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Compactness of Commutators for Singular Integrals on Morrey Spaces

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Abstract. In this paper we characterize the compactness of the commutator $[b, T]$ for the singular integral operator on the Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$. More precisely, we prove that if $b \in \mathrm{VMO}(\mathbb{R}^n)$, the $\mathrm{BMO}(\mathbb{R}^n)$-closure of $C_c^\infty(\mathbb{R}^n)$, then $[b, T]$ is a compact operator on the Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ for $1 < p < \infty$ and $0 < \lambda < n$. Conversely, if $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $[b, T]$ is a compact operator on the $L^{p,\lambda}(\mathbb{R}^n)$ for some p $(1 < p < \infty)$, then $b \in \mathrm{VMO}(\mathbb{R}^n)$. Moreover, the boundedness of a rough singular integral operator T and its commutator $[b, T]$ on $L^{p,\lambda}(\mathbb{R}^n)$ are also given. We obtain a sufficient condition for a subset in Morrey space to be a strongly pre-compact set, which has interest in its own right.

1 Introduction

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ with the area measure $d\sigma$. Suppose that Ω satisfies the following conditions:

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n \backslash \{0\}$, *i.e.*,

$$\Omega(\mu x) = \Omega(x) \quad \text{for any } \mu > 0 \text{ and } x \in \mathbb{R}^n \backslash \{0\}. \tag{1.1}$$

(ii) Ω has mean zero on S^{n-1}, *i.e.*,

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0. \tag{1.2}$$

(iii) $\Omega \in \mathrm{Lip}(S^{n-1})$, *i.e.*,

$$|\Omega(x') - \Omega(y')| \le |x' - y'| \quad \text{for any } x', y' \in S^{n-1}. \tag{1.3}$$

Moreover, here and in the sequel, we assume that $\Omega \neq 0$. Then the Calderón–Zygmund singular integral operator T defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy. \tag{1.4}$$

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For a function $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$, let M_b be the corresponding multiplication operator defined by $M_b f = bf$ for measurable function f. Then the commutator between T and M_b is denoted by

$$[b, T] := M_b T - T M_b = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n}(b(x) - b(y)) f(y)\, dy.$$

It is well known that $[b, T]$ plays a very important role in harmonic analysis and PDEs (see the nice survey articles [8, 25]). Denote

$$\mathrm{BMO}(\mathbb{R}^n) = \{b \in L_{\mathrm{loc}}(\mathbb{R}^n) : \ \|b\|_* := \sup_{\text{cube}} Q \subset \mathbb{R}^n M(b, Q) < \infty\},$$

here and in the sequel,

$$M(b, Q) = \frac{1}{|Q|} \int_Q |b(x) - b_Q|\, dx \quad \text{and} \quad b_Q = \frac{1}{|Q|} \int_Q b(y)\, dy.$$

A famous theorem of Coifman, Rochberg, and Weiss [14] characterized the L^p-boundedness of $[b, R_j]$, where $R_j\,(j = 1, \dots, n,)$ are the Reisz transforms and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Using this characterization, the authors of [14] obtained a decomposition theorem of the real Hardy space $H^1(\mathbb{R}^n)$. Uchiyama [48] and Janson [27] showed that the Reisz transform R_j may be replaced by the Calderón–Zygmund singular integral operator T.

The boundedness result of $[b, T]$ was generalized to other contexts and important applications to some non-linear PDEs were given by Coifman et al. [13]. The characterization of L^p-compactness of $[b, T]$ was obtained by Uchiyama [48]. The interest in the compactness of $[b, T]$ in complex analysis is from the connection between the commutators and the Hankel-type operators. In fact, Beatrous and Li [6] proved the boundedness and compactness characterizations of $[b, T]$ on L^p over some spaces of homogeneous type. Krantz and Li (see [29]) applied the characterization of L^p-compactness of the commutator to give a compactness characterization of Hankel operators on holomorphic Hardy spaces $H^2(D)$, where D is a bounded, strictly pseudoconvex domain in $\mathbb{C}^n$. On the other hand, it is perhaps for this important reason that the L^p-compactness of $[b, T]$ attracted attention among researchers in PDEs. For example, with the aid of the compactness of $[b, T]$, it is easy to derive a Fredholm alternative for equations with VMO coefficients in all L^p spaces for $1 < p < \infty$ (see [26]).

It is well known that the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ (see the definition below), introduced by Morrey in 1938, is connected to certain problems in elliptic PDE [32]. Later, the Morrey spaces were found to have many important applications to the Navier–Stokes equations (see [28, 31, 47]), the Schrödinger equations (see [36, 37, 42, 43]), the elliptic equations with discontinuous coefficients (see [7, 12, 18, 20, 24, 35]) and the potential analysis (see [1, 2]). The Morrey space associated with the heat kernel was studied in [15, 21, 49]. Recently, in [3, 4], the authors set up several functional analyses and potential theory for the Morrey spaces in harmonic analysis.

For $1 \leq p < \infty$, $n \geq 1$ and $0 < \lambda < n$, the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \Big\{ f \in L^p_{\mathrm{loc}} : \|f\|_{p,\lambda} = \sup_{\substack{y\in\mathbb{R}^n \\ r>0}} \Big(\frac{1}{r^\lambda} \int_{B(y,r)} |f(x)|^p \, dx \Big)^{1/p} < \infty \Big\},$$

where $B(y, r)$ denotes the ball centered at y and with radius $r > 0$. The space $L^{p,\lambda}(\mathbb{R}^n)$ becomes a Banach space with norm $\|\cdot\|_{p,\lambda}$. Moreover, for $\lambda = 0$ and $\lambda = n$, the Morrey spaces $L^{p,0}(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n)$ coincide (with equality of norms) with the spaces $L^p(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$, respectively. (See also [38,40,41] for the theory of Morrey spaces with non-doubling measures.)

In 1991, Di Fazio and Ragusa [19] gave a characterization of $L^{p,\lambda}$-boundedness of $[b, T]$ with Ω satisfying (1.2)–(1.3). In 1997, using Janson's idea [27], Ding [16] proved that the commutator $[b, T]$ is a bounded operator on the generalized Morrey space $L^{p,\Phi}(\mathbb{R}^n)$ $(1 < p < \infty)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$ (see [16, 33] for the definition of $L^{p,\Phi}(\mathbb{R}^n)$). Recently, Adams and Xiao [5] gave a new proof about the Morrey spaces boundedness for the commutator of the Riesz potential and developed a regularity theory of commutators for Morrey–Sobolev spaces $I_\alpha(L^{p,\lambda})$.

Like the case on $L^p(\mathbb{R}^n)$, the characterizations of boundedness and compactness of $[b, T]$ on Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ play an important role in PDEs. In fact, the boundedness and compactness of the commutator $[b, T]$ on Morrey spaces had been applied to discuss some regularity problems of solutions of PDEs with VMO coefficients (see [12, 18, 20, 35, 44], for example).

Therefore, it is natural to ask what is the characterization of $L^{p,\lambda}$-compactness of $[b, T]$? The purpose of this paper is to answer this question. In order to compare the results of ours with whose obtained by Uchiyama, let us recall what Uchiyama obtained.

Theorem A ([48]) *Suppose that* Ω *satisfies* (1.1), (1.2), *and* (1.3)

(i) *If* $b \in \mathrm{VMO}(\mathbb{R}^n)$, *then* $[b, T]$ *is compact on* $L^p(\mathbb{R}^n)$ *for all* $1 < p < \infty$.
(ii) *If* $[b, T]$ *is a compact operator on* $L^p(\mathbb{R}^n)$ *for some* p, $1 < p < \infty$, *then* $b \in \mathrm{VMO}(\mathbb{R}^n)$.

Here $\mathrm{VMO}(\mathbb{R}^n)$ *denotes the* BMO-*closure of* $C_c^\infty(\mathbb{R}^n)$, *and* $C_c^\infty(\mathbb{R}^n)$ *is the set of* $C^\infty(\mathbb{R}^n)$ *functions with compact support set.*

On the other hand, recently, we also gave a characterization of compactness for the commutators of Riesz potential on Morrey spaces [11].

Now let us formulate the main results in the present paper as follows.

Theorem 1.1 *Let* $0 < \lambda < n$. *Suppose that* Ω *satisfies* (1.1), (1.2), *and* $\Omega \in L^q(S^{n-1})$ *with* $q > n/(n - \lambda)$ *satisfying*

$$\int_0^1 \frac{\omega_q(\delta)}{\delta} (1 + |\log \delta|) \, d\delta < \infty, \tag{1.5}$$

where $\omega_q(\delta)$ *denotes the integral modulus of continuity of order* q *of* Ω *defined by*

$$\omega_q(\delta) = \sup_{\|\rho\|<\delta} \Big(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^q \, d\sigma(x') \Big)^{1/q}$$

and ρ is a rotation in $\mathbb{R}^n$ and $\|\rho\| = \sup_{x' \in S^{n-1}} |\rho x' - x'|$. If $b \in \mathrm{VMO}(\mathbb{R}^n)$, then the commutator $[b, T]$ is a compact operator on $L^{p,\lambda}(\mathbb{R}^n)$ for $1 < p < \infty$.

