=1}^m \sup_{0<r\leq r_{0/2}} \left(r^{-\nu} \, \|w(x_i + r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_2 - B_1)} \right)$$

is finite.

When $k \geq 2$, we denote by $[\mathcal{C}_\nu^{k,\alpha}(\bar{\Omega}^*(\mathbf{x}))]_{\mathbf{0}}$ to be the subspace of functions $w \in \mathcal{C}_\nu^{k,\alpha}(\bar{\Omega}^*(\mathbf{x}))$ satisfying $w = 0$ on $\partial\Omega$. We recall the following result.

PROPOSITION 3 *[3]. Assume that $\nu < 0$ and $\nu \notin \mathbb{Z}$, then*

$$\begin{array}{rccc} \mathcal{L}_\nu : & [\mathcal{C}_\nu^{2,\alpha}(\bar{\Omega}^*\,(\mathbf{x}))]_{\mathbf{0}} & \longrightarrow & \mathcal{C}_{\nu-2}^{0,\alpha}(\bar{\Omega}^*\,(\mathbf{x})) \\ & w & \longmapsto & \Delta\, w \end{array}$$

is surjective. Denote by $\tilde{\mathcal{G}}_\nu$ the right inverse of $\mathcal{L}_\nu$.

REMARK 1. Observe that when $\nu < 0$, $\nu \notin \mathbb{Z}$, a right inverse is not unique and depends smoothly on the points $x_1, \dots, x_m$, at least locally. Once a right inverse is fixed for one choice of the points $x_1, \dots, x_m$, a right inverse for another choice of points $\tilde{x}_1, \dots, \tilde{x}_m$ close to $x_1, \dots, x_m$ can be obtained by using a simple perturbation argument.

2.3. Harmonic extensions. We study the properties of interior and exterior harmonic extensions. Given $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$, define $H^i(= H^i(\varphi\,;\cdot))$ to be the solution of

$$\begin{cases} \Delta\, H^i = 0 & \text{in } B_1 \\ \quad H^i = \varphi & \text{on } \partial B_1 \end{cases}. \tag{22}$$

We denote by e_1 and e_2 the coordinate functions on S^1.

LEMMA 3 *[3]. If we assume that*

$$\int_{S^1} \varphi \, dv_{S^1} = 0 \qquad \text{and} \qquad \int_{S^1} \varphi \, e_\ell \, dv_{S^1} = 0 \quad \text{for} \quad \ell = 1, 2 \tag{23}$$

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then there exists $c > 0$ such that

$$\|H^i(\varphi;\cdot)\|_{\mathcal{C}_2^{2,\alpha}(\bar{B}_1^*)} \leq c\,\|\varphi\|_{\mathcal{C}^{2,\alpha}(S^1)}.$$

REMARK 2. Observe that, under the first hypothesis of (23), the coefficients of r^0 vanish, hence at least formally, the expansion of H^i involving powers of r which are greater or equal to 1 and under the second hypothesis of (23), the coefficients of r^0 and r^1 vanish, hence the expansion of H^i involving powers of r which are greater or equal to 2. Roughly speaking, when the hypothesis (23) is fulfilled, then $H^i \in \mathcal{C}_2^{2,\alpha}(\bar{B}_1)$ (since $H^i(0) = \partial_1 H^i(0) = \partial_2 H^i(0) = 0$) and then the inequality of Lemma 3 holds.

Given $\tilde{\varphi} \in \mathcal{C}^{2,\alpha}(S^1)$, we define $H^e(= H^e(\tilde{\varphi};\cdot))$ to be the solution of

$$\begin{cases} \Delta H^e = 0 & \text{in } \mathbb{R}^2 - B_1 \\ \quad H^e = \tilde{\varphi} & \text{on } \partial B_1 \end{cases}, \tag{24}$$

which decays at infinity.

DEFINITION 5. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define the space $\mathcal{C}_\nu^{k,\alpha}(\mathbb{R}^2 - B_1)$ as the space of functions $w \in \mathcal{C}_{loc}^{k,\alpha}(\mathbb{R}^2 - B_1)$ for which the following norm

$$\|w\|_{\mathcal{C}_\nu^{k,\alpha}(\mathbb{R}^2 - B_1)} = \sup_{r \geq 1}\left(r^{-\nu}\,\|w(r\,\cdot)\|_{\mathcal{C}_\nu^{k,\alpha}(\bar{B}_2 - B_1)}\right)$$

is finite.

LEMMA 4 *[3]. If we assume that*

$$\int_{S^1} \tilde{\varphi}\, dv_{S^1} = 0, \tag{25}$$

then there exists $c > 0$ such that

$$\|H^e(\tilde{\varphi},\ ;\cdot)\|_{\mathcal{C}_{-1}^{2,\alpha}(\mathbb{R}^2 - B_1)} \leq c\,\|\tilde{\varphi}\|_{\mathcal{C}^{2,\alpha}(S^1)}.$$

REMARK 3. Observe that, under the first hypothesis of (25), the coefficients of r^0 vanish and hence the expansion of H^e involves powers of r which are lower or equal to -1. Roughly speaking, when the hypothesis (25) is fulfilled, then $H^e \in \mathcal{C}_{-1}^{2,\alpha}(\mathbb{R}^2 - B_1)$ and then the inequality of Lemma 4 holds.

If $F \subset L^2(S^1)$ is a space of functions defined on S^1, then we define the space $F^\perp$ to be the subspace of functions of F, which are $L^2(S^1)$-orthogonal to the functions $1, e_1$ and e_2. Then we have the following:

LEMMA 5 *[3]. The mapping*

$$\begin{array}{rccc} \mathcal{P}: & \mathcal{C}^{2,\alpha}(S^1)^\perp & \longrightarrow & \mathcal{C}^{1,\alpha}(S^1)^\perp \\ & \psi & \longmapsto & \partial_r H^i - \partial_r H^e \end{array},$$

where $H^i = H^i(\psi;\cdot)$ and $H^e = H^e(\psi;\cdot)$, is an isomorphism.

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3. The nonlinear interior problem. We are interested in the study of equation

$$\Delta w + \lambda\Big(\frac{\varepsilon}{\tau}\Big)^{2-q} |\nabla w|^q + 2e^w = 0 \tag{26}$$

in $\bar{B}_{R_{\varepsilon,\lambda}}$.

