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RESEARCH ARTICLE

Set superpartitions and superspace duality modules

Brendon Rhoades [1] and Andrew Timothy Wilson [2]

[1]Department of Mathematics, University of California, San Diego, La Jolla, CA, 92093, USA; E-mail: bprhoades@ucsd.edu.
[2]Department of Mathematics, Kennesaw State University, Marietta, GA, 30060, USA; E-mail: andwils342@kennesaw.edu.

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Abstract
The superspace ring Ω_n is a rank n polynomial ring tensored with a rank n exterior algebra. Using an extension of the Vandermonde determinant to Ω_n, the authors previously defined a family of doubly graded quotients $\mathbb{W}_{n,k}$ of Ω_n, which carry an action of the symmetric group $\mathfrak{S}_n$ and satisfy a bigraded version of Poincaré Duality. In this paper, we examine the duality modules $\mathbb{W}_{n,k}$ in greater detail. We describe a monomial basis of $\mathbb{W}_{n,k}$ and give combinatorial formulas for its bigraded Hilbert and Frobenius series. These formulas involve new combinatorial objects called *ordered set superpartitions*. These are ordered set partitions $(B_1 \mid \cdots \mid B_k)$ of $\{1, \ldots, n\}$ in which the nonminimal elements of any block B_i may be barred or unbarred.

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1. Introduction

Let n be a positive integer. *Superspace* of rank n (over the ground field $\mathbb{Q}$) is the tensor product

$$\Omega_n = \mathbb{Q}[x_1, \dots, x_n] \otimes \wedge\{\theta_1, \dots, \theta_n\} \tag{1.1}$$

of a rank n polynomial ring with a rank n exterior algebra. The ring Ω_n carries a 'diagonal' action of the symmetric group $\mathfrak{S}_n$ on n letters, viz.

$$w \cdot x_i = x_{w(i)} \qquad w \cdot \theta_i = \theta_{w(i)} \qquad w \in \mathfrak{S}_n, \;\; 1 \leq i \leq n, \tag{1.2}$$

which turns Ω_n into a bigraded $\mathfrak{S}_n$-module by considering x-degree and θ-degree separately.

The ring Ω_n appears in physics, where the 'bosonic' x_i variables model the states of bosons and the 'fermionic' θ_i variables model the states of fermions [22]. A number of recent papers in algebraic combinatorics consider $\mathfrak{S}_n$-modules constructed with a mix of commuting and anticommuting variables [2, 5, 17, 27, 34]. The Fields Institute Combinatorics Group made the tantalising conjecture (see [34]) that, if $\langle (\Omega_n)_+^{\mathfrak{S}_n} \rangle \subseteq \Omega_n$ denotes the ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term, we have an $\mathfrak{S}_n$-module isomorphism

$$\Omega_n / \langle (\Omega_n)_+^{\mathfrak{S}_n} \rangle \cong \mathbb{Q}[\mathcal{OP}_n] \otimes \mathrm{sign}, \tag{1.3}$$

where $\mathcal{OP}_n$ denotes the family of all ordered set partitions of $[n] = \{1, \dots, n\}$ (with its natural permutation action of $\mathfrak{S}_n$) and sign is the 1-dimensional sign representation of $\mathfrak{S}_n$. Despite significant progress [31], the conjecture (1.3) has remained out of reach; even proving $\Omega_n / \langle (\Omega_n)_+^{\mathfrak{S}_n} \rangle$ has the expected vector space dimension remains open.

The *Vandermonde determinant* $\delta_n \in \mathbb{Q}[x_1, \dots, x_n]$ is the polynomial

$$\delta_n = \varepsilon_n \cdot (x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 x_n^0), \tag{1.4}$$

where $\varepsilon_n = \sum_{w \in \mathfrak{S}_n} \mathrm{sign}(w) \cdot w \in \mathbb{Q}[\mathfrak{S}_n]$ is the antisymmetrising element of the symmetric group algebra. Given positive integers $k \leq n$, the authors defined [27] the following extension $\delta_{n,k}$ of the Vandermonde to superspace:

$$\delta_{n,k} = \varepsilon_n \cdot (x_1^{k-1} \cdots x_{n-k}^{k-1} x_{n-k+1}^{k-1} x_{n-k+2}^{k-2} \cdots x_{n-1}^1 x_n^0 \cdot \theta_1 \cdots \theta_{n-k}). \tag{1.5}$$

When $k = n$, we recover the classical Vandermonde: $\delta_{n,n} = \delta_n$. The $\delta_{n,k}$ may be used to build bigraded $\mathfrak{S}_n$-stable quotient rings $\mathbb{W}_{n,k}$ of Ω_n as follows.

For $1 \leq i \leq n$, the partial derivative operator $\partial/\partial x_i$ acts naturally on the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$ and, by treating the $\theta_1, \dots, \theta_n$ as constants, on the ring Ω_n. We also have a $\mathbb{Q}[x_1, \dots, x_n]$-linear operator $\partial/\partial\theta_i$ on Ω_n defined by

$$\partial/\partial\theta_i : \theta_{j_1} \cdots \theta_{j_r} \mapsto \begin{cases} (-1)^{s-1} \theta_{j_1} \cdots \theta_{j_{s-1}} \theta_{j_{s+1}} \cdots \theta_{j_r} & \text{if } j_s = i \text{ for some } s, \\ 0 & \text{otherwise,} \end{cases} \tag{1.6}$$

where $1 \le j_1, \dots, j_r \le n$ are any distinct indices.[1] These operators satisfy the defining relations of Ω_n, namely

$$(\partial/\partial x_i)(\partial/\partial x_j) = (\partial/\partial x_j)(\partial/\partial x_i), \quad (\partial/\partial x_i)(\partial/\partial \theta_j) = (\partial/\partial \theta_j)(\partial/\partial x_i),$$

$$(\partial/\partial \theta_i)(\partial/\partial \theta_j) = -(\partial/\partial \theta_j)(\partial/\partial \theta_i)$$

for all $1 \le i, j \le n$. Given $f \in \Omega_n$, we, therefore, have a well-defined operator ∂f on Ω_n given by replacing each x_i in f with $\partial/\partial x_i$ and each θ_i in f with $\partial/\partial \theta_i$. This gives rise to an action, denoted $\odot$, of Ω_n on itself:

$$\odot : \Omega_n \times \Omega_n \longrightarrow \Omega_n \qquad f \odot g = (\partial f)(g). \tag{1.7}$$

Definition 1.1 (Rhoades-Wilson [27]). Given positive integers $k \le n$, let $\mathrm{ann}\, \delta_{n,k} \subseteq \Omega_n$ be the annihilator of the superspace Vandermonde $\delta_{n,k}$ under the $\odot$-action:

$$\mathrm{ann}\, \delta_{n,k} = \{f \in \Omega_n \ : \ f \odot \delta_{n,k} = 0\}. \tag{1.8}$$

We let $\mathbb{W}_{n,k}$ be the quotient of Ω_n by this annihilator:

$$\mathbb{W}_{n,k} = \Omega_n / \mathrm{ann}\, \delta_{n,k}. \tag{1.9}$$

When $k = n$, the ring $\mathbb{W}_{n,n}$ may be identified with the singly graded type A coinvariant ring

$$R_n = \mathbb{Q}[x_1, \dots, x_n]/\langle e_1, \dots, e_n \rangle, \tag{1.10}$$

where $e_d = e_d(x_1, \dots, x_n)$ is the degree d elementary symmetric polynomial. Borel [4] proved that R_n presents the cohomology $H^\bullet(\mathcal{F}\ell_n; \mathbb{Q})$ of the variety $\mathcal{F}\ell_n$ of complete flags in $\mathbb{C}^n$. Since $\mathcal{F}\ell_n$ is a smooth compact complex manifold, this means that the ring R_n satisfies Poincaré Duality and the Hard Lefschetz theorem. In this paper, we provide a simple generating set (Definition 4.6, Theorem 4.12) of the ideal $\mathrm{ann}\, \delta_{n,k}$.

The quotient ring $\mathbb{W}_{n,k}$ is a bigraded $\mathfrak{S}_n$-module; we let $(\mathbb{W}_{n,k})_{i,j}$ denote its bihomogeneous piece in x-degree i and θ-degree j. In [27], the following facts were proven about this module. Let $\mathrm{grFrob}(\mathbb{W}_{n,k}; q, z)$ be the bigraded Frobenius image of $\mathbb{W}_{n,k}$, with q tracking x-degree and z tracking θ-degree.

Theorem 1.2 (Rhoades-Wilson [27]). *Let $k \le n$ be positive integers, and let $N = (n-k) \cdot (k-1) + \binom{k}{2}$ and $M = n - k$. We have the following facts concerning the quotient $\mathbb{W}_{n,k}$.*

1. *(Bidegree bound) The bigraded piece $(\mathbb{W}_{n,k})_{i,j}$ is zero unless $0 \le i \le N$ and $0 \le j \le M$.*
2. *(Superspace Poincaré Duality) The vector space $(\mathbb{W}_{n,k})_{N,M} = \mathbb{Q}$ is 1-dimensional and spanned by $\delta_{n,k}$. For any $0 \le i \le N$ and $0 \le j \le M$, the multiplication pairing*

$$(\mathbb{W}_{n,k})_{i,j} \times (\mathbb{W}_{n,k})_{N-i,M-j} \longrightarrow (\mathbb{W}_{n,k})_{N,M} = \mathbb{Q}$$

   is perfect.
3. *(Anticommuting degree zero) The anticommuting degree zero piece of $\mathbb{W}_{n,k}$ is isomorphic to the quotient ring*

$$R_{n,k} = \mathbb{Q}[x_1, \dots, x_n]/\langle e_n, e_{n-1}, \dots, e_{n-k+1}, x_1^k, \dots, x_n^k \rangle, \tag{1.11}$$

   where $e_d = e_d(x_1, \dots, x_n)$ is the degree d elementary symmetric polynomial.

[1]In differential geometry, $\partial/\partial \theta_i$ is called a *contraction* operator.

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4. *(Rotational Duality) The symmetric function* $\mathrm{grFrob}(\mathbb{W}_{n,k};q,z)$ *admits the symmetry*

$$(q^N z^M) \cdot \mathrm{grFrob}(\mathbb{W}_{n,k};q^{-1},z^{-1}) = \omega(\mathrm{grFrob}(\mathbb{W}_{n,k};q,z)). \tag{1.12}$$

Here, ω *is the involution on symmetric functions trading* e_n *and* h_n*.*

The rings $R_{n,k}$ appearing in Theorem 1.2 (3) were introduced by Haglund, Rhoades and Shimozono [13] in their study of the Haglund-Remmel-Wilson *Delta Conjecture* [12] (whose 'rise formulation' was recently proven by D'Adderio and Mellit [6]). Pawlowski and Rhoades [21] proved that $R_{n,k}$ presents the cohomology ring $H^\bullet(X_{n,k};\mathbb{Q})$ of the variety $X_{n,k}$ of n-tuples $(\ell_1,\ldots,\ell_n)$ of lines in $\mathbb{C}^k$ which satisfy $\ell_1+\cdots+\ell_n=\mathbb{C}^k$. The variety $X_{n,k}$ is smooth but not compact. Correspondingly, the Hilbert series of the ring $R_{n,k}$ is not palindromic. Theorem 1.2 (3) implies that the 'superization' $\mathbb{W}_{n,k}$ of $R_{n,k}$ satisfies a bigraded analog of Poincaré Duality. It is for this reason that our title alludes to $\mathbb{W}_{n,k}$ as a 'duality module'.

Theorem 1.2 notwithstanding, the paper [27] left many open questions about the nature of the bigraded $\mathfrak{S}_n$-modules $\mathbb{W}_{n,k}$. Indeed, the dimension of $\mathbb{W}_{n,k}$ was unknown. The purpose of this paper is to elucidate the structure of the duality modules $\mathbb{W}_{n,k}$. In order to do this, we will need the following superspace extensions of set partitions.

Definition 1.3. A *set superpartition* of $[n]$ is a set partition of $[n]$ into nonempty sets $\{B_1,\ldots,B_k\}$ in which the letters $1,\ldots,n$ may be decorated with bars, and in which the minimal element $\min(B_i)$ of any block B_i must be unbarred. An *ordered set superpartition* is a set superpartition $(B_1 \mid \cdots \mid B_k)$ equipped with a total order on its blocks.

As an example,

$$\{\{1,\bar{2},4\},\{3\},\{5,\bar{6}\}\}$$

is a set superpartition of $[6]$ with three blocks. This set superpartition gives rise to $3!$ ordered set superpartitions, one of which is

$$(5,\bar{6} \mid 1,\bar{2},4 \mid 3),$$

where, by convention, we write elements in increasing order within blocks. Roughly speaking, barred letters will correspond algebraically to θ-variables.

We define the following families of ordered set superpartitions

$$\mathcal{OSP}_{n,k} = \{\text{all ordered set superpartitions of } [n] \text{ into } k \text{ blocks}\},$$
$$\mathcal{OSP}_{n,k}^{(r)} = \{\text{all ordered set superpartitions of } [n] \text{ into } k \text{ blocks with } r \text{ barred letters}\}.$$

These sets are counted by

$$\left|\mathcal{OSP}_{n,k}\right| = 2^{n-k}\cdot k!\cdot \mathrm{Stir}(n,k) \quad \text{and} \quad \left|\mathcal{OSP}_{n,k}^{(r)}\right| = \binom{n-k}{r}\cdot k!\cdot \mathrm{Stir}(n,k), \tag{1.13}$$

where $\mathrm{Stir}(n,k)$ is the Stirling number of the second kind counting set partitions of $[n]$ into k blocks.

The algebra of $\mathbb{W}_{n,k}$ is governed by the combinatorics of ordered superpartitions. More precisely, we prove the following.

- The ideal $\mathrm{ann}\,\delta_{n,k} \subseteq \Omega_n$ defining $\mathbb{W}_{n,k}$ has an explicit presentation (Definition 4.6) involving elementary symmetric polynomials in partial variable sets (Theorem 4.12).
- The vector space $\mathbb{W}_{n,k}$ has a basis indexed by $\mathcal{OSP}_{n,k}$ (Theorem 4.12).

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- There are explicit statistics coinv and codinv on $\mathcal{OSP}_{n,k}$ (see Section 3), such that the bigraded Hilbert series of $\mathbb{W}_{n,k}$ is given by

$$\mathrm{Hilb}(\mathbb{W}_{n,k};q,z) = \sum_{r=0}^{n-k} z^r \cdot \sum_{\sigma \in \mathcal{OSP}_{n,k}^{(r)}} q^{\mathrm{coinv}(\sigma)} = \sum_{r=0}^{n-k} z^r \cdot \sum_{\sigma \in \mathcal{OSP}_{n,k}^{(r)}} q^{\mathrm{codinv}(\sigma)}. \tag{1.14}$$

  This bigraded Hilbert series may be computed using a simple recursion (Corollary 4.13).
- The θ-degree pieces of $\mathbb{W}_{n,k}$ are built out of hook-shaped irreducibles. More precisely, if we regard $\mathbb{W}_{n,k}$ as a singly graded module under θ-degree,

$$\mathrm{grFrob}(\mathbb{W}_{n,k};z) = \sum_{(\lambda^{(1)},\dots,\lambda^{(k)})} z^{n-\lambda_1^{(1)}-\cdots-\lambda_1^{(k)}} \cdot s_{\lambda^{(1)}} \cdots s_{\lambda^{(k)}}, \tag{1.15}$$

  where the sum is over all k-tuples $(\lambda^{(1)},\dots,\lambda^{(k)})$ of nonempty hook-shaped partitions which satisfy $|\lambda^{(1)}| + \cdots + |\lambda^{(k)}| = n$ (Corollary 5.22).
- The monomial expansion of $\mathrm{grFrob}(\mathbb{W}_{n,k};q,z)$ is a generating function for the statistics coinv and codinv, extended to a multiset analog of ordered set superpartitions (Theorem 5.20).

Although the codinv interpretation of $\mathrm{grFrob}(\mathbb{W}_{n,k};q,z)$ will implicitly describe this symmetric function as a positive sum of LLT polynomials, we do not have a combinatorial interpretation for its Schur expansion and leave this as an open problem. Our results on $\mathbb{W}_{n,k}$-modules are ‘superizations’ of facts about the rings $R_{n,k}$ proven in [13]. Loosely speaking, ordered set partitions are replaced by ordered set superpartitions in appropriate ways. The *proofs* of these results will be significantly different from those of [13] due to the anticommuting variables.

We analyse the quotient ring $\mathbb{W}_{n,k}$ by considering its isomorphic harmonic subspace $\mathbb{H}_{n,k} \subseteq \Omega_n$. This is the submodule of Ω_n generated by $\delta_{n,k}$ under the $\odot$-action:

$$\mathbb{H}_{n,k} = \{ f \odot \delta_{n,k} \,:\, f \in \Omega_n \}. \tag{1.16}$$

Our analysis of $\mathbb{H}_{n,k}$ involves

- a new total order $\prec$ on monomials in Ω_n (see Section 4) used to describe $\mathbb{H}_{n,k}$ as a graded vector space, and
- a new total order $\lessdot$ on the components of a certain direct sum decomposition $\Omega_n = \bigoplus_{p,q\geq 0} \Omega_n(p,q)$ (both depending on an auxiliary parameter j) used to describe the graded $\mathfrak{S}_n$-structure of $\mathbb{H}_{n,k}$ (see Section 5).

Roughly speaking, the orders $\prec$ and $\lessdot$ arise from the superspace intuition that a product $x_i^j \theta_i$ of an x-variable and the corresponding θ-variable should be given a ‘negative’ exponent weight $-j$. The order $\prec$ restricts to the lexicographical term order on monomials in $\mathbb{Q}[x_1,\dots,x_n]$ but is not a term order on Ω_n in the sense of Gröbner theory. Indeed, the Gröbner theory of important ideals (such as the superspace coinvariant ideal) in Ω_n tends to be messier than that of analogous ideals in $\mathbb{Q}[x_1,\dots,x_n]$. On the other hand, we will see in Section 4 that the $\prec$-leading terms of elements in $\mathbb{H}_{n,k}$ correspond in a natural way to ordered set superpartitions.

It is our hope that the tools in this paper will prove useful in understanding other quotient rings involving Ω_n, such as the superspace coinvariant ring. Indeed, the Fields Group has a conjecture (see [34]) for the bigraded Frobenius image of $\Omega_n / \langle (\Omega_n)_+^{\mathfrak{S}_n} \rangle$ which is equivalent (by work of [13, 14, 27]) to

$$\{z^{n-k}\}\, \mathrm{grFrob}(\Omega_n/\langle(\Omega_n)_+^{\mathfrak{S}_n}\rangle;q,z) = \{z^{n-k}\}\, \mathrm{grFrob}(\mathbb{W}_{n,k};q,z) \quad \text{for all } n,k \geq 0, \tag{1.17}$$

where $\{z^{n-k}\}$ is the operator which extracts the coefficient of z^{n-k}. In our analysis of $\mathbb{W}_{n,k}$, we give an explicit generating set of its defining ideal $\mathrm{ann}\,\delta_{n,k}$. This gives rise (Proposition 6.4) to a side-by-side

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comparison of the θ-degree $n-k$ pieces of $\Omega_n/\langle(\Omega_n)_+^{\mathfrak{S}_n}\rangle$ and $\mathbb{W}_{n,k}$ as quotient modules with explicit relations. Hopefully, this similarity will assist in proving (1.17).

The rest of the paper is organised as follows. In Section 2, we give background material on superspace, $\mathfrak{S}_n$-modules and symmetric functions. Section 3 develops combinatorics of ordered set superpartitions necessary for the algebraic study of the $\mathbb{W}$-modules. Section 4 uses harmonic spaces to give a monomial basis of the modules $\mathbb{W}_{n,k}$ and describe their bigraded Hilbert series. Section 5 uses skewing operators and harmonics to give a combinatorial formula for the bigraded Frobenius image of the $\mathbb{W}$-modules. In Section 6, we conclude with some open problems.

2. Background

2.1. *Alternants in superspace*

Recall that if V is an $\mathfrak{S}_n$-module, a vector $v \in V$ is an *alternant* if

$$w \cdot v = \mathrm{sign}(w) \cdot v \qquad \text{for all } w \in \mathfrak{S}_n. \tag{2.1}$$

The superspace Vandermondes $\delta_{n,k} \in \Omega_n$ are alternants used to construct the quotient rings $\mathbb{W}_{n,k}$. In order to place $\mathbb{W}_{n,k}$ in the proper inductive context, we will need a more general family of alternants and rings.

Definition 2.1. Let $n, k, s \geq 0$ be integers. Define $\delta_{n,k,s} \in \Omega_n$ to be the element

$$\delta_{n,k,s} = \varepsilon_n \cdot (x_1^{k-1} \cdots x_{n-s}^{k-1} x_{n-s+1}^{s-1} \cdots x_{n-1}^{1} x_n^{0} \times \theta_1 \cdots \theta_{n-k}), \tag{2.2}$$

where $\varepsilon_n = \sum_{w \in \mathfrak{S}_n} \mathrm{sign}(w) \cdot w \in \mathbb{Q}[\mathfrak{S}_n]$. Let $\mathrm{ann}\, \delta_{n,k,s} \subseteq \Omega_n$ be the annihilator of $\delta_{n,k,s}$, and define $\mathbb{W}_{n,k,s}$ to be the quotient ring

$$\mathbb{W}_{n,k,s} = \Omega_n / \mathrm{ann}\, \delta_{n,k,s}. \tag{2.3}$$

By convention, if $n < k$, then $\delta_{n,k,s} = 0$. In the special case $s = k$, we have $\mathbb{W}_{n,k,k} = \mathbb{W}_{n,k}$. We only use the ring $\mathbb{W}_{n,k,s}$ in the range $k \geq s$.

2.2. *Harmonics in superspace*

It will be convenient to have a model for $\mathbb{W}_{n,k,s}$ as a subspace rather than a quotient of Ω_n. To this end, we define the *harmonic module* $\mathbb{H}_{n,k,s} \subseteq \Omega_n$ as follows.

Definition 2.2. Let $n, k, s \geq 0$, and consider Ω_n as a module over itself by the $\odot$-action $f \odot g = \partial f(g)$. We define $\mathbb{H}_{n,k,s} \subseteq \Omega_n$ to be the Ω_n-submodule generated by $\delta_{n,k,s}$.

More explicitly, the harmonic module $\mathbb{H}_{n,k,s}$ is the smallest linear subspace of Ω_n containing $\delta_{n,k,s}$, which is closed under the action of the commuting partial derivatives $\partial/\partial x_1, \ldots, \partial/\partial x_n$ as well as the anticommuting partial derivatives $\partial/\partial\theta_1, \ldots, \partial/\partial\theta_n$. The subspace $\mathbb{H}_{n,k,s}$ is a bigraded $\mathfrak{S}_n$-module.

We have a natural inclusion map $\mathbb{H}_{n,k,s} \hookrightarrow \Omega_n$. The composition

$$\mathbb{H}_{n,k,s} \hookrightarrow \Omega_n \twoheadrightarrow \mathbb{W}_{n,k,s} \tag{2.4}$$

of this inclusion with the canonical projection of Ω_n onto $\mathbb{W}_{n,k,s}$ is an isomorphism of bigraded $\mathfrak{S}_n$-modules. We make use of $\mathbb{H}_{n,k,s}$ when we need to consider superspace elements in Ω_n rather than cosets in $\mathbb{W}_{n,k,s}$.

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2.3. Symmetric functions and $\mathfrak{S}_n$-modules

Throughout this paper, we use the following standard q-analogs of numbers, factorials and binomial coefficients:

$$[n]_q = \frac{q^n-1}{q-1} = 1+q+\cdots+q^{n-1}, \quad [n]!_q = [n]_q[n-1]_q\cdots[1]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q\cdot[n-k]!_q}. \tag{2.5}$$

A *partition* λ of n is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ of positive integers which sum to n. We write $\lambda \vdash n$ to mean that λ is a partition of n.

Let $\Lambda = \bigoplus_{n\geq 0}\Lambda_n$ be the ring of symmetric functions in an infinite variable set $\mathbf{x} = (x_1, x_2, \ldots)$ over the ground field $\mathbb{Q}(q,z)$. Bases of the n^{th} graded piece Λ_n of this ring are indexed by partitions $\lambda \vdash n$. We let

$$\{m_\lambda : \lambda \vdash n\}, \quad \{e_\lambda : \lambda \vdash n\}, \quad \{h_\lambda : \lambda \vdash n\} \quad \text{and} \quad \{s_\lambda : \lambda \vdash n\} \tag{2.6}$$

be the *monomial, elementary, homogeneous* and *Schur* bases of Λ_n. Given two partitions λ, μ with $\lambda_i \geq \mu_i$ for all i, we let $s_{\lambda/\mu}$ be the corresponding *skew Schur function*.

A formal power series F in the variable set $\mathbf{x} = (x_1, x_2, \ldots)$ of bounded degree is *quasisymmetric* if the coefficient of $x_1^{a_1}\cdots x_n^{a_n}$ equals the coefficient of $x_{i_1}^{a_1}\cdots x_{i_n}^{a_n}$ for any strictly increasing sequence $i_1 < \cdots < i_n$ of indices. Given a subset $S \subseteq [n-1]$, the *fundamental quasisymmetric function* of degree n is

$$F_{S,n} = \sum_{\substack{i_1\leq\cdots\leq i_n \\ j\in S \Rightarrow i_j < i_{j+1}}} x_{i_1}\cdots x_{i_n}. \tag{2.7}$$

We will encounter the formal power series $F_{S,n}$ exclusively in the case where S is the *inverse descent set* of a permutation $w \in \mathfrak{S}_n$. This is the set

$$\mathrm{iDes}(w) = \{1 \leq i \leq n-1 : w^{-1}(i) > w^{-1}(i+1)\}. \tag{2.8}$$

We let $\langle -,- \rangle$ be the *Hall inner product* on Λ_n obtained by declaring the Schur functions s_λ to be orthonormal. For any $F \in \Lambda$, we have a 'skewing' operator $F^\perp : \Lambda \to \Lambda$ characterised by

$$\langle F^\perp G, H\rangle = \langle G, FH\rangle \tag{2.9}$$

for all $G, H \in \Lambda$. We will make use of the following fact.

Lemma 2.3. *Let $F, G \in \Lambda$ be two homogeneous symmetric functions of positive degree. The following are equivalent.*

1. *We have $F = G$.*
2. *We have $h_j^\perp F = h_j^\perp G$ for all $j \geq 1$.*
3. *We have $e_j^\perp F = e_j^\perp G$ for all $j \geq 1$.*

Lemma 2.3 follows from the fact that either of the sets $\{e_1, e_2, \ldots\}$ or $\{h_1, h_2, \ldots\}$ are algebraically independent generating sets of the ring Λ of symmetric functions.

Irreducible representations of $\mathfrak{S}_n$ are in bijective correspondence with partitions λ of n. If $\lambda \vdash n$ is a partition, let S^λ be the corresponding irreducible $\mathfrak{S}_n$-module. If V is any finite-dimensional $\mathfrak{S}_n$-module, there are unique multiplicities $c_\lambda \geq 0$, such that $V \cong \bigoplus_{\lambda\vdash n} c_\lambda S^\lambda$. The *Frobenius image* of V is the symmetric function

$$\mathrm{Frob}(V) = \sum_{\lambda\vdash n} c_\lambda s_\lambda \in \Lambda_n \tag{2.10}$$

obtained by replacing each irreducible S^λ with the corresponding Schur function s_λ.

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Given two positive integers n, m, we have the corresponding *parabolic subgroup* $\mathfrak{S}_n \times \mathfrak{S}_m \subseteq \mathfrak{S}_{n+m}$ obtained by permuting the first n letters and the last m letters in $[n+m]$ separately. If V is an $\mathfrak{S}_n$-module and W is an $\mathfrak{S}_m$-module, their *induction product* $V \circ W$ is the $\mathfrak{S}_{n+m}$-module given by

$$V \circ W = \mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} (V \otimes W). \tag{2.11}$$

Induction product and Frobenius image are related in that

$$\mathrm{Frob}(V \circ W) = \mathrm{Frob}(V) \cdot \mathrm{Frob}(W). \tag{2.12}$$

Frobenius images interact with the skewing operators $h_j^\perp$ and $e_j^\perp$ in the following way. Let V be an $\mathfrak{S}_n$-module, let $1 \leq j \leq n$ and consider the parabolic subgroup $\mathfrak{S}_j \times \mathfrak{S}_{n-j}$ of $\mathfrak{S}_n$. We have the group algebra elements $\eta_j, \varepsilon_j \in \mathbb{Q}[\mathfrak{S}_j]$

$$\eta_j = \sum_{w \in \mathfrak{S}_j} w \qquad\qquad \varepsilon_j = \sum_{w \in \mathfrak{S}_j} \mathrm{sign}(w) \cdot w \tag{2.13}$$

which symmetrise and antisymmetrise in the **first** j letters, respectively. Since η_j and ε_j commute with permutations in the second parabolic factor $\mathfrak{S}_{n-j}$, the vector spaces

$$\eta_j V = \{\eta_j \cdot v \,:\, v \in V\} \qquad\qquad \varepsilon_j V = \{\varepsilon_j \cdot v \,:\, v \in V\} \tag{2.14}$$

are naturally $\mathfrak{S}_{n-j}$-modules. The Frobenius images of these modules are as follows.

Lemma 2.4. *We have* $\mathrm{Frob}(\eta_j V) = h_j^\perp \mathrm{Frob}(V)$ *and* $\mathrm{Frob}(\varepsilon_j V) = e_j^\perp \mathrm{Frob}(V)$.