Theorem 1.2 *Suppose that Ω satisfies* (1.1), (1.2), *and $\Omega \in \mathrm{Lip}(S^{n-1})$. If $0 < \lambda < n$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, and the commutator $[b, T]$ is a compact operator from $L^{p,\lambda}(\mathbb{R}^n)$ to itself for some p $(1 < p < \infty)$, then $b \in \mathrm{VMO}(\mathbb{R}^n)$.*

Remark 1.3 The conclusion of Theorem 1.2 for $\lambda = 0$ is just Uchiyama's main result in [48]. On the other hand, since the Lipschitz condition (1.3) implies (1.5), we may get the following corollary immediately.

Corolllary 1.4 *Suppose that Ω satisfies* (1.1), (1.2), *and $\Omega \in \mathrm{Lip}(S^{n-1})$. If $0 < \lambda < n$, $1 < p < \infty$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$, then the commutator $[b, T]$ is a compact operator on $L^{p,\lambda}(\mathbb{R}^n)$ if and only if $b \in \mathrm{VMO}(\mathbb{R}^n)$.*

Remark 1.5 If the Lipschitz condition is replaced by the weaker condition, which is the so-called *Hölder condition of log type*:

$$|\Omega(x') - \Omega(y')| \leq \frac{C_1}{\left(\log \frac{2}{|x'-y'|}\right)^{\gamma}} \qquad \text{for any } x', y' \in S^{n-1}, C_1 > 0, \gamma > 1,$$

then Theorem 1.2 and Corollary 1.4 also hold.

Remark 1.6 Recently, Sawano and Shirai [39] proved that if T is bounded on $L^2(\mu)$ and its kernel K satisfies a stronger smoothness condition, then the commutator $[a, T]$ with $a \in BVMO(\mu)$ is a compact operator on the Morrey spaces with non-doubling measures. However, the conditions assumed on the kernel of operator T in [39] are even stronger than condition (1.5). Therefore, in this sense, the conclusion of Theorem 1.1 is an improvement of Theorem 1.6 in [39].

Remark 1.7 In the review of paper [39] in Mathematical Reviews (MR2428477) the reviewer suggested that "It is worthwhile to know how much this sufficient condition is close to being necessary." Our Theorem 1.2 settles this question.

Note that condition (1.5) is weaker than the Lipschitz condition (1.3). Hence, we cannot apply the conclusions of [19] in the proofs of Theorems 1.1 and 1.2. Here we will give the boundedness of a general linear or sublinear operator S and its commutator $[b, S]$ on the Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$, where $[b, S]$ is defined by $[b, S]f(x) = b(x)Sf(x) - S(bf)(x)$ for $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$. The following results have interest in their own right.

Theorem 1.8 *Let $0 < \lambda < n$. Suppose that Ω satisfies* (1.1) *and $\Omega \in L^q(S^{n-1})$ for $q > n/(n - \lambda)$ and S is a linear or sublinear operator satisfying*

$$|Sf(x)| \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)|\, dy. \tag{1.6}$$

(i) *If the operator S is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, then S is also bounded on $L^{p,\lambda}(\mathbb{R}^n)$.*

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(ii) *For $b \in \mathrm{BMO}(\mathbb{R}^n)$, if the commutator $[b, S]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, then $[b, S]$ is also bounded on $L^{p,\lambda}(\mathbb{R}^n)$.*

Note that the Calderón–Zygmund singular integral operator T defined by (1.4) satisfies (1.6). We then immediately get the $L^{p,\lambda}(\mathbb{R}^n)$-boundedness of T and $[b, T]$ by applying the $L^p(\mathbb{R}^n)$-boundedness of T (see [9]) and the $L^p(\mathbb{R}^n)$-boundedness of $[b, T]$ (see [23]), respectively.

Corolllary 1.9 *Let $0 < \lambda < n$. Suppose Ω satisfies (1.1), (1.2), and $\Omega \in L^q(S^{n-1})$ for $q > n/(n - \lambda)$. Then the Calderón–Zygmund singular integral operator T defined by (1.4) and its commutator $[b, T]$ with $b \in \mathrm{BMO}(\mathbb{R}^n)$ are both bounded on $L^{p,\lambda}(\mathbb{R}^n)$ for $1 < p < \infty$.*

Remark 1.10 Obviously, in the conditions of Corollary 1.9 Ω has no any smoothness on the unit sphere S^{n-1}. Therefore, Corollary 1.9 is an improvement of the result in [19].

Remark 1.11 Besides the Calderón–Zygmund singular integral operator, condition (1.6) is satisfied by many interesting operators in harmonic analysis, such as the oscillatory singular integral, the Hardy–Littlewood maximal operator, Carleson's maximal operators and so on. Similar to Corollary 1.9, as some consequences of Theorem 1.8, we may also discuss and obtain the boundedness of these operators mentioned above and their commutators on the Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$.

In the proof of Theorem 1.1, we need the following characterization that a subset in $L^{p,\lambda}(\mathbb{R}^n)$ is a strongly pre-compact set, which is in itself interesting.

Theorem 1.12 *Suppose that $1 \le p < \infty$ and $0 < \lambda < n$. Suppose the subset $\mathcal{G}$ in $L^{p,\lambda}(\mathbb{R}^n)$ satisfies the following conditions:*

(i) *norm boundedness uniformly*

$$\sup_{f \in \mathcal{G}} \|f\|_{p,\lambda} < \infty, \tag{1.7}$$

(ii) *translation continuty uniformly*

$$\lim_{y \to 0} \|f(\cdot + y) - f(\cdot)\|_{p,\lambda} = 0 \quad \textit{uniformly in } f \in \mathcal{G}, \tag{1.8}$$

(iii) *control uniformly away from the origin*

$$\lim_{\alpha \to \infty} \|f \chi_{E_\alpha}\|_{p,\lambda} = 0 \quad \textit{uniformly in } f \in \mathcal{G}, \tag{1.9}$$

where $E_\alpha = \{x \in \mathbb{R}^n : |x| > \alpha\}$. Then $\mathcal{G}$ is strongly pre-compact set in $L^{p,\lambda}(\mathbb{R}^n)$.

Remark 1.13 In the results above, we discuss only the case where $0 < \lambda < n$. As for the case $\lambda = 0$, since $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, some results are well known. In fact, recently, Chen and Ding proved that the commutator $[b, T]$ is a compact operator on $L^p(\mathbb{R}^n)$ if $b \in VMO(\mathbb{R}^n)$ and Ω satisfies (1.1), (1.2), and (1.5) (see [10, Theorem 1.2]). If $\lambda = 0$, then the conclusion of Corollary 1.4 is just Uchiyama's result [48]. Finally, when $\lambda = 0$, Theorem 1.12 is just the famous Frechet–Kolmogorov theorem (see [50]). Therefore, our results obtained in this paper extend some well-known results.

This paper is organized as follows. We prove the main results, Theorem 1.1 and Theorem 1.2, in Sections 2 and 3, respectively. Then in Section 4, we show the $L^{p,\lambda}$-boundedness of the rough operators and its commutators (Theorem 1.8). In the last section, we characterize the strongly pre-compact set in $L^{p,\lambda}(\mathbb{R}^n)$ (Theorem 1.12). Throughout this paper the letter "C" will stand for a positive constant which is independent of the essential variables and not necessarily the same one in each occurrence. As usual, $|E|$ denotes the Lebesgue measure of a measurable set E in $\mathbb{R}^n$ and for $p \geq 1$, $p' = p/(p-1)$ denotes the dual exponent of p.

2 Sufficiency That $[b, T]$ Is a Compact Operator on $L^{p,\lambda}(\mathbb{R}^n)$: Proof of Theorem 1.1

Let us begin by giving two lemmas which will be used in the proof of Theorem 1.1.

Lemma 2.1 *Let $0 < \lambda < n$. Suppose that Ω satisfies* (1.1), (1.2), *and $\Omega \in L^q(S^{n-1})$, where $q > n/(n-\lambda)$. For $\eta > 0$, let*

$$T_\eta f(x) = \int_{|x-y|>\eta} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy.$$

Then for $1 < p < \infty$, $\|T_\eta f\|_{p,\lambda} \leq C\|f\|_{p,\lambda}$, where C is independent of η and f.

Lemma 2.1 is a direct consequence of Theorem 1.8. In fact, the inequality

$$|T_\eta f(x)| \leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy$$

holds uniformly in η. Moreover, T_η is bounded on $L^p(\mathbb{R}^n)$ uniformly in η (see [45]). We invoke the following estimate from [17].

Lemma 2.2 *Suppose that $0 \leq \beta < n$, Ω satisfies* (1.1), *and $\Omega \in L^q(S^{n-1})$, $q \geq 1$. Then there exists a $C > 0$ such that for an $R > 0$ and $x \in \mathbb{R}^n$ with $|x| < R/2$,*

$$\Big(\int_{R<|y|<2R} \Big| \frac{\Omega(y-x)}{|y-x|^{n-\beta}} - \frac{\Omega(y)}{|y|^{n-\beta}} \Big|^q dy\Big)^{1/q} \leq CR^{n/q-(n-\beta)} \Big\{ \frac{|x|}{R} + \int_{|x|/2R}^{|x|/R} \frac{\omega_q(\delta)}{\delta}\, d\delta \Big\}.$$

We now turn to the proof of Theorem 1.1. Without loss of generality, let $\mathcal{F}$ be the unit ball in $L^{p,\lambda}(\mathbb{R}^n)$. By density, we only need to prove that when $b \in C_c^\infty(\mathbb{R}^n)$, the set $\mathcal{G} = \{[b, T]f : f \in \mathcal{F}\}$ is a strongly pre-compact in $L^{p,\lambda}(\mathbb{R}^n)$. Once we accept Theorem 1.12, it is sufficient to show that (1.7)–(1.9) hold uniformly in $\mathcal{G}$.