Given $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ satisfying (23), recall that $u_1 (= u_{\varepsilon=1,\tau=1})$ and the solution h of (18) satisfies (21). Define

$$\mathbf{v} := u_1 + H^i(\varphi, \cdot/R_{\varepsilon,\lambda}) + h.$$

Now we look for a solution of (26) of the form $w = \mathbf{v} + v$. Using the fact that H^i is harmonic, we see that this amounts to solve the equation

$$\begin{aligned} \mathbb{L}\, v = \frac{8}{(1+r^2)^2}\, e^h \left(e^{H^i(\varphi,\cdot/R_{\varepsilon,\lambda})+v} - v - 1\right) + \frac{8}{(1+r^2)^2}\, \left(e^h - 1\right) v \\ + \lambda\Big(\frac{\varepsilon}{\tau}\Big)^{2-q} \left|\nabla[u_1 + H^i(\varphi, \cdot/R_{\varepsilon,\lambda}) + h + v]\right|^q - \lambda\Big(\frac{\varepsilon}{\tau}\Big)^{2-q} |\nabla(u_1 + h)|^q . \end{aligned} \tag{27}$$

We fix

$$\mu \in (1, 2)$$

and denote by $\mathcal{G}_\mu$ to be the right inverse of $\mathbb{L}_\mu$ provided by Proposition 1. To find a solution of (27) it is sufficient to find $v \in \mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)$ solution of

$$v = \mathcal{G}_\mu \circ \mathcal{E}_{R_{\varepsilon,\lambda}} \circ \mathfrak{S}(v), \tag{28}$$

where

$$\begin{aligned} \mathfrak{S}(v) := \frac{8}{(1+r^2)^2}\, e^h \left(e^{H^i(\varphi,\cdot/R_{\varepsilon,\lambda})+v} - v - 1\right) + \frac{8}{(1+r^2)^2}\, (e^h - 1)v \\ + \lambda\Big(\frac{\varepsilon}{\tau}\Big)^{2-q} \left|\nabla\left[u_1 + H^i(\varphi, \cdot/R_{\varepsilon,\lambda}) + h + v\right]\right|^q - \lambda\Big(\frac{\varepsilon}{\tau}\Big)^{2-q} |\nabla(u_1 + h)|^q . \end{aligned}$$

We denote by $\mathcal{N} (:= \mathcal{N}_{\varepsilon,\lambda,\tau,\varphi})$ the nonlinear operator appearing on the right-hand side of equation (28).

Given $\kappa > 1$ (whose value will be fixed later on), we further assume that the functions φ satisfy

$$\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leq \kappa\, r^2_{\varepsilon,\lambda} . \tag{29}$$

Then we have the following result.

LEMMA 6. *Given $\kappa > 1$, $\mu \in (1, 2)$. There exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $c_\kappa > 0$ and $\bar{c}_\kappa > 0$ (which can depend only on κ) such that for all $\varepsilon \in (0, \varepsilon_\kappa)$ and $\lambda \in (0, \lambda_\kappa)$ such that*

$$\|\mathcal{N}(0)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)} \leq c_\kappa r^2_{\varepsilon,\lambda}$$

and

$$\|\mathcal{N}(v_2) - \mathcal{N}(v_1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)} \leq \bar{c}_\kappa\, r^2_{\varepsilon,\lambda}\, \|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)},$$

provided that $v_1, v_2 \in \mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)$ satisfy $\|v_i\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)} \leq 2\, c_\kappa r^2_{\varepsilon,\lambda}$ for $i = 1, 2$.

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Proof. The proof of the first estimate follows from the asymptotic behaviour of H^i together with the assumption on the norm of boundary data φ given by (29). Indeed, let c_κ be a constant depending only on κ (provided ε and λ are chosen small enough), it follows from the estimate of H^i, given by Lemma 3, that

$$\|H^i(\cdot/R_{\varepsilon,\lambda})\|_{\mathcal{C}^{2,\alpha}_2(\bar{B}_{R_{\varepsilon,\lambda}})} \leq c_\kappa\, R_{\varepsilon,\lambda}^{-2}\, ||\varphi||_{\mathcal{C}^{2,\alpha}(S^1)} \leq c_\kappa\, \varepsilon^2.$$

Since for each $x \in \bar{B}_{R_{\varepsilon,\lambda}}$, we have

$$|h(x)| \leq c_\kappa\, r_{\varepsilon,\lambda}^{2+\delta},$$

we prove that $|h(x)| \longrightarrow 0$ as ε and λ tend to 0. Given $\kappa > 0$, there exists $c_\kappa > 0$ such that

$$\left\|(1+|\cdot|^2)^{-2}e^h\left(e^{H^i(\varphi;\cdot/R_{\varepsilon,\lambda})}-1\right)\right\|_{\mathcal{C}^{0,\alpha}_{\mu-2}(\bar{B}_{R_{\varepsilon,\lambda}})} \leq c_\kappa\, \varepsilon^2.$$

On the other hand, making use of Lemma 2, there exists $c_\kappa > 0$ such that

$$\begin{aligned}
&\lambda\left(\frac{\varepsilon}{\tau}\right)^{2-q}\sup_{r\leq R_{\varepsilon,\lambda}} r^{2-\mu}\left|\,\left|\nabla\left[u_1+h+H^i(\varphi,\cdot/R_{\varepsilon,\lambda})\right]\right|^q-|\nabla\left[u_1+h\right]|^q\,\right|\\
&\quad\leq c_\kappa\lambda\left(\frac{\varepsilon}{\tau}\right)^{2-q}\sup_{r\leq R_{\varepsilon,\lambda}} r^{2-\mu}\left(|\nabla u_1|^{q-1}+|\nabla h|^{q-1}+|\nabla H^i(\varphi,\cdot/R_{\varepsilon,\lambda})|^{q-1}\right)|\nabla H^i(\varphi,\cdot/R_{\varepsilon,\lambda})|\\
&\quad\leq c_\kappa r_{\varepsilon,\lambda}^2.
\end{aligned}$$

Making use of Proposition 1 together with (17), we get

$$\|\mathcal{N}(0)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)} \leq c_\kappa r_{\varepsilon,\lambda}^2. \tag{30}$$

To derive the second estimate, we use the fact that, for $v_1, v_2 \in \mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)$ satisfying $\|v_i\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)} \leq 2\, c_\kappa r_{\varepsilon,\lambda}^2$ for $i = 1, 2$, $\mu \in (1, 2)$ and making use of Lemma 2, there exists $c_\kappa > 0$ such that

$$\sup_{r\leq R_{\varepsilon,\lambda}} r^{2-\mu}\, |\mathfrak{S}(v_2)-\mathfrak{S}(v_1)| \leq\, c_\kappa r_{\varepsilon,\lambda}^2 \|v_2-v_1\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)}.$$