The proof of Lemma 2.4, which we omit, uses Frobenius reciprocity. Lemma 2.4 may be generalised by considering the image of V under $\sum_{w \in \mathfrak{S}_j} \chi^\mu(w) \cdot w \in \mathbb{Q}[\mathfrak{S}_j]$, where $\mu \vdash j$ is any partition and $\chi^\mu : \mathfrak{S}_j \to \mathbb{C}$ is the irreducible character; the effect on Frobenius images is the operator $s_\mu^\perp$.

In this paper, we will consider (bi)graded vector spaces and modules. If $V = \bigoplus_{i \geq 0} V_i$ is a graded vector space with each piece V_i finite-dimensional, recall that its *Hilbert series* is given by

$$\mathrm{Hilb}(V; q) = \sum_{i \geq 0} \dim(V_i) \cdot q^i. \tag{2.15}$$

Similarly, if $V = \bigoplus_{i,j \geq 0} V_{i,j}$ is a bigraded vector space, we have the *bigraded Hilbert series*

$$\mathrm{Hilb}(V; q, z) = \sum_{i,j \geq 0} \dim(V_{i,j}) \cdot q^i z^j. \tag{2.16}$$

If $V = \bigoplus_{i \geq 0} V_i$ is a graded $\mathfrak{S}_n$-module, its *graded Frobenius image* is

$$\mathrm{grFrob}(V; q) = \sum_{i \geq 0} \mathrm{Frob}(V_i) \cdot q^i. \tag{2.17}$$

Extending this, if $V = \bigoplus_{i,j \geq 0} V_{i,j}$ is a bigraded $\mathfrak{S}_n$-module, its *bigraded Frobenius image* is

$$\mathrm{grFrob}(V; q, z) = \sum_{i,j \geq 0} \mathrm{Frob}(V_{i,j}) \cdot q^i z^j. \tag{2.18}$$

2.4. Ordered set superpartitions

We will show that the duality modules $\mathbb{W}_{n,k}$ are governed by the combinatorics of ordered set superpartitions in $\mathcal{OSP}_{n,k}$. The more general modules $\mathbb{W}_{n,k,s}$ of Definition 2.1 which we will use to inductively describe the $\mathbb{W}_{n,k}$ are controlled by the following more general combinatorial objects.

Definition 2.5. For $n, k, s \geq 0$, we let $\mathcal{OSP}_{n,k,s}$ be the family of k-tuples $(B_1 \mid \cdots \mid B_k)$ of sets of positive integers, such that

- we have the disjoint union decomposition $[n] = B_1 \sqcup \cdots \sqcup B_k$,
- the first s sets $B_1, \ldots, B_s$ are nonempty,
- the elements of $B_1, \ldots, B_k$ may be barred or unbarred,
- the minimal elements $\min B_1, \ldots, \min B_s$ of the first s sets are unbarred.

We denote by $\mathcal{OSP}^{(r)}_{n,k,s} \subseteq \mathcal{OSP}_{n,k,s}$ the subfamily of $\sigma \in \mathcal{OSP}_{n,k,s}$ with r barred elements.

Note that, when $s = k$, $\mathcal{OSP}_{n,k} = \mathcal{OSP}_{n,k,s}$. We refer to elements $\sigma = (B_1 \mid \cdots \mid B_k) \in \mathcal{OSP}_{n,k,s}$ as *ordered set superpartitions*, despite the fact that any of the last $k - s$ sets $B_{s+1}, \ldots, B_k$ in σ could be empty and that the minimal elements of $B_{s+1}, \ldots, B_k$ (if they exist) may be barred.

3. Ordered set superpartitions

3.1. The statistics coinv *and* codinv

In this section, we define two statistics on $\mathcal{OSP}_{n,k,s}$. The first of these is an extension of the classical inversion statistic (or rather, its complement) on permutations in $\mathfrak{S}_n$. Given $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$, its *coinversion code* is the sequence $(c_1, \ldots, c_n)$, where c_i is the number of entries in the set $\{i+1, i+2, \ldots, n\}$ which appear to the right of i in π. The *coinversion number* of π is the sum $\operatorname{coinv}(\pi) = c_1 + \cdots + c_n$. We generalise these concepts as follows.

Definition 3.1. Let $\sigma = (B_1 \mid \cdots \mid B_k) \in \mathcal{OSP}_{n,k,s}$ be an ordered set superpartition. The *coinversion code* is the length n sequence $\operatorname{code}(\sigma) = (c_1, \ldots, c_n)$ over the alphabet $\{0, 1, 2, \ldots, \bar{0}, \bar{1}, \bar{2}, \ldots\}$ whose a^{th} entry c_a is defined as follows. Suppose that a lies in the i^{th} block B_i of σ.

- The entry c_a is barred if and only if a is barred in σ.
- If $a = \min B_i$ and $i \leq s$, then

$$c_a = \left|\{i+1 \leq j \leq s \,:\, \min B_j > a\}\right|.$$

- If a is barred, then

$$c_a = \begin{cases} |\{1 \leq j \leq i-1 \,:\, \min B_j < a\}| & \text{if } i \leq s, \\ |\{1 \leq j \leq s \,:\, \min B_j < a\}| + (i - s - 1) & \text{if } i > s. \end{cases}$$

- Otherwise, we set

$$c_a = |\{i+1 \leq j \leq s \,:\, \min B_j > a\}| + (i-1).$$

The *coinversion number* of σ is the sum

$$\operatorname{coinv}(\sigma) = c_1 + \cdots + c_n$$

of the entries $(c_1, \ldots, c_n)$ in $\operatorname{code}(\sigma)$.

For example, consider the ordered set superpartition

$$\sigma = (2, \bar{5} \mid 3, 6, \bar{8}, 9 \mid \varnothing \mid \bar{1}, \bar{4}, 7 \mid \varnothing) \in \mathcal{OSP}_{9,5,2}.$$

The coinversion code of σ is given by

$$\operatorname{code}(\sigma) = (c_1, \ldots, c_9) = (\bar{1}, 1, 0, \bar{3}, \bar{0}, 1, 3, \bar{1}, 1),$$

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so that

$$\operatorname{coinv}(\sigma) = 1 + 1 + 0 + 3 + 0 + 1 + 3 + 1 + 1 = 11.$$

Bars on the entries $c_1, \ldots, c_n$ are ignored when calculating $\operatorname{coinv}(\sigma) = c_1 + \cdots + c_n$.

The coinversion code $(c_1, \ldots, c_n)$ can be visualised by considering the *column diagram* notation for ordered set superpartitions. Given $\sigma = (B_1 \mid \cdots \mid B_k) \in \mathcal{OSP}_{n,k,s}$, we draw the entries of B_i in the i^{th} column. The entries of B_i fill the i^{th} column according to the following rules.

- For $1 \leq i \leq k$, the barred entries of B_i start at height 1 and fill up in increasing order.
- For $1 \leq i \leq s$, the unbarred entries of B_i start at height 0 and fill down in increasing order.
- For $s + 1 \leq i \leq k$, the unbarred entries of B_i start at height -1 and fill down in increasing order; we also place a $\bullet$ at height 0 in these columns.

In our example, the column diagram of

$$(2, \bar{5} \mid 3, 6, \bar{8}, 9 \mid \varnothing \mid \bar{1}, \bar{4}, 7 \mid \varnothing) \in \mathcal{OSP}_{9,5,2}.$$

is given by

$$\begin{array}{r|c|c|c|c|c|}
2 & & & & \bar{4} & \\
1 & \bar{5} & \bar{8} & & \bar{1} & \\
0 & 2 & 3 & \bullet & \bullet & \bullet \\
-1 & & 6 & & 7 & \\
-2 & & 9 & & & \\
\\
 & 1 & 2 & 3 & 4 & 5 \;,
\end{array}$$

where the column indices are shown below and the heights of entries are shown on the left. The three $\bullet$'s at the height 0 level correspond to the fact that the blocks B_3, B_4, B_5 are allowed to be empty for $\sigma \in \mathcal{OSP}_{9,5,2}$. Given $\sigma \in \mathcal{OSP}_{n,k,s}$, we have the following column diagram interpretation of the a^{th} letter c_a of $\operatorname{code}(\sigma) = (c_1, \ldots, c_n)$.

- If a appears at height 0, then c_a counts the number of height zero entries to the right of a which are $> a$.
- If a appears at negative height in column i, then c_a is $i - 1$, plus the number of height 0 entries to the right of a which are $> a$.
- If a appears at positive height, then c_a counts the number of height 0 entries to the left of a which are $< a$, plus the number of $\bullet$'s to the left of a.

We will see that codes $(c_1, \ldots, c_n)$ of elements $\sigma \in \mathcal{OSP}_{n,k,s}$ correspond to a monomial basis of the quotient ring $\mathbb{W}_{n,k,s}$. In order to obtain the bigraded Frobenius image of $\mathbb{W}_{n,k,s}$ in terms of coinv, we use diagrams to define a formal power series as follows.

The *reading word* of $\sigma \in \mathcal{OSP}_{n,k,s}$, denoted $\operatorname{read}(\sigma)$ is the permutation in $\mathfrak{S}_n$ obtained by reading the column diagram of σ from top to bottom, and, within each row, **from right to left** (ignoring any bars on letters). If σ is our example ordered set partition above, we have

$$\operatorname{read}(\sigma) = 418532769 \in \mathfrak{S}_9.$$

Definition 3.2. Let $n, k, s \geq 0$ be integers, and let $0 \leq r \leq n - s$. We define a quasisymmetric function $C^{(r)}_{n,k,s}(\mathbf{x}; q)$ by the formula

$$C^{(r)}_{n,k,s}(\mathbf{x}; q) = \sum_{\sigma \in \mathcal{OSP}^{(r)}_{n,k,s}} q^{\operatorname{coinv}(\sigma)} \cdot F_{\operatorname{iDes}(\operatorname{read}(\sigma)),n}(\mathbf{x}). \tag{3.1}$$

Also define a quasisymmetric function $C_{n,k,s}(\mathbf{x};q,z)$ by

$$C_{n,k,s}(\mathbf{x};q,z)=\sum_{r=0}^{n-s}C^{(r)}_{n,k,s}(\mathbf{x};q)\cdot z^r. \tag{3.2}$$

We may avoid the use of the F's in the definition of the C-functions by allowing repeated entries in our column diagrams. Let $\mathcal{CT}_{n,k,s}$ be the family of column tableaux which are all fillings of finite subsets of the infinite strip $[k]\times\infty$ with the alphabet $\{\bullet,1,2,\dots,\bar{1},\bar{2},\dots\}$, such that

- The height 0 row is filled with a sequence of s unbarred numbers, followed by a sequence of $k-s$ $\bullet$'s.
- The filled cells form a contiguous sequence within each column.
- Entries at positive height are barred, while entries at negative height are unbarred.
- Unbarred numbers weakly increase going down, and barred numbers strictly increase going up.
- An unbarred number at height 0 is strictly smaller than any barred number above it.

An example column tableau $\tau\in\mathcal{CT}_{13,5,2}$ is shown below.

$$\begin{array}{r|c|c|c|c|c|}
2 & \bar{4} & \bar{6} & & \bar{4} & \\
1 & \bar{3} & \bar{5} & & \bar{1} & \\
0 & 2 & 3 & \bullet & \bullet & \bullet \\
-1 & & 3 & & 7 & 1 \\
-2 & & 4 & & & 1 \\
 & 1 & 2 & 3 & 4 & 5\ .
\end{array}$$

The coinversion number $\mathrm{coinv}(\tau)$ extends naturally to column tableaux $\tau\in\mathcal{CT}_{n,k,s}$: given an entry a in such a tableau, we may compute its contribution c_a to coinv as before, and sum over all entries. For example, the column tableau depicted above has coinversion number

$$3+1+0+1+1+0+0+1+4+3+1+4+1=20,$$

where the entries in the sum on the left are processed in reading order, that is the first value in the sum is 3 because the $\bar{4}$ in column 4 contributes 3 to the coinversion number. Let $\mathbf{x}^\tau=x_1^{\alpha_1}x_2^{\alpha_2}\cdots$, where α_i is the number of $i's$ in τ. In our example, we have $\mathbf{x}^\tau=x_1^3x_2x_3^3x_4^3x_5x_6x_7$. The exponent sequence $\alpha=(\alpha_1,\alpha_2,\dots)$ of $\mathbf{x}^\tau$ is called the *content* of τ. Let $\mathcal{CT}^{(r)}_{n,k,s}\subseteq\mathcal{CT}_{n,k,s}$ be the subfamily of column tableaux with r barred letters.

Observation 3.3. *The formal power series $C^{(r)}_{n,k,s}(\mathbf{x};q)$ and $C_{n,k,s}(\mathbf{x};q,z)$ are given by*

$$C^{(r)}_{n,k,s}(\mathbf{x};q)=\sum_{\tau\in\mathcal{CT}^{(r)}_{n,k,s}}q^{\mathrm{coinv}(\tau)}\mathbf{x}^\tau\quad\textit{and}\quad C_{n,k,s}(\mathbf{x};q,z)=\sum_{\tau\in\mathcal{CT}_{n,k,s}}q^{\mathrm{coinv}(\tau)}z^{\#\textit{ of bars in }\tau}\mathbf{x}^\tau.$$

This observation follows from the definition of the fundamental quasisymmetric functions. The formulas for the C-functions in Observation 3.3 are more aesthetic but less efficient than those in Definition 3.2. It will turn out that $C_{n,k,s}(\mathbf{x};q,z)$ is the bigraded Frobenius image $\mathrm{grFrob}(\mathbb{W}_{n,k,s};q,z)$. For reasons related to skewing recursions, the refinement $C^{(r)}_{n,k,s}(\mathbf{x};q)$ will be convenient to consider. At this point, it is not clear that the C-functions are even symmetric. Their symmetry (and Schur positivity) will follow from an alternative description in terms of another statistic on $\mathcal{OSP}_{n,k,s}$.

Let $\sigma\in\mathcal{OSP}_{n,k,s}$. We augment the column diagram of σ by placing infinitely many $+\infty$'s below the entries in every column (we drop the + in diagrams for the sake of compactness). Furthermore, we

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regard every • in the column diagram as being filled with a 0. Ignoring bars, a pair of entries $a < b$ form a *diagonal coinversion pair* if

- ∘ a appears to the left of and at the same height as b, or
- ∘ a appears to the right of and at height one less than b.

Schematically, these conditions have the form

$$\boxed{a} \quad \cdots \quad \boxed{b} \qquad \text{and} \qquad \begin{array}{|c|}\hline b \\ \hline \\ \hline\end{array} \quad \cdots \quad \begin{array}{|c|}\hline \\ \hline a \\ \hline\end{array}$$

for $a < b$.

Definition 3.4. For $\sigma \in \mathcal{OSP}_{n,k,s}$, the *diagonal coinversion number* $\operatorname{codinv}(\sigma)$ is the total number of diagonal coinversion pairs in the augmented column diagram of σ.

When $n = k = s$, codinv and coinv are identical. Although these statistics clearly differ in general, it will turn out that codinv is equidistributed with coinv on $\mathcal{OSP}_{n,k,s}$ and even on the subsets $\mathcal{OSP}^{(r)}_{n,k,s}$ obtained by restricting to r barred letters. In analogy with the C-functions, we define a quasisymmetric function attached to codinv.

Definition 3.5. Let $n, k, s \geq 0$ be integers, and let $0 \leq r \leq n - s$. Define a quasisymmetric function $D^{(r)}_{n,k,s}(\mathbf{x}; q)$ by the formula

$$D^{(r)}_{n,k,s}(\mathbf{x}; q) = \sum_{\sigma \in \mathcal{OSP}^{(r)}_{n,k,s}} q^{\operatorname{codinv}(\sigma)} \cdot F_{\mathrm{iDes}(\mathrm{read}(\sigma)),n}(\mathbf{x}). \tag{3.3}$$

Also define

$$D_{n,k,s}(\mathbf{x}; q, z) = \sum_{r=0}^{n-s} D^{(r)}_{n,k,s}(\mathbf{x}; q) \cdot z^r. \tag{3.4}$$

Like the C-functions, the D-functions may be expressed in terms of infinite sums over column tableaux.

Observation 3.6. *The formal power series $D^{(r)}_{n,k,s}(\mathbf{x}; q)$ and $D_{n,k,s}(\mathbf{x}; q, z)$ are given by*

$$D^{(r)}_{n,k,s}(\mathbf{x}; q) = \sum_{\tau \in \mathcal{CT}^{(r)}_{n,k,s}} q^{\operatorname{codinv}(\tau)} \mathbf{x}^{\tau} \quad \textit{and} \quad D_{n,k,s}(\mathbf{x}; q, z) = \sum_{\tau \in \mathcal{CT}_{n,k,s}} q^{\operatorname{codinv}(\tau)} z^{\textit{\# of bars in } \tau} \mathbf{x}^{\tau}.$$

We use the theory of LLT polynomials to show that the D-functions are symmetric and Schur positive.

Proposition 3.7. *The quasisymmetric functions $D^{(r)}_{n,k,s}(\mathbf{x}; q)$ and $D_{n,k,s}(\mathbf{x}; q, z)$ are symmetric and Schur positive.*

Proof. We prove this fact by writing $D_{n,k,s}(\mathbf{x}; q, z)$ as a positive linear combination of *LLT polynomials* [18]. We use the version of LLT polynomials employed by Haglund, Haiman and Loehr [11].

A *skew diagram* is a set of cells in the first quadrant given by μ/ν for some partitions $\mu \supseteq \nu$. Given a tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)})$ of skew diagrams, the LLT polynomial indexed by $\boldsymbol{\lambda}$ is

$$\mathrm{LLT}_{\boldsymbol{\lambda}}(\mathbf{x}; q) = \sum q^{\operatorname{inv}(T)} \mathbf{x}^T,$$

where the sum is over all semistandard fillings of the skew diagrams of $\boldsymbol{\lambda}$, $\mathbf{x}^T$ is the product, where x_i appears as many times as i appears in the filling T, and $\operatorname{inv}(T)$ is the following statistic. Given a cell u with coordinates (x, y) in one of the skew shapes $\lambda^{(i)}$, where the leftmost cell at height 0 has coordinates

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(0,0), the *content* of u is $c(u) = x - y$. Then $\mathrm{inv}(T)$ is the number of pairs of cells $u \in \lambda^{(i)}$, $v \in \lambda^{(j)}$ with $i < j$, such that

- $c(u) = c(v)$ and $T(u) > T(v)$, or
- $c(u) + 1 = c(v)$ and $T(u) < T(v)$.

LLT polynomials are Schur-positive symmetric functions [10, 11, 18]. We claim that we can decompose $D^{(r)}_{n,k,s}(\mathbf{x};q)$ into LLT polynomials:

$$D^{(r)}_{n,k,s}(\mathbf{x};q) = \sum_{\boldsymbol{\lambda}} q^{\mathrm{stat}(\boldsymbol{\lambda})}\mathrm{LLT}_{\boldsymbol{\lambda}}(\mathbf{x};q), \tag{3.5}$$

where stat is a fixed statistic depending on $\boldsymbol{\lambda}$ and the sum is over all tuples of skew diagrams $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ satisfying

- if $i \leq k - s$, then $\lambda^{(i)} = \mu^{(i)}/(1)$ for a hook shape $\mu^{(i)}$,
- if $i > k - s$, then $\lambda^{(i)}$ is a single nonempty hook shape,
- $\sum_{i=1}^{k} |\lambda^{(i)}| = n$ and
- $\boldsymbol{\lambda}$ has r total cells of negative content.

We define nonnegative integer sequences $\alpha, \beta \in \mathbb{N}^k$ so that

- $\lambda^{(i)} = (\alpha_i + 1, 1^{\beta_i})/(1)$ for $i \leq k - s$ and
- $\lambda^{(i)} = (\alpha_i + 1, 1^{\beta_i})$ for $i > k - s$.

Given a specific semistandard filling T that contributes to $\mathrm{LLT}_{\boldsymbol{\lambda}}(\mathbf{x};q)$ for such a $\boldsymbol{\lambda}$, we will create a column tableau $\tau \in \mathcal{CT}^{(r)}_{n,k,s}$, such that τ will have exactly α_i entries at negative height and β_i entries at positive height in column $k - i + 1$. The sequences α and β will completely determine how many diagonal coinversions involving an ∞ or a $\bullet$, respectively, appear in τ. In Figure 1, there are six such diagonal coinversions involving an ∞ and seven involving a $\bullet$. In general, this number is

$$\mathrm{stat}(\boldsymbol{\lambda}) = \sum_{1 \leq i < j \leq k} \left(|\alpha_i - \alpha_j| - \chi(\alpha_i > \alpha_j)\right) + \sum_{i=1}^{k-s} |\{j > i : \beta_j \neq 0\}|,$$

where $\chi(p)$ is 1 if p is true and 0 if p is false. Note that $\mathrm{stat}(\boldsymbol{\lambda})$ can be computed from α and β only.

Now, given a specific semistandard filling T of $\boldsymbol{\lambda}$, we map the unique entry in $\lambda^{(i)}$ with content j to column $k - i + 1$ and row $-j$ in τ. Every pair which contributes to $\mathrm{inv}(T)$ now contributes a diagonal coinversion between integers (not $\bullet$ or ∞) in τ. In Figure 1, there are 10 such diagonal coinversions and the filling T has $\mathrm{inv}(T) = 10$. Since this map is bijective, we have the desired result. □

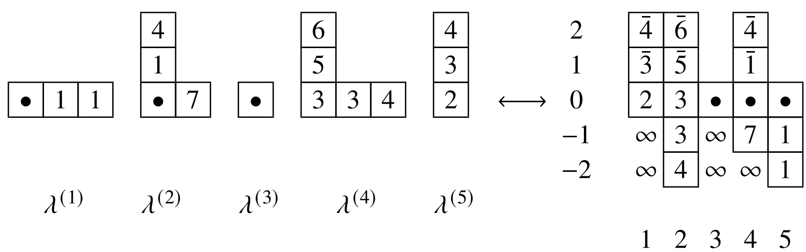

Figure 1. *We depict an example of the correspondence in the proof of Proposition 3.7. Each of the five skew diagrams on the left is justified so that its bottom left entry has content 0. In this example, we have* $\alpha = (2, 1, 0, 2, 0)$, $\beta = (0, 2, 0, 2, 2)$ *and* $\mathrm{stat}(\boldsymbol{\lambda}) = 6 + 7 = 13$

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3.2. Equality of the C- and D-functions

Our first main result states that the C-functions and D-functions coincide. This given, Proposition 3.7 implies that the C-functions are symmetric and Schur positive. We know of no direct proof of either of these facts.

Theorem 3.8. *For any integers $n, k, s \geq 0$ with $k \geq s$ and any $0 \leq r \leq n - s$, we have the equality of formal power series*

$$C^{(r)}_{n,k,s}(\mathbf{x}; q) = D^{(r)}_{n,k,s}(\mathbf{x}; q). \tag{3.6}$$

Consequently, we have

$$C_{n,k,s}(\mathbf{x}; q, z) = D_{n,k,s}(\mathbf{x}; q, z). \tag{3.7}$$

Our proof of Theorem 3.8 is combinatorial and uses the column tableau forms of the formal power series $C^{(r)}_{n,k,s}(\mathbf{x}; q)$ and $D^{(r)}_{n,k,s}(\mathbf{x}; q)$ in Observations 3.3 and 3.6. The idea is to consider building up a general column tableau $\tau \in \mathcal{CT}^{(r)}_{n,k,s}$ from the column tableau consisting of $k - s$ empty columns by successively adding larger entries and showing that the statistics coinv and codinv satisfy the same recursions.

Proof. Let τ be a column tableau with k columns and $k - s\bullet$'s whose entries are $< N$. We consider building a larger column tableau involving N's from τ by the following three-step process.

1. Placing $\bar{N}$'s on top of some subset of the k columns of τ.
2. Placing N's at height 0 between and on either side of the s columns of τ without a $\bullet$.
3. Placing N's at negative heights below some multiset of columns of the resulting figure.

We track the behavior of coinv and codinv as we perform this procedure, starting by placing the $\bar{N}$'s on top of columns.

Placing a $\bar{N}$ on top of column i of τ increases the statistic coinv by $i - 1$. We reflect this fact by giving column i the *barred* coinv *label* of $i - 1$. The barred coinv labels of the column tableau are shown in bold and barred below.

$$\begin{array}{r|ccccc}
3 & \bar{\mathbf{0}} & \bar{\mathbf{1}} & & \bar{\mathbf{3}} & \\
2 & \bar{4} & \bar{6} & & \bar{4} & \\
1 & \bar{3} & \bar{5} & \bar{\mathbf{2}} & \bar{1} & \bar{\mathbf{4}} \\
0 & 2 & 3 & \bullet & \bullet & \bullet \\
-1 & & 3 & & 7 & 1 \\
-2 & & 4 & & & 1 \\
\hline
 & 1 & 2 & 3 & 4 & 5\,.
\end{array}$$

The barred coinv labels give rise to a bijection

$$\iota^{\mathrm{coinv}}_{\bar{N}} : \left\{ \begin{array}{c} k\text{-column tableaux } \tau \text{ with} \\ k - s \bullet \text{'s and all entries } < N \end{array} \right\} \times \left\{ \begin{array}{c} \text{subsets } S \text{ of} \\ \{0, 1, \ldots, k-1\} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} k\text{-column tableaux } \tau' \text{ with } k - s \bullet \text{'s,} \\ \text{such that all entries of } \tau' \text{ are } \leq N \text{ and} \\ \text{the only } N\text{'s in } \tau' \text{ are barred} \end{array} \right\} \tag{3.8}$$

by letting $\iota^{\mathrm{coinv}}_{\bar{N}}(\tau, S)$ be the tableau τ' obtained by placing a $\bar{N}$ on top of every column with barred coinv label in S. If $\iota^{\mathrm{coinv}}_{\bar{N}} : (\tau, S) \mapsto \tau'$, then $\mathrm{coinv}(\tau') = \mathrm{coinv}(\tau) + \sum_{i \in S} i$.

Next, we consider the effect of $\bar{N}$ insertion on codinv. We bijectively label the k columns of τ with the k *barred* codinv *labels* $0, 1, \ldots, k-1$ (in that order) in descending order of maximal height and, within columns of the same maximal height, proceeding from right to left. The barred codinv labels in our example are shown below.

$$\begin{array}{r|ccccc}
3 & \bar{\mathbf{2}} & \bar{\mathbf{1}} & & \bar{\mathbf{0}} & \\
2 & \bar{4} & \bar{6} & & \bar{4} & \\
1 & \bar{3} & \bar{5} & \bar{\mathbf{4}} & \bar{1} & \bar{\mathbf{3}} \\
0 & 2 & 3 & \bullet & \bullet & \bullet \\
-1 & & 3 & & 7 & 1 \\
-2 & & 4 & & & 1 \\
\\
 & 1 & 2 & 3 & 4 & 5
\end{array}.$$

The barred codinv labels give a bijection

$$\iota_{\bar{N}}^{\mathrm{codinv}} : \left\{ \begin{matrix} k\text{-column tableaux } \tau \text{ with} \\ k-s \bullet \text{'s and all entries } < N \end{matrix} \right\} \times \left\{ \begin{matrix} \text{subsets } S \text{ of} \\ \{0, 1, \ldots, k-1\} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} k\text{-column tableaux } \tau' \text{ with } k-s \bullet \text{'s,} \\ \text{such that all entries of } \tau' \text{ are } \le N \text{ and} \\ \text{the only } N\text{'s in } \tau' \text{ are barred} \end{matrix} \right\}, \tag{3.9}$$

where $\iota_{\bar{N}}^{\mathrm{codinv}}(\tau, S)$ is obtained from τ by placing a $\bar{N}$ on top of every column with barred codinv label indexed by S. If $\iota_{\bar{N}}^{\mathrm{codinv}} : (\tau, S) \mapsto \tau'$, then $\mathrm{codinv}(\tau') = \mathrm{codinv}(\tau) + \sum_{i \in S} i$.

We move on to Step 2 of our insertion procedure: creating new columns by placing N's at height 0. A single 'height 0 insertion map' ι_0 of this kind on column tableaux has the same effect on coinv and codinv.

If τ is a column tableau with s nonbullet letters at height zero, N's can be placed (with repetition) in any of the $s+1$ places between and on either side of these nonbullet letters. This gives rise to a bijection

$$\iota_0 : \left\{ \begin{matrix} k\text{-column tableaux } \tau' \text{ with } k-s \bullet \text{'s,} \\ \text{such that all entries of } \tau' \text{ are } \le N \text{ and} \\ \text{the only } N\text{'s in } \tau' \text{ are barred} \end{matrix} \right\} \times \left\{ \begin{matrix} \text{finite multisets } S \\ \text{drawn from } \{0, 1, \ldots, s\} \end{matrix} \right\} \longrightarrow \bigsqcup_{K \ge k} \left\{ \begin{matrix} K\text{-column tableaux } \tau'' \text{ with } k-s \bullet \text{'s,} \\ \text{such that all entries of } \tau'' \text{ are } \le N \text{ and} \\ \text{no } N\text{'s in } \tau'' \text{ have negative height} \end{matrix} \right\}, \tag{3.10}$$

where $\iota_0(\tau', S)$ is obtained from τ' by inserting m_i copies of N after column i of τ, where m_i is the multiplicity of i in S. Suppose $\iota_0(\tau', S) = \tau''$. The number K of columns of τ'' is related to the number k of columns of τ' by $K = k + |S|$. Furthermore, if there are b unbarred entries in τ' of negative height, we have

$$\mathrm{coinv}(\tau'') = \mathrm{coinv}(\tau') + b \cdot |S| + \sum_{i \in S} i \quad \text{and} \quad \mathrm{codinv}(\tau'') = \mathrm{codinv}(\tau') + b \cdot |S| + \sum_{i \in S} i. \tag{3.11}$$

In other words, the height zero insertion map ι_0 has the same effect on coinv and codinv.