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Notice that $b \in C_c^\infty(\mathbb{R}^n)$. Applying Corollary 1.9, we have

$$\sup_{f\in\mathcal{F}} \|[b,T]f\|_{p,\lambda} \le C\|b\|_* \sup_{f\in\mathcal{F}} \|f\|_{p,\lambda} \le C\|b\|_* < \infty.$$

On the other hand, suppose that $\beta > 1$ taken so large that $\operatorname{supp} b \subset \{x : |x| \le \beta\}$. Recall that $q > n/(n-\lambda)$, for any $0 < \varepsilon < 1$, we take $\alpha > \beta$ such that $(\alpha-\beta)^{n(1-q)} < \varepsilon^q$. Below we show that for every $t \in \mathbb{R}^n$ and $r > 0$, $q > 1$,

$$\Big\{ \frac{1}{r^\lambda} \int_{B(t,r)} |[b,T]f|^p \chi_{E_\alpha}(x)\,dx \Big\}^{1/p} < C\varepsilon \|\Omega\|_{L^q(S^{n-1})}. \tag{2.1}$$

In fact, for any $x \in E_\alpha = \{x \in \mathbb{R}^n : |x| > \alpha\}$ and every $f \in \mathcal{F}$, without loss of generality, we may assume $q < p$. Then we have

$$\begin{aligned}
|[b,T]f(x)| &= \Big| \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} (b(x) - b(y)) f(y)\,dy \Big| \\
&\le C\|b\|_\infty \int_{|y|\le\beta} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)|\,dy \\
&\le C\Big(\int_{|x-y|\le\beta} \frac{|\Omega(y)|^q}{|y|^{nq}} |f(x-y)|^q\,dy \Big)^{1/q}.
\end{aligned}$$

Then for every $t \in \mathbb{R}^n$ and $r > 0$, by the Minkowski inequality and the choice of α, we get

$$\begin{aligned}
&\Big\{ \frac{1}{r^\lambda} \int_{B(t,r)} |[b,T]f|^p \chi_{E_\alpha}(x)\,dx \Big\}^{1/p} \\
&\qquad \le C\Big\{ \frac{1}{r^\lambda} \int_{B(t,r)} \Big(\int_{|x-y|\le\beta} \frac{|\Omega(y)|^q}{|y|^{nq}} |f(x-y)|^q d\,y \Big)^{p/q} \chi_{E_\alpha}(x)\,dx \Big\}^{1/p} \\
&\qquad \le C\|f\|_{p,\lambda} \Big\{ \int_{|y|>\alpha-\beta} \frac{|\Omega(y)|^q}{|y|^{nq}}\,dy \Big\}^{1/q} \\
&\qquad < C\varepsilon \|\Omega\|_{L^q(S^{n-1})} \|f\|_{p,\lambda} \\
&\qquad \le C\varepsilon \|\Omega\|_{L^q(S^{n-1})}.
\end{aligned}$$

Thus, we get (2.1), which shows that (1.9) holds for $[b,T]$ in $\mathcal{G}$ uniformly. Finally, to finish the proof of Theorem 1.1, it remains to show that the translation continuity condition (1.8) holds for the commutator $[b,T]$ in $\mathcal{G}$ uniformly. We need to prove that for any $0 < \varepsilon < 1/2$, if $|z|$ is sufficiently small depending only on ε, then for every $f \in \mathcal{F}$,

$$\|[b,T]f(\cdot) - [b,T]f(\cdot + z)\|_{p,\lambda} \le C\varepsilon.$$

Then for $z \in \mathbb{R}^n$ we write

$$\begin{aligned}
&[b,T]f(x+z)-[b,T]f(x)\\
&\quad=\int_{|x-y|>e^{1/\varepsilon}|z|}\frac{\Omega(x-y)}{|x-y|^n}\,[b(x+z)-b(x)]f(y)\,dy\\
&\qquad+\int_{|x-y|>e^{1/\varepsilon}|z|}\Big(\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x+z-y)}{|x+z-y|^n}\Big)[b(y)-b(x+z)]f(y)\,dy\\
&\qquad+\int_{|x-y|\le e^{1/\varepsilon}|z|}\frac{\Omega(x-y)}{|x-y|^n}\,[b(y)-b(x)]f(y)\,dy\\
&\qquad-\int_{|x-y|\le e^{1/\varepsilon}|z|}\frac{\Omega(x+z-y)}{|x+z-y|^n}[b(y)-b(x+z)]f(y)\,dy\\
&\quad:=\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3-\mathrm{J}_4\,.
\end{aligned}$$

Since $b \in C_c^\infty(\mathbb{R}^n)$, we have $|b(x)-b(x+z)| \le C\|\nabla b\|_\infty|z|$. Now $\Omega \in L^q(S^{n-1})$ and $q > n/(n-\lambda)$, hence, applying Lemma 2.1, we get

$$\|\,\mathrm{J}_1\,\|_{p,\lambda} \le C|z|\|f\|_{p,\lambda} < C|z|. \tag{2.2}$$

As for J_2, for every $t \in \mathbb{R}^n$ and $r > 0$, using Lemma 2.2 and the Minkowski inequality, we get

$$\begin{aligned}
&\Big(\frac{1}{r^\lambda}\int_{B(t,r)}|\,\mathrm{J}_2\,|^p\,dx\Big)^{1/p}\\
&\quad\le 2\|b\|_\infty\Big(\frac{1}{r^\lambda}\int_{B(t,r)}\Big(\int_{|y|>e^{1/\varepsilon}|z|}|f(x-y)|\Big|\frac{\Omega(y)}{|y|^n}-\frac{\Omega(y+z)}{|y+z|^n}\Big|\,dy\Big)^p dx\Big)^{1/p}\\
&\quad\le C\|f\|_{p,\lambda}\int_{|y|>e^{1/\varepsilon}|z|}\Big|\frac{\Omega(y)}{|y|^n}-\frac{\Omega(y+z)}{|y+z|^n}\Big|\,dy\\
&\quad\le C\|f\|_{p,\lambda}\sum_{k=0}^{\infty}\Big\{\frac{|z|}{2^k e^{1/\varepsilon}|z|}+\int_{\frac{|z|}{2^{k+1}e^{1/\varepsilon}|z|}}^{\frac{|z|}{2^k e^{1/\varepsilon}|z|}}\frac{\omega(\delta)}{\delta}\,\mathrm{d}\delta\Big\}\\
&\quad\le C\|f\|_{p,\lambda}\sum_{k=0}^{\infty}\Big\{\frac{1}{2^k e^{1/\varepsilon}}+\frac{1}{1+k+1/\varepsilon}\int_{\frac{1}{2^{k+1}e^{1/\varepsilon}}}^{\frac{1}{2^k e^{1/\varepsilon}}}\frac{\omega(\delta)}{\delta}(1+|\log\delta|)\,\mathrm{d}\delta\Big\}\\
&\quad\le C(e^{-1/\varepsilon}+\varepsilon)\|f\|_{p,\lambda}\le C\varepsilon.
\end{aligned}$$

Thus, we have

$$\|\,\mathrm{J}_2\,\|_{p,\lambda} \le C\varepsilon. \tag{2.3}$$

Regarding J_3, we have $|b(x)-b(y)| \le C\|\nabla b\|_\infty|x-y|$ by $b \in C_c^\infty(\mathbb{R}^n)$. Thus

$$|\,\mathrm{J}_3\,| \le C\int_{|x-y|\le e^{1/\varepsilon}|z|}|\Omega(x-y)||x-y|^{-n+1}|f(y)|\,dy.$$

By the Minkowski inequality, for every $t \in \mathbb{R}^n$ and $r > 0$, we have

$$\Big\{ \frac{1}{r^\lambda} \int_{B(t,r)} |\,\mathrm{J}_3\,|^p\, dx \Big\}^{1/p} \le C \int_{|y| \le e^{1/\varepsilon}|z|} |\Omega(y)||y|^{-n+1}\, dy \Big(\frac{1}{r^\lambda} \int_{B(t-y,r)} |f(x)|^p\, dx \Big)^{1/p} \le C e^{1/\varepsilon} |z| \|f\|_{p,\lambda}.$$

Thus

$$\|\,\mathrm{J}_3\,\|_{p,\lambda} < C e^{1/\varepsilon}|z|. \tag{2.4}$$

Finally, by $|b(x+z) - b(y)| \le C\|\nabla b\|_\infty |x+z-y|$ we have

$$|\,\mathrm{J}_4\,| \le C \int_{|x-y| \le e^{1/\varepsilon}|z|} |\Omega(x+z-y)||x+z-y|^{-n+1}|f(y)|\, dy.$$

Using a similar method, it is easy to check that

$$\|\,\mathrm{J}_4\,\|_{p,\lambda} < C(e^{1/\varepsilon}|z| + |z|). \tag{2.5}$$

From (2.2), (2.3), (2.4), and (2.5), and taking $|z|$ to be sufficiently small, we can get

$$\|[b,T]f(\cdot) - [b,T]f(\cdot + z)\|_{p,\lambda} \le \|\,\mathrm{J}_1\,\|_{p,\lambda} + \|\,\mathrm{J}_2\,\|_{p,\lambda} + \|\,\mathrm{J}_3\,\|_{p,\lambda} + \|\,\mathrm{J}_4\,\|_{p,\lambda} \le C\varepsilon.$$

Therefore, we show that the translation continuity (1.8) holds for the commutator $[b,T]$ in $\mathcal{G}$ uniformly and this completes the proof of Theorem 1.1.