Similarly, making use of Proposition 1 together with (17), there exists $\bar{c}_\kappa > 0$ such that

$$\|\mathcal{N}(v_2)-\mathcal{N}(v_1)\|_{\mathcal{C}^{2,\alpha}_\mu(\bar{B}_{R_{\varepsilon,\lambda}})} \leq \bar{c}_\kappa\, r_{\varepsilon,\lambda}^2\, \|v_2-v_1\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)}. \tag{31}$$

□

Reducing $\varepsilon_\kappa > 0$ and $\lambda_\kappa > 0$, if necessary, we can assume that

$$\bar{c}_\kappa r_{\varepsilon,\lambda}^2 \leq \frac{1}{2}$$

for all $\varepsilon \in (0, \varepsilon_\kappa)$ and $\lambda \in (0, \lambda_\kappa)$. Then (30) and (31) are enough to show that

$$v \longmapsto \mathcal{N}(v)$$

is a contraction from the ball

$$\left\{v \in \mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\ :\ \|v\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2\right\}$$

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into itself and hence has a unique fixed point $v(=\bar{v}_{\varepsilon,\lambda,\tau,\varphi})$ in this set. This fixed point is a solution of (28) in $\mathbb{R}^2$. We summarize this in the following proposition.

PROPOSITION 4. *Given $\kappa > 1$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$ and $c_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ for all τ in some fixed compact subset of $[\tau_-, \tau^+] \subset (0, \infty)$ and for a given φ satisfying (23)–(29), there exists a unique $v(:= \bar{v}_{\varepsilon,\lambda,\tau,\varphi})$ solution of (28) such that*

$$w := u_1 + H^i(\varphi, \cdot/R_{\varepsilon,\lambda}) + h + \bar{v}_{\varepsilon,\lambda,\tau,\varphi}$$

solves (26) *in $\bar{B}_{R_{\varepsilon,\lambda}}$. In addition,*

$$\|v\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)} \leq 2\, c_\kappa\, r^2_{\varepsilon,\lambda}.$$

Observe that the function $v(:= \bar{v}_{\varepsilon,\lambda,\tau,\varphi})$ obtained as a fixed point for contraction mappings depends continuously on parameter τ.

4. The nonlinear exterior problem.

Recall that $G(\cdot, \tilde{x})$ denotes the unique solution of

$$-\Delta G(\cdot, \tilde{x}) = 8\,\pi\,\delta_{\tilde{x}}$$

in Ω, with $G(\cdot, \tilde{x}) = 0$ on $\partial\Omega$. In addition, the following decomposition holds

$$G(x, \tilde{x}) = -4\,\log|x - \tilde{x}| + H(x, \tilde{x}),$$

where $x \longmapsto H(x, \tilde{x})$ is a smooth function. Here we will give an estimate of the gradient of $H(x, \tilde{x})$ without proof (see [**25**] and more details in [**21**, Lemma 2.1). There exists a constant $c > 0$ so that

$$|\nabla_x H(x, \tilde{x})| \leq c \log|x - \tilde{x}|.$$

Let $\tilde{\mathbf{x}} := (\tilde{x}_1, \ldots, \tilde{x}_m)$ be close enough to $\mathbf{x} := (x_1, \ldots, x_m)$, $\tilde{\eta} := (\tilde{\eta}^1, \ldots, \tilde{\eta}^m) \in \mathbb{R}^m$ be close to 0 and $\tilde{\varphi} := (\tilde{\varphi}^1, \ldots, \tilde{\varphi}^m) \in (\mathcal{C}^{2,\alpha}(S^1))^m$ satisfying (25). We define

$$\tilde{\mathbf{v}} := \sum_{i=1}^m (1 + \tilde{\eta}^i) G(\cdot, \tilde{x}_i) + \sum_{i=1}^m \chi_{r_0}(\cdot - \tilde{x}_i)\, H^e(\tilde{\varphi}^i; (\cdot - \tilde{x}_i)/r_{\varepsilon,\lambda}), \tag{32}$$

where χ_{r_0} is a cutoff function identically equal to 1 in $B_{r_0/2}$ and identically equal to 0 outside B_{r_0}. We would like to find a solution of the equation

$$\Delta\, u + \lambda|\nabla u|^q + \rho^2 e^u = 0, \tag{33}$$

which is defined in $\bar{\Omega}_{r_{\varepsilon,\lambda}}(\tilde{\mathbf{x}}) := \bar{\Omega} - \cup_{i=1}^m B_{r_{\varepsilon,\lambda}}(\tilde{x}_i)$ and is a perturbation of $\tilde{\mathbf{v}}$. Writing $v = \tilde{\mathbf{v}} + \tilde{v}$, this amounts to solve

$$-\Delta\, \tilde{v} = \rho^2 e^{\tilde{\mathbf{v}} + \tilde{v}} + \lambda|\nabla(\tilde{\mathbf{v}} + \tilde{v})|^q + \Delta\, \tilde{\mathbf{v}}.$$

We need to define an auxiliary weighted space.

DEFINITION 6. Let $\bar r \in (0, r_0/2)$, $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define the Hölder weighted space $\mathcal{C}^{k,\alpha}_\nu(\bar\Omega_{\bar r}(\mathbf{x}))$ as the set of functions $w \in \mathcal{C}^{k,\alpha}(\bar\Omega_{\bar r}(\mathbf{x}))$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu(\bar\Omega_{\bar r}(\mathbf{x}))} := \|w\|_{\mathcal{C}^{k,\alpha}(\bar\Omega_{r_0/2}(\mathbf{x}))} + \sum_{i=1}^m \sup_{r\in[\bar r, r_0/2)} \left(r^{-\nu}\, \|w(x_i + r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar B_2 - B_1)}\right).$$

is finite

For all $\sigma \in (0, r_0/2)$ and all $Y = (y_1, \dots, y_m) \in \Omega^m$ such that $\|X - Y\| \le r_0/2$, where $X = (x_1, \dots, x_m)$, we denote by

$$\tilde{\mathcal{E}}_{\sigma,Y} : \mathcal{C}^{0,\alpha}_\nu(\bar\Omega_\sigma(Y)) \longrightarrow \mathcal{C}^{0,\alpha}_\nu(\bar\Omega^*(Y))$$