Finally, let τ'' be a column tableau with K columns with entries $\le N$ in which there are no N's with negative height. To perform Step 3 of our insertion process, we insert N's at the bottom of some multiset of columns of τ''. We track the effect on coinv and codinv as before.

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We label the columns of τ'' from left-to-right with the *unbarred* coinv *labels* $0, 1, \dots, K$. An example of this labeling is shown below in bold. Since we add unbarred letters on the bottom of a tableau, we show the unbarred labels there as well.

$$\begin{array}{r|c|c|c|c|c|}
2 & \bar{4} & \bar{6} & & \bar{4} & \\
1 & \bar{3} & \bar{5} & & \bar{1} & \\
0 & 2 & 3 & \bullet & \bullet & \bullet \\
-1 & \mathbf{0} & 3 & \mathbf{2} & 7 & 1 \\
-2 & & 4 & & \mathbf{3} & 1 \\
-3 & & \mathbf{1} & & & \mathbf{4} \\
\\
 & 1 & 2 & 3 & 4 & 5\,.
\end{array}$$

The unbarred coinv labels give a bijection

$$\iota_N^{\mathrm{coinv}} : \left\{ \begin{array}{c} K\text{-column tableaux } \tau'' \text{ with } k-s \bullet\text{'}, \\ \text{such that all entries of } \tau'' \text{ are } \le N \text{ and} \\ \text{no } N\text{'s in } \tau'' \text{ have negative height} \end{array} \right\} \times \left\{ \begin{array}{c} \text{finite multisets } S \\ \text{drawn from } \{0, 1, \dots, K-1\} \end{array} \right\} \longrightarrow$$
$$\left\{ \begin{array}{c} K\text{-column tableaux } \tau''' \text{ with } k-s \bullet\text{'s}, \\ \text{such that all entries in } \tau''' \text{ are } \le N \end{array} \right\}, \tag{3.12}$$

where $\iota_N^{\mathrm{coinv}}(\tau'', S)$ is obtained by placing m_i copies of N below the column with label i, where m_i is the multiplicity of i in S. If $\iota_N^{\mathrm{coinv}} : (\tau'', S) \mapsto \tau'''$, then $\mathrm{coinv}(\tau''') = \mathrm{coinv}(\tau'') + \sum_{i \in S} i$.

The effect of inserting N's at negative height on codinv may be described as follows. We label the columns of τ bijectively with the *unbarred* codinv *labels* $0, 1, \dots, k-1$ (in that order) starting at columns of lesser maximal depth and, within columns of the same maximal depth, proceeding from right to left. The unbarred codinv labels in our example as shown below.

$$\begin{array}{r|c|c|c|c|c|}
2 & \bar{4} & \bar{6} & & \bar{4} & \\
1 & \bar{3} & \bar{5} & & \bar{1} & \\
0 & 2 & 3 & \bullet & \bullet & \bullet \\
-1 & \mathbf{1} & 3 & \mathbf{0} & 7 & 1 \\
-2 & & 4 & & \mathbf{2} & 1 \\
-3 & & \mathbf{4} & & & \mathbf{3} \\
\\
 & 1 & 2 & 3 & 4 & 5\,.
\end{array}$$

The unbarred codinv labels give a bijection

$$\iota_N^{\mathrm{codinv}} : \left\{ \begin{array}{c} K\text{-column tableaux } \tau'' \text{ with } k-s \bullet\text{'s}, \\ \text{such that all entries of } \tau'' \text{ are } \le N \text{ and} \\ \text{no } N\text{'s in } \tau'' \text{ have negative height} \end{array} \right\} \times \left\{ \begin{array}{c} \text{finite multisets } S \\ \text{drawn from } \{0, 1, \dots, K-1\} \end{array} \right\} \longrightarrow$$
$$\left\{ \begin{array}{c} K\text{-column tableaux } \tau''' \text{ with } k-s \bullet\text{'s}, \\ \text{such that all entries in } \tau''' \text{ are } \le N \end{array} \right\} \tag{3.13}$$

as follows. Given (τ'', S), we process the entries of S **in weakly increasing order** and, given an entry i, we place an N at the bottom of the column with unbarred codinv label i. Recalling that we consider every unfilled cell below a column to have the label $+\infty$ for the purpose of calculating codinv, we see that if $\iota_N^{\mathrm{codinv}} : (\tau'', S) \mapsto \tau'''$, then $\mathrm{codinv}(\tau''') = \mathrm{codinv}(\tau'') + \sum_{i \in S} i$. In summary, the ι-maps show that the C-functions and D-functions satisfy the same recursion, which establishes the theorem. □

Although the D-functions are more directly seen to be symmetric and Schur positive, in order to describe a recursive formula for the action of $h_j^\perp$, it will be more convenient to use the C-functions instead (Lemma 3.14). This formula will involve the more refined $C^{(r)}_{n,k,s}(\mathbf{x};q)$ rather than their coarsened versions $C_{n,k,s}(\mathbf{x};q,z)$.

3.3. Substaircase shuffles

Given $\sigma \in \mathcal{OSP}_{n,k,s}$, $\mathrm{code}(\sigma) = (c_1,\dots,c_n)$ is a length n sequence over the alphabet $\{0,1,2,\dots,\bar{0},\bar{1},\bar{2},\dots\}$. In this subsection, we show that $\sigma \mapsto \mathrm{code}(\sigma)$ is injective and characterise its image. We begin by defining a partial order on sequences of (potentially barred) nonnegative integers.

Definition 3.9. $(c_1,\dots,c_n) \le (b_1,\dots,b_n)$ if, for every i, $c_i \le b_i$ in value and that c_i is barred if and only if b_i is barred.

The image of the classical coinversion code map on permutations in $\mathfrak{S}_n$ is given by words $(c_1,\dots,c_n)$ which are $\le$ the 'staircase' word $(n-1,n-2,\dots,1,0)$. For ordered set partitions of $[n]$ into k blocks with no barred letters, this result was generalised in [26], where the appropriate notion of 'staircase' is given by a shuffle of two sequences. We extend these definitions to barred letters as follows.

Recall that a *shuffle* of two sequences $(a_1,\dots,a_r)$ and $(b_1,\dots,b_s)$ is an interleaving $(c_1,\dots,c_{r+s})$ of these sequences which preserves the relative order of the a's and the b's. The following collection $\mathcal{SS}^{(r)}_{n,k,s}$ of words will turn out to be the image of the map code on $\mathcal{OSP}^{(r)}_{n,k,s}$.

Definition 3.10. Let $n,k,s \ge 0$ be integers. For $0 \le r \le n-s$, a *staircase* with r barred letters is a length n word obtained by shuffling

$$(\overline{k-s-1})^{r_0}(s-1)(\overline{k-s})^{r_1}(s-2)(\overline{k-s+1})^{r_2}(s-3)\cdots 0(\overline{k-1})^{r_s} \quad \text{and} \quad (k-1)^{n-r-s},$$

where $r_0+r_1+\cdots+r_s = r$. If $k=s$, we insist that $r_0 = 0$ in the above expression. Let $\mathcal{SS}^{(r)}_{n,k,s}$ be the family of words $(c_1,\dots,c_n)$ which are $\le$ some staircase $(b_1,\dots,b_n)$. We also let

$$\mathcal{SS}_{n,k,s} = \mathcal{SS}^{(0)}_{n,k,s} \sqcup \mathcal{SS}^{(1)}_{n,k,s} \sqcup \cdots \sqcup \mathcal{SS}^{(n-s)}_{n,k,s} \tag{3.14}$$

and refer to words in $\mathcal{SS}_{n,k,s}$ as *substaircase*.

Definition 3.10 implies that

$$\mathcal{SS}_{n,k,s} = \varnothing \qquad \text{if } n < s. \tag{3.15}$$

We give an example to clarify these concepts.

Example 3.11. Consider the case $n=5$, $k=3$, $s=2$ and $r=2$. The staircases with r barred letters are the shuffles of any of the six words

$$(\bar{0},\bar{0},1,0),\ (\bar{0},1,\bar{1},0),\ (\bar{0},1,0,\bar{2}),\ (1,\bar{1},\bar{1},0),\ (1,\bar{1},0,\bar{2}),\ (1,0,\bar{2},\bar{2})$$

with the single-letter word (2). For example, if we shuffle (2) into the second sequence from the left, we get the five staircases

$$(2,\bar{0},1,\bar{1},0),\ (\bar{0},2,1,\bar{1},0),\ (\bar{0},1,2,\bar{1},0),\ (\bar{0},1,\bar{1},2,0),\ (\bar{0},1,\bar{1},0,2).$$

The leftmost sequence displayed above contributes

$$(2,\bar{0},1,\bar{1},0),\ (1,\bar{0},1,\bar{1},0),\ (0,\bar{0},1,\bar{1},0),\ (2,\bar{0},0,\bar{1},0),\ (1,\bar{0},0,\bar{1},0),\ (0,\bar{0},0,\bar{1},0),$$

$$(2,\bar{0},1,\bar{0},0),\ (1,\bar{0},1,\bar{0},0),\ (0,\bar{0},1,\bar{0},0),\ (2,\bar{0},0,\bar{0},0),\ (1,\bar{0},0,\bar{0},0),\ (0,\bar{0},0,\bar{0},0)$$

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to $\mathcal{SS}^{(2)}_{5,3,2}$. These are the 12 sequences which have the same bar pattern as, and are componentwise $\leq$ to, the staircase $(2, \bar{0}, 1, \bar{1}, 0)$.

The previous example shows that applying Definition 3.10 to obtain the set of words in $\mathcal{SS}_{n,k,s}$ can be involved. The following lemma gives a simple recursive definition of substaircase sequences of length n in terms of substaircase sequences of length $n-1$. This recursion gives an efficient way to calculate $\mathcal{SS}_{n,k,s}$ and will be useful in our algebraic analysis of $\mathbb{W}_{n,k,s}$ in Section 4. It will turn out that words in $\mathcal{SS}_{n,k,s}$ index a monomial basis of $\mathbb{W}_{n,k,s}$.

Lemma 3.12. *Let $n, k, s \geq 0$ be integers with $k \geq s$, and let $0 \leq r \leq n-s$. The set $\mathcal{SS}^{(r)}_{n,k,s}$ has the disjoint union decomposition*

$$\mathcal{SS}^{(r)}_{n,k,s} = \bigsqcup_{a=0}^{k-s-1} \left\{ (\bar{a}, c_2, \ldots, c_n) \,:\, (c_2, \ldots, c_n) \in \mathcal{SS}^{(r-1)}_{n-1,k,s} \right\} \sqcup \tag{3.16}$$

$$\bigsqcup_{a=0}^{s-1} \left\{ (a, c_2, \ldots, c_n) \,:\, (c_2, \ldots, c_n) \in \mathcal{SS}^{(r)}_{n-1,k,s-1} \right\} \sqcup \tag{3.17}$$

$$\bigsqcup_{a=s}^{k-1} \left\{ (a, c_2, \ldots, c_n) \,:\, (c_2, \ldots, c_n) \in \mathcal{SS}^{(r)}_{n-1,k,s} \right\}, \tag{3.18}$$

and the set $\mathcal{SS}_{n,k,s}$ has the disjoint union decomposition

$$\mathcal{SS}_{n,k,s} = \bigsqcup_{a=0}^{k-s-1} \left\{ (\bar{a}, c_2, \ldots, c_n) \,:\, (c_2, \ldots, c_n) \in \mathcal{SS}_{n-1,k,s} \right\} \sqcup \tag{3.19}$$

$$\bigsqcup_{a=0}^{s-1} \left\{ (a, c_2, \ldots, c_n) \,:\, (c_2, \ldots, c_n) \in \mathcal{SS}_{n-1,k,s-1} \right\} \sqcup \tag{3.20}$$

$$\bigsqcup_{a=s}^{k-1} \left\{ (a, c_2, \ldots, c_n) \,:\, (c_2, \ldots, c_n) \in \mathcal{SS}_{n-1,k,s} \right\}. \tag{3.21}$$

Proof. The second disjoint union decomposition for $\mathcal{SS}_{n,k,s}$ follows from the first disjoint union decomposition for $\mathcal{SS}^{(r)}_{n,k,s}$ by taking the (disjoint) union over all r, so we focus on the first decomposition.

Given $(c_1, c_2, \ldots, c_n) \in \mathcal{SS}^{(r)}_{n,k,s}$, there exists some word $(b_1, b_2, \ldots, b_n)$ obtained by shuffling

$$(\overline{k-s-1})^{r_0}(s-1)(\overline{k-s})^{r_1}(s-2)(\overline{k-s+1})^{r_2}(s-3)\cdots 0(\overline{k-1})^{r_s},$$

where $r_0+r_1+\cdots+r_s = r$ with the constant sequence $(k-1)^{n-r-s}$, such that $(c_1, c_2, \ldots, c_n)$ has the same bar pattern as $(b_1, b_2, \ldots, b_n)$ and $c_i \leq b_i$ for all i. There are three possibilities for the first entry c_1.

- If $c_1 = \bar{a}$ is barred, then we must have $r_0 > 0$ and $b_1 = \overline{k-s-1}$. It follows that $0 \leq a \leq k-s-1$. Furthermore, the word $(b_2, \ldots, b_n)$ is a shuffle of

$$(\overline{k-s-1})^{r_0-1}(s-1)(\overline{k-s})^{r_1}(s-2)(\overline{k-s+1})^{r_2}(s-3)\cdots 0(\overline{k-1})^{r_s} \quad \text{and} \quad (k-1)^{n-r-s}.$$

  This implies that $(c_2, \ldots, c_n) \in \mathcal{SS}^{(r-1)}_{n-1,k,s}$. Conversely, given $(c_2, \ldots, c_n) \in \mathcal{SS}^{(r-1)}_{n-1,k,s}$ and $0 \leq a \leq k-s-1$, we see that $(\bar{a}, c_2, \ldots, c_n) \in \mathcal{SS}^{(r)}_{n,k,s}$ by prepending the letter $\overline{k-s-1}$ to any $(n-1,k,s)$-staircase $(b_2, \ldots, b_n) \geq (c_2, \ldots, c_n)$.
- If $c_1 = a$ is unbarred with $0 \leq a \leq s-1$, the first letter b_1 of $(b_1, b_2, \ldots, b_n)$ must also be unbarred and $r_0 = 0$. This means that $b_1 = s-1$ or $k-1$. If $k > s$ and $b_1 = k-1$, we may interchange the b_1 with the unique occurrence of $s-1$ in $(b_1, b_2, \ldots, b_n)$ to get a new staircase $(b'_1, b'_2, \ldots, b'_n)$ which

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shares the color pattern of $(c_1, c_2, \ldots, c_n)$ and satisfies $c_i \le b_i'$ for all i. We may, therefore, assume that $b_1 = s-1$. This means that the sequence $(b_2, \ldots, b_n)$ is a shuffle of the words

$$(\overline{k-s})^{r_1}(s-2)(\overline{k-s+1})^{r_2}(s-3)\cdots 0(\overline{k-1})^{r_s} \quad \text{and} \quad (k-1)^{n-r-s}$$

and $r_1 + r_2 + \cdots + r_s = r$. Therefore, we have $(c_2, \ldots, c_n) \in \mathcal{SS}^{(r)}_{n-1,k,s-1}$. Conversely, given $(c_2, \ldots, c_n) \in \mathcal{SS}^{(r)}_{n-1,k,s-1}$, one sees that $(a, c_2, \ldots, c_n) \in \mathcal{SS}^{(r)}_{n,k,s}$ by prepending an $s-1$ to any $(n-1, k, s-1)$-staircase $(b_2, \ldots, b_n) \ge (c_2, \ldots, c_n)$.

- Finally, if $c_1 = a$ is unbarred and $c_1 \ge s$, the first letter b_1 of $(b_1, b_2, \ldots, b_n)$ must be unbarred and $\ge s$. This implies that $b_1 = k-1$, so that $s \le a < k-1$. Furthermore, the sequence $(b_2, \ldots, b_n)$ is a shuffle of the words

$$(\overline{k-s-1})^{r_0}(s-1)(\overline{k-s})^{r_1}(s-2)(\overline{k-s+1})^{r_2}(s-3)\cdots 0(\overline{k-1})^{r_s} \quad \text{and} \quad (k-1)^{n-r-s-1},$$

where $r_0 + r_1 + \cdots + r_s = r$. This means that $(c_2, \ldots, c_n) \in \mathcal{SS}^{(r)}_{n-1,k,s}$. Conversely, for any $(c_2, \ldots, c_n) \in \mathcal{SS}^{(r)}_{n-1,k,s}$, we see that $(a, c_2, \ldots, c_n) \in \mathcal{SS}^{(r)}_{n,k,s}$ by prepending a $k-1$ to any $(n-1, k, s)$-staircase $(b_2, \ldots, b_n) \ge (c_2, \ldots, c_n)$.

The three bullet points above show that $\mathcal{SS}^{(r)}_{n,k,s}$ is a union of the claimed sets of words. The disjointness of this union follows since the first letters of the words in these sets are distinct. □

We are ready to state the main result of this subsection: the codes of ordered set superpartitions are precisely the substaircase words. The key idea of the proof is to invert the map code : $\sigma \mapsto (c_1, \ldots, c_n)$, sending an ordered set partition to its coinversion code. The inverse map $\iota : (c_1, \ldots, c_n) \mapsto \sigma$ is a variant on the insertion maps in the proof of Theorem 3.8.

Theorem 3.13. *Let $n, k, s \ge 0$ be integers with $k \ge s$, and let $r \le n-k$. The coinversion code map gives a well-defined bijection*

$$\mathrm{code} : \mathcal{OSP}^{(r)}_{n,k,s} \longrightarrow \mathcal{SS}^{(r)}_{n,k,s}.$$

Proof. Our first task is to show that the function code is well-defined, that is that $\mathrm{code}(\sigma) \in \mathcal{SS}^{(r)}_{n,k,s}$ for any $\sigma \in \mathcal{OSP}^{(r)}_{n,k,s}$. To this end, let $\sigma = (B_1 \mid \cdots \mid B_k) \in \mathcal{OSP}^{(r)}_{n,k,s}$ with $\mathrm{code}(\sigma) = (c_1, \ldots, c_n)$. We associate a staircase $(b_1, \ldots, b_n)$ to σ as follows. Write the minimal elements $\min B_1, \ldots, \min B_s$ in increasing order $i_1 < \cdots < i_s$, and set $b_{i_j} = s - j$ for each $j \in \{1, \ldots, s\}$. If $1 \le i \le n$ and i is barred in σ, write $b_i = \overline{k-s-1+m_i}$, where $m_i = |\{1 \le j \le s : i_j < i\}|$. Finally, if $1 \le i \le n$, i is unbarred in σ, and i is not minimal in any of the first s blocks $B_1, \ldots, B_s$, set $b_i = k-1$.

As an example of these concepts, consider $\sigma = (5, 7 \mid 1 \mid 3, \bar{4}, \bar{8} \mid \varnothing \mid \bar{2}, 6) \in \mathcal{OSP}^{(3)}_{8,5,3}$. The associated staircase is $(b_1, \ldots, b_8) = (2, \bar{2}, 1, \bar{3}, 0, 4, 4, \bar{4})$. We have $\mathrm{code}(\sigma) = (c_1, \ldots, c_8) = (1, \bar{2}, 0, \bar{1}, 0, 4, 0, \bar{2})$, which has the same bar pattern as, and is componentwise $\le$, the sequence $(b_1, \ldots, b_8)$.

By construction, b is indeed a staircase. Furthermore, b_i is barred if and only if c_i is barred. It is also the case that $c_i \le b_i$ for $1 \le i \le n$. One way to see this is to note that b is the unique set superpartition τ that has the same positive-height entries, the same zero-height entries and the same negative-height entries as σ but with its code maximised in every coordinate. We can construct τ by writing the zero-height entries in increasing order from left to right and placing the other entries as far right as possible. In our example, we have $\tau = (1 \mid 3 \mid 5 \mid \varnothing \mid \bar{2}, \bar{4}, 6, 7, \bar{8}) \in \mathcal{OSP}^{(3)}_{8,5,3}$.

We leave it for the reader to verify that $(c_1, \ldots, c_n)$ has the same bar pattern as $(b_1, \ldots, b_n)$ and $(c_1, \ldots, c_n) \le (b_1, \ldots, b_n)$ componentwise. This shows that the function code in the statement is well defined.

We show that code is a bijection by constructing its inverse map

$$\iota : \mathcal{SS}^{(r)}_{n,k,s} \longrightarrow \mathcal{OSP}^{(r)}_{n,k,s} \tag{3.22}$$

as follows. The map ι starts with a sequence $(\varnothing \mid \cdots \mid \varnothing)$ of k copies of the empty set and builds up an ordered set superpartition by insertion.

Given a sequence $(B_1 \mid \cdots \mid B_k)$ of k possibly empty subsets of the alphabet $\{1, 2, \ldots, \bar{1}, \bar{2}, \ldots\}$, we assign the blocks B_i the *unbarred labels* $0, 1, 2, \ldots, k-1$ as follows. Moving from right to left, we assign the labels $0, 1, \ldots, j-1$ to the j empty blocks among $B_1, \ldots, B_s$. Then, moving from left to right, we assign the unlabeled blocks the labels $j, j+1, \ldots, k-1$. An example of unbarred labels when $k = 7$ and $s = 4$ is as follows:

$$(3, \bar{4}_{\,2} \mid \varnothing_{\,1} \mid \varnothing_{\,0} \mid 1, 5_{\,3} \mid \varnothing_{\,4} \mid \bar{2}_{\,5} \mid \varnothing_{\,6});$$

we draw unbarred labels below their blocks. The *barred* labels $\bar{0}, \bar{1}, \ldots,$ as assigned to the blocks B_i, where either $i > s$ or $B_i \neq \varnothing$ by moving left to right. In our example, the barred labels are

$$(3, \bar{4}^{\,\bar{0}} \mid \varnothing \mid \varnothing \mid 1, 5^{\,\bar{1}} \mid \varnothing^{\,\bar{2}} \mid \bar{2}^{\,\bar{3}} \mid \varnothing^{\,\bar{4}}),$$

where the blocks $B_2 = B_3 = \varnothing$ do not receive a barred label because $2, 3 \leq s = 4$. Barred labels are written above their blocks.

Let $(c_1, \ldots, c_n) \in \mathcal{SS}^{(r)}_{n,k,s}$. To define $\iota(c_1, \ldots, c_n)$, we start with the sequence $(B_1 \mid \cdots \mid B_k) = (\varnothing \mid \cdots \mid \varnothing)$ of k empty blocks and iteratively insert i into the block with unbarred label c_i (if c_i is unbarred) or insert $\bar{i}$ into the block with barred label c_i (if c_i is barred). For example, if $(n, k, s) = (8, 5, 3)$ and $(c_1, \ldots, c_8) = (1, \bar{2}, 0, \bar{1}, 0, 4, 0, \bar{2})$, we perform the following insertion procedure.

i	c_i	$(B_1 \mid \cdots \mid B_s)$
1	1	$(\varnothing_{\,1} \mid 1_{\,2}{}^{\bar{0}} \mid \varnothing_{\,0} \mid \varnothing_{\,3}{}^{\bar{1}} \mid \varnothing_{\,4}{}^{\bar{2}})$
2	$\bar{2}$	$(\varnothing_{\,1} \mid 1_{\,2}{}^{\bar{0}} \mid \varnothing_{\,0} \mid \varnothing_{\,3}{}^{\bar{1}} \mid \bar{2}_{\,4}{}^{\bar{2}})$
3	0	$(\varnothing_{\,0} \mid 1_{\,1}{}^{\bar{0}} \mid 3_{\,2}{}^{\bar{1}} \mid \varnothing_{\,3}{}^{\bar{2}} \mid \bar{2}_{\,4}{}^{\bar{3}})$
4	$\bar{1}$	$(\varnothing_{\,0} \mid 1_{\,1}{}^{\bar{0}} \mid 3, \bar{4}_{\,2}{}^{\bar{1}} \mid \varnothing_{\,3}{}^{\bar{2}} \mid \bar{2}_{\,4}{}^{\bar{3}})$
5	0	$(5_{\,0}{}^{\bar{0}} \mid 1_{\,1}{}^{\bar{1}} \mid 3, \bar{4}_{\,2}{}^{\bar{2}} \mid \varnothing_{\,3}{}^{\bar{3}} \mid \bar{2}_{\,4}{}^{\bar{4}})$
6	4	$(5_{\,0}{}^{\bar{0}} \mid 1_{\,1}{}^{\bar{1}} \mid 3, \bar{4}_{\,2}{}^{\bar{2}} \mid \varnothing_{\,3}{}^{\bar{3}} \mid \bar{2}, 6_{\,4}{}^{\bar{4}})$
7	0	$(5, 7_{\,0}{}^{\bar{0}} \mid 1_{\,1}{}^{\bar{1}} \mid 3, \bar{4}_{\,2}{}^{\bar{2}} \mid \varnothing_{\,3}{}^{\bar{3}} \mid \bar{2}, 6_{\,4}{}^{\bar{4}})$
8	$\bar{2}$	$(5, 7_{\,0}{}^{\bar{0}} \mid 1_{\,1}{}^{\bar{1}} \mid 3, \bar{4}, \bar{8}_{\,2}{}^{\bar{2}} \mid \varnothing_{\,3}{}^{\bar{3}} \mid \bar{2}, 6_{\,4}{}^{\bar{4}})$

The above table shows that

$$\iota : (1, \bar{2}, 0, \bar{1}, 0, 4, 0, \bar{2}) \mapsto (5, 7 \mid 1 \mid 3, \bar{4}, \bar{8} \mid \varnothing \mid \bar{2}, 6).$$

To verify that ι is well defined, we need to check that for any $(c_1, \ldots, c_n) \in \mathcal{SS}^{(r)}_{n,k,s}$, the sequence $\iota(c_1, \ldots, c_n) = (B_1 \mid \cdots \mid B_k)$ of sets is a valid element of $\mathcal{OSP}^{(r)}_{n,k,s}$. That is, the first s sets $B_1, \ldots, B_s$ must be nonempty. Indeed, there are s indices $1 \leq i_1 < \cdots < i_s \leq n$, such that $c_{i_1}, \ldots, c_{i_s}$ are unbarred and $c_{i_j} \leq s - j$. From the definition of our labeling, it follows that, for any j, at least j minimal elements of the nonempty sets in the list $B_1, \ldots, B_s$ are $\leq i_j$. Taking $j = s$, we see that all s of the sets $B_1, \ldots, B_s$ are nonempty. The fact that code and ι are mutually inverse follows from the definition of our labeling. $\square$

3.4. *Skewing formula*

To show that the function $C_{n,k,s}(\mathbf{x}; q, z)$ is the bigraded Frobenius image $\mathrm{grFrob}(\mathbb{W}_{n,k,s}; q, z)$, we will show that the image of both of these symmetric functions under the operator $h_j^{\perp}$ for $j \geq 1$ satisfy the same recursive formula. Lemma 2.3 will then imply that these functions coincide.

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We need a better understanding of the rings $\mathbb{W}_{n,k,s}$ before we prove an $h_j^\perp$-recursion for them, but we can prove the relevant recursion for the C-functions now (using their coinv formulation). This recursion is stated more naturally in terms of the more refined $C^{(r)}_{n,k,s}(\mathbf{x};q)$ functions rather than their coarsened versions $C_{n,k,s}(\mathbf{x};q,z)$.

Lemma 3.14. *Let $n,k,s \geq 0$ with $k \geq s$, and let $0 \leq r \leq n-s$. For any $j \geq 1$, we have*

$$h_j^\perp C^{(r)}_{n,k,s}(\mathbf{x};q) = \sum_{\substack{0 \leq a,b \leq j \\ a \leq r,\, b \leq s}} q^{\binom{j-a-b}{2}+(s-b)a} \times \begin{bmatrix} k-s-1+a+b \\ a \end{bmatrix}_q \cdot \begin{bmatrix} s \\ b \end{bmatrix}_q \cdot \begin{bmatrix} k-s \\ j-a-b \end{bmatrix}_q \cdot C^{(r-j+a+b)}_{n-j,k,s-b}(\mathbf{x};q). \tag{3.23}$$

Proof. It is well known that the homogeneous and monomial symmetric functions are dual bases of the ring of symmetric functions. That is, we have

$$\langle h_\lambda, m_\mu \rangle = \begin{cases} 1 & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases} \tag{3.24}$$

for any partitions λ and μ. Therefore, if $\alpha = (\alpha_1, \alpha_2, \ldots)$ is any composition of n with $h_\alpha = h_{\alpha_1} h_{\alpha_2} \cdots$, we have

$$\langle h_\alpha, C^{(r)}_{n,k,s}(\mathbf{x};q) \rangle = \sum_\tau q^{\mathrm{coinv}(\tau)}, \tag{3.25}$$

where the sum is over all column tableaux $\tau \in \mathcal{CT}^{(r)}_{n,k,s}$ with α_i copies of i for all i. Specialising Equation (3.25) at $\alpha_1 = j$ and applying the adjoint property gives

$$\langle h_{\alpha'}, h_j^\perp C^{(r)}_{n,k,s}(\mathbf{x};q) \rangle = \sum_\tau q^{\mathrm{coinv}(\tau)}, \tag{3.26}$$

where $\alpha' = (\alpha_2, \alpha_3, \ldots)$ is the composition of $n-j$ obtained by removing α_1 from α and the right-hand sides of Equations (3.25) and (3.26) are the same. The strategy is to show that the right-hand sides of Equations (3.23) and (3.26) coincide. This will be achieved combinatorially with a procedure which inserts j new **smallest** entries in a column tableau. Our insertion process will be somewhat more involved than the similar procedure in the proof of Theorem 3.8, where we inserted new largest entries instead. Fix nonnegative integers j, a and b satisfying $0 \leq a, b \leq j$, $a \leq r$ and $b \leq s$. Consider a column tableau $\sigma \in \mathcal{CT}^{(r-j+a+b)}_{n-j,k,s-b}$. We increment by 1 every entry in σ to obtain another filling $\tau \in \mathcal{CT}^{(r-j+a+b)}_{n-j,k,s-b}$. We will, in total, introduce j 1's and $\bar{1}$'s into the tableau τ. Since we must add $j-a-b$ new $\bar{1}$'s, b1's at height 0 and j1's overall to obtain a tableau in $\mathcal{CT}^{(r)}_{n,k,s}$, we will need to add a unbarred 1's at negative heights. We begin by inserting these a negative height 1's.