3 Necessity that $[b,T]$ Is a Compact Operator on $L^{p,\lambda}(\mathbb{R}^n)$: Proof of Theorem 1.2

We first recall some known facts.

Lemma 3.1 ([45]) *If $b \in \mathrm{BMO}(\mathbb{R}^n)$, $C_2 > C_1 > 2$, Q is a cube centered at x_0 and of diameter q, then there exist positive constants C_3, C_4, C_5 (depending on C_1, C_2 and b), such that*

$$\big|\{C_1 q < |x - x_0| < C_2 q : |b(x) - b_Q| > \nu + C_3\}\big| \le C_4 |Q| e^{-C_5 \nu} \quad (0 < \nu < \infty).$$

Lemma 3.2 ([46]) *Suppose that $f(x)$ is a measurable function on $\mathbb{R}^n$. Denote $\lambda_f(s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}|$ for $s > 0$ and $f^*(t) = \inf\{s : \lambda_f(s) \le t\}$ for $t > 0$. Then for any measurable set E and $1 \le p < \infty$,*

$$\int_E |f(x)|^p\, dx \le \int_0^{|E|} |f^*(t)|^p\, dt.$$

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Lemma 3.3 ([48]) *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then $b \in \mathrm{VMO}(\mathbb{R}^n)$ if and only if b satisfies the following three conditions:*

(i) $\lim_{a\to 0} \sup_{|Q|=a} M(b, Q) = 0$;
(ii) $\lim_{a\to\infty} \sup_{|Q|=a} M(b, Q) = 0$;
(iii) $\lim_{|x|\to\infty} M(b, Q + x) = 0$ *for each* Q.

To prove Theorem 1.2, we need the following result.

Lemma 3.4 *Suppose that $b \in \mathrm{BMO}(\mathbb{R}^n)$ with $\|b\|_* = 1$. If for some $0 < \zeta < 1$ and a cube Q with its center at c_Q and radius $\ell(Q)$, b is not a constant on cube Q and satisfies*

$$M(b, Q) = \frac{1}{|Q|}\int_Q |b(y) - b_Q|\, dy > \zeta, \tag{3.1}$$

then for the function f_Q defined by

$$f_Q = \ell(Q)^{(\lambda-n)/p}\Big(\operatorname{sgn}(b - b_Q) - \frac{1}{|Q|}\int_Q \operatorname{sgn}(b - b_Q)\Big)\chi_Q, \tag{3.2}$$

there exist constants $\beta, \gamma_1, \gamma_2$, and γ_3 satisfying $\gamma_2 > \gamma_1 > 2 > \beta > 0$ and $\gamma_3 > 0$, such that

$$\int_{\gamma_1\ell(Q)<|x-c_Q|<\gamma_2\ell(Q)} |[b, T]f_Q(y)|^p\, dy \ge \gamma_3^p \ell(Q)^\lambda, \tag{3.3}$$

$$\int_{|x-c_Q|>\gamma_2\ell(Q)} |[b, T]f_Q(y)|^p\, dy \le \frac{\gamma_3^p}{4^p}\ell(Q)^\lambda. \tag{3.4}$$

Moreover, for all measurable subsets $E \subset \{x : \gamma_1\ell(Q) < |x - c_Q| < \gamma_2\ell(Q)\}$, satisfying $|E|/|Q| < \beta^n$

$$\int_E |[b, T]f_Q(y)|^p dy \le \frac{\gamma_3^p}{4^p}\ell(Q)^\lambda. \tag{3.5}$$

Proof Denote $\alpha_0 = |Q|^{-1}\int_Q \operatorname{sgn}(b(y) - b_Q)dy$. Since $\int_Q (b(y) - b_Q)\, dy = 0$. It is easy to check that $|\alpha_0| < 1$ and f_Q satisfies

$$f_Q(y)(b(y) - b_Q) > 0, \tag{3.6}$$

$$\int_{\mathbb{R}^n} f_Q(y)dy = 0, \tag{3.7}$$

$$|f_Q(y)| \le 2|Q|^{-(n-\lambda)/(np)}, \quad \text{for } y \in Q. \tag{3.8}$$

Moreover, for any $t \in \mathbb{R}^n$,

$$\Big(\frac{1}{r^\lambda}\int_{B(t,r)} |f_Q(x)|^p\, dx\Big)^{1/p} \le \begin{cases} C\Big(\dfrac{r}{\ell(Q)}\Big)^{(n-\lambda)/p} \le C & 0 < r \le \ell(Q), \\ \Big(\dfrac{1}{r^\lambda}\displaystyle\int_Q |f_Q(x)|^p\, dx\Big)^{1/p} \le C\Big(\dfrac{\ell(Q)}{r}\Big)^{\lambda/p} \le C & r > \ell(Q) > 0. \end{cases} \tag{3.9}$$

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Thus, $\|f_Q\|_{L^{p,\lambda}} \le C$, where C is independent of r and t.

First, we prove (3.3) and (3.4). For $i = 1, 2$, A_i denotes the positive constant depending only on $\Omega, p, n, \lambda, \zeta$ and $A_k (1 \le k < i)$. Since Ω satisfies (1.2) (noting that $\Omega \ne 0$), there exists an A_1 such that $0 < A_1 < 1$ and

$$\sigma(\{x' \in S^{n-1} : \Omega(x') \ge 2A_1\}) > 0.$$

By condition (1.3), it is easy to see that $\Lambda := \{x' \in S^{n-1} : \Omega(x') \ge 2A_1\}$ is a closed set.

Claim 3.5 *If $x' \in \Lambda$ and $y' \in S^{n-1}$ satisfy $|x' - y'| \le A_1$, then $\Omega(y') \ge A_1$.*

In fact, since $|\Omega(x') - \Omega(y')| \le |x' - y'| \le A_1$ and $\Omega(x') \ge 2A_1$, we therefore get $\Omega(y') \ge A_1$ and Claim 3.5 is justified. Taking $A_2 > 2/A_1$, if $y \in Q$, then we have $|x - c_Q| > A_2 |y - c_Q|$ for $x \in (A_2 Q)^c \cap \{x : (x - c_Q)' \in \Lambda\}$. Thus

$$|(x - c_Q)' - (x - y)'| \le \frac{2|y - c_Q|}{|x - c_Q|} \le \frac{2}{A_2} < A_1.$$

Applying Claim 3.5, we get $\Omega((x - y)') \ge A_1$. Thus, for $x \in (A_2 Q)^c \cap \{x : (x - c_Q)' \in \Lambda\}$, by (3.1), (3.2), and (3.6), and noting that $|x - c_Q| \simeq |x - y|$, we have

$$
\begin{aligned}
(3.10)\quad & |T((b - b_Q) f_Q)(x)| \\
&= |Q|^{-1/p+\lambda/(np)} \int_Q \frac{\Omega(x - y)}{|x - y|^n} (b(y) - b_Q)[\operatorname{sgn}(b(y) - b_Q) - \alpha_0]\, dy \\
&\ge C |Q|^{-1/p+\lambda/(np)} |x - c_Q|^{-n} \int_Q (|b(y) - b_Q| - \alpha_0 (b(y) - b_Q))\, dy \\
&= C |Q|^{-1/p+\lambda/(np)} |x - c_Q|^{-n} \int_Q |b(y) - b_Q|\, dy \\
&\ge C \zeta |Q|^{1/p' + \lambda/(np)} |x - c_Q|^{-n}.
\end{aligned}
$$

On the other hand, for $x \in (A_2 Q)^c$, by $\Omega \in L^\infty(S^{n-1})$, (3.2), and (3.8), it is easy to check that

$$(3.11) \qquad |T((b - b_Q) f_Q)(x)| \le C |Q|^{1/p' + \lambda/(np)} |x - c_Q|^{-n}.$$

By (3.7) we have

$$
\begin{aligned}
(3.12)\quad & |(b(x) - b_Q) T(f_Q)(x)| \le |b(x) - b_Q| \left| \int_{\mathbb{R}^n} f_Q(y) \Big(\frac{\Omega(x - y)}{|x - y|^n} - \frac{\Omega(x - c_Q)}{|x - c_Q|^n} \Big)\, dy \right| \\
&\le C \frac{\ell(Q) |b(x) - b_Q| |Q|^{1/p' + \lambda/(np)}}{|x - c_Q|^{n+1}}.
\end{aligned}
$$

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Note that the constants appearing in (3.10)∼(3.12) are only dependent on n, p, and b. Since $|b_{2Q} - b_Q| \le C\|b\|_* = C$, we have

$$\Big(\int_{2^s\ell(Q)<|x-y_j|<2^{s+1}\ell(Q)} |b(x) - b_Q|^p\,dx\Big)^{1/p} \le Cs2^{sn/p}|Q|^{1/p}.$$