the extension operator defined by $\tilde{\mathcal{E}}_{\sigma,Y}(f) = f$ in $\bar\Omega_\sigma(Y)$

$$\tilde{\mathcal{E}}_{\sigma,Y}(f)(y_i + x) = \tilde\chi\left(\frac{|x|}{\sigma}\right) f\left(y_i + \sigma\frac{x}{|x|}\right)$$

in $B_\sigma(y_i) - B_{\sigma/2}(y_i)$, for each $i = 1, \dots, m$ and $\tilde{\mathcal{E}}_{\sigma,Y}(f) = 0$ in each $B_{\sigma/2}(y_i)$, where $t \longmapsto \tilde\chi(t)$ is a cutoff function identically equal to 1 for $t \ge 1$ and identically equal to 0 for $t \le 1/2$. It is easy to check that there exists a constant $c = c(\nu) > 0$ only depending on ν such that

$$\|\tilde{\mathcal{E}}_{\sigma,Y}(w)\|_{\mathcal{C}^{0,\alpha}_\nu(\bar\Omega^*(Y))} \le c\,\|w\|_{\mathcal{C}^{0,\alpha}_\nu(\bar\Omega_\sigma(Y))}. \tag{34}$$

We fix

$$\nu \in (-1, 0),$$

and denote by $\tilde{\mathcal{G}}_\nu : \mathcal{C}^{0,\alpha}_{\nu-2}(\bar\Omega^*(\tilde{\mathbf{x}})) \longrightarrow \mathcal{C}^{2,\alpha}_\nu(\bar\Omega^*(\tilde{\mathbf{x}}))$ the right inverse of Δ provided by Proposition 3 with $\bar\Omega^*(\tilde{\mathbf{x}}) = \bar\Omega - \{\tilde x_1, \dots, \tilde x_m\}$. Clearly, it is enough to find $\tilde v \in \mathcal{C}^{2,\alpha}_\nu(\bar\Omega^*(\tilde{\mathbf{x}}))$ solution of

$$\tilde v = \tilde{\mathcal{G}}_\nu \circ \tilde{\mathcal{E}}_{r_{\varepsilon,\lambda},\tilde{\mathbf{x}}}\left(\rho^2 e^{\tilde{\mathbf{v}}+\tilde v} + \lambda|\nabla(\tilde{\mathbf{v}} + \tilde v)|^q + \Delta\,\tilde{\mathbf{v}}\right) = \tilde{\mathcal{G}}_\nu \circ \tilde{\mathcal{E}}_{r_{\varepsilon,\lambda},\tilde{\mathbf{x}}} \circ \tilde{\mathfrak{R}}(\tilde v), \tag{35}$$

where

$$\tilde{\mathfrak{R}}(\tilde v) = \rho^2 e^{\tilde{\mathbf{v}}+\tilde v} + \lambda|\nabla(\tilde{\mathbf{v}} + \tilde v)|^q + \Delta\,\tilde{\mathbf{v}}.$$

We denote by $\tilde{\mathcal{N}}(:= \tilde{\mathcal{N}}_{\varepsilon,\lambda,\tilde\eta,\tilde{\mathbf{x}},\tilde\varphi})$ the nonlinear operator that appears on the right-hand side of equation (35).

Given $\kappa > 1$, we assume that the point $\tilde{\mathbf{x}} = (\tilde x_1, \dots, \tilde x_m)$, the function $\tilde\varphi = (\tilde\varphi^1, \dots, \tilde\varphi^m)$ and the parameter $\tilde\eta = (\tilde\eta^1, \dots, \tilde\eta^m)$ satisfy

$$|\tilde x_i - x_i| \le \kappa\, r_{\varepsilon,\lambda}, \tag{36}$$

$$\|\tilde\varphi^i\|_{\mathcal{C}^{2,\alpha}} \le \kappa\, r^2_{\varepsilon,\lambda} \tag{37}$$

and

$$|\tilde{\eta}^i| \le \kappa r_{\varepsilon,\lambda}^2. \tag{38}$$

Then the following result holds.

LEMMA 7. *Given $\kappa > 1$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$, $c_\kappa > 0$ and $\bar{c}_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$ and $\lambda \in (0, \lambda_\kappa)$ we have,*

$$\|\tilde{\mathcal{N}}(0)\|_{\mathcal{C}_\nu^{2,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \le c_\kappa\, r_{\varepsilon,\lambda}^2$$

and

$$\|\tilde{\mathcal{N}}(\tilde{v}_2) - \tilde{\mathcal{N}}(\tilde{v}_1)\|_{\mathcal{C}_\nu^{2,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \le \bar{c}_\kappa\, r_{\varepsilon,\lambda}^2\, \|\tilde{v}_2 - \tilde{v}_1\|_{\mathcal{C}_\nu^{2,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))},$$

provided $\tilde{v}_1, \tilde{v}_2 \in \mathcal{C}_\nu^{2,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))$ and satisfying $\|\tilde{v}_i\|_{\mathcal{C}_\nu^{2,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \le 2\, c_\kappa\, r_{\varepsilon,\lambda}^2$ for $i = 1, 2$.

Proof. The proof of the first estimate follows from the asymptotic behaviour of H^e together with the assumption on the norm of boundary data $\tilde{\varphi}^i$ given by (37). Indeed, let c_κ be a constant depending only on κ (provided ε and λ are chosen small enough), it follows from the estimate of H^e, given by lemma 4, that

$$|H^e(\tilde{\varphi}^i; (x - \tilde{x}_i)/r_{\varepsilon,\lambda})| \le c_\kappa r_{\varepsilon,\lambda}^3 r^{-1}. \tag{39}$$

Recalling that $\tilde{\mathcal{N}}(\tilde{v}) = \tilde{\mathcal{G}}_\nu \circ \tilde{\mathfrak{R}}(\tilde{v})$, we will estimate $\tilde{\mathcal{N}}(0)$ in different subregions of $\bar{\Omega}^*(\tilde{\mathbf{x}})$.