Since the unbarred entries must be weakly increasing down each column, at this stage, we can only insert a negative height 1 by placing it just below one of the $k-(s-b)$-many $\bullet$'s. If we insert such an entry into column i, it will contribute $i-1$ to coinv. In other words, we have a bijection

$$\begin{aligned} \xi_{<0} : &\left\{\tau \in \mathcal{CT}^{(r-j+a+b)}_{n-j,k,s-b} : \tau \text{ has no 1's}\right\} \times \\ &\qquad \{\text{finite multisets } S \text{ of size } a \text{ drawn from } \{s-b, s-b+1, \ldots, k-1\}\} \longrightarrow \\ &\qquad\qquad \left\{\tau' \in \mathcal{CT}^{(r-j+a+b)}_{n-j+a,k,s-b} : \tau' \text{ has } a \text{1's at negative heights and no other 1's}\right\}, \end{aligned} \tag{3.27}$$

where $\tau' = \xi_{<0}(\tau, S)$ is obtained by iteratively placing a 1 at height -1 in column $i+1$ for each occurrence of i in S. Since $\mathrm{coinv}(\tau') = \mathrm{coinv}(\tau) + \sum_{i \in S} i$, this map contributes $q^{(s-b)a} \begin{bmatrix} k-s-1+a+b \\ a \end{bmatrix}_q$ to (3.26).

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Next, we insert b 1's at height 0. This will be the most complicated part of the procedure, as we need to be careful not to violate the necessary inequalities in each column. Our map will be a bijection

$$\xi_0 : \left\{\tau' \in \mathcal{CT}^{(r-j+a+b)}_{n-j+a,k,s-b} : \tau' \text{ has } a\,1\text{'s at negative heights and no other 1's}\right\}\times$$
$$\{\text{finite multisets } S \text{ of size } b \text{ drawn from } \{0,1,\dots,s-b\}\} \longrightarrow$$
$$\left\{\tau'' \in \mathcal{CT}^{(r-j+a+b)}_{n-j+a+b,k,s} : \tau'' \text{ has } a\,1\text{'s at negative heights, } b \text{ 1's at height 0 and no other 1's}\right\}. \quad (3.28)$$

Given τ' and S, we begin by simply replacing the b leftmost $\bullet$'s in τ' with 1's. Since there are $k-(s-b)$ bullets and $k \geq s$, this can always be done. Furthermore, since τ' has no $\bar{1}$'s, this replacement does not violate any necessary column inequalities. Suppose $S = \{i_1 \geq \ldots \geq i_b\}$. Initialise $j = 1$. For $j = 1$ to b, we repeat the following algorithm i_j times:

1. Suppose the j^{th} 1 at height 0 is in column ℓ, and let $m > 1$ be the height 0 entry in column $\ell - 1$.
2. Switch the height 0 entries in columns ℓ and $\ell - 1$.
3. Move every barred entry $\leq m$ in column ℓ and every unbarred entry $< m$ in column ℓ to column $\ell - 1$ (moving entries up and down as necessary so that the diagram is justified as usual).

We claim that each loop of this algorithm increments coinv, yields a valid column tableau and is invertible. The latter two statements are true by construction. Clearly, each loop creates one new coinversion between the 1 at height 0 and the entry m. We check that this is the only change to the total number of coinversions.

- If u is some other height 0 entry, u has the same number of height 0 entries to its right which are $> u$.
- Suppose u is at negative height. The only case that might have affected u's contribution to coinv is if $u < m$ starts in column ℓ. Then u is moved to column $\ell - 1$, but, due to m, there is a new height 0 entry to the right of u that is $> u$.
- Suppose u is at positive height. If $u \leq m$, then m and u did not contribute a coinversion, so moving u to the left does not affect the total coinv. If $u > m$, then $u > 1$ and u contributes the same number of coinversions.

After repeating this procedure i_j times for $j = 1$ to b, we have $\operatorname{coinv}(\tau'') = \operatorname{coinv}(\tau') + \sum_{i\in S} i$, yielding the $\left[{s \atop b}\right]_q$ term in (3.14).

Finally, we insert the $j - a - b\,\bar{1}$'s into τ''. A $\bar{1}$ can only be placed above a $\bullet$ in τ'', of which there are now $k - s$, and only, at most, once in each column. Placing $\bar{1}$ over the $i^{\text{th}}\bullet$ from the left creates $i - 1$ new coinversions, one with each bullet strictly to the left. Therefore, we have a bijection

$$\xi_{>0} : \left\{\tau'' \in \mathcal{CT}^{(r-j+a+b)}_{n-j+a+b,k,s} : \tau'' \text{ has } a \text{ 1's at negative heights,} b \text{ 1's at height 0 and no other 1's}\right\}\times$$
$$\{\text{finite sets } S \text{ of size } j-a-b \text{ drawn from } \{0,1,\dots,k-s-1\}\} \longrightarrow$$
$$\left\{\tau''' \in \mathcal{CT}^{(r)}_{n,k,s} : \tau''' \text{ has } a\,1\text{'s at negative heights, } b \text{ 1's at height 0 } j-a-b\ \bar{1}\text{'s}\right\} \quad (3.29)$$

which places a $\bar{1}$ above $\bullet$ number $i+1$ from the left for every $i \in S$. This bijection satisfies $\operatorname{coinv}(\tau''') = \operatorname{coinv}(\tau'') + \sum_{i\in S} i$ and contributes the factor $q^{\binom{j-a-b}{2}}\left[{k-s \atop j-a-b}\right]_q$ to (3.14).

In Figure 2, we depict an example for $n = 18$, $k = 5$, $s = 4$, $j = 6$, $a = 3$, $b = 2$, where we have already incremented each entry so that no 1's or $\bar{1}$'s appear. When applying $\xi_{<0}$, ξ_0 and $\xi_{>0}$, we take $S = \{2, 2, 4\}$, $\{1, 2\}$ and $\{0\}$, respectively. The initial tableau has 17 coinv:

- 1 at height 0, between the 3 and the 4,
- $1 + 1 + 3 + 4 + 4 = 13$ from negative height entries, all coming from the column of each of these entries and
- 3 from positive height entries (the 3 at height 0 with $\bar{6}$ and $\bar{7}$, and the $\bar{2}$ with the $\bullet$ to its left)

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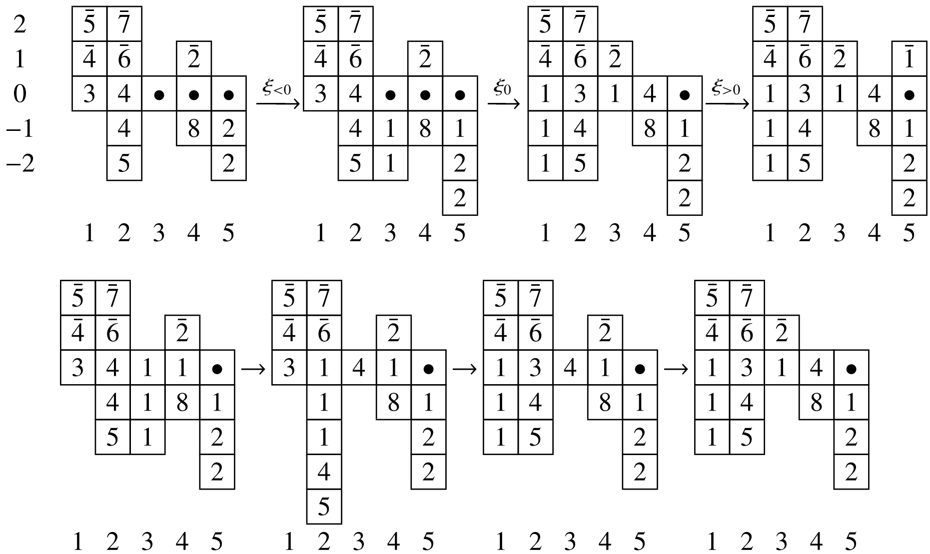

Figure 2. *We show an example of the bijections described in the proof of Lemma 3.14. We take $S = \{2, 2, 4\}$, $\{1, 2\}$ and $\{0\}$ in the three maps, respectively. On the second line, the iterations making up ξ_0 are shown in finer detail.*

After the maps in Figure 2, the increases in coinv correspond to the sums of the elements in each S. Respectively, the coinv increases to $25 = 17 + (2 + 2 + 4)$, then $28 = 25 + (1 + 2)$ and then $28 = 28 + 0$. $\square$

4. Quotient presentation, monomial basis and Hilbert series

4.1. The superlex order on monomials

We identify length n words over $\{0, 1, 2, \ldots, \bar{0}, \bar{1}, \bar{2}, \ldots\}$ with monomials in Ω_n by

$$(c_1, \ldots, c_n) \leftrightarrow x_1^{c_1} \cdots x_n^{c_n} \times \theta_I \qquad (I = \{1 \leq i \leq n \ : \ c_i \text{ is barred}\}). \tag{4.1}$$

Here, we adopt the shorthand notation $\theta_I = \theta_{i_1} \cdots \theta_{i_k}$, where $I = \{i_1 < \cdots < i_k\} \subseteq [n]$. For example, inside the ring Ω_5 we have

$$(2, \bar{2}, \bar{0}, 0, \bar{1}) \leftrightarrow (x_1^2 x_2^2 x_5) \times (\theta_{235}).$$

We sometimes compress this expression further by using exponent notation for the x_i variables, for example,

$$(x_1^2 x_2^2 x_5) \times (\theta_{235}) = x^{22001} \theta_{235}.$$

We refer to the word $(c_1, \ldots, c_n)$ as the *exponent sequence* of its corresponding superspace monomial in Ω_n. The monomials corresponding to substaircase sequences will be important.

Definition 4.1. Let $n, k, s \geq 0$ be integers with $k \geq s$, and let $r \leq n - s$. We let $\mathcal{M}_{n,k,s}$ (respectively, $\mathcal{M}_{n,k,s}^{(r)}$) be the family of monomials in Ω_n corresponding to the substaircase sequences in $\mathcal{SS}_{n,k,s}$ (resp. $\mathcal{SS}_{n,k,s}^{(r)}$).

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Orders on words $(c_1, \ldots, c_n)$ give orders on monomials in Ω_n. The following total order will be crucial in our work.

Definition 4.2 (Superlex Order). Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two length n words over the alphabet $\{0, 1, 2, \ldots, \bar{0}, \bar{1}, \bar{2}, \ldots\}$. We write $a \prec b$ if there exists an index $1 \leq j \leq n$, such that

- $a_i = b_i$ for all $i < j$ and
- $a_j < b_j$ under the order

$$\cdots < \bar{3} < \bar{2} < \bar{1} < \bar{0} < 0 < 1 < 2 < 3 < \cdots$$

  on individual letters.

We say that two monomials $m, m' \in \Omega_n$ satisfy $m \prec m'$ if their exponent sequences satisfy this relation. Given any nonzero superspace element $f \in \Omega_n$, we let $\mathrm{in}(f)$ be the $\prec$-maximal superspace monomial which appears in f with nonzero coefficient.

In the absence of θ-variables, superspace monomials in Ω_n reduce to classical monomials in $\mathbb{Q}[x_1, \ldots, x_n]$, and Definition 4.2 is the lexicographical order on monomials in this ring. The classical lex order is a *monomial order* in the sense of Gröbner theory:

- there is no infinite descending chain $m_1 \succ m_2 \succ m_3 \succ \cdots$ of monomials in $\mathbb{Q}[x_1, \ldots, x_n]$ (i.e. $\prec$ is a *well order* on these monomials) and
- if m_1, m_2 and m_3 are monomials in $\mathbb{Q}[x_1, \ldots, x_n]$ with $m_1 \prec m_2$, then $m_1 m_3 \prec m_2 m_3$.

Both of these properties fail when we introduce θ-variables. We have $\theta_1 \succ x_1\theta_1 \succ x_1^2\theta_1 \succ \cdots$ and $x_1 \prec x_1^2$, whereas $x_1\theta_1 \succ x_1^2\theta_1$.

Despite its failure to be a monomial order, the superlex order $\prec$ of Definition 4.2 will be important for our work. It is our belief that this relates to the inscrutable Gröbner theory of important superspace ideals, such as the superspace coinvariant ideal in [34].

4.2. A dimension lower bound

Recall that $\mathbb{H}_{n,k,s}$ is the Ω_n-submodule (under the $\odot$-action) of Ω_n generated by the Vandermonde $\delta_{n,k,s}$. Therefore, the space $\mathbb{H}_{n,k,s}$ is spanned by elements of the form $m \odot \delta_{n,k,s}$, where $m \in \Omega_n$ is a monomial.

In this subsection, we derive a lower bound for the Hilbert series of $\mathbb{H}_{n,k,s}$ (and $\mathbb{W}_{n,k,s}$) by showing that every substaircase monomial in $\mathcal{M}_{n,k,s}$ arises as the initial term $\mathrm{in}(m \odot \delta_{n,k,s})$ for some monomial $m \in \Omega_n$. In fact, we show that the map $m \mapsto \mathrm{in}(m \odot \delta_{n,k,s})$ is an involution on the set $\mathcal{M}_{n,k,s}$.

Proposition 4.3. *Let $n, k, s \geq 0$ be integers with $k \geq s$. The map*

$$\iota : \mathcal{M}_{n,k,s} \longrightarrow \mathcal{M}_{n,k,s} \tag{4.2}$$

given by $\iota(m) = \mathrm{in}(m \odot \delta_{n,k,s})$ is a well-defined involution.

The well-definedness assertion in Proposition 4.3 means that $m \odot \delta_{n,k,s}$ is nonzero whenever $m \in \mathcal{M}_{n,k,s}$ and that $\mathrm{in}(m \odot \delta_{n,k,s}) \in \mathcal{M}_{n,k,s}$. We will see that for any $f \in \Omega_n$ with $f \odot \delta_{n,k,s} \neq 0$, we have $\mathrm{in}(f \odot \delta_{n,k,s}) \in \mathcal{M}_{n,k,s}$. Before we prove Proposition 4.3, it will be helpful to state an observation about $m \odot M$ for arbitrary monomials m and M in Ω_n.

Observation 4.4. *Suppose m and M are monomials in Ω_n with exponent sequences u and U, respectively. Let $m^* = m \odot M$. Then m^* is nonzero if and only if*

- *$u_i \leq U_i$ after ignoring bars for every i and*
- *if u_i is barred, then U_i is barred.*

If m^* is nonzero, then its exponent sequence u^* has entries given by $u_i^* = U_i - u_i$ and u_i^* is barred if and only if U_i is barred and u_i is not barred.

Proof of Proposition 4.3. Fix some $m \in \mathcal{M}_{n,k,s}$, and let u be the exponent sequence of m. We use Algorithm 1 below to construct the exponent sequence U of the monomial M which appears in $\delta_{n,k,s}$, such that $m \odot M \doteq \iota(m)$, where $\doteq$ denotes equality up to a nonzero scalar. To begin a running example, if $n = 9$, $k = 7$, $s = 4$ and $u = (\bar{1}, 3, 0, 5, 2, 0, \bar{2}, \bar{5}, 0)$, Algorithm 1 returns

$$U = (\bar{6}, 3, 2, \bar{6}, \bar{6}, 1, \bar{6}, \bar{6}, 0).$$

Algorithm 1 Constructing U from u.

1: $i \leftarrow 1, \ell \leftarrow s, U \leftarrow \varnothing$
2: **while** $i \le n$ **do**
3:     **if** u_i is barred or $u_i \ge \ell$, **then**
4:         $U_i \leftarrow \overline{k-1}$
5:     **else**
6:         $\ell \leftarrow \ell - 1$
7:         $U_i \leftarrow \ell$
8:     **end if**
9:     $i \leftarrow i + 1$
10: **end while**

Since u is substaircase, U is indeed a shuffle of $s-1, s-2, \ldots, 0$ and $(\overline{k-1})^{n-s}$, so M appears as a monomial in $\delta_{n,k,s}$. We claim that M is the largest monomial in $\delta_{n,k,s}$ with respect to superlexicographic order, such that $m \odot M \neq 0$. Suppose U' is the exponent sequence of another monomial M' in $\delta_{n,k,s}$, such that $U' \succ U$. Then there exists an index j, such that $U'_i = U_i$ for all $i < j$ and $U'_j \succ U_j$. Since U and U' are shuffles of $s-1, s-2, \ldots, 1, 0$ with $(\overline{k-1})^{n-s}$, this implies that U'_j is not barred and $U_j = \overline{k-1}$. By lines 3 and 4 in Algorithm 1, $U_j = \overline{k-1}$ implies that either u_j is barred or $u_j \ge L$, where

$$L = \min\{\text{unbarred entries in } U_1 \ldots U_{j-1}\}.$$

If u_j is barred and U'_j is unbarred, $m \odot M' = 0$ by Observation 4.4. Otherwise, $u_j \ge L$ and L is strictly greater than all the potential unbarred values of U'_j, which, again, implies $m \odot M' = 0$. Therefore, M is the largest monomial in $\delta_{n,k,s}$, such that $m \odot M \neq 0$.

Now suppose that $M'' \prec M$ is a monomial in $\delta_{n,k,s}$ and $m \odot M'' \neq 0$. We aim to show that $m \odot M'' \prec m \odot M$. Suppose that U'' is the exponent sequence of M'' and that j is the first index at which U'' and U differ, so $U''_j \prec U_j$. Since the only barred entry in either U'' or U is $\overline{k-1}$, U_j must be unbarred. By Algorithm 1, u_j must be unbarred, and since $m \odot M'' \neq 0$, we conclude that U''_j is unbarred. Since Algorithm 1 always uses the largest available unbarred entry from the set $\{s, s-1, \ldots, 0\}$ in line 7, $U''_j < U_j$. It follows from Observation 4.4 that the entry at index j in $m \odot M''$ is strictly less than the entry at index j in $m \odot M$ and that both entries are unbarred. Hence, $m \odot M'' \prec m \odot M$ and

$$m \odot M = \operatorname{in}(m \odot \delta_{n,k,s}) = \iota(m).$$

Next, we show that $\iota(m) \in \mathcal{M}_{n,k,s}$. Let $m^* = \iota(m)$ and u^* be the exponent sequence of m^*. In Algorithm 2, we use u and U to construct an (n, k, s)-staircase v that certifies $m^* \in \mathcal{M}_{n,k,s}$, in that for every i, $u^*_i \le v_i$ and v_i is barred if and only if u^*_i is barred. Continuing our running example, if $n = 9$, $k = 7$, $s = 4$, $u = (\bar{1}, 3, 0, 5, 2, 0, \bar{2}, \bar{5}, 0)$ and $U = (\bar{6}, 3, 2, \bar{6}, \bar{6}, 1, \bar{6}, \bar{6}, 0)$, we can compute

$$u^* = (5, 0, 2, \bar{1}, \bar{4}, 1, 4, 1, 0).$$

From u and U, Algorithm 2 produces

$$v = (6, 3, 2, \bar{4}, \bar{4}, 1, 6, 6, 0),$$

and u^* is indeed 'below' v. Note that, although the stage of Algorithm 2 involving the set A seems complicated, we are simply letting v_i be the unique barred entry that is allowed at that index in an (n, k, s)-staircase.

Algorithm 2 Creating an (n, k, s)-staircase v from u and U.

1: $i \leftarrow 1, \ell = s, v \leftarrow \varnothing$
2: **while** $i \leq n$ **do**
3:     **if** u_i and U_i are both barred, **then**
4:         $v_i \leftarrow k - 1$
5:     **else if** U_i is not barred, **then**
6:         $\ell \leftarrow \ell - 1$
7:         $v_i \leftarrow \ell$
8:     **else**
9:         $A \leftarrow \{j \in v_1 \ldots v_{i-1} : j \in \{0, 1, \ldots, s-1\},\ j \text{ not barred}\}$
10:         **if** $A = \varnothing$, **then**
11:             $v_i = \overline{k - s - 1}$
12:         **else**
13:             $v_i = \overline{k - \min A - 1}$
14:         **end if**
15:     **end if**
16:     $i \leftarrow i + 1$
17: **end while**

By construction, v_i is barred if and only if u_i^* is barred. In the first 'if' clause of Algorithm 2, we must have $u_i^* \leq v_i$ simply because $k - 1$ is the largest possible entry of u^*. In the second such clause, if we have $u_i^* > v_i$, since $u_i^* = U_i - u_i$, we must have $U_i - u_i > v_i$ and $U_i > v_i$. This is impossible, since Algorithms 1 and 2 always use the largest unused element ℓ of $\{0, 1, \ldots, s-1\}$ in line 7 of each respective algorithm. In the third such clause, we must have $U_i = \overline{k-1}$ and u_i is not barred. If $A = \emptyset$, then line 3 in Algorithm 1 implies $u_i \geq s$. Hence, $u_i^* \leq k - s - 1$ and u_i^* is barred, so u_i^* and v_i are both barred and $u_i^* \leq v_i$. If $A \neq \varnothing$, suppose for contradiction that $u_i^* > v_i$. Then

$$\begin{aligned} u_i^* = U_i - u_i &> v_i \\ k - 1 - u_i &> k - \min A - 1 \\ u_i &< \min A, \end{aligned}$$

so u_i is an unbarred entry that is less than $\min A$. Then Algorithm 1 would have used line 7 to set $U_i = \min A - 1$, since at that stage, we would have $\ell > u_i$. Therefore, $m^* \in \mathcal{M}_{n,k,s}$.

Finally, we show that ι is an involution. From the argument above, we know $\iota(m) = m^* \in \mathcal{M}_{n,k,s}$. Suppose M^* is the monomial we obtain by running Algorithm 1 on m^*. Then M^* is the monomial in $\delta_{n,k,s}$, such that $m^* \odot M^* \doteq \iota(m^*)$. We claim that $M^* = M$. If we can show this, then we will have

$$\iota(\iota(m)) \doteq m^* \odot M^* = m^* \odot M \doteq (m \odot M) \odot M \doteq m,$$

so $\iota(\iota(m)) = m$. In our running example, where $n = 9$, $k = 7$, $s = 4$, $u = (\bar{1}, 3, 0, 5, 2, 0, \bar{2}, \bar{5}, 0)$, $U = (\bar{6}, 3, 2, \bar{6}, \bar{6}, 1, \bar{6}, \bar{6}, 0)$, $u^* = (5, 0, 2, \bar{1}, \bar{4}, 1, 4, 1, 0)$ and $v = (6, 3, 2, \bar{4}, \bar{4}, 1, 6, 6, 0)$, we get

$$U^* = (\bar{6}, 3, 2, \bar{6}, \bar{6}, 1, \bar{6}, \bar{6}, 0) = U.$$

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To prove that $U^* = U$ in every case, consider running Algorithm 1 on m and m^* in parallel. We let ℓ^* be the value in the algorithm creating M^* corresponding to ℓ and let U^* be the exponent sequence of M^*. This process has the following loop invariants.

Claim. *After every loop of Algorithm 1, we have $\ell^* = \ell$ and $U_i^* = U_i$.*

We prove the Claim by considering the i^{th} loop of Algorithm 1 and analysing the possible values of u_i.

- Suppose u_i is barred. Then $U_i = \overline{k-1}$ and $u_i^* = k - 1 - u_i$. Since u is substaircase, $u_i \leq k - \ell - 1$, so

$$u_i^* = k - 1 - u_i \geq k - 1 - (k - \ell - 1) = \ell.$$

  Hence, $U_i^* = \overline{k-1} = U_i$.
- Suppose u_i is unbarred and $u_i \geq \ell$. Then $U_i = \overline{k-1}$, u_i^* is barred and $U_i^* = \overline{k-1} = U_i$.
- Finally, suppose u_i is unbarred and $u_i < \ell$. Then we let $U_i = \ell - 1$ and get

$$u_i^* = \ell - u_i - 1 = \ell^* - u_i - 1 < \ell^*.$$

  Then the algorithm decrements both ℓ and ℓ^* and sets $U_i^* = \ell^* = \ell = U_i$.

This proves the Claim and the proposition. □

Proposition 4.3 gives the following lower bound on the dimension of $\mathbb{W}_{n,k,s}$. In Theorem 4.12, we will prove that this lower bound is tight.

Lemma 4.5. *The set $\mathcal{M}_{n,k,s}$ descends to a linearly independent subset of $\mathbb{W}_{n,k,s}$, and we have* $\dim \mathbb{W}_{n,k,s} = \dim \mathbb{H}_{n,k,s} \geq |\mathcal{M}_{n,k,s}| = |\mathcal{OSP}_{n,k,s}|$.

Proof. Proposition 4.3 shows that $\{m \odot \delta_{n,k,s} \; : \; m \in \mathcal{M}_{n,k,s}\}$ is a linearly independent subset of $\mathbb{H}_{n,k,s}$, so that $\dim \mathbb{W}_{n,k,s} = \dim \mathbb{H}_{n,k,s} \geq |\mathcal{M}_{n,k,s}|$. The equality $|\mathcal{M}_{n,k,s}| = |\mathcal{OSP}_{n,k,s}|$ follows from Theorem 3.13. □

4.3. Statement of monomial basis and quotient presentation

The goal of this section is twofold: giving an explicit generating set of the defining ideal $\operatorname{ann} \delta_{n,k,s}$ of $\mathbb{W}_{n,k,s}$ and proving that $\mathcal{M}_{n,k,s}$ descends to a basis of $\mathbb{W}_{n,k,s}$. The proofs of these two results will be deeply intertwined.

We begin by defining an ideal $I_{n,k,s} \subseteq \Omega_n$, which will turn out to equal $\operatorname{ann} \delta_{n,k,s}$. If $S \subseteq [n]$ and $d \geq 0$, we let $e_d(S)$ be the elementary symmetric polynomial of degree d in the variable set $\{x_i \; : \; x \in S\}$. For example, we have $e_2(134) = x_1x_3 + x_1x_4 + x_3x_4$. We adopt the convention $e_d(S) = 0$ whenever $d > |S|$.

Definition 4.6. Given integers $n, k, s \geq 0$ with $k \geq s$, define $I_{n,k,s} \subseteq \Omega_n$ to be the ideal

$$I_{n,k,s} = \langle x_1^k, x_2^k, \ldots, x_n^k \rangle + \langle x_1^j\theta_1 + x_2^j\theta_2 + \cdots + x_n^j\theta_n \; : \; j \geq k - s \rangle + \\ \langle e_d(S) \cdot \theta_T \; : \; S \sqcup T = [n], \; d > |S| - s, d > 0 \rangle. \quad (4.3)$$

In the third summand, the pair (S, T) ranges over all disjoint union decompositions $S \sqcup T$ of $[n]$.

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For example, suppose $(n, k, s) = (4, 3, 2)$. The ideal $I_{4,3,2} \subseteq \Omega_4$ is generated by the variable powers $x_1^3, x_2^3, x_3^3, x_4^3$, the elements $x_1^j\theta_1 + x_2^j\theta_2 + x_3^j\theta_3 + x_4^j\theta_4$, where $j \geq 1$ and the polynomials

$$
\begin{aligned}
&e_4(1234), e_3(1234),\\
&\quad e_3(123)\cdot\theta_4, e_2(123)\cdot\theta_4, e_3(124)\cdot\theta_3, e_2(124)\cdot\theta_3, e_3(134)\cdot\theta_2, e_2(134)\cdot\theta_2, e_3(234)\cdot\theta_1,\\
&\qquad e_2(234)\cdot\theta_1, e_2(12)\cdot\theta_{34}, e_1(12)\cdot\theta_{34}, e_2(13)\cdot\theta_{24}, e_1(13)\cdot\theta_{24}, e_2(14)\cdot\theta_{23},\\
&\qquad e_1(14)\cdot\theta_{23}, e_2(23)\cdot\theta_{14}, e_1(23)\cdot\theta_{14}, e_2(24)\cdot\theta_{13}, e_1(24)\cdot\theta_{13}, e_2(34)\cdot\theta_{12},\\
&\qquad\quad e_1(34)\cdot\theta_{12}, e_1(4)\cdot\theta_{123}, \theta_{123}, e_1(3)\cdot\theta_{124}, \theta_{124}, e_1(2)\cdot\theta_{134}, \theta_{134},\\
&\qquad\quad e_1(1)\cdot\theta_{234}, \theta_{234}.
\end{aligned}
$$

There is redundancy in this list of generators, for example, the generator $e_1(4)\cdot\theta_{123} = x_4\cdot\theta_{123}$ is a multiple of the generator θ_{123}.