Taking $v > \max\{A_1, 16\}$, by (3.12) we obtain

$$\begin{aligned}
(3.13)\qquad \Big(\int_{|x-c_Q|>v\ell(Q)} &|(b(x) - b_Q)T(f_Q)(x)|^p\,dx\Big)^{\frac{1}{p}} \\
&\le C|Q|^{1/p\,'+\lambda/(np)}\ell(Q) \\
&\quad\times \sum_{s=[\log_2 v]}^{\infty} \Big(\int_{2^s\ell(Q)<|x-c_Q|<2^{s+1}\ell(Q)} \frac{|b(x) - b_Q|^p}{|x-y_j|^{p(n+1)}}\,dx\Big)^{\frac{1}{p}} \\
&\le C|Q|^{\lambda/(np)} \sum_{s=[\log_2 v]}^{\infty} 2^{-s(n-n/p+1-1/2)} \\
&\le C|Q|^{\lambda/(np)} v^{-(n-n/p+1/2)}.
\end{aligned}$$

Then for $u > v > \max\{A_1, 16\}$, using (3.10) and (3.13), we get

$$\begin{aligned}
(3.14)\quad \Big(\int_{\{v\ell(Q)<|x-c_Q|<u\ell(Q)\}} &|[b,T]f_Q(x)|^p\,dx\Big)^{\frac{1}{p}} \\
&\ge \Big(\int_{\{v\ell(Q)<|x-y_j|<u\ell(Q)\}} |T((b-b_Q)f_Q)(x)|^p\,dx\Big)^{\frac{1}{p}} \\
&\quad - \Big(\int_{|x-y_j|>v\ell(Q)} |(b(x) - b_Q)T(f_Q)(x)|^p\,dx\Big)^{\frac{1}{p}} \\
&\ge C\zeta|Q|^{\lambda/(np)}(v^{-np+n} - u^{-np+n})^{1/p} \\
&\quad - C|Q|^{\lambda/(np)} v^{(-n+n/p-1/2)}.
\end{aligned}$$

Similarly, from (3.11) and (3.13), we have

$$\begin{aligned}
(3.15)\quad \Big(\int_{|x-c_Q|>u\ell(Q)} |[b,T]f_j(x)|^p\,dx\Big)^{\frac{1}{p}} &\le C|Q|^{\lambda/(np)}u^{(n-np)/p} \\
&\quad + C|Q|^{\lambda/(np)}u^{(n/p-n-1/2)}.
\end{aligned}$$

Once again, the constants appearing in (3.13)∼(3.15) are independent of f_Q and Q. Since $n/p < n$, by (3.14) and (3.15), it is easy to see that there exist constants $\gamma_2 > \gamma_1 > 2$ and $\gamma_3 > 0$, which are dependent only on p, n, ζ, λ, and b, such that (3.3) and (3.4) hold for any f_Q and Q.

We now verify (3.5). Let $E \subset \{x : \gamma_1 \ell(Q) < |x - c_Q| < \gamma_2 \ell(Q)\}$ be an arbitrary measurable set. Then by (3.11), (3.12), and the Minkowski inequality, we have

$$
\begin{aligned}
\Big(\int_E |[b,T] f_Q(x)|^p dx\Big)^{1/p} &\le C|Q|^{1/p'+\lambda/(np)}\Big(\int_E |x-c_Q|^{-p(n)}\,dx\Big)^{1/p} \\
&\quad + C\ell(Q)|Q|^{1/p'+\lambda/(np)}\Big(\int_E \frac{|b(x)-b_Q|^p}{|x-c_Q|^{p(n+1)}}\,dx\Big)^{1/p} \\
&\le C|Q|^{\lambda/(np)}\Big\{\frac{|E|^{1/p}}{|Q|^{1/p}} + \Big(\frac{1}{|Q|}\int_E |b(x)-b_Q|^p\,dx\Big)^{1/p}\Big\}.
\end{aligned} \tag{3.16}
$$

Let $h_Q(x) = b(x) - b_Q$. For $0 < \omega < \infty$, denote

$$
\lambda_{h_Q}(\omega) = |\{x : \gamma_1\ell(Q) < |x-c_Q| < \gamma_2\ell(Q) \text{ and } |h_Q(x)| > \omega\}|.
$$

Then by Lemma 3.1, there exist positive constants C_3, C_4, C_5 (dependent on γ_1, γ_2, and b only) such that $\lambda_{h_Q}(\omega + C_3) \le C_4|Q|e^{-C_5\omega}$. Hence, $\lambda_{h_Q}(\omega) \le C_4|Q|e^{-C_5(\omega - C_3)}$. For $t > 0$, let $h_Q^*(t) = \inf\{\omega : \lambda_{h_Q}(\omega) \le t\}$. Then when $0 < t < C_4|Q|$,

$$
h_Q^*(t) \le \frac{1}{C_5}\log\frac{C_4|Q|}{t} + C_3. \tag{3.17}
$$

Recall $E \subset \{x : \gamma_1\ell(Q) < |x-c_Q| < \gamma_2\ell(Q)\}$. Applying Lemma 3.2 and (3.17), if $|E| < C_4|Q|$, we have

$$
\begin{aligned}
\frac{1}{|Q|}\int_E |b(x)-b_Q|^p\,dx &\le \frac{1}{|Q|}\int_0^{|E|} |h_Q^*(t)|^p\,dt \\
&\le CC_4\int_0^{|E|/(C_4|Q|)}\Big(C_3 - \frac{1}{C_5}\log t\Big)^p dt \\
&\le C\frac{|E|}{|Q|}\Big(1 + \log\frac{C_4|Q|}{|E|}\Big)^{[p]+1},
\end{aligned} \tag{3.18}
$$

where C is independent of C_4. Combining (3.16) with (3.18), if we take

$$
\beta < \min\{C_4^{1/n}, \gamma_2\},
$$

then (3.5) holds. ■

Proof of Theorem 1.2 We will use a reduction to absurdity to prove Theorem 1.2. That is, we will show that if $b \in \mathrm{BMO}(\mathbb{R}^n)$ and b fails one of the conditions (i), (ii), or (iii) in Lemma 3.3, then the commutator $[b, T]$ is not a compact operator from $L^{p,\lambda}(\mathbb{R}^n)$ to itself. To this end, we choose a bounded sequence $\{f_j\}_{j=1}^{\infty}$ in $L^{p,\lambda}(\mathbb{R}^n)$ and show that there exists a subsequence $\{[b,T]f_{j_k}\}_{k=1}^{\infty}$ in $\{[b,T]f_j\}_{j=1}^{\infty}$ such that $\{[b,T]f_{j_k}\}_{k=1}^{\infty}$ has no convergent subsequence in $L^{p,\lambda}(\mathbb{R}^n)$. Without loss of generality, we assume $\|b\|_* = 1$.

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First, we assume that b does not satisfy Lemma 3.3(i). Then there exist $0 < \zeta < 1$ and a sequence of cubes $\{Q_j(y_j, d_j) := Q_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} d_j = 0$ such that for every j

$$M(b, Q_j) = |Q_j|^{-1} \int_{Q_j} |b(y) - b_{Q_j}|\, dy > \zeta.$$

For Q_j ($j = 1, 2, \dots$) and b, we denote by f_j the function f_{Q_j} defined by (3.2). Thus, $\{f_j\}_{j=1}^{\infty}$ satisfies (3.6)–(3.8) if replacing Q by Q_j. In particular, $\{\|f_j\|_{L^{p,\lambda}}\}_{j=1}^{\infty}$ is bounded uniformly by (3.9). Hence the sequence $\{[b, T]f_j\}_{j=1}^{\infty}$ is also a bounded set in $L^{p,\lambda}(\mathbb{R}^n)$ by Corollary 1.9.

Since $\lim_{j\to\infty} d_j = 0$, we may assume that the sequence $\{d_j\}$ satisfies

$$d_{j+1}/d_j < \beta/\gamma_2. \tag{3.19}$$

Below we need only to show that there exists a constant $\delta > 0$, independent of f_j, such that for any $j, m \in \mathbb{N}$,

$$\|[b, T]f_j - [b, T]f_{j+m}\|_{L^{p,\lambda}} \geq \delta. \tag{3.20}$$

For fixed $j, m \in \mathbb{N}$, denote

$$G = \{x : \gamma_1 d_j < |x - y_j| < \gamma_2 d_j\}, \quad G_1 = G\backslash\{x : |x - y_{j+m}| \leq \gamma_2 d_{j+m}\},$$
$$G_2 = \{x : |x - y_{j+m}| > \gamma_2 d_{j+m}\},$$

where γ_1 and γ_2 are from Lemma 3.4. Note that $G_1 \subset B(y_j, \gamma_2 d_j) \cap G_2$. Hence, we have

$$\Big(\int_{B(y_j,\gamma_2 d_j)} |[b, T]f_j - [b, T]f_{j+m}|^p\, dx\Big)^{1/p} \geq \Big(\int_{G_1} |[b, T]f_j|^p\, dx\Big)^{1/p} - \Big(\int_{G_2} |[b, T]f_{j+m}|^p\, dx\Big)^{1/p}.$$