- In $B_{r_0}(\tilde{x}_i)$, we have $\chi_{r_0}(x - \tilde{x}_i) = 1$ and $\Delta\tilde{\mathbf{v}} = 0$ so that

$$\begin{aligned}|\tilde{\mathfrak{R}}(0)| &\le c_\kappa \varepsilon^2 |x - \tilde{x}_i|^{-4(1+\tilde{\eta}^i)} \prod_{\ell=1, \ell\neq i}^{m} |x - \tilde{x}_\ell|^{-4(1+\tilde{\eta}^\ell)} + c_\kappa \lambda |\nabla \tilde{\mathbf{v}}|^{\mathbf{q}} \le \mathbf{c}_\kappa \varepsilon^{\mathbf{2}} \mathbf{r}^{-\mathbf{4}(\mathbf{1}+\tilde{\eta}^{\mathbf{i}})} \\ &\quad + c_\kappa \lambda \left|4(1 + \tilde{\eta}^i) r^{-1} + (1 + \tilde{\eta}^i)|\nabla_x H(x, \tilde{x})| + |\nabla H^e(\tilde{\varphi}^i; (\cdot - \tilde{x}_i)/r_{\varepsilon,\lambda})|\right|^q \\ &\le c_\kappa \varepsilon^2 r^{-4(1+\tilde{\eta}^i)} + c_\kappa \lambda \left((1 + \tilde{\eta}^i) r^{-1} + (1 + \tilde{\eta}^i)|\log r| + r_{\varepsilon,\lambda}^3 r^{-2}\right)^q.\end{aligned}$$

  Hence, for $\nu \in (-1, 0)$ and $\tilde{\eta}^i$ small enough, we get

$$||\tilde{\mathfrak{R}}(0)||_{\mathcal{C}_{\nu-2}^{0,\alpha}(\bigcup_{i=1}^m B_{r_0}(\tilde{x}_i))} \le \sup_{r_{\varepsilon,\lambda} \le r \le r_0/2} r^{2-\nu} |\tilde{\mathfrak{R}}(0)| \le c_\kappa \varepsilon^2 r_{\varepsilon,\lambda}^{-2} + c_\kappa \lambda.$$

- In $\bar{\Omega} - B_{r_0}(\tilde{x}_i)$, we have $\chi_{r_0}(x - \tilde{x}_i) = 0$ and $\Delta\tilde{\mathbf{v}} = 0$. Thus,

$$|\tilde{\mathfrak{R}}(0)| \le c_\kappa \varepsilon^2 \prod_{\ell=1}^{m} e^{(1+\tilde{\eta}^\ell)\, G(x, \tilde{x}_\ell)} + c_\kappa \lambda \left((1 + \tilde{\eta}^i) r^{-1} + (1 + \tilde{\eta}^i)|\log r| + r_{\varepsilon,\lambda}^3 r^{-2}\right)^q.$$

  So for $\nu \in (-1, 0)$ we have

$$||\tilde{\mathfrak{R}}(0)||_{\mathcal{C}_{\nu-2}^{0,\alpha}(\bar{\Omega} - \bigcup_{i=1}^m B_{r_0}(\tilde{x}_i))} \le \sup_{r_0 \le r} r^{2-\nu} |\tilde{\mathfrak{R}}(0)| \le c_\kappa \varepsilon^2 + c_\kappa \lambda.$$

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- In $B_{r_0}(\tilde{x}_i) - B_{r_0/2}(\tilde{x}_i)$, using the estimate (39), we have

$$\begin{aligned}|\tilde{\mathfrak{R}}(0)| &\le c_\kappa\, \varepsilon^2 r^{-4(1+\tilde{\eta}^i)} + c_\kappa \lambda \left((1+\tilde{\eta}^i) r^{-1} + (1+\tilde{\eta}^i)|\log r| + r_{\varepsilon,\lambda}^3 r^{-2} \right)^q \\ &\quad + \sum_{i=1}^m |[\Delta, \chi_{r_0}(x-\tilde{x}_i)]|\, |H^e(\tilde{\varphi}^i; (x-\tilde{x}_i)/r_{\varepsilon,\lambda})| \\ &\le c_\kappa\, (\varepsilon^2 + \lambda \left((1+\tilde{\eta}^i) r^{-1} + (1+\tilde{\eta}^i)|\log r| + r_{\varepsilon,\lambda}^3 r^{-2} \right)^q + r^{-1} r_{\varepsilon,\lambda}^3),\end{aligned}$$

where

$$[\Delta, \chi_{r_0}]w = \Delta w \chi_{r_0} + w \Delta \chi_{r_0} + 2\nabla w \cdot \nabla \chi_{r_0}.$$

Then,

$$||\tilde{\mathfrak{R}}(0)||_{\mathcal{C}^{0,\alpha}_{\nu-2}((B_{r_0}(\tilde{x}_i) - \bigcup_{i=1}^m B_{r_0/2}(\tilde{x}_i)))} \le \sup_{r_0/2 \le r \le r_0} r^{2-\nu} |\tilde{\mathfrak{R}}(0)| \le c_\kappa r_{\varepsilon,\lambda}^2 + c_\kappa \lambda.$$

So,

$$\|\tilde{\mathfrak{R}}(0)\|_{\mathcal{C}^{0,\alpha}_{\nu-2}((\Omega - \bigcup_{i=1}^m B_{r_0}(\tilde{x}_i)))} \le c_\kappa\, r_{\varepsilon,\lambda}^2. \tag{40}$$

Making use of Proposition 3 together with (34) we conclude that

$$\|\tilde{\mathcal{N}}(0)\|_{\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\mathbf{x}))} \le c_\kappa\, r_{\varepsilon,\lambda}^2. \tag{41}$$

For the proof of the second estimate, let $\tilde{v}_1$ and $\tilde{v}_2 \in C^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{x}}))$ satisfying $\|\tilde{v}_i\|_{\mathcal{C}^{2,\alpha}_\mu(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \le 2\, c_\kappa r_{\varepsilon,\lambda}^2$ for $i = 1, 2$, we have

$$|\tilde{\mathfrak{R}}(\tilde{v}_2) - \tilde{\mathfrak{R}}(\tilde{v}_1)| \le c_\kappa \varepsilon^2 e^{\tilde{\mathbf{v}}} \left| (e^{\tilde{v}_2} - e^{\tilde{v}_1}) \right| + c_\kappa \lambda \Big| |\nabla(\tilde{\mathbf{v}} + \tilde{v}_2)|^q - |\nabla(\tilde{\mathbf{v}} + \tilde{v}_1)|^q | \Big|.$$