We adopt the temporary notation

$$\mathbb{U}_{n,k,s} = \Omega_n / I_{n,k,s} \tag{4.4}$$

for the quotient of $\mathbb{U}_{n,k,s}$ by $I_{n,k,s}$. Since $I_{n,k,s}$ is $\mathfrak{S}_n$-stable and bihomogeneous, the quotient $\mathbb{U}_{n,k,s}$ is a bigraded $\mathfrak{S}_n$-module. It will turn out that $\mathbb{U}_{n,k,s}$ and $\mathbb{W}_{n,k,s}$ coincide. The following lemma shows that $\mathbb{W}_{n,k,s}$ is a quotient of $\mathbb{U}_{n,k,s}$.

Lemma 4.7. *Let $n, k, s \geq 0$ with $k \geq s$. We have the containment $I_{n,k,s} \subseteq \operatorname{ann}\delta_{n,k,s}$.*

Proof. For every generator g of the ideal $I_{n,k,s}$, we check that $g \odot \delta_{n,k,s} = 0$. We handle the three different kinds of generators g of $I_{n,k,s}$ separately, using $\doteq$ for equality up to a nonzero scalar.

Case 1. *$g = x_i^k$ for some i.*

Since no power of $x_i \geq k$ appears in

$$\delta_{n,k,s} = \varepsilon_n \cdot (x_1^{k-1}\cdots x_{n-s}^{k-1}x_{n-s+1}^{s-1}\cdots x_{n-1}^1 x_n^0 \times \theta_1\cdots\theta_{n-s}),$$

it follows that $x_i^k \odot \delta_{n,k,s} = 0$.

Case 2. *$g = x_1^j\theta_1 + \cdots + x_n^j\theta_n$ for some $j \geq k-s$.*

We compute

$$
\begin{aligned}
g \odot \delta_{n,k,s} &= g \odot \varepsilon_n \cdot (x_1^{k-1}\cdots x_{n-s}^{k-1}x_{n-s+1}^{s-1}\cdots x_{n-1}^1 x_n^0 \times \theta_1\cdots\theta_{n-s}) && (4.5)\\
&= \varepsilon_n \cdot (g \odot (x_1^{k-1}\cdots x_{n-s}^{k-1}x_{n-s+1}^{s-1}\cdots x_n^0 \times \theta_1\cdots\theta_{n-s})) && (4.6)\\
&\doteq \varepsilon_n \cdot \left(\sum_{i=1}^{n-s}(-1)^{i-1}x_1^{k-1}\cdots x_i^{k-j-1}\cdots x_{n-s}^{k-1}x_{n-s+1}^{s-1}\cdots x_n^0 \times \theta_1\cdots\widehat{\theta_i}\cdots\theta_{n-s}\right) && (4.7)\\
&= \sum_{i=1}^{n-s}(-1)^{i-1}\varepsilon_n \cdot \left(x_1^{k-1}\cdots x_i^{k-j-1}\cdots x_{n-s}^{k-1}x_{n-s+1}^{s-1}\cdots x_n^0 \times \theta_1\cdots\widehat{\theta_i}\cdots\theta_{n-s}\right) && (4.8)\\
&= 0. && (4.9)
\end{aligned}
$$

The first equality is the definition of $\delta_{n,k,s}$. The second equality follows because $g \odot (-)$ is an $\mathfrak{S}_n$-invariant differential operator, and so commutes with the action of $\mathbb{Q}[\mathfrak{S}_n]$ on Ω_n. The third equality is the effect of applying $g \odot (-)$ and the fourth is rearrangement. The final equality holds since the hypothesis $j \geq s$ implies $k - j - 1 \leq s - 1$, so that ε_n annihilates the given superspace monomial.

Case 3. *$g = e_d(S)\cdot\theta_T$ for some $S \sqcup T = [n]$ with $d > |S| - s$.*

Let $\delta_{|S|,k,s}(S)$ denote the superspace Vandermonde $\delta_{|S|,k,s}$ defined in the set $\{x_i, \theta_i \,:\, i \in S\}$ of commuting and anticommuting variables indexed by S. Applying the operator $\theta_T \odot (-)$ to $\delta_{n,k,s}$, we see that

$$\theta_T \odot \delta_{n,k,s} \doteq \prod_{j \in T} x_j^{k-1} \times \delta_{|S|,k,s}(S). \tag{4.10}$$

Therefore, we have

$$g \odot \delta_{n,k,s} \doteq e_d(S) \odot \left(\prod_{j \in T} x_j^{k-1} \times \delta_{|S|,k,s}(S) \right) = \prod_{j \in T} x_j^{k-1} \times \left(e_d(S) \odot \delta_{|S|,k,s}(S) \right) = 0, \tag{4.11}$$

where the second equality follows from the disjointness of the sets S and T and the final equality is a consequence of [27, Lemma 3.2]. □

4.4. The harmonic space $I_{n,k,s}^{\perp}$

Throughout this section, we adopt the notation

$$\Omega'_{n-1} = \mathbb{Q}[x_2, \ldots, x_n] \otimes \wedge\{\theta_2, \ldots, \theta_n\} \tag{4.12}$$

for rank $n-1$ superspace over the commuting variables $x_2, \ldots, x_n$ and the anticommuting variables $\theta_2, \ldots, \theta_n$.

Lemma 4.7 gives the containment $I_{n,k,s} \subseteq \operatorname{ann} \delta_{n,k,s}$. We consider the harmonic space

$$I_{n,k,s}^{\perp} = \{f \in \Omega_n \,:\, g \odot f = 0 \text{ for all } g \in I_{n,k,s}\} \tag{4.13}$$

to the ideal $I_{n,k,s}$. The composite map

$$I_{n,k,s}^{\perp} \hookrightarrow \Omega_n \twoheadrightarrow \mathbb{U}_{n,k,s} \tag{4.14}$$

is an isomorphism of bigraded $\mathfrak{S}_n$-modules. Lemma 4.7 gives the containment $I_{n,k,s} \subseteq \operatorname{ann} \delta_{n,k,s}$, which implies

$$\mathbb{H}_{n,k,s} \subseteq I_{n,k,s}^{\perp}. \tag{4.15}$$

We will see (Theorem 4.12) that these spaces coincide, but it will be convenient to work with the *a priori* larger space $I_{n,k,s}^{\perp}$ in this section.

In order to show the containment (4.15) is in fact an equality, we bound the dimension of $I_{n,k,s}^{\perp}$ from above. To do this, we inductively show (Lemma 4.11) that the leading terms of nonzero polynomials $f \in I_{n,k,s}^{\perp}$ belong to the family $\mathcal{M}_{n,k,s}$ of substaircase monomials. For the sake of this induction, we prove lemmata about the expansions of polynomials $f \in I_{n,k,s}^{\perp}$ in terms of the initial variables x_1 and θ_1. The first of these is a simple degree bound.

Lemma 4.8. *Let $n, k, s \geq 0$ with $k \geq s$, and let $f \in I_{n,k,s}^{\perp}$. Then f may be expressed uniquely as*

$$f = x_1^0 g_0 + x_1^1 g_1 + \cdots + x_1^{k-1} g_{k-1} + x_1^0 \theta_1 h_0 + x_1^1 \theta_1 h_1 + \cdots + x_1^{k-1} \theta_1 h_{k-1} \tag{4.16}$$

for some elements $g_0, g_1, \ldots, g_{k-1}, h_0, h_1, \ldots, h_{k-1} \in \Omega'_{n-1}$.

Proof. The element x_1^k is a generator of $I_{n,k,s}$. Since $f \in I_{n,k,s}^{\perp}$, we have $x_1^k \odot f = 0$, so that no power of $x_1 \geq k$ may appear in *f*. □

By the definition of the order $\prec$ on superspace monomials, if $f \in I_{n,k,s}^{\perp}$ is nonzero and if

$$f = x_1^0 g_0 + x_1^1 g_1 + \cdots + x_1^{k-1} g_{k-1} + x_1^0 \theta_1 h_0 + x_1^1 \theta_1 h_1 + \cdots + x_1^{k-1} \theta_1 h_{k-1}$$

as in Lemma 4.8, the $\prec$-leading term of f is given by

$$\mathrm{in}(f) = \begin{cases} x_1^i \cdot \mathrm{in}(g_i) & \text{if } g_j \neq 0 \text{ for some } j, \text{ and if } i \text{ is maximal with } g_i \neq 0, \\ x_1^i \theta_1 \cdot \mathrm{in}(h_i) & \text{if } g_j = 0 \text{ for all } j, \text{ and if } i \text{ is minimal with } h_i \neq 0. \end{cases}$$

With Lemma 3.12 in mind, we aim to relate the harmonicity of $f \in I_{n,k,s}^{\perp}$ to showing that g_i and h_i lie in appropriate harmonic spaces within Ω'_{n-1}. Our starting point is the following vanishing result for harmonics divisible by θ_1.

Lemma 4.9. *Let $n, k, s \geq 0$ be integers with $k \geq s$, and suppose $f \in I_{n,k,s}^{\perp}$ is a multiple of θ_1:*

$$f = x_1^0 \theta_1 h_0 + x_1^1 \theta_1 h_1 + \cdots + x_1^{k-1} \theta_1 h_{k-1}$$

for some $h_0, h_1, \ldots, h_{k-1} \in \Omega'_{n-1}$. We have $h_{k-s} = h_{k-s+1} = \cdots = h_{k-1} = 0$.

The vanishing conditions in Lemma 4.9 will turn out to mirror the 'top branches' (3.16) and (3.19) of the disjoint union decompositions in Lemma 3.12.

Proof. The generator $x_1^{k-s}\theta_1 + x_2^{k-s}\theta_2 + \cdots + x_n^{k-s}\theta_n \in I_{n,k,s}$ annihilates f under the $\odot$-action. This gives an equation of the form

$$0 = C_{k-s} x_1^0 h_{k-s} + C_{k-s+1} x_1^1 h_{k-s+1} + \cdots + C_{k-1} x_1^{s-1} h_{k-1} + (\text{a multiple of } \theta_1), \tag{4.17}$$

where the C's are nonzero constants. Taking the coefficients of $x_1^0, x_1^1, \ldots, x_1^{s-1}$, we see that $h_{k-s} = h_{k-s+1} = \cdots = h_{k-1} = 0$. □

Lemma 4.9 may be enhanced to consider harmonics $f \in I_{n,k,s}^{\perp}$ whose decomposition contains terms $x_1^i g_i$ for only relatively low values of i.

Lemma 4.10. *Let $n, k, s \geq 0$ be integers with $k \geq s$, and let $f \in I_{n,k,s}^{\perp}$. Assume that*

$$f = x_1^0 g_0 + x_1^1 g_1 + \cdots + x_1^i g_i + x_1^0 \theta_1 h_0 + x_1^1 \theta_1 h_1 + \cdots + x_1^{k-1} \theta_1 h_{k-1}$$

for some $0 \leq i \leq k-1$. Then $h_{k-s+i+1} = h_{k-s+i+2} = \cdots = h_{k-1} = 0$.

In light of Lemma 4.8, Lemma 4.10 gives no information in the range $s \leq i \leq k$. In accordance with the 'middle branches' (3.17) and (3.20) of the disjoint union decompositions of Lemma 3.12, Lemma 4.10 will be useful in the range $0 \leq i \leq s-1$.

Proof. The space $I_{n,k,s}^{\perp}$ is closed under the operator $x_1^{i+1} \odot (-)$. Applying this operator to f, we get the harmonic polynomial

$$x_1^{i+1} \odot f = C_{i+1} x_1^0 \theta_1 h_{i+1} + C_{i+2} x_1^1 \theta_1 h_{i+2} + \cdots + C_{k-1} x_1^{k-i-2} \theta_1 h_{k-1} \in I_{n,k,s}^{\perp}, \tag{4.18}$$

where the C's are nonzero constants. Applying Lemma 4.9 to this polynomial, we see that $h_{k-s+i+1} = h_{k-s+i+2} = \cdots = h_{k-1} = 0$. □

The following lemma characterises the leading terms of nonzero elements in $I_{n,k,s}^{\perp}$. The recursive description of the set $\mathcal{M}_{n,k,s}$ of substaircase monomials given in Lemma 3.12 is crucial to its proof.

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Lemma 4.11. *Let $n, k, s \geq 0$ with $k \geq s$, and let $f \in I_{n,k,s}^{\perp}$ be nonzero. The $\prec$-leading term of f is a substaircase monomial, that is*

$$\mathrm{in}(f) \in \mathcal{M}_{n,k,s}. \tag{4.19}$$

Proof. We proceed by induction on n. When $n = 1$, the space $I_{1,k,s}^{\perp}$ and the family of monomials $\mathcal{M}_{1,k,s}$ have three flavors depending on the value of s. If $s = 0$, then

$$\mathcal{M}_{1,k,0} = \{x_1^{k-1}\theta_1, \ldots, x_1\theta_1, \theta_1, x_1^{k-1}, \ldots, x_1^1, 1\}, \tag{4.20}$$

and we have $I_{1,k,0}^{\perp} = \operatorname{span} \mathcal{M}_{1,k,0}$, so the result follows. If $s = 1$, then $I_{1,k,1}^{\perp}$ is the ground field $\mathbb{Q}$ (independent of the value of k) and $\mathcal{M}_{1,k,1} = \{1\}$, so the result holds. If $s > 1$, then $e_0 = 1 \in I_{1,k,s}$, so that $I_{1,k,s} = \Omega_1$ and $I_{1,k,s}^{\perp} = 0$; since $\mathcal{M}_{1,k,s} = \varnothing$, the result is true in this case as well. Going forward, we let $n > 1$ and that the lemma has been established for all smaller values of n, and all values of k and s.

By Lemma 4.8, we may write f uniquely as

$$f = x_1^0 g_0 + x_1^1 g_1 + \cdots + x_1^{k-1} g_{k-1} + x_1^0\theta_1 h_0 + x_1^1\theta_1 h_1 + \cdots + x_1^{k-1}\theta_1 h_{k-1} \tag{4.21}$$

for some $g_0, g_1, \ldots, g_{k-1}, h_0, h_1, \ldots, h_{k-1} \in \Omega'_{n-1}$. Our analysis breaks up into cases depending on the vanishing properties of these elements of Ω'_{n-1}.

Case 1. *We have $g_0 = g_1 = \cdots = g_{k-1} = 0$.*

In this case, f is a multiple of θ_1. Lemma 4.9 applies, and we may write

$$f = x_1^i\theta_1 h_i + x_1^{i+1}\theta_1 h_{i+1} + \cdots + x_1^{k-s-1}\theta_1 h_{k-s-1} \tag{4.22}$$

for some $0 \leq i < k - s$ with $h_i \neq 0$. We have

$$\mathrm{in}(f) = x_1^i\theta_1 \mathrm{in}(h_i). \tag{4.23}$$

By (3.16) and (3.19) in Lemma 3.12, it is enough to show that $h_i \in (I'_{n-1,k,s})^{\perp}$. Here, $I'_{n-1,k,s} \subseteq \Omega'_{n-1}$ is the image of $I_{n-1,k,s}$ under the map $x_j \mapsto x_{j+1}, \theta_j \mapsto \theta_{j+1}$. We show that each generator of $I'_{n-1,k,s}$ annihilates h_i.

For $2 \leq j \leq n$, we have $x_j^k \odot f = 0$, so that

$$0 = x_j^k \odot f = x_1^i\theta_1[(x_j^k) \odot h_i] + x_1^{i+1}\theta_1[(x_j^k) \odot h_{i+1}] + \cdots + x_1^{k-s-1}\theta_1[(x_j^k) \odot h_{k-s-1}]. \tag{4.24}$$

Taking the coefficient of $x_1^i\theta_1$ shows that $x_j^k \odot h_i = 0$.

Given $d \geq k - s$, we have $(x_1^d\theta_1 + x_2^d\theta_2 + \cdots + x_n^d\theta_n) \odot f = 0$, so that

$$0 = \sum_{j=i}^{k-s-1} x_1^j\theta_1 \times (x_2^d\theta_2 + \cdots + x_n^d\theta_n) \odot h_j + (\text{terms not involving } \theta_1). \tag{4.25}$$

Taking the coefficient of $x_1^i\theta_1$, we see that $(x_2^d\theta_2 + \cdots + x_n^d\theta_n) \odot h_j = 0$.

Finally, let $S \sqcup T = \{2, \ldots, n\}$ be a disjoint union decomposition, and let $d > |S| - s$. We aim to show that $(e_d(S)\theta_T) \odot h_i = 0$. Indeed, if we let $\hat{T} = T \cup \{1\}$, we have $(e_d(S)\theta_{\hat{T}}) \odot f = 0$ since $f \in I_{n,k,s}^{\perp}$. This means

$$0 = x_1^i[(e_d(S)\theta_T) \odot h_i] + x_1^{i+1}[(e_d(S)\theta_T) \odot h_{i+1}] + \cdots + x_1^{k-s-1}[(e_d(S)\theta_T) \odot h_{k-s-1}], \tag{4.26}$$

and taking the coefficient of x_1^i shows $(e_d(S)\theta_T) \odot h_i = 0$.

The previous three paragraphs imply $h_i \in (I'_{n-1,k,s})^{\perp}$, so Lemma 3.12 and induction on n complete the proof of Case 1.

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Case 2. *At least one g_j is nonzero, but $g_j = 0$ for all $j \geq s$.*
In this case, Lemma 4.10 applies, and we may write

$$f = x_1^0 g_0 + x_1^1 g_1 + \cdots + x_1^i g_i + x_1^0 \theta_1 h_0 + x_1^1 \theta_1 h_1 + \cdots + x_1^{k-s-i} \theta_1 h_{k-s+i} \tag{4.27}$$

for some $0 \leq i < s$, where $g_i \neq 0$. We have

$$\mathrm{in}(f) = x_1^i \mathrm{in}(g_i), \tag{4.28}$$

and, inspired by the 'middle branches' (3.17) and (3.20) of Lemma 3.12, we verify that $g_i \in (I'_{n-1,k,s-1})^\perp$, where the prime, again, denotes increasing variable indices by one. We check that g_i is annihilated by every generator of $I_{n-1,k,s-1}$. The verification that $x_j^k \odot g_i = 0$ is analogous to that in Case 1 and is omitted.

Let $d \geq k - s + 1$. We aim to show that $(x_2^d \theta_2 + \cdots + x_n^d \theta_n) \odot g_i = 0$. Indeed, since $f \in I_{n,k,s}^\perp$, we have

$$\begin{aligned} 0 &= (x_1^d \theta_1 + x_2^d \theta_2 + \cdots + x_n^d \theta_n) \odot f \\ &= \sum_{j=0}^{i} x_1^j (x_2^d \theta_2 + \cdots + x_n^d \theta_n) \odot g_j + \sum_{j'=0}^{k-s+i} x_1^{j'} \theta_1 (x_2^d \theta_2 + \cdots + x_n^d \theta_n) \odot h_j + \sum_{j''=0}^{k-s+i-d} C_{j''} x_1^{j''} h_{j''+d}, \end{aligned} \tag{4.29}$$

where the C's are nonzero constants. Since $d \geq k - s + 1$, the final sum does not involve the power x_1^i. Taking the coefficient of x_1^i shows that $(x_2^d \theta_2 + \cdots + x_n^d \theta_n) \odot g_i = 0$.

Finally, let $S \sqcup T = \{2, \ldots, n\}$ be a disjoint union decomposition and let $d > |S| - s + 1$. We need to show $(e_d(S)\theta_T) \odot g_i = 0$. If we let $\hat{S} = S \cup \{1\}$, we have the relation

$$e_d(\hat{S}) = e_d(S) + x_1 e_{d-1}(S) \tag{4.30}$$

and $e_d(\hat{S})\theta_T$ is a generator of $I_{n,k,s}$. The equation $(e_d(\hat{S})\theta_T) \odot f) = 0$ has the form

$$0 = \sum_{j=0}^{i} [(e_d(S) + x_1 e_{d-1}(S))\theta_T] \odot (x_1^j g_j) + (\text{a multiple of } \theta_1). \tag{4.31}$$

Taking the coefficient of x_1^i shows that $(e_d(S)\theta_T) \odot g_i = 0$, as required.

The previous two paragraphs show that $g_i \in (I'_{n-1,k,s-1})^\perp$. Lemma 3.12 and induction on n complete the proof of Case 2.

Case 3. *At least one g_j is nonzero for some $j \geq s$.*
We apply Lemma 4.8 to write

$$f = x_1^0 g_0 + x_1^1 g_1 + \cdots + x_1^i g_i + x_1^0 \theta_1 h_0 + x_1^1 \theta_1 h_1 + \cdots + x_1^{k-1} \theta_1 h_{k-1} \tag{4.32}$$

with $g_i \neq 0$ for some $i \geq s$. We verify $g_i \in (I'_{n-1,k,s})^\perp$ by showing that every generator of $I'_{n-1,k,s}$ annihilates g_i. Generators of the form x_j^k are handled as before.

Let $d \geq k - s$. Since $f \in I_{n,k,s}^\perp$, we have

$$0 = (x_1^d \theta_1 + x_2^d \theta_2 + \cdots + x_n^d \theta_n) \odot f \tag{4.33}$$

$$= \sum_{j=0}^{i} x_1^j [(x_2^d \theta_2 + \cdots + x_n^d \theta_n) \odot g_j] + \sum_{j'=0}^{k-d-1} C_{j'} x_1^{j'} h_{j'+d} + (\text{a multiple of } \theta_1), \tag{4.34}$$

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where the C's are constants. Since $i \geq s > k - d - 1$, taking the coefficient of x_1^i shows that $(x_2^d \theta_2 + \cdots + x_n^d \theta_n) \odot g_i = 0$.

Finally, let $S \sqcup T = \{2, \ldots, n\}$ be a disjoint union decomposition and let $d > |S| - s$. We show that $(e_d(S)\theta_T) \odot g_i = 0$ as follows. If we set $\hat{S} = S \cup \{1\}$, we calculate

$$x_1^i e_d(S) = x_1^{i-1} e_{d+1}(\hat{S}) - x_1^{i-1} e_{d+1}(S) \tag{4.35}$$
$$= x_1^{i-1} e_{d+1}(\hat{S}) - x_1^{i-2} e_{d+2}(\hat{S}) + x_1^{i-2} e_{d+2}(S) \tag{4.36}$$
$$= \cdots$$
$$= x_1^{i-1} e_{d+1}(\hat{S}) - x_1^{i-2} e_{d+2}(\hat{S}) + \cdots + (-1)^{i-1} e_{d+i}(\hat{S}) + (-1)^i e_{d+i}(S) \tag{4.37}$$
$$= x_1^{i-1} e_{d+1}(\hat{S}) - x_1^{i-2} e_{d+2}(\hat{S}) + \cdots + (-1)^{i-1} e_{d+i}(\hat{S}), \tag{4.38}$$

where the last equality used the conditions $i \geq s$ and $d > |S| - s$, so that $e_{d+i}(S) = 0$. Multiplying both sides by θ_T, we get

$$x_1^i e_d(S)\theta_T = x_1^{i-1} e_{d+1}(\hat{S})\theta_T - x_1^{i-2} e_{d+2}(\hat{S})\theta_T + \cdots + (-1)^{i-1} e_{d+i}(\hat{S})\theta_T \in I_{n,k,s} \tag{4.39}$$

since every term on the right-hand side is a generator of $I_{n,k,s}$. Since $f \in I_{n,k,s}^{\perp}$, this means

$$0 = (x_1^i e_d(S)\theta_T) \odot f = C(e_d(S)\theta_T) \odot g_i + (\text{a multiple of } \theta_1), \tag{4.40}$$

where C is a nonzero constant. This implies that $(e_d(S)\theta_T) \odot g_i = 0$.

The previous arguments show that $g_i \in (I'_{n-1,k,s})^{\perp}$. The fact that $\mathrm{in}(f) = x_1^i \mathrm{in}(g_i)$, the 'lower branches' (3.18) and (3.21) of the disjoint union decompositions in Lemma 3.12 and induction on n all complete the proof of Case 3, and of the lemma. □

4.5. *Proof of quotient presentation and monomial basis*

We are ready to prove that the ideal $I_{n,k,s}$ and the annihilator $\mathrm{ann}\,\delta_{n,k,s}$ coincide, so that $\mathbb{W}_{n,k,s} = \mathbb{U}_{n,k,s}$, and that the substaircase monomials $\mathcal{M}_{n,k,s}$ descend to a basis of $\mathbb{W}_{n,k,s}$. Thanks to our lemmata, this is a quick argument.

Theorem 4.12. *Let $n, k, s \geq 0$ be integers with $k \geq s$. The ideal $I_{n,k,s}$ is the annihilator of the superspace Vandermonde $\delta_{n,k,s}$:*

$$\mathrm{ann}\,\delta_{n,k,s} = I_{n,k,s}. \tag{4.41}$$

Consequently, the quotient rings $\mathbb{W}_{n,k,s}$ and $\mathbb{U}_{n,k,s}$ coincide:

$$\mathbb{W}_{n,k,s} = \mathbb{U}_{n,k,s}. \tag{4.42}$$

Furthermore, the set $\mathcal{M}_{n,k,s}$ descends to a monomial basis of $\mathbb{W}_{n,k,s}$.

Proof. We bound the dimension of $I_{n,k,s}^{\perp}$ from above using Lemma 4.11 and an argument appearing in the work of Rhoades, Yu and Zhao [28, Section 4.4] on harmonic spaces. Let $N = |\mathcal{M}_{n,k,s}|$ be the number of substaircase monomials. We claim that $\dim I_{n,k,s}^{\perp} \leq N$. Given $f_1, \ldots f_N, f_{N+1} \in I_{n,k,s}^{\perp}$, we have

$$f = c_1 f_1 + \cdots + c_N f_N + c_{N+1} f_{N+1} \in I_{n,k,s}^{\perp} \tag{4.43}$$

for any scalars $c_1, \ldots, c_N, c_{N+1} \in \mathbb{Q}$. We may select c_i not all zero so that the coefficient of m vanishes on the right-hand side of Equation (4.43) for all $m \in \mathcal{M}_{n,k,s}$. By Lemma 4.11, this forces $f = 0$, so that Equation (4.43) shows $\{f_1, \ldots, f_N, f_{N+1}\}$ is linearly dependent.

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We have the chain of equalities

$$N \geq \dim I_{n,k,s}^{\perp} = \dim \mathbb{U}_{n,k,s} \geq \dim \mathbb{W}_{n,k,s} \geq N, \tag{4.44}$$

where we applied Lemma 4.7 to get that $\mathbb{U}_{n,k,s}$ projects onto $\mathbb{W}_{n,k,s}$ and Lemma 4.5 to get $\dim \mathbb{W}_{n,k,s} \geq N$. This proves that $\operatorname{ann} \delta_{n,k,s} = I_{n,k,s}$ and $\mathbb{W}_{n,k,s} = \mathbb{U}_{n,k,s}$. Lemma 4.5 implies that $\mathcal{M}_{n,k,s}$ descends to a monomial basis of $\mathbb{W}_{n,k,s}$. ◻

To illustrate Theorem 4.12, let $(n,k,s) = (3,2,2)$. The staircases in this case are

$$(1,1,0), \quad (1,0,1), \quad (1,\bar{0},0), \quad (1,0,\bar{1}),$$

so that

$$\mathcal{M}_{3,2,2} = \{1, x_1, x_2, x_3, x_1x_2, x_1x_3, \theta_2, \theta_3, x_1\theta_2, x_1\theta_3, x_3\theta_3, x_1x_3\theta_3\}$$

and Theorem 4.12 asserts that $\mathcal{M}_{3,2,2}$ descends to a basis of $\mathbb{W}_{3,2,2}$. In particular, the bigraded Hilbert series $\operatorname{Hilb}(\mathbb{W}_{3,2,2}; q, z)$ is given by

$$\operatorname{Hilb}(\mathbb{W}_{3,2,2}; q, z) = 1 + 3q + 2q^2 + 2z + 3qz + q^2z.$$

We may display this Hilbert series as a matrix

$$\operatorname{Hilb}(\mathbb{W}_{3,2,2}; q, z) = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

by letting rows track θ-degree and columns track x-degree. The 180° rotational symmetry of this matrix is guaranteed by the Rotational Duality of Theorem 1.2. The larger example

$$\operatorname{Hilb}(\mathbb{W}_{6,3,2}; q, z) = \begin{pmatrix} 1 & 6 & 21 & 50 & 90 & 125 & 134 & 105 & 55 & 15 \\ 6 & 35 & 119 & 273 & 463 & 575 & 511 & 301 & 105 & 20 \\ 15 & 84 & 274 & 580 & 853 & 853 & 580 & 274 & 84 & 15 \\ 20 & 105 & 301 & 511 & 575 & 463 & 273 & 119 & 35 & 6 \\ 15 & 55 & 105 & 134 & 125 & 90 & 50 & 21 & 6 & 1 \end{pmatrix}$$

is easy to compute with the following recursion.