Since $G_1 = G - (G_2^c \cap G)$, by (3.3) and (3.4) we get

$$\begin{aligned}\Big(\int_{B(y_j,\gamma_2 d_j)} &|[b, T]f_j - [b, T]f_{j+m}|^p\, dx\Big)^{1/p} \\ &\geq \Big(\int_G |[b, T]f_j|^p\, dx - \int_{G_2^c\cap G} |[b, T]f_j|^p dx\Big)^{1/p} \\ &\quad - \Big(\int_{G_2} |[b, T]f_{j+m}|^p\, dx\Big)^{1/p} \\ &\geq \Big(\gamma_3^p |Q_j|^{\lambda/n} - \int_{G_2^c\cap G} |[b, T]f_j|^p dx\Big)^{\frac{1}{p}} - \frac{\gamma_3}{4}\, |Q_{j+m}|^{\lambda/(np)}.\end{aligned} \tag{3.21}$$

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By (3.19),

$$(3.22)\qquad \frac{|G_2^c \cap G|}{|Q_j|} \leq \frac{\gamma_2^n d_{j+m}^n}{d_j^n} < \gamma_2^n \Big(\frac{\beta^n}{\gamma_2^n}\Big)^m < \gamma_2^n \Big(\frac{\beta^n}{\gamma_2^n}\Big) = \beta^n.$$

By (3.22) and applying (3.5) for $E := G_2^c \cap G$, we have

$$(3.23)\qquad \int_{G_2^c \cap G} |[b,T]f_j|^p\,dx \leq \Big(\frac{\gamma_3}{4}\Big)^p |Q_j|^{\lambda/n}.$$

By (3.21) and (3.23) and note that $|Q_{j+m}| < |Q_j|$ for any $m \in \mathbb{N}$ (by (3.19)), there exists $\delta_0 = \delta_0(\gamma_3, p, n) > 0$ such that

$$\Big(\int_{B(y_j,\gamma_2 d_{j_k})} |[b,T]f_j - [b,T]f_{j+m}|^p\,dx\Big)^{\frac{1}{p}} \geq \delta_0 |Q_j|^{\lambda/(np)}.$$

Thus,

$$\Big(\frac{1}{d_j^\lambda}\int_{B(y_j,\gamma_2 d_j)} |[b,T]f_j - [b,T]f_{j+m}|^p\,dx\Big)^{\frac{1}{p}} \geq \delta,$$

where $\delta = \delta(\delta_0, n, q, \lambda)$ and δ is independent of m. We therefore get (3.20). Hence, $[b,T]$ is not a compact operator from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$. This contradiction shows that b must satisfy Lemma 3.3(i).

To finish the proof of Theorem 1.2 it remains to show that b must satisfy conditions (ii) and (iii) in Lemma 3.3. For simplicity, we verify only condition (iii). As done above, we show that (3.20) still holds if b fails for Lemma 3.3(iii).

In fact, in this case there exist a cube Q with its diameter d and a sequence $\{y_j\}$ with $\lim_{j\to\infty} |y_j| = \infty$, such that (3.1) holds for the sequence $\{Q_j := Q + y_j\}$. Thus, by Lemma 3.4, (3.3)$\sim$(3.5) still hold for the function sequence $\{f_j\}$ defined by (3.2). Now we denote $B_j = \{x \in \mathbb{R}^n : |x - y_j| < \gamma_2 d\}$. Since $\lim - j \to \infty |y_j| = \infty$, we may choose $\{y_j\}$ such that $B_j \cap B_k = \varnothing$ for $l \neq k$. Now let f_j be the function associated with Q_j defined by (3.2). With the same definitions of the sets G, G_1, G_2 above, we see that $G_1 = G - G_2^c = G$ by $B_j \cap B_{j+m} = \varnothing$. Thus, for any $j, m \in \mathbb{N}$, by (3.3) and (3.4) we get

$$\begin{aligned}\Big(\int_{B_j} |[b,T]f_j - [b,T]f_{j+m}|^p\,dx\Big)^{1/p} \\ &\geq \Big(\int_G |[b,T]f_j|^p dx\Big)^{1/p} - \Big(\int_{G_2} |[b,T]f_{j+m}|^p dx\Big)^{1/p} \\ &\geq \gamma_3 |Q|^{\lambda/(np)} - \frac{\gamma_3}{4}|Q|^{\lambda/(np)} \geq \frac{\gamma_3}{4}|Q|^{\lambda/(np)}.\end{aligned}$$

Hence, we still have

$$\|[b,T]f_j - [b,T]f_{j+m}\|_{L^{p,\lambda}} \geq \delta.$$

This is inconsistent with the compactness of $[b,T]$ from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$. So, b also satisfies Lemma 3.3(iii). ■

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4 $L^{p,\lambda}$-Boundedness of the Rough Operators and Its Commutators: Proof of Theorem 1.8

Let us first give the boundedness of the rough maximal operator $\mathcal{M}_\Omega$ on the Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$, which is defined by

$$\mathcal{M}_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\le r} |\Omega(y')f(x-y)|\,dy,$$

where $\Omega \in L^1(S^{n-1})$. The following lemma is well known (see [30, Theorem 2.3.8]) and gives the weighted boundedness of $\mathcal{M}_\Omega$ on $L^p(\omega)$.

Lemma 4.1 *Suppose that $1 < p < \infty$ and Ω satisfies (1.1) w ith $\Omega \in L^q(S^{n-1})$ for $q > 1$. If $\omega \ge 0$ and satisfies $\omega^{q'} \in A_p$, where A_p denotes the Muckenhoupt weight class, then $\mathcal{M}_\Omega$ is bounded on $L^p(\omega)$.*

Lemma 4.2 *Let $0 < \lambda < n, 1 < p < \infty$ and $\Omega \in L^q(S^{n-1})$ for $q > 1$. Then there is an $\varepsilon > 0$ such that for any $k \in \mathbb{N}$ and $f \in L^{p,\lambda}(\mathbb{R}^n)$,*

$$\int_{B(t,r)} |\mathcal{M}_\Omega f_k(x)|^p\,dx \le C2^{-k\varepsilon} r^\lambda \|f\|^p_{L^{p,\lambda}}, \tag{4.1}$$

where $B(t,r)$ is an arbitrary fixed ball, $f_k = f\chi_{2^{k+1}B\setminus 2^kB}$ and C is independent of k, t, r, and f.

Proof Denote by f^* the Hardy–Littlewood maximal function of f, which is defined by

$$f^*(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\le r} |f(x-y)|\,dy.$$

Then by the relationship between f^* and A_p weights, we know that $(\chi_B^*)^{\theta q'} \in A_p$ for any $p, q > 1$ and $0 < \theta < 1/q'$ (see [22]). Then by Lemma 4.1, we obtain

$$\int_B |\mathcal{M}_\Omega f_k(x)|^p dx \le C \int_{\mathbb{R}^n} |\mathcal{M}_\Omega f_k(x)|^p (\chi_B^*)^\theta\,dx \le C \int_{\mathbb{R}^n} |f_k(x)|^p (\chi_B^*)^\theta\,dx.$$

Note that $\chi_B^*(x) \sim 2^{-kn}$ when $x \in 2^{k+1}B\setminus 2^kB$ and invoke the following fact (see [34]): for $0 < \delta < 1$, $0 < \lambda < n$, and $1 < p < \infty$, there is a $C > 0$ such that for any $f \in L^{p,\lambda}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(x)|^p (\chi_B^*(x))^\delta\,dx \le Cr^\lambda \|f\|^p_{L^{p,\lambda}}.$$

Hence, we take $0 < \delta < \theta$. Then

$$\begin{aligned}\int_B |\mathcal{M}_\Omega f_k(x)|^p\,dx &\le C2^{-kn(\theta-\delta)} \int_{\mathbb{R}^n} |f_k(x)|^p (\chi_B^*(x))^\delta\,dx \\ &\le C2^{-kn(\theta-\delta)} r^\lambda \|f\|^p_{L^{p,\lambda}}.\end{aligned}$$

Thus, Lemma 4.2 follows by setting $\varepsilon = n(\theta - \delta)$. ■

Proof of Theorem 1.8 (i) Fixing $t \in \mathbb{R}^n$ and $r > 0$, we abbreviate $B = B(t, r)$. For $f \in L^{p,\lambda}(\mathbb{R}^n)$, we write

$$f(y) = f(y)\chi_{2B}(y) + \sum_{k=1}^{\infty} f(y)\chi_{2^{k+1}B\setminus 2^k B}(y) := \sum_{k=0}^{\infty} f_k(y). \tag{4.2}$$

Thus, for $k = 0$, we have

$$\int_B |Sf_0(x)|^p\,dx \le \|Sf_0\|_{L^p}^p \le C\|f_0\|_{L^p}^p = C\int_{2B} |f(y)|^p\,dy \le C(2r)^\lambda \|f\|_{L^{p,\lambda}}^p. \tag{4.3}$$

For $k > 0$, by (4.1) we get

$$\int_B |Sf_k(x)|^p\,dx \le C\int_B |\mathcal{M}_\Omega f_k(x)|^p\,dx \le C2^{-k\varepsilon} r^\lambda \|f\|_{L^{p,\lambda}}^p, \tag{4.4}$$

where C is independent of f and k. Thus, by (4.3) and (4.4) we have

$$\begin{aligned}\Big(\frac{1}{r^\lambda}\int_B |Sf(x)|^p\,dx\Big)^{1/p} &\le C\sum_{k=0}^{\infty}\Big(\frac{1}{r^\lambda}\int_B |Sf_k(x)|^p\,dx\Big)^{1/p}\\ &\le C\|f\|_{L^{p,\lambda}}\Big(1+\sum_{k=1}^{\infty} 2^{-k\varepsilon/p}\Big) \le C\|f\|_{L^{p,\lambda}}.\end{aligned}$$

Hence $\|Sf\|_{L^{p,\lambda}} \le C\|f\|_{L^{p,\lambda}}$.