Then for all small $\tilde{\eta}^i \in \mathbb{R}_+$, making use of Lemma 2, there exists a positive constant c_κ such that

$$\begin{aligned}|\tilde{\mathfrak{R}}(\tilde{v}_2) - \tilde{\mathfrak{R}}(\tilde{v}_1)| &\le c_\kappa \varepsilon^2 |x - \tilde{x}_i|^{-4(1+\tilde{\eta}^i)}\, |\tilde{v}_2 - \tilde{v}_1| + c_\kappa \lambda \Big(|\nabla(\tilde{\mathbf{v}} + \tilde{v}_2)|^q - |\nabla(\tilde{\mathbf{v}} + \tilde{v}_1)|^q \Big) \\ &\le c_\kappa \varepsilon^2 r^{-4(1+\tilde{\eta}^i)}\ |\tilde{v}_2 - \tilde{v}_1| + c_\kappa \lambda\, (|\nabla \tilde{\mathbf{v}}|^{q-1} + |\nabla \tilde{v}_1|^{q-1} \\ &\quad + |\nabla \tilde{v}_2|^{q-1}) |\nabla(\tilde{v}_2 - \tilde{v}_1)|.\end{aligned}$$

So for $\tilde{\eta}^i$ small enough and using the estimate (34), there exists $\bar{c}_\kappa$ (depending on κ) such that

$$\|\tilde{\mathcal{N}}(\tilde{v}_2) - \tilde{\mathcal{N}}(\tilde{v}_1)\|_{\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \le \bar{c}_\kappa\, r_{\varepsilon,\lambda}^2 \|\tilde{v}_2 - \tilde{v}_1\|_{\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{x}}))}. \tag{42}$$

Then we get the second estimate. □

Reducing $\lambda_\kappa > 0$ and $\varepsilon_\kappa > 0$ if necessary, we can assume that

$$\bar{c}_\kappa r_{\varepsilon,\lambda}^2 \le \frac{1}{2}$$

for all $\lambda \in (0, \lambda_\kappa)$ and $\varepsilon \in (0, \varepsilon_\kappa)$. Then (41) and (42) are enough to show that

$$\tilde{v} \longmapsto \tilde{\mathcal{N}}(\tilde{v})$$

is a contraction from the ball

$$\{\tilde v \in \mathcal{C}^{2,\alpha}_\nu(\mathbb{R}^2) \ : \ \|\tilde v\|_{\mathcal{C}^{2,\alpha}_\nu(\mathbb{R}^2)} \le 2c_\kappa r^2_{\varepsilon,\lambda}\}$$

into itself and hence has a unique fixed point $\tilde v (:= \bar v_{\varepsilon,\lambda,\tilde\eta,\tilde{\mathbf{x}},\tilde\varphi})$ in this set. This fixed point is a solution of (35). We summarize this in the following proposition.

PROPOSITION 5. *Given $\kappa > 1$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$ and $c_\kappa > 0$ (depending on κ) such that for all $\varepsilon \in (0, \varepsilon_\kappa)$ and $\lambda \in (0, \lambda_\kappa)$, for all set of parameters $\tilde\eta^i$ satisfying (38), the points $\tilde x_i$ satisfying (36) and function $\tilde\varphi$ satisfying (25)–(37), there exists a unique $\tilde v (:= \tilde v_{\varepsilon,\lambda,\tilde\eta,\tilde{\mathbf{x}},\tilde\varphi})$ solution of (35) such that*

$$\tilde u := \sum_{i=1}^m (1+\tilde\eta^i) G(\cdot, \tilde x_i) + \sum_{i=1}^m \chi_{r_0}(\cdot - \tilde x_i)\, H^e(\tilde\varphi^i; (\cdot - \tilde x_i)/r_{\varepsilon,\lambda}) + \tilde v_{\varepsilon,\lambda,\tilde\eta,\tilde{\mathbf{x}},\tilde\varphi}$$

solves (33) in $\bar\Omega_{r_{\varepsilon,\lambda}}(\tilde{\mathbf{x}})$. In addition,

$$\|\tilde v\|_{\mathcal{C}^{2,\alpha}_\nu(\bar\Omega^*(\mathbf{x}))} \le 2\, c_\kappa\, r^2_{\varepsilon,\lambda}.$$

As in the previous section, observe that the function $\tilde v (:= \tilde v_{\varepsilon,\lambda,\tilde\eta,\tilde{\mathbf{x}},\tilde\varphi})$ being obtained as a fixed point for contraction mapping, depends smoothly on the parameters $\tilde\eta$ and the points $\tilde{\mathbf{x}}$.

5. The nonlinear Cauchy-data matching. Keeping the notations of the previous sections, we gather the results of Propositions 4 and 5. Assume that $\tilde{\mathbf{x}} := (\tilde x_1, \ldots, \tilde x_m) \in \Omega^m$ is given close to $\mathbf{x} := (x_1, \ldots, x_m)$ and satisfies (36). Assume also that $\tau := (\tau_1, \ldots, \tau_m) \in [\tau_-, \tau^+]^m \subset (0, \infty)^m$ is given (the values of τ_- and τ^+ will be fixed shortly). First, we consider some set of boundary data $\varphi := (\varphi^1, \ldots, \varphi^m) \in (\mathcal{C}^{2,\alpha}(S^1))^m$ satisfying (23). We set

$$R^i_{\varepsilon,\lambda} = \tau_i r_{\varepsilon,\lambda}/\varepsilon$$

and recall that

$$\rho^2 = \frac{8\,\varepsilon^2}{(1+\varepsilon^2)^2}.$$

According to the result of Proposition 4, we can find v^i_{int} a solution of

$$\Delta u + \lambda |\nabla u|^q + \rho^2 e^u = 0 \tag{43}$$

in each $B_{r_{\varepsilon,\lambda}}(\tilde x_i)$ that can be decomposed as

$$\begin{aligned} v^i_{\mathrm{int}}(x) = {} & u_{\varepsilon,\tau_i}(x - \tilde x_i) + h(R^i_{\varepsilon,\lambda}(x - \tilde x_i)/r_{\varepsilon,\lambda}) + H^i(\varphi^i; (x - \tilde x_i)/r_{\varepsilon,\lambda}) \\ & + \bar v_{\varepsilon,\lambda,\tau_i,\varphi^i}(R^i_{\varepsilon,\lambda}(x - \tilde x_i)/r_{\varepsilon,\lambda}), \end{aligned}$$

where the function $v^i = \bar v_{\varepsilon,\lambda,\tau_i,\varphi^i}$ satisfies

$$\|v^i\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)} \le 2\, c_\kappa r^2_{\varepsilon,\lambda}\,. \tag{44}$$