Corollary 4.13. *Suppose $n, k, s \geq 0$ are integers with $k \geq s$. We have*

$$\operatorname{Hilb}(\mathbb{W}_{n,k,s}; q, z) = \sum_{r=0}^{n-s} z^r \cdot \sum_{\sigma \in \mathcal{OSP}_{n,k,s}^{(r)}} q^{\operatorname{coinv}(\sigma)} = \sum_{r=0}^{n-s} z^r \cdot \sum_{\sigma \in \mathcal{OSP}_{n,k,s}^{(r)}} q^{\operatorname{codinv}(\sigma)}.$$

This bigraded Hilbert series satisfies the recursion

$$\operatorname{Hilb}(\mathbb{W}_{n,k,s}; q, z) = (z + q^s) \cdot [k-s]_q \cdot \operatorname{Hilb}(\mathbb{W}_{n-1,k,s}; q, z) + [s]_q \cdot \operatorname{Hilb}(\mathbb{W}_{n-1,k,s-1}; q, z).$$

Proof. This follows from Lemma 3.12 and Theorem 4.12. ◻

Together with the initial conditions

$$\operatorname{Hilb}(\mathbb{W}_{n,k,s}; q, z) = 0 \qquad \text{if } s > n \tag{4.45}$$

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and

$$\mathrm{Hilb}(\mathbb{W}_{1,k,s};q,z) = \begin{cases} (1+z)\cdot[k]_q & s=0 \\ 1 & s=1, \end{cases} \tag{4.46}$$

Corollary 4.13 determines $\mathrm{Hilb}(\mathbb{W}_{n,k,s};q,z)$ completely. It is predicted [27, Conjecture 6.5] that the matrices $\mathrm{Hilb}(\mathbb{W}_{n,k,s};q,z)$ have unimodal rows and columns. In the case $(n,k,s)=(n,n,n)$, the ring $\mathbb{W}_{n,n,n}$ is the cohomology $H^\bullet(\mathcal{F}\ell_n;\mathbb{Q})$ of the flag variety $\mathcal{F}\ell_n$ and is a consequence of the Hard Lefschetz property for this smooth and compact complex manifold. Corollary 4.13 has been used to verify this conjecture for all triples $n \geq k \geq s$ with $n \leq 9$.

Since the composition of maps $\mathbb{H}_{n,k,s} \hookrightarrow \Omega_n \twoheadrightarrow \mathbb{W}_{n,k,s}$ is an isomorphism, any basis of the harmonic space $\mathbb{H}_{n,k,s}$ descends to a basis of $\mathbb{W}_{n,k,s}$. A basis of $\mathbb{W}_{n,k,s}$ obtained in this way is a *harmonic basis*. Harmonic bases of quotients of the polynomial ring $\mathbb{Q}[x_1,\ldots,x_n]$ have received significant attention [3, 30] and are useful because working with them does not involve computationally expensive operations with cosets. The space $\mathbb{W}_{n,k,s}$ admits the following harmonic basis.

Corollary 4.14. *Let $n,k,s \geq 0$ be integers with $k \geq s$. We have the following equality of subspaces of Ω_n.*

$$\mathbb{H}_{n,k,s} = I_{n,k,s}^{\perp}. \tag{4.47}$$

Furthermore, the set

$$\{m \odot \delta_{n,k,s} \;:\; m \in \mathcal{M}_{n,k,s}\} \tag{4.48}$$

is a basis of $\mathbb{H}_{n,k,s}$, and, therefore, descends to a harmonic basis of $\mathbb{W}_{n,k,s}$.

Proof. The equality $\mathbb{H}_{n,k,s} = I_{n,k,s}^{\perp}$ is immediate from Theorem 4.12. For any $c_1,\ldots,c_N \in \mathbb{Q}$ and any monomials $m_1,\ldots,m_N$, we have

$$c_1(m_1 \odot \delta_{n,k,s}) + \cdots + c_N(m_N \odot \delta_{n,k,s}) = (c_1 m_1 + \cdots + c_N m_N) \odot \delta_{n,k,s}, \tag{4.49}$$

so that a linear dependence in the subset $\{m_1 \odot \delta_{n,k,s},\ldots,m_N \odot \delta_{n,k,s}\}$ of Ω_n induces a linear dependence in $\{m_1,\ldots,m_N\}$ modulo $\mathrm{ann}\,\delta_{n,k,s}$. Theorem 4.12 implies $\{m \odot \delta_{n,k,s} \;:\; m \in \mathcal{M}_{n,k,s}\}$ is linearly independent and a dimension count finishes the proof. □

The harmonic basis of Corollary 4.14 will be used to calculate the bigraded Frobenius image of $\mathbb{W}_{n,k,s}$.

5. Frobenius image

5.1. *A tensor product decomposition of Ω_n*

The goal of this section is to prove that the graded Frobenius image $\mathrm{grFrob}(\mathbb{W}_{n,k,s};q,z)$ has the combinatorial expansion $C_{n,k,s}(\mathbf{x};q,z)$. Since $\mathbb{W}_{n,k,s} \cong \mathbb{H}_{n,k,s}$ as bigraded $\mathfrak{S}_n$-modules, we will often use the harmonic space $\mathbb{H}_{n,k,s}$ to avoid working with cosets. The combinatorial recursion of Lemma 3.14 necessitates restricting these spaces to a given θ-degree.

Definition 5.1. Given $r \geq 0$, let $\mathbb{W}_{n,k,s}^{(r)} \subseteq \mathbb{W}_{n,k,s}$ and $\mathbb{H}_{n,k,s}^{(r)} \subseteq \mathbb{H}_{n,k,s}$ be the subspaces of homogeneous θ-degree r.

$\mathbb{W}_{n,k,s}^{(r)}$ and $\mathbb{H}_{n,k,s}^{(r)}$ are isomorphic singly graded $\mathfrak{S}_n$-modules under x-degree. Our goal is to show

$$\mathrm{grFrob}(\mathbb{W}_{n,k,s}^{(r)};q) = \mathrm{grFrob}(\mathbb{H}_{n,k,s}^{(r)};q) = C_{n,k,s}^{(r)}(\mathbf{x};q). \tag{5.1}$$

Lemma 2.3 allows us to prove Equation (5.1) by showing both sides satisfy the same recursion under an appropriate family of skewing operators. The combinatorial side $C^{(r)}_{n,k,s}$ was handled by Lemma 3.14; we must now consider the representation theoretic side. In order to avoid repeating hypotheses, we fix the following:

Notation. For the remainder of this section, we fix integers $n \geq s \geq 0$, $k \geq s$, $n - s \geq r \geq 0$ and $1 \leq j \leq n$.

The group algebra $\mathbb{Q}[\mathfrak{S}_j]$ and its antisymmetrising and symmetrising elements ε_j and η_j act on the first j indices of Ω_n. Given the relationship

$$\mathrm{grFrob}(\eta_j \mathbb{H}^{(r)}_{n,k,s}; q) = h_j^{\perp} \mathrm{grFrob}(\mathbb{H}^{(r)}_{n,k,s}; q), \tag{5.2}$$

and Lemma 3.14, we would like a recursive understanding of the $\mathfrak{S}_{n-j}$-modules $\eta_j \mathbb{H}^{(r)}_{n,k,s}$. This will be accomplished by finding a strategic basis $\mathcal{C}$ of $\eta_j \mathbb{H}^{(r)}_{n,k,s}$ (see Definition 5.11 below).

We will need to distinguish between the first j and last $n - j$ indices appearing in Ω_n. To this end, we make the tensor product identification

$$\Omega_n = \Omega_j \otimes \Omega_{n-j} = \left(\mathbb{Q}[x_1, \ldots, x_j] \otimes \wedge\{\theta_1, \ldots, \theta_j\}\right) \otimes \left(\mathbb{Q}[x_{j+1}, \ldots, x_n] \otimes \wedge\{\theta_{j+1}, \ldots, \theta_n\}\right). \tag{5.3}$$

We make use of the following simple properties of this decomposition.

Proposition 5.2. *Consider the tensor product decomposition $\Omega_n = \Omega_j \otimes \Omega_{n-j}$.*

1. *If $f, f' \in \Omega_j$ and $g, g' \in \Omega_{n-j}$ have homogeneous θ-degree, then*

$$(f \otimes g) \odot (f' \otimes g') = \pm (f \odot f') \otimes (g \odot g'),$$

where the sign is $(-1)^{\theta\text{-degree of } g + \theta\text{-degree of } f'}$.

2. *If $u \in \mathfrak{S}_j$, $v \in \mathfrak{S}_{n-j}$, $f \in \Omega_j$ and $g \in \Omega_{n-j}$, then the action of $u \times v \in \mathfrak{S}_j \times \mathfrak{S}_{n-j} \subseteq \mathfrak{S}_n$ on $f \otimes g$ is given by*

$$(u \times v) \cdot (f \otimes g) = (u \cdot f) \otimes (v \cdot g).$$

3. *If $f \in \varepsilon_j \Omega_j$ and $g \in \Omega_{n-j}$, then*

$$(f \otimes g) \odot \delta_{n,k,s} \in \eta_j \mathbb{H}_{n,k,s}.$$

Proof. Items (1) and (2) are straightforward and left to the reader. For Item (3), let $u \in \mathfrak{S}_j$. We calculate

$$\begin{aligned} u \cdot \left[(f \otimes g) \odot \delta_{n,k,s}\right] &= [u \cdot f \otimes g] \odot \left[u \cdot \delta_{n,k,s}\right] &(5.4)\\ &= [\mathrm{sign}(u) f \otimes g] \odot \left[\mathrm{sign}(u) \delta_{n,k,s}\right] &(5.5)\\ &= \mathrm{sign}(u)^2 \times \left[(f \otimes g) \odot \delta_{n,k,s}\right] &(5.6)\\ &= (f \otimes g) \odot \delta_{n,k,s}. &(5.7) \end{aligned}$$

The first equality uses (2), the second equality uses $f \in \varepsilon_j \Omega_j$ and $\delta_{n,k,s} \in \varepsilon_j \Omega_n$ and the third equality is bilinearity. ◻

Proposition 5.2 (3) gives rise to a 'duality' between the images of the $\mathbb{W}$-modules under ε_j and the images of the $\mathbb{H}$-modules under η_j. We state this duality as follows.

Proposition 5.3. *Let $\mathcal{A} \subseteq \varepsilon_j \Omega_n$ be a subset of homogeneous θ-degree r. Define a subset $\mathcal{A}^{\vee} \subseteq \mathbb{H}^{(n-s-r)}_{n,k,s}$ by*

$$\mathcal{A}^{\vee} = \{f \odot \delta_{n,k,s} \,:\, f \in \mathcal{A}\}.$$

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1. *$\mathcal{A}$ descends to a linearly independent subset of $\varepsilon_j \mathbb{W}^{(r)}_{n,k,s}$ if and only if $\mathcal{A}^\vee$ is linearly independent in $\eta_j \mathbb{H}^{(n-s-r)}_{n,k,s}$.*
2. *$\mathcal{A}$ descends to a spanning subset of $\varepsilon_j \mathbb{W}^{(r)}_{n,k,s}$ if and only if $\mathcal{A}^\vee$ spans $\eta_j \mathbb{H}^{(n-s-r)}_{n,k,s}$.*
3. *$\mathcal{A}$ descends to a basis of $\varepsilon_j \mathbb{W}^{(r)}_{n,k,s}$ if and only if $\mathcal{A}^\vee$ is a basis of $\eta_j \mathbb{H}^{(n-s-r)}_{n,k,s}$.*

Proof. Proposition 5.2 (3) shows that $\mathcal{A}^\vee$ is indeed a subset of $\eta_j \mathbb{H}^{(n-s-r)}_{n,k,s}$. The isomorphism $\mathbb{W}_{n,k,s} = \Omega_n / \mathrm{ann}\, \delta_{n,k,s} \xrightarrow{\sim} \mathbb{H}_{n,k,s}$ given by $f \mapsto f \odot \delta_{n,k,s}$ implies Items (1) through (3). □

5.2. A spanning subset of $\varepsilon_j \mathbb{W}^{(r)}_{n,k,s}$

Proposition 5.3 allows us to move back and forth between the alternating subspace $\varepsilon_j \mathbb{W}_{n,k,s}$ and the invariant subspace $\eta_j \mathbb{H}_{n,k,s}$. The following subset $\mathcal{B} \subseteq \varepsilon_j \Omega_n$ will turn out to descend to a basis of $\varepsilon_j \mathbb{W}^{(r)}_{n,k,s}$.

Definition 5.4. Define a subset $\mathcal{B} \subseteq \Omega_n$ by

$$\mathcal{B} = \bigsqcup_{\substack{a,b \geq 0 \\ a \leq r,\ b \leq s}} \bigsqcup_{\mathbf{i}} \left\{ \varepsilon_j \cdot (x_1^{i_1} \cdots x_j^{i_j} \cdot \theta_1 \cdots \theta_a) \otimes m \ :\ m \in \mathcal{M}^{(r-a)}_{n-j,k,s-b} \right\}, \tag{5.8}$$

where the index $\mathbf{i} = (i_1, \ldots, i_j)$ of the inner disjoint union ranges over all length j integer sequences whose first a, next b and final $j - a - b$ entries satisfy the conditions

$$0 \leq i_1 \leq \cdots \leq i_a \leq k - s - 1 + b, \quad 0 \leq i_{a+1} < \cdots < i_{a+b} \leq s - 1, \quad s \leq i_{a+b+1} < \cdots < i_j \leq k - 1.$$

The subset $\mathcal{B} \subseteq \Omega_n$ depends on n, k, s, r and j, but we suppress this dependence to reduce notational clutter. We note that, since every $\mathfrak{S}_j$-orbit consists of a unique representative $\mathbf{i}$ that meets these conditions and the indices a and b are determined by this representative, the unions in the definition are indeed disjoint. Thanks to our tensor product notation, the monomials m appearing in Definition 5.4 are automatically elements of $\Omega_{n-j} = \mathbb{Q}[x_{j+1}, \ldots, x_n] \otimes \wedge\{\theta_{j+1}, \ldots, \theta_n\}$. The conditions on a, b and the sequences $\mathbf{i} = (i_1, \ldots, i_j)$ appearing in Definition 5.4 may look complicated, but they combinatorially correspond to the sum and q-binomial coefficients in the skewing recursion of Lemma 3.14 satisfied by the C-functions. Algebraically, they are obtained by applying ε_j to every monomial in $\mathcal{M}^{(r)}_{n,k,s}$ and removing 'obvious' linear dependencies.

Lemma 5.5. *The subset $\mathcal{B}$ of Ω_n descends to a spanning set of $\varepsilon_j \mathbb{W}^{(r)}_{n,k,s}$.*

Proof. By Theorem 4.12, we know that $\mathcal{M}^{(r)}_{n,k,s}$ descends to a basis for $\mathbb{W}^{(r)}_{n,k,s}$. This implies that

$$\varepsilon_j \mathcal{M}^{(r)}_{n,k,s} = \left\{ \varepsilon_j \cdot m_0 \ :\ m_0 \in \mathcal{M}^{(r)}_{n,k,s} \right\} \tag{5.9}$$

descends to a spanning set of $\varepsilon_j \mathbb{W}^{(r)}_{n,k,s}$. We proceed to remove linear dependencies from the set $\varepsilon_j \mathcal{M}^{(r)}_{n,k,s}$.

Let $m_0 \in \mathcal{M}^{(r)}_{n,k,s}$. There exists a permutation $w \in \mathfrak{S}_j$, such that $w \cdot m_0 = x_1^{i_1} \cdots x_j^{i_j} \theta_1 \cdots \theta_a \otimes m$, where $0 \leq i_1 \leq \cdots \leq i_a \leq k-1$, $0 \leq i_{a+1} \leq \cdots \leq i_j \leq k-1$. We have

$$\varepsilon_j \cdot (w \cdot m_0) = \mathrm{sign}(w) \varepsilon_j \cdot m_0 = \pm \varepsilon_j \cdot m_0. \tag{5.10}$$

Furthermore, the action of ε_j annihilates $w \cdot m_0$ unless $i_{a+1} < \cdots < i_j$. It follows that $\varepsilon_j \mathbb{W}^{(r)}_{n,k,s}$ is spanned by

$$\bigsqcup_{\substack{a,b \geq 0 \\ a \leq r,\, b \leq s}} \left\{ \varepsilon_j \cdot (x_1^{i_j} \cdots x_j^{i_1} \cdot \theta_{j-a+1} \cdots \theta_j) \otimes m \; : \; \begin{array}{c} (x_1^{i_1} \cdots x_j^{i_j} \cdot \theta_1 \cdots \theta_a) \otimes m \in \mathcal{M}^{(r)}_{n,k,s}, \\ 0 \leq i_1 \leq \cdots \leq i_a \leq k-1, \\ 0 \leq i_{a+1} < \cdots < i_{a+b} \leq s-1, \\ s \leq i_{a+b+1} < \cdots < i_j \leq k-1, \end{array} \right\}, \tag{5.11}$$

where the parameter b tracks the index at which the sequence $0 \leq i_{a+1} < \cdots < i_j \leq k-1$ exceeds the value s. The 'variable reversal'

$$x_1^{i_1} \cdots x_j^{i_j} \cdot \theta_1 \cdots \theta_a \rightsquigarrow x_1^{i_j} \cdots x_j^{i_1} \cdot \theta_{j-a+1} \cdots \theta_j$$

in (5.11) relative to the definition of $\mathcal{B}$ only introduces a sign upon application of ε_j.

For which monomials $m \in \Omega_{n-j}$ and sequences $(i_1, \ldots, i_j)$ do we have the containment

$$(x_1^{i_j} \cdots x_j^{i_1} \cdot \theta_{j-a+1} \cdots \theta_j) \otimes m \in \mathcal{M}^{(r)}_{n,k,s}?$$

The lemma is reduced to the following claim.

Claim. *Given sequences* $0 \leq i_1 \leq \cdots \leq i_a \leq k-1, 0 \leq i_{a+1} < \cdots < i_{a+b} \leq s-1$ *and* $s \leq i_{a+b+1} < \cdots < i_j \leq k-1$ *and a monomial* $m \in \Omega_{n-j}$*, we have*

$$(x_1^{i_j} \cdots x_j^{i_1} \cdot \theta_{j-a+1} \cdots \theta_j) \otimes m \in \mathcal{M}^{(r)}_{n,k,s} \text{ if and only if } m \in \mathcal{M}^{(r-a)}_{n-j,k,s-b} \text{ and } i_a \leq k-s-1+b.$$

The reason for this 'reversal' in (5.11) is to make the monomials in the claim divisible by staircase monomials. The claim follows from Lemma 3.12 and induction; we leave details to the reader. □

5.3. $\mathcal{B}$ *is linearly independent*

The goal of this technical subsection is to show that the set $\mathcal{B}$ in Definition 5.4 is linearly independent in Ω_n. By virtue of Lemma 5.5, this implies that $\mathcal{B}$ descends to a basis of $\varepsilon_j \mathbb{W}^{(r)}_{n,k,s}$. To do this, we prove that

$$\mathcal{B}^\vee = \{ f \odot \delta_{n,k,s} \; : \; f \in \mathcal{B} \} \subseteq \eta_j \Omega_n \tag{5.12}$$

is linearly independent and apply Proposition 5.3.

We will show that $\mathcal{B}^\vee$ is linearly independent by considering a strategic basis of the space $\eta_j \Omega_n$ and showing that the expansions of the $f \odot \delta_{n,k,s}$ in this basis satisfy a triangularity condition. Both our basis and the triangularity condition will be defined in terms of the superlex order $\prec$.

Definition 5.6. A monomial $m \in \Omega_n$ with exponent sequence $u = (u_1, \ldots, u_n)$ is j-increasing if $u_1 \leq \cdots \leq u_j$ under the order

$$\cdots < \bar{3} < \bar{2} < \bar{1} < \bar{0} < 0 < 1 < 2 < 3 < \cdots$$

on letters defining superlex order $\prec$.

Each orbit of the $\mathfrak{S}_j$-action on the first j letters of exponent sequences $(u_1, \ldots, u_n)$ of monomials in Ω_n has a unique j-increasing representative. In terms of the operator η_j, we have the following.

Observation 5.7. *The nonzero elements in* $\{\eta_j \cdot m : m \text{ is } j\text{-increasing}\}$ *form a basis for* $\eta_j \Omega_n$.

The word 'nonzero' is necessary in Observation 5.7. Indeed, when $n = 3$ and $j = 2$, the word $(\bar{0}, \bar{0}, 0)$ is j-increasing and $\eta_2 \cdot \theta_1 \theta_2 = 0$.

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For any $f \in \mathcal{B}$, the element $f \odot \delta_{n,k,s} \in \eta_j \Omega_n$ may be uniquely expanded in the basis of Observation 5.7. The next definition extracts a useful 'leading term' of this expansion. This is a variant of the superlex order which incorporates the parameter j.

Definition 5.8. Given any nonzero $g \in \eta_j \Omega_n$, let $\mathrm{in}_j(g)$ be the j-increasing monomial m, such that

- $\eta_j \cdot m$ appears in the expansion of g in the basis of Observation 5.7 with nonzero coefficient, and
- for any j-increasing monomial m', such that $\eta_j \cdot m'$ appears in this expansion with nonzero coefficient, we have $m \geq m'$.

As an example of these notions, for $n = 5$, $k = 4$, $s = 2$ and $j = 3$, we consider the superpolynomial $f \in \mathcal{B}$ given by

$$f = \varepsilon_3 \cdot (x^{00110}\theta_1) = (x_3 - x_2)x_4\theta_1 + (x_1 - x_3)x_4\theta_2 + (x_2 - x_1)x_4\theta_3. \tag{5.13}$$

Then

$$\begin{aligned} f \odot \delta_{5,4,2} = &- 18\eta_3 \cdot (x^{00323}\theta_{45}) + 54\eta_3 \cdot (x^{21320}\theta_{14}) + 18\eta_3 \cdot (x^{20303}\theta_{15}) \\ &- 54\eta_3 \cdot (x^{20321}\theta_{14}) + 18\eta_3 \cdot (x^{30320}\theta_{14}) - 18\eta_3 \cdot (x^{32300}\theta_{12}). \end{aligned} \tag{5.14}$$

We have written the terms in the final expression so that each monomial that appears is j-increasing and these monomials decrease in superlex order from left to right. Therefore

$$\mathrm{in}_j(f \odot \delta_{n,k,s}) = x^{00323}\theta_{45}.$$

The most important property of in_j is as follows.

Lemma 5.9. *The map*

$$\mathrm{in}_j : \mathcal{B}^\vee \longrightarrow \{j\textit{-increasing monomials in } \Omega_n\}$$

is injective.

Proof. We give a method for finding $\mathrm{in}_j(f \odot \delta_{n,k,s})$ for any $f \in \mathcal{B}$. Suppose that

$$f = \varepsilon_j \cdot \left(x_1^{i_1} \cdots x_j^{i_j} \theta_1 \cdots \theta_a\right) \otimes m \in \mathcal{B}$$

so $m \in \mathcal{M}^{(r-a)}_{n-j,k,s-b}$ and

$$0 \leq i_1 \leq \cdots \leq i_a \leq k - s - 1 + b,\ 0 \leq i_{a+1} < \cdots < i_{a+b} \leq s - 1,\ s \leq i_{a+b+1} < \cdots < i_j \leq k - 1.$$

We define a reordering $(h_1, \ldots, h_j)$ of the sequence $(i_1, \ldots, i_j)$ by

$$(h_1, \ldots, h_j) = (i_{a+b+1}, i_{a+b+2}, \ldots, i_{j-1}, i_j, i_{a+b}, i_{a+b-1}, \ldots, i_2, i_1).$$

The sequence $(h_1, \ldots, h_j)$ satisfies

$$\begin{aligned} &s \leq h_1 < \cdots < h_{j-a-b} \leq k - 1,\ s - 1 \geq h_{j-a-b+1} > \cdots > h_{j-a} \geq 0,\ k - s - 1 + b \\ &\geq h_{j-a+1} \geq \cdots \geq h_j \geq 0. \end{aligned}$$

Furthermore, let $t = (t_1, \ldots, t_n)$ be the exponent sequence of the entire monomial

$$\left(x_1^{h_1} \cdots x_j^{h_j} \theta_{j-a+1} \cdots \theta_j\right) \otimes m.$$

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We construct a monomial M with exponent sequence u as follows:

- $u_1 = \cdots = u_{j-a-b} = \overline{k-1}$,
- $(u_{j-a-b+1}, \ldots, u_{j-a}) = (s-1, s-2, \ldots, s-b)$,
- $u_{j-a+1} = \cdots = u_j = \overline{k-1}$ and
- For $p = j+1$ to n, u_p is $\overline{k-1}$ if t_p is barred or greater than the largest unused element of $\{0, 1, \ldots, s-b-1\}$. Otherwise, we let u_p be the largest unused element of $\{0, 1, 2 \ldots, s-b-1\}$.

The last bullet point above is an instance of Algorithm 1 in the proof of Proposition 4.3. Since m is substaircase, this algorithm successfully produces a monomial M appearing in $\delta_{n,k,s}$.

Claim. $\mathrm{in}_j(f \odot \delta_{n,k,s}) \doteq \left(x_1^{h_1} \cdots x_j^{h_j} \theta_1 \cdots \theta_a \otimes m\right) \odot M$.

We check this construction for the example $f = \varepsilon_3(x^{00110}\theta_1) \in \mathcal{B}$ given in (5.13), where $n = 5$, $k = 4$, $s = 2$ and $j = 3$. Then $a = 1$, $b = 2$, $t = (1, 0, \bar{0}, 1, 0)$, $u = (1, 0, \bar{3}, \bar{3}, \bar{3})$ and

$$x^{10010}\theta_3 \odot x^{10333}\theta_{345} \doteq x^{00323}\theta_{45},$$

which agrees with the computation of $\mathrm{in}_j(f \otimes \delta_{5,4,2})$ in (5.14).

Now we prove the claim. Let $v = (v_1, \ldots, v_n)$ be the exponent sequence of the monomial $(x_1^{h_1} \cdots x_j^{h_j} \theta_{j-a+1} \cdots \theta_j \otimes m) \odot M$ appearing in the claim. First, we check that v is indeed j-increasing. From the definitions of t and u, the first $j-a-b$ entries of v are all barred and

$$k-s-1 \geq v_1 > \cdots > v_{j-a-b} \geq 0.$$

The next $a+b$ entries of v are all unbarred and satisfy

$$0 \leq v_{j-a-b+1} \leq \cdots \leq v_{j-a} < s-b \leq v_{j-a+1} \leq \cdots \leq v_j \leq k-1,$$

so v is j-increasing.

Let $v' = (v'_1, \ldots, v'_n)$ be the $\prec$-maximal j-increasing exponent sequence whose monomial appears in $f \odot \delta_{n,k,s}$. There are monomials appearing in f and $\delta_{n,k,s}$ that yield v' under the left $\odot$ action of f on $\delta_{n,k,s}$. Denote the exponent sequences of these monomials by $t' = (t'_1, \ldots, t'_n)$ and $u' = (u'_1, \ldots, u'_n)$, respectively. Then t' is obtained from t by some rearrangement of its first j entries. We show that $v = v'$ as follows.

The sequence $(t'_1, \ldots, t'_j)$ has $j-a-b$ unbarred entries $> s$. Thus, the sequence $(v'_1, \ldots, v'_j)$ must have at least $j-a-b$ barred entries. Since v' is j-increasing, the first $j-a-b$ entries of v' must be barred, which forces

$$u'_1 = \cdots = u'_{j-a-b} = \overline{k-1}, \text{ which agrees with } u_1 = \cdots = u_{j-a-b} = \overline{k-1}.$$

Since $v \leq v'$, and v' is j-increasing, the barred entries $v'_1, \ldots, v'_{j-a-b}$ are all $< k-s$. We see that the first $j-a-b$ entries of t' are unbarred and $> s$. Since the first j entries of t' are a rearrangement of the first j entries of t, this forces the first j entries of t' to be $t_1, \ldots, t_{j-a-b}$ (in some order). Since v' is j-increasing, we have

$$(t'_1, \ldots, t'_{j-a-b}) = (t_1, \ldots, t_{j-a-b}) \text{ so that } (v'_1, \ldots, v'_{j-a-b}) = (v_1, \ldots, v_{j-a-b}).$$

The above paragraph forces $(t'_{j-a-b+1}, \ldots, t'_j)$ to be a rearrangement of $(t_{j-a-b+1}, \ldots, t_j)$. Both of these sequences contain

- b unbarred entries $< s$, all of which are unique, and
- a barred entries, which are all $< k-s-1$.

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The fact that v' is the superlex maximal j-increasing exponent sequence appearing in $f \odot \delta_{n,k,s}$ has the following consequences.

1. Every entry in the subsequence $(v'_{j-a-b+1}, \ldots, v'_j)$ is unbarred (since $v \preceq v'$). Consequently, the corresponding b entries of u' are unbarred and the corresponding a entries of u' are barred.
2. The corresponding b entries in u' must be $s-1, s-2, \ldots, s-b$ (in some order), and the corresponding a entries of u' must be $\overline{k-1}$.
3. The b entries must come before the a entries.

Items (2) and (3) above imply

$$(u'_{j-a+1}, \ldots, u'_j) = (\overline{k-1}, \ldots, \overline{k-1}) = (u_{j-a+1}, \ldots, u_j),$$

and since v' is j-increasing, we have

$$(t'_{j-a+1}, \ldots, t'_j) = (t_{j-a+1}, \ldots, t_j) \text{ so that } (v'_{j-a+1}, \ldots, v'_j) = (v_{j-a+1}, \ldots, v_n).$$

We see that the three pairs of length b sequences

$$\begin{cases} (u'_{j-a-b+1}, \ldots, u'_{j-a}) \text{ and } (u_{j-a-b+1}, \ldots, u_{j-a}), \\ (t'_{j-a-b+1}, \ldots, t'_{j-a}) \text{ and } (t_{j-a-b+1}, \ldots, t_{j-a}), \\ (v'_{j-a-b+1}, \ldots, v'_{j-a}) \text{ and } (v_{j-a-b+1}, \ldots, v_{j-a}) \end{cases}$$

are rearrangements of each other and $(u_{j-a-b+1}, \ldots, u_{j-a}) = (s-1, s-2, \ldots, s-b)$. We may apply a simultaneous permutation of $(u'_{j-a-b+1}, \ldots, u'_{j-a})$ and $(t'_{j-a-b+1}, \ldots, t'_{j-a})$ to get

$$(u'_{j-a-b+1}, \ldots, u'_{j-a}) = (s-1, s-2, \ldots, s-b) = (u_{j-a-b+1}, \ldots, u_{j-a})$$

without affecting $(v'_{j-a-b+1}, \ldots v'_{j-a})$. Since v' is j-increasing, and the entries in $(t'_{j-a-b+1}, \ldots, t'_{j-a})$ are distinct, we have $t'_{j-a-b+1} > \cdots > t'_{j-a}$, which implies

$$(t'_{j-a-b+1}, \ldots, t'_{j-a}) = (t_{j-a-b+1}, \ldots, t_{j-a}) \text{ so that } (v'_{j-a-b+1}, \ldots, v'_{j-a}) = (v_{j-a-b+1}, \ldots, v_{j-a}).$$

The last two paragraphs show that the first j entries of v' and v coincide. The fact that the last $n-j$ entries of v' and v coincide follows from an argument similar to the proof of Proposition 4.3; it is omitted here. This completes the proof of the claim.