(ii) For any $t \in \mathbb{R}^n$ and $r > 0$, let $B = B(t, r)$ and write f as in (4.2). By the L^p-boundedness of $[b, S]$, we obtain

$$\int_B |[b, S]f_0(x)|^p\,dx \le C\|f_0\|_{L^p}^p \le C(2r)^\lambda \|f\|_{L^{p,\lambda}}^p.$$

For $k > 0$ and $x \in B$, we write

$$\begin{aligned}|[b, S]f_k(x)| &\le \frac{C}{(2^k r)^n}\int_{2^{k+1}B} |b(x) - b_r||\Omega(x-y)f_k(y)|\,dy\\ &\quad + \frac{C}{(2^k r)^n}\int_{2^{k+1}B} |b_r - b_{2^{k+1}r}||\Omega(x-y)f_k(y)|\,dy\\ &\quad + \frac{C}{(2^k r)^n}\int_{2^{k+1}B} |b(y) - b_{2^{k+1}r}||\Omega(x-y)f_k(y)|\,dy\\ &:= \mathrm{I}_1(x) + \mathrm{I}_2(x) + \mathrm{I}_3(x),\end{aligned}$$

where and in what follows, for $\delta > 0$, b_δ is defined by

$$b_\delta = \frac{1}{|B(t,\delta)|}\int_{B(t,\delta)} b(y)\,dy.$$

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By the well-known fact that for any $r > 0$ and $k \in \mathbb{N}$, $|b_{2^{k+1}r} - b_r| \le C_n(k+1)\|b\|_*$ (see [45]), we obtain $I_2(x) \le C(k+1)\|b\|_* \mathcal{M}_\Omega f_k(x)$. From Lemma 4.1, it follows that there exists $\varepsilon_1 > 0$, independent of f, r, and k, such that

$$\int_B I_2(x)^p\,dx \le C(k+1)^p 2^{-k\varepsilon_1}\|b\|_*^p r^\lambda \|f\|_{L^{p,\lambda}}^p. \tag{4.5}$$

For $I_3(x)$, we choose $1 < u < \min\{p,q\}$. By Hölder's inequality, we have

$$\begin{aligned} I_3(x) &\le C\Big(\frac{1}{(2^k r)^n}\int_{2^{k+1}B}|b(y)-b_{2^{k+1}r}|^{u'}\,dy\Big)^{1/u'} \\ &\quad\times\Big(\frac{1}{(2^k r)^n}\int_{2^{k+1}B}|\Omega(x-y)|^u|f_k(y)|^u\,dy\Big)^{1/u} \\ &\le C\|b\|_*(\mathcal{M}_{|\Omega|^u}(|f_k|^u)(x))^{1/u}. \end{aligned}$$

Noting that $|\Omega|^u \in L^{q/u}(S^{n-1})$, by Lemma 4.1, there exists $\varepsilon_2 > 0$, independent of f, B, and k, such that

$$\int_B I_3(x)^p\,dx \le C\|b\|_*^p\int_{B(t,r)}(\mathcal{M}_{|\Omega|^u}(|f_k|^u)(x))^{p/u}\,dx \le C2^{-k\varepsilon_2}\|b\|_*^p r^\lambda\|f\|_{L^{p,\lambda}}^p. \tag{4.6}$$

By (4.5) and (4.6), we have

$$\sum_{k=1}^\infty\Big(\int_B I_2(x)^p\,dx\Big)^{1/p} + \sum_{k=1}^\infty\Big(\int_B I_3(x)^p\,dx\Big)^{1/p} \le C\|b\|_* r^{\lambda/p}\|f\|_{L^{p,\lambda}}. \tag{4.7}$$

Finally, we give the estimate of $I_1(x)$. First we consider the case where $p \ge q'$. We have $\Omega \in L^{p'}(S^{n-1})$ in this case. By Hölder's inequality, we have for $x \in B$,

$$\begin{aligned} I_1(x) &\le C|b(x)-b_r|\Big(\frac{1}{(2^k r)^n}\int_{2^{k+1}B\setminus 2^k B}|\Omega(x-y)|^{p'}dy\Big)^{1/p'} \\ &\quad\times\Big(\frac{1}{(2^k r)^n}\int_{2^{k+1}B}|f_k(y)|^p\,dy\Big)^{1/p} \\ &\le C|b(x)-b_r|(2^k r)^{-n/p}\,\|f\|_{L^{p,\lambda}}(2^{k+1}r)^{\lambda/p}. \end{aligned}$$

Thus, we get

$$\sum_{k=1}^\infty\Big(\int_B I_1(x)^p\,dx\Big)^{1/p} \le Cr^{\lambda/p}\|b\|_*\|f\|_{L^{p,\lambda}}, \tag{4.8}$$

since $0 < \lambda < n$.

For the case where $1 < p < q'$, we choose $u > 1$ and $\frac{1}{q} < s < 1$ such that

$$\frac{1}{pu} + \frac{1}{q} < 1 \quad\text{and}\quad \frac{1}{pu} + \frac{1}{sq} = 1.$$

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Thus, we have $pu' > sq$. Since $1/(pu) + 1/(sq) = 1$, for $x \in B$, by Hölder's inequality we have

$$\begin{aligned}
\mathrm{I}_1(x) &\le C|b(x) - b_r|\Big(\frac{1}{(2^k r)^n} \int_{2^{k+1}B} |f_k(y)|^p \,dy\Big)^{1/(pu)} \\
&\quad \times \Big(\frac{1}{(2^k r)^n} \int_{2^{k+1}B} |\Omega(x-y)|^{sq} |f_k(y)|^{sq/u'} \,dy\Big)^{1/(sq)} \\
&\le \frac{C|b(x)-b_r|}{(2^k r)^{n/(pu)}} (2^{k+1} r)^{\lambda/(pu)} \|f\|_{L^{p,\lambda}}^{1/u} (\mathcal{M}_{|\Omega|^{sq}}(|f_k|^{sq/u'})(x))^{1/(sq)}.
\end{aligned}$$

By $|\Omega|^{sq} \in L^{1/s}(S^{n-1})$ and $pu'/(sq) > 1$, applying Hölder's inequality, and Lemma 4.1, we obtain

$$\begin{aligned}
\int_B \mathrm{I}_1(x)^p \,dx
&\le \frac{C}{(2^k r)^{n/u}} (2^{k+1} r)^{\lambda/u} \|f\|_{L^{p,\lambda}}^{p/u} \int_B |b(x) - b_r|^p (\mathcal{M}_{|\Omega|^{sq}}(|f_k|^{sq/u'})(x))^{p/(sq)} \,dx \\
&\le \frac{C}{(2^k r)^{n/u}} (2^{k+1} r)^{\lambda/u} \|f\|_{L^{p,\lambda}}^{p/u} \Big(\int_B |b(x) - b_r|^{pu} \,dx\Big)^{1/u} \\
&\quad \times \Big(\int_B (\mathcal{M}_{|\Omega|^{sq}}(|f_k|^{sq/u'})(x))^{pu'/(sq)} \,dx\Big)^{1/u'} \\
&\le \frac{C\|b\|_*^p}{2^{kn/u}} (2^{k+1} r)^{\lambda/u} \|f\|_{L^{p,\lambda}}^{p/u} 2^{-k\varepsilon_0} r^{\lambda/u'} \| |f|^{sq/u'} \|_{L^{pu'/(sq),\lambda}}^{p/(sq)},
\end{aligned}$$

where $\varepsilon_0 > 0$ is independent of k, $B(t,r)$, and f. Noting that

$$\| |f|^{sq/u'} \|_{L^{pu'/(sq),\lambda}}^{p/(sq)} = \|f\|_{L^{p,\lambda}}^{p/u'},$$

we have

$$\int_B \mathrm{I}_1(x)^p \,dx \le C 2^{-k(n-\lambda)/u - k\varepsilon_0} r^\lambda \|b\|_*^p \|f\|_{L^{p,\lambda}}^p.$$

Therefore, for $1 < p < q'$ we have

$$\begin{aligned}
\sum_{k=1}^\infty \Big(\int_B I_1(x)^p \,dx\Big)^{1/p} &\le C \sum_{k=1}^\infty 2^{-k[(n-\lambda)/u + \varepsilon_0]/p} r^{\lambda/p} \|b\|_* \|f\|_{L^{p,\lambda}} \\
&\le C r^{\lambda/p} \|b\|_* \|f\|_{L^{p,\lambda}}.
\end{aligned} \tag{4.9}$$

Then (4.8) and (4.9) show that for $1 < p < \infty$ and $q > n/(n-\lambda)$,

$$\sum_{k=1}^\infty \Big(\int_B I_1(x)^p \,dx\Big)^{1/p} \le C r^{\lambda/p} \|b\|_* \|f\|_{L^{p,\lambda}}. \tag{4.10}$$

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From (4.7) and (4.10), we get

$$\sum_{k=1}^{\infty}\Big(\int_B |[b,S]|f_k(x)|^p\,dx\Big)^{1/p} \le Cr^{\lambda/p}\|b\|_*\|f\|_{L^{p,\lambda}}.$$