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Similarly, given some boundary data $\tilde\varphi = (\tilde\varphi^1, \dots, \tilde\varphi^m) \in (\mathcal{C}^{2,\alpha}(S^1))^m$ satisfying (25), some parameters $\tilde\eta := (\tilde\eta^1, \dots, \tilde\eta^m) \in \mathbb{R}^m$ satisfying (38), provided $\varepsilon \in (0, \varepsilon_\kappa)$ and $\lambda \in (0, \lambda_\kappa)$, we use the result of Proposition 5 to find a solution $v_{\rm ext}$ of (43), which can be decomposed as

$$v_{\rm ext} = \sum_{i=1}^m (1+\tilde\eta^i) G(\cdot, \tilde x_i) + \sum_{i=1}^m \chi_{r_0}(\cdot - \tilde x_i)\, H^e(\tilde\varphi^i; (\cdot - \tilde x_i)/r_{\varepsilon,\lambda}) + \tilde v_{\varepsilon,\lambda,\tilde\eta,\tilde{\mathbf{x}},\tilde\varphi}$$

in $\bar\Omega_{r_{\varepsilon,\lambda}}$ where the function $\tilde v (:= \tilde v_{\varepsilon,\lambda,\tilde\eta,\tilde{\mathbf{x}},\tilde\varphi}) \in \mathcal{C}^{2,\alpha}_\nu(\bar\Omega^*(\tilde{\mathbf{x}}))$ satisfies

$$\|\tilde v\|_{\mathcal{C}^{2,\alpha}_\nu(\bar\Omega^*(\tilde{\mathbf{x}}))} \le 2\, c_\kappa\, r^2_{\varepsilon,\lambda}. \tag{45}$$

It remains to determine the parameters and functions in such a way that the function which is equal to $v^i_{\rm int}$ in $\cup_{i=1}^m B_{r_{\varepsilon,\lambda}}(\tilde x_i)$ and is equal to $v_{\rm ext}$ in $\bar\Omega_{r_{\varepsilon,\lambda}}(\tilde{\mathbf{x}}))$ is smooth. This amounts to find the boundary data and the parameters so that for each $i = 1, \dots, m$

$$v^i_{\rm int} = v_{\rm ext} \quad \text{and} \quad \partial_r v^i_{\rm int} = \partial_r v_{\rm ext} \tag{46}$$

on $\partial B_{r_{\varepsilon,\lambda}}(\tilde x_i)$. Assuming we have already done so, this provides for each ε and λ small enough a function $v_{\varepsilon,\lambda} \in \mathcal{C}^{2,\alpha}$ (which is obtained by patching together the functions $v^i_{\rm int}$ and the function $v_{\rm ext}$) solution of $-\Delta u - \lambda|\nabla u|^q = \rho^2 e^u$, and the elliptic regularity theory implies that this solution is in fact smooth. This will complete the proof of our result since, as ε and λ tend to 0, the sequence of solutions that we have obtained satisfies the required properties, namely away from the points x_i the sequence $v_{\varepsilon,\lambda}$ converges to $\sum_i G(\cdot, x_i)$.

Before we proceed, the following remarks are due. First, it will be convenient to observe that the function u_{ε,τ_i} can be expanded as

$$u_{\varepsilon,\tau_i}(x) = -2\log\tau_i - 4\log|x| + \mathcal{O}\left(\frac{\varepsilon^2\tau_i^{-2}}{|x|^2}\right) \tag{47}$$

near $\partial B_{r_{\varepsilon,\lambda}}$. The function

$$\sum_{\ell=1}^m (1+\tilde\eta^\ell)\, G(x, \tilde x_\ell),$$

which appear in the expression of $v_{\rm ext}$, can be expanded as

$$\sum_{\ell=1}^m (1+\tilde\eta^\ell)\, G(x+\tilde x_i, \tilde x_\ell) = -4(1+\tilde\eta^i)\,\log|x| + \mathcal{F}_i(\tilde{\mathbf{x}}; \tilde x_i) + \nabla\mathcal{F}_i(\tilde{\mathbf{x}}; \tilde x_i)\cdot x + \mathcal{O}(r^2_{\varepsilon,\lambda}) \tag{48}$$

near $\partial B_{r_{\varepsilon,\lambda}}(\tilde x_i)$. Here we have defined

$$\mathcal{F}_i(\tilde{\mathbf{x}}; \cdot) := H(\tilde x_i, \cdot) + \sum_{\ell\ne i} G(\tilde x_\ell, \cdot).$$

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Thus, for x near $\partial B_{r_{\varepsilon,\lambda}}$, we have

$$
\begin{aligned}
&(v^i_{\rm int} - v_{\rm ext})(x)\\
&\quad = -2\log\tau_i + 4\tilde\eta^i \log|x-\tilde x_i| + h\left(R^i_{\varepsilon,\lambda}(x-\tilde x_i)/r_{\varepsilon,\lambda}\right) + H^i(\varphi^i;(x-\tilde x_i)/r_{\varepsilon,\lambda})\\
&\qquad -H^e(\tilde\varphi^i;(x-\tilde x_i)/r_{\varepsilon,\lambda})\\
&\qquad -\left((1+\tilde\eta^i)H(x,\tilde x_i) + \sum_{\ell=1,\ell\neq i}^m (1+\tilde\eta^\ell)G(x,\tilde x_\ell)\right) + \mathcal{O}\left(\frac{\varepsilon^2\tau_i^{-2}}{|x-\tilde x_i|^2}\right) + \mathcal{O}(r^2_{\varepsilon,\lambda})\\
&\quad = -2\log\tau_i + 4\tilde\eta^i\log|x| - \left((1+\tilde\eta^i)H(\tilde x_i,\tilde x_i) + \sum_{\ell=1,\ell\neq i}^m (1+\tilde\eta^\ell)G(\tilde x_i,\tilde x_\ell)\right)\\
&\qquad + \mathcal{O}(|x-\tilde x_i|^2) + \mathcal{O}\left(\frac{\varepsilon^2\tau_i^{-2}}{|x-\tilde x_i|^2}\right) + \mathcal{O}(r^2_{\varepsilon,\lambda})\\
&\quad = -2\log\tau_i + 4\tilde\eta^i\log r_{\varepsilon,\lambda} - \mathcal{F}_i(\tilde x_i,\tilde{\mathbf{x}}) + \mathcal{O}(r^2_{\varepsilon,\lambda}),
\end{aligned}
\tag{49}
$$

where $\tilde{\mathbf{x}} = (\tilde x_1,\dots,\tilde x_m)$.