Our claim implies that $\text{in}_j(f \odot \delta_{n,k,s})$ is the monomial associated to $v = (v_1, \ldots, v_n)$. We show how to recover the exponent sequence $u = (u_1, \ldots, u_n)$ of the monomial M appearing in the claim. In turn, this allows us to recover $x_1^{i_1} \cdots x_j^{i_j} \theta_1 \cdots \theta_a$ and m, such that

$$f = \varepsilon_j \cdot \left(x_1^{i_1} \cdots x_j^{i_j} \theta_1 \cdots \theta_a\right) \otimes m,$$

completing the proof of the lemma.

Since $m \in \mathcal{M}^{(r-a)}_{n-j,k,s-b}$ and the construction of the last $n-j$ entries of u and v follows the algorithms in Proposition 4.3, that proposition proves we can recover the last $n-j$ entries of u and the last $n-j$ entries of t from the last $n-j$ entries of v. Since r is a global parameter and m has degree $r-a$ in the θ_i variables, knowing the last $n-j$ entries of t recovers a. This gives $u_{j-a+1} = \cdots = u_j = \overline{k-1}$ and, since we know these entries of v, we recover t_{j-a+1} through t_j. By construction, v begins with $j-a-b$ barred entries. Since we know j and a, we learn b. Then we know that the first $j-a$ entries of u are $j-a-b$ copies of $\overline{k-1}$ followed by the sequence $(s-1)(s-2)\ldots(s-b)$. From this information and the corresponding entries of v, we can recover t_1 through t_{j-a}. Now we know all of t. Since we know all of v, this determines all of u. $\square$

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Finally, we put everything together to get our desired basis of $\varepsilon_j \mathbb{W}_{n,k,s}^{(r)}$.

Lemma 5.10. *The subset of $\mathcal{B} \subseteq \Omega_n$ descends to a basis of $\varepsilon_j \mathbb{W}_{n,k,s}^{(r)}$.*

Proof. Lemma 5.9 implies that $\mathcal{B}^\vee$ is linearly independent in Ω_n, so Proposition 5.3 shows that $\mathcal{B}$ is linearly independent in $\mathbb{W}_{n,k,s}^{(r)}$. Now apply Lemma 5.5. □

5.4. The $\mathcal{C}$ basis of $\eta_j \mathbb{H}_{n,k,s}^{(r)}$

Lemma 5.10 gives a basis $\mathcal{B}$ of $\varepsilon_j \mathbb{W}_{n,k,s}^{(r)}$. The corresponding harmonic basis is as follows.

Definition 5.11. Let $\mathcal{C} \subseteq \Omega_n$ be the disjoint union

$$\mathcal{C} = \bigsqcup_{\substack{a,b \geq 0 \\ a \leq n-s-r,\ b \leq s}} \bigsqcup_{\mathbf{i}} \left\{ \left[\varepsilon_j \cdot (x_1^{i_1} \cdots x_j^{i_j} \cdot \theta_1 \cdots \theta_a) \otimes m \right] \odot \delta_{n,k,s} \ :\ m \in \mathcal{M}_{n-j,k,s-b}^{(n-s-r-a)} \right\}, \tag{5.15}$$

where the index $\mathbf{i} = (i_1, \ldots, i_j)$ of the inner disjoint union ranges over all length j integer sequences whose first a, next b and final $j-a-b$ entries satisfy the conditions

$$0 \leq i_1 \leq \cdots \leq i_a \leq k-s-1+b, \quad 0 \leq i_{a+1} < \cdots < i_{a+b} \leq s-1, \quad s \leq i_{a+b+1} < \cdots < i_j \leq k-1.$$

The subset $\mathcal{C} \subseteq \Omega_n$ depends on n, k, s, r and j, but we suppress this dependence to avoid notational clutter.

Definition 5.11 is formulated so that $\mathcal{C} \subseteq \mathbb{H}_{n,k,s}^{(r)}$. The following result justifies the disjointness of the unions appearing in Definition 5.11.

Lemma 5.12. *The family $\mathcal{C} \subseteq \mathbb{H}_{n,k,s}^{(r)}$ is a basis of the invariant space $\eta_j \mathbb{H}_{n,k,s}^{(r)}$.*

Proof. Apply Proposition 5.3 and Lemma 5.10. □

Thanks to its avoidance of cosets, the $\mathcal{C}$ basis will be more convenient for us going forward.

5.5. The $\lessdot$ order on bidegrees

Our goal is to use the basis $\mathcal{C}$ of Lemma 5.12 to show that $\eta_j \mathbb{H}_{n,k,s}^{(r)}$ is isomorphic as a graded $\mathfrak{S}_{n-j}$-module to a direct sum of graded shifts of smaller $\mathbb{H}$-modules in a way that matches the recursion of Lemma 3.14. To do this, we introduce a strategic direct sum decomposition of $\Omega_n = \Omega_j \otimes \Omega_{n-j}$, place a total order $\lessdot$ on the pieces of this decomposition and examine the $\lessdot$-lowest components of the elements in $\mathcal{C}$.

Our direct sum decomposition of $\Omega_n = \Omega_j \otimes \Omega_{n-j}$ is defined as follows. The second factor Ω_{n-j} of the tensor product $\Omega_n = \Omega_j \otimes \Omega_{n-j}$ is a bigraded algebra

$$\Omega_{n-j} = \bigoplus_{p,q \geq 0} (\Omega_{n-j})_{p,q}, \tag{5.16}$$

where

$$(\Omega_{n-j})_{p,q} = \mathbb{Q}[x_{j+1}, \ldots, x_n]_p \otimes \wedge^q \{\theta_{j+1}, \ldots, \theta_n\}. \tag{5.17}$$

We, therefore, have a direct sum decomposition

$$\Omega_n = \bigoplus_{p,q \geq 0} \Omega_n(p,q) \tag{5.18}$$

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of the larger algebra Ω_n, where we set

$$\Omega_n(p,q) = \Omega_j \otimes (\Omega_{n-j})_{p,q}. \tag{5.19}$$

In other words, the bigrading $\Omega_n(-,-)$ on Ω_n is obtained by focusing on commuting and anticommuting degree in the last $n-j$ indices only. In particular, we have the following important observation.

Observation 5.13. *Although the direct sum decomposition $\Omega_n = \bigoplus_{p,q\geq 0} \Omega_n(p,q)$ is not stable under the action of $\mathfrak{S}_n$, it is stable under the parabolic subgroup $\mathfrak{S}_j \times \mathfrak{S}_{n-j}$ and the further subgroup $\mathfrak{S}_{n-j}$.*

The decomposition (5.18) will be used to study the action of $\mathfrak{S}_{n-j}$ on Ω_n, and, in particular, on the basis $\mathcal{C}$ of $\eta_j \mathbb{H}^{(r)}_{n,k,s}$. We will need a nonstandard notion of 'lowest degree component' for elements of $\mathcal{C}$. To this end, we introduce a total order $\triangleleft$ on the summands $\Omega_n(p,q)$ appearing in (5.18) as follows.

Definition 5.14. Given two pairs of nonnegative integers (p,q) and (p',q'), write $(p,q) \triangleleft (p',q')$ if

- we have $q' < q$, or
- we have $q' = q$ and $p' > p$.

We also use the symbol $\triangleleft$ to denote the induced order on the summands $\Omega_n(p,q)$ of the direct sum (5.18).

If there were no θ-variables, Definition 5.14 would be the usual degree order induced from the second factor of the tensor decomposition $\mathbb{Q}[x_1,\ldots,x_n] = \mathbb{Q}[x_1,\ldots,x_j] \otimes \mathbb{Q}[x_{j+1},\ldots,x_n]$. The order $\triangleleft$ first compares θ-degrees in the 'opposite' order and breaks ties by comparing x-degrees in the classical order. This superisation of polynomial degree order should be compared with the superisation of the lexicographical term order $\prec$ in Definition 4.2 in which anticommuting variables involve a similar 'reversal'.

5.6. *Factoring the $\triangleleft$-lowest components of the $\mathcal{C}$ basis*

We consider elements in the $\mathcal{C}$ basis of $\eta_j \mathbb{H}^{(r)}_{n,k,s}$ with respect to the direct sum decomposition $\Omega_n = \bigoplus_{p,q\geq 0} \Omega_n(p,q)$ of (5.18). While these elements are almost always inhomogeneous members of this direct sum, their $\triangleleft$-lowest components have a useful recursive structure.

Lemma 5.15. *Let $\left[\varepsilon_j \cdot (x_1^{i_1}\cdots x_j^{i_j}\cdot\theta_1\cdots\theta_a)\otimes m\right] \odot \delta_{n,k,s} \in \mathcal{C}$, where $m \in \mathcal{M}^{(n-s-r-a)}_{n-j,k,s-b}$ and the sequence $\mathbf{i} = (i_1,\ldots,i_j)$ satisfies*

$$0 \leq i_{a+1} < \cdots < i_{a+b} < s \leq i_{a+b+1} < \cdots < i_j \leq k-1 \quad \textit{and} \quad 0 \leq i_1 \leq \cdots \leq i_a \leq k-s-1+b.$$

This element of $\mathcal{C}$ has an expansion under $\triangleleft$ of the form

$$\left[\varepsilon_j \cdot (x_1^{i_1}\cdots x_j^{i_j}\cdot\theta_1\cdots\theta_a)\otimes m\right] \odot \delta_{n,k,s} = \pm f_{a,b,\mathbf{i}} \otimes [m \odot \delta_{n-j,k,s-b}] + \text{greater terms under } \triangleleft, \tag{5.20}$$

where

$$f_{a,b,\mathbf{i}} = \left[\varepsilon_j \cdot (x_1^{i_1}\cdots x_j^{i_j}\cdot\theta_1\cdots\theta_a)\right] \odot \left[\varepsilon_j \cdot m_{a,b,\mathbf{i}}\right] \tag{5.21}$$

and

$$m_{a,b,\mathbf{i}} = x_1^{k-1}\cdots x_a^{k-1} x_{a+1}^{s-b} x_{a+2}^{s-b+1} \cdots x_{a+b}^{s-1} x_{a+b+1}^{k-1} \cdots x_j^{k-1} \cdot \theta_1\theta_2\cdots\theta_a \cdot \theta_{a+b+1}\theta_{a+b+2}\cdots\theta_j. \tag{5.22}$$

The polynomial $f_{a,b,\mathbf{i}} \in \Omega_j$ is $\mathfrak{S}_j$-invariant and nonzero.

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Remark 5.16. The polynomial $f_{a,b,\mathbf{i}}$ is bihomogeneous of x-degree

$$(k-1)\cdot(j-b)+b\cdot(s-b)+\binom{b}{2}-i_1-\cdots-i_j \tag{5.23}$$

and θ-degree $j-a-b$. The $\mathfrak{S}_j$-invariance of $f_{a,b,\mathbf{i}}$ is justified by Proposition 5.2. Observe that $f_{a,b,\mathbf{i}}$ depends on a, b and $\mathbf{i}$ but is independent of $m \in \mathcal{M}^{(n-s-r-a)}_{n-j,k,s-b}$.

To better understand Lemma 5.15, we analyse its statement in the case of the classical Vandermonde δ_n.

Example 5.17. In the special case $(n,k,s)=(n,n,n)$ and $j=1$, Lemma 5.15 follows from the following expansion for the Vandermonde determinant δ_n in the variable x_1:

$$\delta_n=\varepsilon_n\cdot(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1x_n^0)=\sum_{d=0}^{n-1}(-1)^{n-d+1}x_1^d\otimes\varepsilon_{n-1}\cdot(x_2^{n-1}\cdots x_{n-d}^{d+1}x_{n-d+1}^{d-1}\cdots x_n^0). \tag{5.24}$$

Since we have no anticommuting variables, the order $\lhd$ is simply the degree order on the last $n-j=n-1$ variables $x_2,\ldots,x_n$. Therefore, the $d=n-1$ term

$$x_1^{n-1}\otimes\varepsilon_{n-1}\cdot(x_2^{n-2}\cdots x_n^0)=x_1^{n-1}\otimes\delta_{n-1} \tag{5.25}$$

is the lowest $\lhd$-degree component of this expansion. The condition $x_1^{c_1}x_2^{c_2}\cdots x_n^{c_n}\in\mathcal{M}_{n,n,n}$ means that $(c_1,c_2,\ldots,c_n)\le(n-1,n-2,\ldots,0)$ componentwise. If we set $m=x_2^{c_2}\cdots x_n^{c_n}$, applying $x_1^{c_1}x_2^{c_2}\cdots x_n^{c_n}=(\varepsilon_1\cdot x_1^{c_1})\otimes m$ to δ_n under the $\odot$-action yields

$$[(\varepsilon_1\cdot x_1^{c_1})\otimes m]\odot\delta_n=Cx_1^{n-c_1}\otimes[m\odot\delta_{n-1}]+\text{greater terms under }\lhd, \tag{5.26}$$

where C is a nonzero constant.

Proof of Lemma 5.15. By definition, the superspace Vandermonde $\delta_{n,k,s}$ is the antisymmetrisation

$$\delta_{n,k,s}=\varepsilon_n\cdot(x_1^{p_1}\cdots x_n^{p_n}\theta_1^{q_1}\cdots\theta_n^{q_n}), \tag{5.27}$$

where the exponent sequences are given by $(p_1,\ldots,p_n)=((k-1)^{n-s},s-1,s-2,\ldots,1,0)$ and $(q_1,\ldots,q_n)=(1^{n-s},0^s)$. We expand Equation (5.27) in a fashion compatible with the tensor product decomposition $\Omega_n=\Omega_j\otimes\Omega_{n-j}$. For any system of right coset representatives w for $\mathfrak{S}_j\times\mathfrak{S}_{n-j}$ inside $\mathfrak{S}_n$, Equation (5.27) reads

$$\begin{aligned}\delta_{n,k,s}&=\\ &\sum_w\pm\Big[\varepsilon_j\cdot(x_1^{p_{w^{-1}(1)}}\cdots x_j^{p_{w^{-1}(j)}}\theta_1^{q_{w^{-1}(1)}}\cdots\theta_j^{q_{w^{-1}(j)}})\Big]\\ &\otimes\Big[\varepsilon_{n-j}\cdot(x_{j+1}^{p_{w^{-1}(j+1)}}\cdots x_n^{p_{w^{-1}(n)}}\theta_{j+1}^{q_{w^{-1}(j+1)}}\cdots\theta_n^{q_{w^{-1}(n)}})\Big],\end{aligned} \tag{5.28}$$

where $\pm$ is the sign of the coset representative w. Equation (5.28) is true for any system of coset representatives, but we restrict our choice somewhat in the next paragraph.

Our aim is to apply the operator

$$\Big\{\Big[\varepsilon_j\cdot(x_1^{i_1}\cdots x_j^{i_j}\theta_1\cdots\theta_a)\Big]\otimes m\Big\}\odot(-)=\Big\{\Big[\varepsilon_j\cdot(x_1^{i_1}\cdots x_j^{i_j}\theta_1\cdots\theta_a)\Big]\odot(-)\Big\}\otimes\{m\odot(-)\} \tag{5.29}$$

to Equation (5.28) and extract the lowest term under $\lhd$ for which the first tensor factor does not vanish. A summand of Equation (5.28) indexed by a permutation w for which fewer than a entries in $(q_{w^{-1}(1)},\ldots,q_{w^{-1}(j)})$ equal 1 will be annihilated by the first tensor factor of the operator (5.29).

We may, therefore, restrict our attention to those summands in which at least a of the entries in $(q_{w^{-1}(1)}, \dots, q_{w^{-1}(j)})$ equal 1; the corresponding entries in $(p_{w^{-1}(1)}, \dots, p_{w^{-1}(j)})$ will equal $k-1$. We now assume the coset representatives w in Equation (5.28) are, such that

$$p_{w^{-1}(1)} = \cdots = p_{w^{-1}(a)} = k-1 \text{ and } 0 \leq p_{w^{-1}(a+1)} \leq \cdots \leq p_{w^{-1}(j)} \leq k-1.$$

When $k = s$, we further assume that

$$(q_{w^{-1}(1)}, \dots, q_{w^{-1}(j)}) \text{ has the form } (1^a, 0^{j-a-c}, 1^c) \text{ for some } a+c \leq j$$

(this follows from the assumption on the p's when $k > s$). There are typically many choices of coset representatives w which achieve this.

We apply the operator (5.29) to each term

$$\left[\varepsilon_j \cdot (x_1^{p_{w^{-1}(1)}} \cdots x_j^{p_{w^{-1}(j)}} \theta_1^{q_{w^{-1}(1)}} \cdots \theta_j^{q_{w^{-1}(j)}})\right] \otimes \left[\varepsilon_{n-j} \cdot (x_{j+1}^{p_{w^{-1}(j+1)}} \cdots x_n^{p_{w^{-1}(n)}} \theta_{j+1}^{q_{w^{-1}(j+1)}} \cdots \theta_n^{q_{w^{-1}(n)}})\right] \tag{5.30}$$

of Equation (5.28). In the first tensor factor, we get

$$\left[\varepsilon_j \cdot (x_1^{i_1} \cdots x_j^{i_j} \theta_1 \cdots \theta_a)\right] \odot \left[\varepsilon_j \cdot (x_1^{p_{w^{-1}(1)}} \cdots x_j^{p_{w^{-1}(j)}} \theta_1^{q_{w^{-1}(1)}} \cdots \theta_j^{q_{w^{-1}(j)}})\right]. \tag{5.31}$$

The element of Ω_j in Equation (5.31) is certainly $\mathfrak{S}_j$-invariant but can vanish for some coset representatives w.

In order to minimise the application (5.31) of (5.29) to (5.30) under the order $\triangleleft$, we select our coset representative w in (5.31), such that

1. the expression (5.31) is a nonzero element of Ω_j, and subject to this
2. the expression (5.31) has the smallest θ-degree possible, and subject to this
3. the expression (5.31) has the largest x-degree possible.

For Item (1), we see that (5.31) is nonzero if and only if we have the componentwise inequalities

$$(i_1, \dots, i_j) \leq (p_{w^{-1}(1)}, \dots, p_{w^{-1}(j)}) \text{ and } (1^a, 0^{j-a}) \leq (q_{w^{-1}(1)}, \dots, q_{w^{-1}(j)}). \tag{5.32}$$

Subject to this, in order to satisfy Item (2), we must have

$$(q_{w^{-1}(1)}, \dots, q_{w^{-1}(j)}) = (1^a, 0^b, 1^{j-a-b}). \tag{5.33}$$

Subject to both (5.32) and (5.33), to satisfy Item (3), we must have

$$(p_{w^{-1}(1)}, \dots, p_{w^{-1}(j)}) = ((k-1)^a, s-b, s-b+1, \dots, s-1, (k-1)^{j-a-b}). \tag{5.34}$$

Said differently, Items (1)–(3) are satisfied precisely when

$$x_1^{p_{w^{-1}(1)}} \cdots x_j^{p_{w^{-1}(j)}} \theta_1^{q_{w^{-1}(1)}} \cdots \theta_j^{q_{w^{-1}(j)}} = m_{a,b,\mathbf{i}}, \tag{5.35}$$

so that Equation (5.31) equals $f_{a,b,\mathbf{i}}$.

Let w be the unique coset representative which satisfies (5.32), (5.33) and (5.34). For this coset representative, the exponent sequence $(p_{w^{-1}(j+1)}, \dots, p_{w^{-1}(n)})$ appearing in the second tensor factor of (5.30) is a rearrangement of $((k-1)^{n-s-j+b}, s-b-1, \dots, 1, 0)$. This shows that

$$\varepsilon_{n-j} \cdot (x_{j+1}^{p_{w^{-1}(j+1)}} \cdots x_n^{p_{w^{-1}(n)}} \theta_{j+1}^{q_{w^{-1}(j+1)}} \cdots \theta_n^{q_{w^{-1}(n)}}) = \pm\delta_{n-j,k,s-b} \tag{5.36}$$

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for this choice of w. In summary, the application of (5.29) to $\delta_{n,k,s}$ has the form

$$\left\{\left[\varepsilon_j \cdot (x_1^{i_1} \cdots x_j^{i_j} \theta_1 \cdots \theta_a)\right] \otimes m\right\} \odot \delta_{n,k,s} \\ = \pm f_{a,b,\mathbf{i}} \otimes (m \odot \delta_{n-j,k,s-b}) + \text{greater terms under } \lessdot, \tag{5.37}$$

which is what we wanted to show. □

We give an example to illustrate the statement and proof of Lemma 5.15 in the presence of θ-variables.

Example 5.18. Suppose $(n, k, s) = (7, 7, 4)$ so the superspace Vandermonde is

$$\delta_{7,7,4} = \varepsilon_7 \cdot (x^{6663210} \theta_{123}) \in \Omega_7.$$

We record the exponent sequence of the x-variables by $(p_1, \ldots, p_7) = (6, 6, 6, 3, 2, 1, 0)$ and that of the θ-variables by $(q_1, \ldots, q_7) = (1, 1, 1, 0, 0, 0, 0)$.

We take $j = 3$, so that we factor superspace elements according to $\Omega_7 = \Omega_j \otimes \Omega_{n-j} = \Omega_3 \otimes \Omega_4$. For any system of right coset representatives of $\mathfrak{S}_3 \times \mathfrak{S}_4$ in $\mathfrak{S}_7$, the Vandermonde $\delta_{7,7,4}$ expands as

$$\delta_{7,7,4} = \sum_w \pm \left[\varepsilon_3 \cdot (x_1^{p_{w^{-1}(1)}} x_2^{p_{w^{-1}(2)}} x_3^{p_{w^{-1}(3)}} \theta_1^{q_{w^{-1}(1)}} \theta_2^{q_{w^{-1}(2)}} \theta_3^{q_{w^{-1}(3)}})\right] \\ \otimes \left[\varepsilon_4 \cdot (x_4^{p_{w^{-1}(4)}} \cdots x_7^{p_{w^{-1}(7)}} \theta_4^{q_{w^{-1}(4)}} \cdots \theta_7^{q_{w^{-1}(7)}})\right],$$

where $\pm$ is the sign of the coset representative w.

Let $m \in \Omega_4 \subseteq \Omega_3 \otimes \Omega_4$ be an arbitrary monomial in $x_4, \ldots, x_7, \theta_4, \ldots, \theta_7$. We consider applying the operator

$$\left\{\left[\varepsilon_3 \cdot (x^{325}\theta_1)\right] \otimes m\right\} \odot (-) = \left\{\left[\varepsilon_3 \cdot (x^{325}\theta_1)\right] \odot (-)\right\} \otimes \{m \odot (-)\}$$

to $\delta_{7,7,4}$ by applying it to each term

$$\pm \left[\varepsilon_3 \cdot (x_1^{p_{w^{-1}(1)}} x_2^{p_{w^{-1}(2)}} x_3^{p_{w^{-1}(3)}} \theta_1^{q_{w^{-1}(1)}} \theta_2^{q_{w^{-1}(2)}} \theta_3^{q_{w^{-1}(3)}})\right] \otimes \left[\varepsilon_4 \cdot (x_4^{p_{w^{-1}(4)}} \cdots x_7^{p_{w^{-1}(7)}} \theta_4^{q_{w^{-1}(4)}} \cdots \theta_7^{q_{w^{-1}(7)}})\right]$$

of its expansion in $\Omega_3 \otimes \Omega_4$. We choose our coset representatives w so that

$$q_{w^{-1}(2)} \le q_{w^{-1}(3)} \text{ and } p_{w^{-1}(2)} \le p_{w^{-1}(3)}$$

in each term.

Focusing on the first tensor factor, the evaluation

$$\left[\varepsilon_3 \cdot (x^{325}\theta_1)\right] \odot \left[\varepsilon_3 \cdot (x_1^{p_{w^{-1}(1)}} x_2^{p_{w^{-1}(2)}} x_3^{p_{w^{-1}(3)}} \theta_1^{q_{w^{-1}(1)}} \theta_2^{q_{w^{-1}(2)}} \theta_3^{q_{w^{-1}(3)}})\right]$$

is $\mathfrak{S}_3$-invariant, and is nonzero if and only if we have the componentwise inequalities

$$(3, 2, 5) \le (p_{w^{-1}(1)}, p_{w^{-1}(2)}, p_{w^{-1}(3)}) \text{ and } (1, 0, 0) \le (q_{w^{-1}(1)}, q_{w^{-1}(2)}, q_{w^{-1}(3)}).$$

To minimise under the order $\lessdot$ subject to these conditions, we take the unique coset representative w for which

$$(p_{w^{-1}(1)}, p_{w^{-1}(2)}, p_{w^{-1}(3)}) = (6, 3, 6) \text{ and } (q_{w^{-1}(1)}, q_{w^{-1}(2)}, q_{w^{-1}(3)}) = (1, 0, 1).$$

For this choice of w, in the second tensor factor, we have

$$\varepsilon_4 \cdot (x_4^{p_{w^{-1}(4)}} \cdots x_7^{p_{w^{-1}(7)}} \theta_4^{q_{w^{-1}(4)}} \cdots \theta_7^{q_{w^{-1}(7)}}) = \pm\varepsilon_4 \cdot (x_4^6 x_5^2 x_6^1 x_7^0 \theta_4) = \pm\delta_{4,7,3},$$

so that

$$m \odot \left[\varepsilon_4 \cdot (x_4^{p_{w^{-1}(4)}} \cdots x_7^{p_{w^{-1}(7)}} \theta_4^{q_{w^{-1}(4)}} \cdots \theta_7^{q_{w^{-1}(7)}})\right] = \pm m \odot \delta_{4,7,3}.$$

This yields an expansion

$$\begin{aligned} &\left\{\left[\varepsilon_3 \cdot (x^{325}\theta_1)\right] \otimes m\right\} \odot \delta_{7,7,4} = \\ &\qquad \pm \left[\varepsilon_3 \cdot (x^{325}\theta_1)\right] \odot \left[\varepsilon_3 \cdot (x^{626}\theta_{13})\right] \otimes [m \odot \delta_{4,7,3}] + \text{greater terms under } \lessdot, \end{aligned}$$

where the $\mathfrak{S}_3$-invariant in the first tensor factor is nonzero.

5.7. *A recursion for* $\eta_j \mathbb{H}_{n,k,s}^{(r)}$

We are ready to state our recursion for the graded $\mathfrak{S}_{n-j}$-module $\eta_j \mathbb{H}_{n,k,s}^{(r)}$. Our crucial tools are Observation 5.13 and Lemma 5.15. Given any graded $\mathfrak{S}_{n-j}$-module $V = \bigoplus_i V_i$ and any integer i_0, let $V\{-i_0\}$ denote the same $\mathfrak{S}_{n-j}$ module with degree shifted up by i_0. That is, the i^{th} graded piece of $V\{-i_0\}$ is given by

$$(V\{-i_0\})_i = V_{i-i_0}. \tag{5.38}$$

Lemma 5.19. *Let $n, k, s \geq 0$ with $k \geq s$, and let $0 \leq r \leq n - s$. Let $1 \leq j \leq n$. There holds an isomorphism of graded $\mathfrak{S}_{n-j}$-modules*

$$\eta_j \mathbb{H}_{n,k,s}^{(r)} \cong \bigoplus_{\substack{a,b \geq 0 \\ a \leq n-s-r \\ b \leq s}} \left[\bigoplus_{\mathbf{i}} \mathbb{H}_{n-j,k,s-b}^{(r-j+a+b)}\left\{i_1 + \cdots + i_j - \binom{b}{2} - b \cdot (s-b) - (k-1) \cdot (j-b)\right\}\right], \tag{5.39}$$

where the inner direct sum is over all j-tuples $\mathbf{i} = (i_1, \ldots, i_j)$ of nonnegative integers, such that

$$0 \leq i_{a+1} < \cdots < i_{a+b} < s \leq i_{a+b+1} < \cdots < i_j \leq k-1$$

and

$$0 \leq i_1 \leq \cdots \leq i_a \leq k - s - 1 + b.$$

Equivalently, we have the symmetric function identity

$$\begin{aligned} &h_j^\perp \mathrm{grFrob}(\mathbb{H}_{n,k,s}^{(r)}; q) = \\ &\sum_{\substack{a,b \geq 0 \\ a \leq n-s-r \\ b \leq s}} q^{\binom{j-a-b}{2} + (s-b)a} \times \begin{bmatrix} k-s-1+a+b \\ a \end{bmatrix}_q \cdot \begin{bmatrix} s \\ b \end{bmatrix}_q \cdot \begin{bmatrix} k-s \\ j-a-b \end{bmatrix}_q \cdot \mathrm{grFrob}(\mathbb{H}_{n-j,k,s-b}^{(r-j+a+b)}; q). \end{aligned} \tag{5.40}$$

Proof. The basis $\mathcal{C}$ of $\eta_j \mathbb{H}_{n,k,s}^{(r)}$ in Lemma 5.12 admits a disjoint union decomposition

$$\bigsqcup_{\substack{a,b \geq 0 \\ a \leq n-s-r \\ b \leq s}} \bigsqcup_{\mathbf{i}} \left\{\left[\varepsilon_j \cdot (x_1^{i_1} \cdots x_j^{i_j} \cdot \theta_1 \cdots \theta_a) \otimes m\right] \odot \delta_{n,k,s} \;:\; m \in \mathcal{M}_{n-j,k,s-b}^{(n-s-r-a)}\right\}, \tag{5.41}$$

where the tuples $\mathbf{i} = (i_1, \ldots, i_j)$ satisfy the conditions in the statement of the theorem. Lemma 5.15 shows that, for each triple $(a, b, \mathbf{i})$ indexing the disjoint union (5.41), there is a single nonzero invariant

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$f_{a,b,\mathbf{i}} \in (\Omega_j)^{\mathfrak{S}_j}$, such that

$$\pm\Big[\varepsilon_j \cdot (x_1^{i_1} \cdots x_j^{i_j} \cdot \theta_1 \cdots \theta_a) \otimes m\Big] \odot \delta_{n,k,s} = f_{a,b,\mathbf{i}} \otimes (m \odot \delta_{n-j,k,s-b}) + \text{ greater terms under } \triangleleft \tag{5.42}$$

for all $m \in \mathcal{M}_{n-j,k,s-b}^{(n-s-r-a)}$. The sign in Equation (5.42) may depend on m but the invariant $f_{a,b,\mathbf{i}}$ does not. The leading term $f_{a,b,\mathbf{i}} \otimes (m \odot \delta_{n-j,k,s-b})$ in (5.42) lies in a single bihomogeneous piece

$$(p, q) = (p(a, b, \mathbf{i}), q(a, b, \mathbf{i})) \tag{5.43}$$

of the decomposition $\Omega_n = \bigoplus_{p,q \geq 0} \Omega_n(p, q)$ defining $\triangleleft$.