Next, in (46) all functions are defined on $\partial B_{r_{\varepsilon,\lambda}}(\tilde x_i)$, but it will be convenient to solve the following equations:

$$
\left(v^i_{\rm int} - v_{\rm ext}\right)(\tilde x_i + r_{\varepsilon,\lambda}\,\cdot) = 0 \quad \text{and} \quad \partial_r(v^i_{\rm int} - v_{\rm ext})(\tilde x_i + r_{\varepsilon,\lambda}\,\cdot) = 0 \tag{50}
$$

on S^1. Here all functions are considered as functions of $y\in S^1$ and we have simply used the change of variables $x = \tilde x_i + r_{\varepsilon,\lambda} y$ to parameterize $\partial B_{r_{\varepsilon,\lambda}}(\tilde x_i)$.

Since the boundary data we have chosen satisfy (23) and (25), we can decompose

$$
\varphi^i = \varphi^i_0 + \varphi^i_1 + \varphi^{i,\perp} \quad \text{and} \quad \tilde\varphi^i = \tilde\varphi^i_0 + \tilde\varphi^i_1 + \tilde\varphi^{i,\perp},
$$

where $\varphi^i_0, \tilde\varphi^i_0 \in \mathbb{E}_0 = \mathbb{R}$ are constant functions on S^1, $\varphi^i_1, \tilde\varphi^i_1$ belong to $\mathbb{E}_1 = ker(\Delta_{S^1} + 1) = Span\{e_1, e_2\}$ and $\varphi^{i,\perp}, \tilde\varphi^{i,\perp}$ are $L^2(S^1)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

Projecting equation (50) over $\mathbb{E}_0$ will yield the system

$$
\begin{cases}
-2\log\tau_i + 4\tilde\eta^i\log r_{\varepsilon,\lambda} - \mathcal{F}_i(\tilde x_i,\tilde{\mathbf{x}}) + \mathcal{O}(r^2_{\varepsilon,\lambda}) = 0\\
\qquad\qquad\qquad\qquad\qquad\qquad 4\tilde\eta^i + \mathcal{O}(r^2_{\varepsilon,\lambda}) = 0
\end{cases}. \tag{51}
$$

Let us briefly comment on how these equations are obtained. These simply come from (50) when expansions (47) and (48) are used, together with the expression of H^i and H^e given in Lemmas 3 and 4, and also the estimates (44) and (45). The system (51) can be readily simplified into

$$
\frac{1}{\log r_{\varepsilon,\lambda}}\left[2\log\tau_i + \mathcal{F}_i(\tilde x_i,\tilde{\mathbf{x}})\right] = \mathcal{O}(r^2_{\varepsilon,\lambda}) \quad \text{and} \quad \tilde\eta^i = \mathcal{O}(r^2_{\varepsilon,\lambda}).
$$

We are now in a position to define τ_- and τ^+ since, according to above, as ε and λ tend to 0, we expect that $\tilde x_i$ will converge to x_i and τ_i will converge to τ_i^* satisfying

$$
2\log\tau_i^* = -\mathcal{F}_i(x_i,\mathbf{x}),
$$

and hence it is enough to choose τ_- and τ^+ in such a way that

$$
2\log(\tau_-) < -\sup_i \mathcal{F}_i(x_i,\mathbf{x}) \leq -\inf_i \mathcal{F}_i(x_i,\mathbf{x}) < 2\log(\tau^+).
$$

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We now consider the L^2-projection of (50) over $\mathbb{E}_1$. Given a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, \partial_{x_2} f)$ with the element of $\mathbb{E}_1$

$$\bar{\nabla} f = \sum_{i=1}^{2} \partial_{x_i} f\, e_i.$$

With these notations in mind, we obtain equations

$$\bar{\nabla}\mathcal{F}_i(\tilde{x}_i, \tilde{\mathbf{x}}) = \mathcal{O}(r^2_{\varepsilon,\lambda}) \quad \text{and} \quad \varphi_1^i = \mathcal{O}(r^2_{\varepsilon,\lambda}). \tag{52}$$

Finally, we consider the L^2-projection onto $L^2(S^1)^\perp$. This yields the system

$$\begin{cases} \varphi^{i,\perp} - \tilde{\varphi}^{i,\perp} + \mathcal{O}(r^2_{\varepsilon,\lambda}) = 0 \\ \partial_r\big(H^{i,\perp} - H^{e,\perp}\big) + \mathcal{O}(r^2_{\varepsilon,\lambda}) = 0 \end{cases}. \tag{53}$$

Thanks to the result of Lemma 5, this last system can be rewritten as

$$\varphi^{i,\perp} = \mathcal{O}(r^2_{\varepsilon,\lambda}) \quad \text{and} \quad \tilde{\varphi}^{i,\perp} = \mathcal{O}(r^2_{\varepsilon,\lambda}).$$

If we define the parameters $\mathbf{t} = (t_i) \in \mathbb{R}^m$ by

$$t_i = \frac{1}{\log r_{\varepsilon,\lambda}}\Big[2\log\tau_i + \mathcal{F}_i(\tilde{x}_i, \tilde{x})\Big], \quad \forall\, 1 \le i \le m,$$

then the system that we have to solve reads

$$\Big(\mathbf{t}, \tilde{\eta}, \varphi_0, \tilde{\varphi}_0, \varphi_1, \tilde{\varphi}_1, \bar{\nabla}\mathcal{F}(\tilde{\mathbf{x}}, \mathbf{x}), \varphi^\perp, \tilde{\varphi}^\perp\Big) = \mathcal{O}(r^2_{\varepsilon,\lambda}), \tag{54}$$

where, as usual, the term $\mathcal{O}(r^2_{\varepsilon,\lambda})$ depends nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of ε and λ) time $r^2_{\varepsilon,\lambda}$, provided $\varepsilon \in (0, \varepsilon_\kappa)$ and $\lambda \in (0, \lambda_\kappa)$. Then the nonlinear mapping, which appears on the right-hand side of (54), is continuous and compact. In addition, reducing ε_κ and λ_κ if necessary, this nonlinear mapping sends the ball of radius $\kappa\, r^2_{\varepsilon,\lambda}$ (for the natural product norm) into itself, provided κ is fixed large enough. Applying Schauder's Fixed Theorem in the ball of radius $\kappa r^2_{\varepsilon,\lambda}$ in the product space where the entries live yield the existence of a solution of equation (54), and this completes the proof of Theorem 1. □

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