The observations of the previous paragraph give rise to a filtration of $\eta_j \mathbb{H}_{n,k,s}^{(r)}$. For any $p, q \geq 0$, we set

$$\begin{aligned}&\left(\eta_j \mathbb{H}_{n,k,s}^{(r)}\right)_{\trianglelefteq(p,q)} = \\ &\operatorname{span} \bigsqcup_{(p(a,b,\mathbf{i}),q(a,b,\mathbf{i})) \trianglelefteq (p,q)} \left\{\Big[\varepsilon_j \cdot (x_1^{i_1} \cdots x_j^{i_j} \cdot \theta_1 \cdots \theta_a) \otimes m\Big] \odot \delta_{n,k,s} \;:\; m \in \mathcal{M}_{n-j,k,s-b}^{(n-s-r-a)}\right\},\end{aligned} \tag{5.44}$$

where $(p(a, b, \mathbf{i}), q(a, b, \mathbf{i}))$ is defined as in (5.43). Analogously, we define

$$\begin{aligned}&\left(\eta_j \mathbb{H}_{n,k,s}^{(r)}\right)_{\triangleleft(p,q)} = \\ &\operatorname{span} \bigsqcup_{(p(a,b,\mathbf{i}),q(a,b,\mathbf{i})) \triangleleft (p,q)} \left\{\Big[\varepsilon_j \cdot (x_1^{i_1} \cdots x_j^{i_j} \cdot \theta_1 \cdots \theta_a) \otimes m\Big] \odot \delta_{n,k,s} \;:\; m \in \mathcal{M}_{n-j,k,s-b}^{(n-s-r-a)}\right\}\end{aligned} \tag{5.45}$$

and the corresponding quotient space

$$\left(\eta_j \mathbb{H}_{n,k,s}^{(r)}\right)_{=(p,q)} = \left(\eta_j \mathbb{H}_{n,k,s}^{(r)}\right)_{\trianglelefteq(p,q)} \Big/ \left(\eta_j \mathbb{H}_{n,k,s}^{(r)}\right)_{\triangleleft(p,q)}. \tag{5.46}$$

By Observation 5.13, the decomposition $\Omega_n = \bigoplus_{p,q \geq 0} \Omega_n(p, q)$ is $\mathfrak{S}_{n-j}$-stable. Therefore, the three vector spaces (5.44), (5.45) and (5.46) are graded $\mathfrak{S}_{n-j}$-modules under x-degree. Furthermore, we have an isomorphism

$$\eta_j \mathbb{H}_{n,k,s}^{(r)} \cong \bigoplus_{p,q \geq 0} \left(\eta_j \mathbb{H}_{n,k,s}^{(r)}\right)_{=(p,q)}. \tag{5.47}$$

What does the summand $\left(\eta_j \mathbb{H}_{n,k,s}^{(r)}\right)_{=(p,q)}$ in (5.47) look like? The expansion (5.42) implies an isomorphism of graded $\mathfrak{S}_{n-j}$-modules

$$\left(\eta_j \mathbb{H}_{n,k,s}^{(r)}\right)_{=(p,q)} \cong \bigoplus_{(p(a,b,\mathbf{i}),q(a,b,\mathbf{i}))=(p,q)} \mathbb{V}_{a,b,\mathbf{i}}, \tag{5.48}$$

where

$$\mathbb{V}_{a,b,\mathbf{i}} = \operatorname{span}\left\{f_{a,b,\mathbf{i}} \otimes (m \odot \delta_{n-j,k,s-b}) \;:\; m \in \mathcal{M}_{n-j,k,s-b}^{(n-s-r-a)}\right\}. \tag{5.49}$$

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In light of Corollary 4.14, the space $\mathbb{V}_{a,b,\mathbf{i}}$ affords a copy of the $\mathfrak{S}_{n-j}$-module $\mathbb{H}_{n-j,k,s-b}^{(r-j+a+b)}$ with degree shifted up by

$$x\text{-degree of } f_{a,b,\mathbf{i}} = \binom{b}{2} + b \cdot (s-b) + (k-1) \cdot (j-b) - i_1 - \cdots - i_j. \tag{5.50}$$

This completes the proof. □

5.8. *The $\mathfrak{S}_n$-structure of $\mathbb{W}_{n,k,s}$*

All of the pieces are in place for us to prove our combinatorial formula for $\mathrm{grFrob}(\mathbb{W}_{n,k,s}; q, z)$.

Theorem 5.20. *Let $n, k \geq s \geq 0$ be integers. The bigraded Frobenius image $\mathrm{grFrob}(\mathbb{W}_{n,k,s}; q, z)$ has the following combinatorial expressions in terms of the statistics* coinv *and* codinv.

$$\mathrm{grFrob}(\mathbb{W}_{n,k,s}; q, z) = C_{n,k,s}(\mathbf{x}; q, z) = D_{n,k,s}(\mathbf{x}; q, z). \tag{5.51}$$

Proof. It suffices to show

$$\mathrm{grFrob}(\mathbb{H}_{n,k,s}^{(r)}; q) = C_{n,k,s}^{(r)}(\mathbf{x}; q) \tag{5.52}$$

for all r. Lemmas 2.3 and 2.4 reduce this task to proving

$$\mathrm{grFrob}(\eta_j \mathbb{H}_{n,k,s}^{(r)}; q) = h_j^{\perp} C_{n,k,s}^{(r)}(\mathbf{x}; q) \tag{5.53}$$

for any $j \geq 1$. By Lemmas 3.14 and 5.19, both sides of Equation (5.53) satisfy the same recursion and we are done by induction on n. □

As an example of Theorem 5.20, let $(n, k, s) = (3, 2, 2)$. We list the 12 ordered set superpartitions $\sigma \in \mathcal{OSP}_{3,2,2}$ together with their coinversion numbers, reading words and inverse descent sets.

σ	$\mathrm{coinv}(\sigma)$	$\mathrm{read}(\sigma)$	$\mathrm{iDes}(\mathrm{read}(\sigma))$
$(1, 2 \mid 3)$	2	312	$\{2\}$
$(1, 3 \mid 2)$	1	213	$\{1\}$
$(2, 3 \mid 1)$	0	123	$\varnothing$
$(1 \mid 2, 3)$	2	213	$\{1\}$
$(2 \mid 1, 3)$	1	123	$\varnothing$
$(3 \mid 1, 2)$	1	132	$\{2\}$
σ	$\mathrm{coinv}(\sigma)$	$\mathrm{read}(\sigma)$	$\mathrm{iDes}(\mathrm{read}(\sigma))$
$(1, \bar{2} \mid 3)$	1	231	$\{1\}$
$(1, \bar{3} \mid 2)$	1	321	$\{1, 2\}$
$(2, \bar{3} \mid 1)$	0	312	$\{2\}$
$(1 \mid 2, \bar{3})$	2	321	$\{1, 2\}$
$(2 \mid 1, \bar{3})$	1	312	$\{2\}$
$(3 \mid 1, \bar{2})$	0	213	$\{1\}$

This leads to the expression

$$F_{\varnothing,3} + q \cdot (F_{\varnothing,3} + F_{1,3} + F_{2,3}) + q^2 \cdot (F_{1,3} + F_{2,3}) + z \cdot (F_{1,3} + F_{2,3}) + qz \cdot (F_{1,3} + F_{2,3} + F_{12,3}) + q^2 z \cdot (F_{12,3})$$

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for $\mathrm{grFrob}(\mathbb{W}_{3,2,2}; q, z)$ which has Schur expansion

$$s_3 + q \cdot (s_3 + s_{2,1}) + q^2 \cdot s_{2,1} + z \cdot s_{2,1} + qz \cdot (s_{2,1} + s_{1,1,1}) + q^2 z \cdot s_{1,1,1}.$$

As with the bigraded Hilbert series, the bigraded Frobenius image is more attractive in matrix format

$$\mathrm{grFrob}(\mathbb{W}_{3,2,2}; q, z) = \begin{pmatrix} s_3 & s_3 + s_{2,1} & s_{2,1} \\ s_{2,1} & s_{2,1} + s_{1,1,1} & s_{1,1,1} \end{pmatrix},$$

where the Rotational Duality of Theorem 1.2 becomes apparent.

Remark 5.21. In the case $k = s$, the family $\mathcal{OSP}_{n,k,k}$ admits an involution ι sending an ordered set superpartition $\sigma = (B_1 \mid \cdots \mid B_k) \in \mathcal{OSP}_{n,k,k}$ to $\iota(\sigma) = (\overline{B_k} \mid \cdots \mid \overline{B_1})$, where $\overline{B_i}$ is obtained from B_i by switching the barred/unbarred status of every nonminimal element of B_i. It may be checked that ι complements both the statistic coinv and the subset iDes in the sense that

$$\mathrm{coinv}(\sigma) + \mathrm{coinv}(\iota(\sigma)) = \binom{k}{2} + (n-k) \cdot (k-1) \quad \text{and} \quad \mathrm{iDes}(\mathrm{read}(\sigma)) \sqcup \mathrm{iDes}(\mathrm{read}(\iota(\sigma))) = [n-1]$$

for any $\sigma \in \mathcal{OSP}_{n,k,k}$. The Rotational Duality statement of Theorem 1.2 is, therefore, consistent with Theorem 5.20.

The authors do not know a combinatorial formula for the Schur expansion of $\mathrm{grFrob}(\mathbb{W}_{n,k,s}; q, z)$. On the other hand, if we consider $\mathbb{W}_{n,k,s}$ as a singly graded $\mathfrak{S}_n$-module under θ-degree, we have a simple formula for this Schur expansion. Recall that a partition λ is a *hook* if it has the form $\lambda = (a, 1^{m-a})$ for some a. The anticommutative graded pieces of $\mathbb{W}_{n,k,s}$ are built out of hook shapes.

Corollary 5.22. *Consider $\mathbb{W}_{n,k,s}$ as a singly graded $\mathfrak{S}_n$-module under θ-degree. Then*

$$\mathrm{grFrob}(\mathbb{W}_{n,k,s}; z) = \sum_{(\lambda^{(1)},\ldots,\lambda^{(k)})} z^{n-\lambda_1^{(1)}-\cdots-\lambda_1^{(k)}} \cdot s_{\lambda^{(1)}} \cdots s_{\lambda^{(s)}} \cdot s_{\lambda^{(s+1)}/(1)} \cdots s_{\lambda^{(k)}/(1)}, \tag{5.54}$$

where $(\lambda^{(1)}, \ldots, \lambda^{(k)})$ ranges over k-tuples of nonempty hooks with a total of

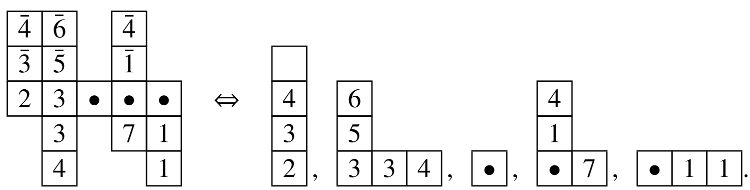
$$|\lambda^{(1)}| + \cdots + |\lambda^{(k)}| = n + k - s$$

boxes.

Observe that the skew partitions $\lambda^{(s+1)}/(1), \ldots, \lambda^{(k)}/(1)$ indexing the last $k - s$ factors in Corollary 5.22 can be empty and we have the factorisation $s_{(a,1^{m-a})/(1)} = h_{a-1} \cdot e_{m-a}$. When $(n, k, s) = (4, 3, 2)$, Corollary 5.22 says that the piece of $\mathbb{W}_{4,3,2}$ of anticommuting degree 1 has Frobenius image

$$s_{2,1} \cdot s_1 + s_{1,1} \cdot s_2 + s_2 \cdot s_{1,1} + s_1 \cdot s_{2,1} + s_{1,1} \cdot s_1 \cdot h_1 + s_1 \cdot s_{1,1} \cdot h_1 + s_2 \cdot s_1 \cdot e_1 + s_1 \cdot s_2 \cdot e_1 + s_1 \cdot s_1 \cdot (h_1 e_1).$$

Proof. We have a natural bijection between column tableaux and tuples of hook-shaped semistandard tableaux, where a $\bullet$ at height zero corresponds to a skewed-out box, viz.



Now apply Theorem 5.20. ◻

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6. Conclusion

In this paper, we gave a combinatorial formula (Theorem 5.20) for the bigraded Frobenius image $\mathrm{grFrob}(\mathbb{W}_{n,k,s}; q, z)$ of a family of quotients $\mathbb{W}_{n,k,s}$ of the superspace ring $\mathbf{\Omega}_n$. The following problem remains open.

Problem 6.1. Find the Schur expansion of $\mathrm{grFrob}(\mathbb{W}_{n,k,s}; q, z)$.

A solution to Problem 6.1 would refine Corollary 5.22. We remark that symmetric functions admit the operation of *superisation* which has the plethystic definition $f[\mathbf{x}] \mapsto f[\mathbf{x} - \mathbf{y}]$, where $\mathbf{x}$ and $\mathbf{y}$ are two infinite alphabets; see [20] for more details. Although superisation can be used to yield (for example) the bigraded Frobenius image of $\mathbf{\Omega}_n$ from that of $\mathbb{Q}[x_1, \ldots, x_n]$, we do not know of a way to use superisation to solve Problem 6.1.

One possible way to solve Problem 6.1 would be to define a statistic on ordered set superpartitions which extends the *major index* statistic on permutations. In the context of ordered set partitions, two such extensions are available: maj and minimaj [1, 12, 23, 24]. If such an extension were found, an insertion argument of Wilson [33] (in the case of maj) or the crystal techniques of Benkart, Colmenarejo, Harris, Orellana, Panova, Schilling and Yip [1] (in the case of minimaj) could perhaps be used to solve Problem 6.1.

The substaircase basis of $\mathbb{W}_{n,k,s}$ of Theorem 4.12 is tied to the inversion-like statistics coinv and codinv on ordered set superpartitions. Steinberg proved [29] that the set of *descent monomials* $\{d_w : w \in \mathfrak{S}_n\}$ form a basis for the classical coinvariant ring $\mathbb{Q}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n\rangle$, where

$$d_w = \prod_{\substack{1 \leq i \leq n-1 \\ w(i) > w(i+1)}} x_{w(1)} \cdots x_{w(i)}. \tag{6.1}$$

This basis was studied further by Garsia and Stanton [8] in the context of Stanley-Reisner theory and extended to the context of $R_{n,k}$ by Haglund, Rhoades and Shimozono [13].

Problem 6.2. Define an extension of the major index statistic to ordered set superpartitions in $\mathcal{OSP}_{n,k,s}$ and a companion descent monomial basis of $\mathbb{W}_{n,k,s}$.

In the context of $R_{n,k}$, Meyer used the descent monomial basis to refine the formulas for $\mathrm{grFrob}(R_{n,k}; q)$ of Haglund, Rhoades and Shimozono [13, 19]. This refinement matched a symmetric function arising from the crystal-theoretic machinery of Benkart et al. [1]. A solution to Problem 6.2 might lead to similar refinements in the $\mathbb{W}$-module context.

There are several results about the anticommuting degree zero piece of $\mathbb{W}_{n,k}$ one could try to push to the entire module $\mathbb{W}_{n,k}$. We mention a couple briefly. One could look for a Hecke action on all of superspace, extending work of Huang, Rhoades and Scrimshaw [15]. One could also hope to generalise the geometric discoveries of Pawlowski and Rhoades to the entire module $\mathbb{W}_{n,k}$ [21]. In particular, it would be interesting to see if this larger module allows for the definition of a Schubert basis with nonnegative structure constants, eliminating the negative constants that can appear in [21].

We close with a discussion of the superspace coinvariant problem [34] which was a key motivation of this work. As mentioned in the Introduction, it is equivalent to the following.

Conjecture 6.3. *Let $\langle (\mathbf{\Omega}_n)_+^{\mathfrak{S}_n} \rangle \subseteq \mathbf{\Omega}_n$ be the ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term. For any k, we have*

$$\{z^{n-k}\}\, \mathrm{grFrob}(\mathbf{\Omega}_n / \langle (\mathbf{\Omega}_n)_+^{\mathfrak{S}_n} \rangle; q, z) = \{z^{n-k}\}\, \mathrm{grFrob}(\mathbb{W}_{n,k}; q, z), \tag{6.2}$$

where q tracks x-degree and z tracks θ-degree.

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Let $k \leq n$ be positive integers, and let $\Omega_n^{(n-k)}$ be the piece of Ω_n of homogeneous θ-degree $n-k$, viewed as a free $\mathbb{Q}[x_1, \ldots, x_n]$-module of rank $\binom{n}{k}$. We define two submodules $A_{n,k}, B_{n,k} \subseteq \Omega^{(n-k)}$ in terms of generating sets as follows. Let $\mathcal{E} = \sum_{i=1}^n \frac{\partial}{\partial x_i} \theta_i$ be the Euler operator.

$$A_{n,k} = \langle e_j \cdot \theta_T \;:\; j \geq 1,\; T \subseteq [n],\; |T| = n-k \rangle + \langle \mathcal{E}(p_j) \cdot \theta_{T'} \;:\; j \geq 1,\; T' \subseteq [n],\; |T'| = n-k-1 \rangle \tag{6.3}$$

$$\begin{aligned} B_{n,k} = &\langle e_j(S) \cdot \theta_T \;:\; j \geq 1,\; S \sqcup T = [n],\; |T| = n-k \rangle + \\ &\langle \mathcal{E}(p_j) \cdot \theta_{T'} \;:\; j \geq 1,\; T' \subseteq [n],\; |T'| = n-k-1 \rangle + \langle x_i^k \cdot \theta_T \;:\; T \subseteq [n],\; |T| = n-k \rangle. \end{aligned} \tag{6.4}$$

Both of the submodules $A_{n,k}, B_{n,k} \subseteq \Omega_n^{(n-k)}$ are homogeneous under the x-grading and stable under the $\mathfrak{S}_n$-action. While their generating sets are similar, they differ as follows:

- The generators $e_j \cdot \theta_T$ of $A_{n,k}$ involve elementary symmetric polynomials in the full variable set $\{x_1, \ldots, x_n\}$, whereas the corresponding generators $e_j(S) \cdot \theta_T$ of $B_{n,k}$ involve a ‘separation of variables’ with $S \sqcup T = [n]$.
- The submodule $B_{n,k}$ has generators of the form $x_i^k \cdot \theta_T$ which are not present in $A_{n,k}$.

Proposition 6.4. *Let $n \geq k$ be positive integers, and let $\langle (\Omega_n)_+^{\mathfrak{S}_n} \rangle \subseteq \Omega_n$ be the ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term. Also let $I_{n,k} = \operatorname{ann} \delta_{n,k} \subseteq \Omega_n$ be the defining ideal of $\mathbb{W}_{n,k}$. We have*

$$\langle (\Omega_n)_+^{\mathfrak{S}_n} \rangle \cap \Omega_n^{(n-k)} = A_{n,k} \qquad I_{n,k} \cap \Omega_n^{(n-k)} = B_{n,k}. \tag{6.5}$$

Therefore, Conjecture 6.3 is equivalent to the isomorphism of graded $\mathfrak{S}_n$-modules

$$\Omega_n^{(n-k)} / A_{n,k} \cong \Omega_n^{(n-k)} / B_{n,k}. \tag{6.6}$$

Proof. The equality $\langle (\Omega_n)_+^{\mathfrak{S}_n} \rangle \cap \Omega_n^{(n-k)} = A_{n,k}$ follows from a beautiful result of Solomon [30]: the ideal $\langle (\Omega_n)_+^{\mathfrak{S}_n} \rangle \subseteq \Omega_n$ is generated by $e_1, e_2, \ldots, e_n$ together with $\mathcal{E}(p_1), \mathcal{E}(p_2), \ldots, \mathcal{E}(p_n)$. In fact, either list of polynomials $e_1, \ldots, e_n$ and $p_1, \ldots, p_n$ could be replaced by any algebraically independent set of generators of the ring $\mathbb{Q}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ of symmetric polynomials and this assertion would remain true.

The second equality $I_{n,k} \cap \Omega_n^{(n-k)} = B_{n,k}$ may be deduced as follows. From the definition of $I_{n,k}$, we see that $I_{n,k} \cap \Omega_n^{(n-k)}$ is generated by $B_{n,k}$ together with generators of the form

$$e_j(S) \cdot \theta_T \;:\; S \cup T = [n],\; |T| = n-k,\; j > |S| - k, \tag{6.7}$$

where the union $S \cup T = [n]$ need not be disjoint. If we consider such a generator $e_j(S) \cdot \theta_T$, we may write the set S as a disjoint union $S = S_1 \sqcup S_2$, where $S_1 = S - T$ and $S_2 = S \cap T$. The identity

$$e_j(S) \cdot \theta_T = e_j(S_1 \sqcup S_2) \cdot \theta_T = \sum_{a+b=j} e_a(S_1) \cdot e_b(S_2) \cdot \theta_T \tag{6.8}$$

and the assumption $j > |S| - k = |S_1| + |S_2| - k$ imply that all nonvanishing terms $e_a(S_1) \cdot e_b(S_2) \cdot \theta_T$ in this sum satisfy $a > 0$ and are, therefore, members of $B_{n,k}$. □

Proposition 6.4 gives a side-by-side comparison of the $\mathfrak{S}_n$-modules on either side of the conjecture (1.17) as explicit quotients of the same free $\mathbb{Q}[x_1, \ldots, x_n]$-module. The authors hope that this will help shed light on the ring of superspace coinvariants.

In the case of only commuting variables, the theory of *orbit harmonics* (see [25] for an exposition) has been successfully applied to discover and study a variety of combinatorial quotient rings [7, 9, 16, 17, 32] including the ring $R_{n,k}$ [13]. Conjecture 6.3 would ideally be attacked by extending orbit harmonics to the fermionic setting, with the ideals in Proposition 6.4 arising in quotient rings thereby. Such an extension would ideally incorporate the quotient rings in [5, 17] which involve fermionic variables.

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References

[1] G. Benkart, L. Colmenarejo, P. Harris, R. Orellana, G. Panova, A. Schilling and M. Yip, 'A minimaj-preserving crystal on ordered multiset partitions', *Adv. Appl. Math.* **85** (2018), 96–115.
[2] F. Bergeron, 'Multivariate diagonal coinvariant spaces for complex reflection groups', *Adv. Math.* **239** (2013), 97–108.
[3] N. Bergeron and A. Garsia, 'On certain spaces of harmonic polynomials'. In: *Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications (Tampa, FL, 1991), Contemp. Math*. **138** (Amer. Math. Soc., Providence, RI, 1992), pp. 51–86.
[4] A. Borel, 'Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compass', *Ann. of Math.* **57** (1953), 115–207.
[5] S. Billey, B. Rhoades and V. Tewari, 'Boolean product polynomials, Schur positivity, and Chern plethysm', *Int. Math. Res. Not. IMRN* **2021**(21) (2019), 16636–16670.
[6] M. D'Adderio and A. Mellit, 'A proof of the compositional Delta conjecture', *Adv. Math.* **402** (2022), 108342.
[7] A. M. Garsia and C. Procesi, 'On certain graded S_n-modules and the q-Kostka polynomials', *Adv. Math.* **94**(1) (1992), 82–138.
[8] A. M. Garsia and D. Stanton, 'Group actions on Stanley-Reisner rings and invariants of permutation groups', *Adv. Math.* **51**(2) (1984), 107–201.
[9] S. Griffin, 'Ordered set partitions, Garsia-Procesi modules, and rank varieties', *Trans. Amer. Math. Soc.* **374** (2021), 2609–2660.
[10] I. Grojnowski and M. Haiman, 'Affine Hecke algebras and positivity of LLT and Macdonald polynomials', Preprint, 2007. https://math.berkeley.edu/~mhaiman/ftp/llt-positivity/new-version.pdf.
[11] J. Haglund, M. Haiman and N. Loehr, 'A combinatorial formula for Macdonald polynomials', *J. Amer. Math. Soc.* **18** (2005), 735–761.
[12] J. Haglund, J. Remmel and A. T. Wilson, 'The Delta Conjecture', *Trans. Amer. Math. Soc.* **370** (2018), 4029–4057.
[13] J. Haglund, B. Rhoades and M. Shimozono, 'Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture', *Adv. Math.* **329** (2018), 851–915.
[14] J. Haglund, B. Rhoades and M. Shimozono, 'Hall-Littlewood expansions of Schur delta operators at $t = 0$', *Sém. Loth. Combin.* **B79c** (2019).
[15] J. Huang, B. Rhoades and T. Scrimshaw, 'Hall-Littlewood polynomials and a Hecke action on ordered set partitions', *Proc. Amer. Math. Soc.* **147** (2019), 1839–1850.
[16] B. Kostant, 'Lie group representations on polynomial rings', *Bull. Amer. Math. Soc.* **69(**4**)** (1963), 518–526.
[17] D. Kroes and B. Rhoades, 'Packed words and quotient rings', *Discrete Math.* **345**(9) (2022), 112945.
[18] A. Lascoux, B. Leclerc and J.-Y. Thibon, 'Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties', *J. Math. Phys.* **38**(2) (1997), 1041–1068.
[19] K. Meyer, 'Descent representations for generalized coinvariant algebras', *Algebraic Combin.* **3**(4) (2020), 805–830.
[20] J.-C. Novelli and J.-Y. Thibon, 'Superization and (q, t)-specialization in combinatorial Hopf algebras', *Electron. J. Combin.* **16**(2), Preprint, 2009, arXiv:0803.1816v1.
[21] B. Pawlowski and B. Rhoades, 'A flag variety for the Delta Conjecture', *Trans. Amer. Math. Soc.* **374** (2019), 8195–8248.
[22] M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory*, (Addison-Wesley, Reading, MA, 1995).
[23] J. Remmel and A. T. Wilson, 'An extension of MacMahon's Equidistribution Theorem to ordered set partitions', *J. Combin. Theory Ser. A*. **34** (2015), 242–277.
[24] B. Rhoades, 'Ordered set partition statistics and the Delta Conjecture', *J. Combin. Theory Ser. A* **154** (2018), 172–217.
[25] B. Rhoades, 'Generalizations of the flag variety tied to the Macdonald-theoretic delta operators', Chapter for *Handbook of Combinatorial Algebraic Geometry*, Preprint, 2022, arXiv:2204.03386.
[26] B. Rhoades and A. T. Wilson, 'Line configurations and r-Stirling partitions', *J. Comb.* **10**(3) (2019), 411–431.

[27] B. Rhoades and A. T. Wilson, 'Vandermondes in superspace', *Trans. Amer. Math. Soc.* **373**(6) (2020), 4483–4516.

[28] B. Rhoades, T. Yu and Z. Zhao, 'Harmonic bases for generalized coinvariant algebras', *Electron. J. Combin.* **4**(4) (2020), P4.16.

[29] R. Steinberg, 'On a theorem of Pittie', *Topology*. **14** (1975), 173–177.

[30] L. Solomon, 'Invariants of finite reflection groups', *Nagoya Math. J.* **22** (1963), 57–64.

[31] J. Swanson and N. Wallach, 'Harmonic differential forms for pseudo-reflection groups I. Semi-invariants', *J. Combin. Theory Ser. A.* **182** (2021), 105474.

[32] T. Tanisaki , 'Defining ideals of the closures of conjugacy classes and representations of the Weyl groups', *Tohoku Math. J.* **33**(4) (1982), 575–585.

[33] A. T. Wilson, 'An extension of MacMahon's Equidistribution Theorem to ordered multiset partitions', *Electron. J. Combin.* **23**(1) (2016), P1.5.

[34] M. Zabrocki, 'A module for the Delta Conjecture', Preprint, 2019, arXiv:1902.08966.