Thus

$$\begin{aligned}\Big(\frac{1}{r^\lambda}\int_{B(a,r)}|[b,S]f(x)|^p\,dx\Big)^{1/p} &\le C\sum_{k=0}^{\infty}\Big(\frac{1}{r^\lambda \nu}\int_{B(a,r)}|[b,S]f_k(x)|^p\,dx\Big)^{1/p}\\ &\le C\|b\|_*\|f\|_{L^{p,\lambda}}.\end{aligned}$$

Hence $\|[b,S]f\|_{L^{p,\lambda}} \le C\|b\|_*\|f\|_{L^{p,\lambda}}$. This finishes the proof of Theorem 1.8. ■

5 The Characterization of Pre-Compact Set in $L^{p,\lambda}$: Proof of Theorem 1.12

Fix $a > 0$; we define the mean value of f in $\mathcal{G}$ by

$$M_a f(x) = \frac{1}{a^n}\int_{|y|\le a} f(x+y)\,dy.$$

By Hölder's inequality and the Fubini–Tonelli theorem, for $1 \le p < \infty$, we have

$$\begin{aligned}&\Big(\frac{1}{r^\lambda}\int_{B(t,r)}|M_a f(x) - f(x)|^p\,dx\Big)^{1/p}\\ &\le \Big\{\frac{1}{r^\lambda}\int_{B(t,r)}\Big(\frac{1}{a^n}\int_{|y|\le a}|f(x+y)-f(x)|\,dy\Big)^p dx\Big\}^{1/p}\\ &\le C\Big(\frac{1}{r^\lambda}\int_{B(t,r)}\frac{1}{a^n}\int_{|y|\le a}|f(x+y)-f(x)|^p\,dydx\Big)^{1/p}\\ &= C\Big(\frac{1}{a^n}\int_{|y|\le a}dy\frac{1}{r^\lambda}\int_{B(t,r)}|f(x+y)-f(x)|^p\,dx\Big)^{1/p}\\ &\le C\sup_{|y|\le a}\|f(\cdot+y)-f(\cdot)\|_{p,\lambda}.\end{aligned}$$

Thus

$$\|M_a f - f\|_{p,\lambda} \le C\sup_{|y|\le a}\|f(\cdot+y)-f(\cdot)\|_{p,\lambda}. \tag{5.1}$$

By (5.1) and (1.7), (1.8), we get

$$\lim_{a\to 0}\|M_a f - f\|_{p,\lambda} = 0 \quad \text{uniformly in } f\in\mathcal{G} \tag{5.2}$$

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and $\{M_a f : f \in \mathcal{G}\} \subset L^{p,\lambda}(\mathbb{R}^n)$ satisfies $\sup_{f\in\mathcal{G}} \|M_a f\|_{L^{p,\lambda}} \le C$. By (1.9), for any $0 < \varepsilon < 1$, there exist $N > 0$ and α such that $1 < \varepsilon^{-N}/4 < \alpha^{n/p} < \varepsilon^{-N}/2$ and for every $f \in \mathcal{G}$

$$\|f\chi_{E_\alpha}\|_{L^{p,\lambda}} < \varepsilon/8. \tag{5.3}$$

Now we prove that for each fixed a, the set $\{M_a f : f \in \mathcal{G}\}$ is a strongly pre-compact set in $\mathcal{C}(E_\alpha^c)$, where $E_\alpha^c = \{x \in \mathbb{R}^n : |x| \le \alpha\}$ and $\mathcal{C}(E_\alpha^c)$ denotes the continuous function space on E_α^c with uniform norm. By the Ascoli–Arzelà theorem, it suffices to show that $\{M_a f : f \in \mathcal{G}\}$ is bounded and equicontinuous in $\mathcal{C}(E_\alpha^c)$. In fact, applying Hölder's inequality and (1.7) for $f \in \mathcal{G}$ and $x \in E_\alpha^c$, we have

$$\begin{aligned}|M_a f(x)| &\le \Big\{\frac{1}{a^n}\int_{|y|\le a} |f(x+y)|^p\,dy\Big\}^{1/p} = \Big\{\frac{1}{a^n}\int_{|y-x|\le a}|f(y)|^p\,dy\Big\}^{1/p}\\ &\le C\|f\|_{L^{p,\lambda}} \le C.\end{aligned}$$

Obviously, the constant C is independent of f and x here. On the other hand, for any $x_1, x_2 \in E_\alpha^c$

$$\begin{aligned}|(M_a f)(x_1) - (M_a f)(x_2)| &\le \frac{1}{a^n}\int_{|y|\le a}|f(x_1+y) - f(x_2+y)|\,dy\\ &\le \Big\{\frac{1}{a^n}\int_{|y|\le a}|f(x_1+y)-f(x_2+y)|^p\,dy\Big\}^{1/p}\\ &\le \|f(\cdot + x_2 - x_1) - f(\cdot)\|_{L^{p,\lambda}}.\end{aligned} \tag{5.4}$$

Thus, (5.4) and (1.8) show the equicontinuity of $\{M_a f : f \in \mathcal{G}\}$.

Next we show that for small enough a, the set $\{M_a f : f \in \mathcal{G}\}$ is also a strongly pre-compact set in $L^{p,\lambda}(\mathbb{R}^n)$. To do this, we need only to prove that the set $\{M_a f : f \in \mathcal{G}\}$ is a totally bounded set in $L^{p,\lambda}(\mathbb{R}^n)$ since $L^{p,\lambda}(\mathbb{R}^n)$ is a Banach space. Because the set $\{M_a f : f \in \mathcal{G}\}$ is a totally bounded set in $\mathcal{C}(E_\alpha^c)$, hence for the above ε and N, there exist $\{f_1, f_2, \dots, f_m\} \subset \mathcal{G}$, such that $\{M_a f_1, M_a f_2, \dots, M_a f_m\}$ is a finite ε^{N+1}-net in $\{M_a f : f \in \mathcal{G}\}$ in the norm of $\mathcal{C}(E_\alpha^c)$. We then know that for any $f \in \mathcal{G}$, there is $1 \le j \le m$ such that

$$\sup_{y\in E_\alpha^c} |(M_a f)(y) - (M_a f_j)(y)| < \varepsilon^{N+1}. \tag{5.5}$$

Below we show that $\{M_a f_1, M_a f_2, \dots, M_a f_m\}$ is also a finite ε-net of $\{M_a f : f \in \mathcal{G}\}$ in the norm of $L^{p,\lambda}(\mathbb{R}^n)$ if a is small enough. Clearly, we need only to show that for any $f \in \mathcal{G}$, $r > 0$ and $t \in \mathbb{R}^n$, there exists f_j $(1 \le j \le m)$ such that

$$I := \Big\{\frac{1}{r^\lambda}\int_{B(t,r)} |(M_a f)(x) - (M_a f_j)(x)|^p\,dx\Big\}^{1/p} < \varepsilon. \tag{5.6}$$

The estimate of (5.6) will be divided into three cases.

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Case 1: $B(t, r) \subset E_\alpha^c$. We have

$$I = \left\{ \frac{1}{r^\lambda} \int_{B(t,r)\cap E_\alpha^c} |(M_a f)(x) - (M_a f_j)(x)|^p \, dx \right\}^{1/p}.$$

If $r \leq 1$, then by (5.5) we have $I \leq r^{(n-\lambda)/p} \varepsilon^{N+1} < \varepsilon$. If $r > 1$. Then still by (5.5) we get

$$I \leq \left\{ \int_{E_\alpha^c} |(M_a f)(x) - (M_a f_j)(x)|^p \, dx \right\}^{1/p} \leq \alpha^{n/p} \varepsilon^{N+1} < \varepsilon.$$

Case 2: $B(t, r) \subset E_\alpha$. In this case,

$$I = \left\{ \frac{1}{r^\lambda} \int_{B(t,r)} |(M_a f)(x) - (M_a f_j)(x)|^p \chi_{E_\alpha} \, dx \right\}^{1/p}.$$

Applying the Minkowski inequality, (5.2) and (5.3), for $a > 0$ small enough, we have

$$\begin{aligned} I &\leq \|M_a f - f\|_{p,\lambda} + \left(\frac{1}{r^\lambda} \int_{B(t,r)} |f(x) - f_j(x)|^p \chi_{E_\alpha} \, dx \right)^{1/p} + \|M_a f_j - f_j\|_{L^{p,\lambda}} \\ &\leq \|M_a f - f\|_{p,\lambda} + \|f \chi_{E_\alpha}\|_{L^{p,\lambda}} + \|f_j \chi_{E_\alpha}\|_{L^{p,\lambda}} + \|M_a f_j - f_j\|_{p,\lambda} \\ &< \varepsilon. \end{aligned}$$

Case 3: $B(t, r) \cap E_\alpha^c \neq \varnothing$ and $B(t, r) \cap E_\alpha \neq \varnothing$. The conclusion (5.6) in this case may be deduced from Case 1 and Case 2. In fact,

$$\begin{aligned} I &\leq \left\{ \frac{1}{r^\lambda} \int_{B(t,r)} |(M_a f)(x) - (M_a f_j)(x)|^p \chi_{E_\alpha} \, dx \right\}^{1/p} \\ &\quad + \left\{ \frac{1}{r^\lambda} \int_{B(t,r)} |(M_a f)(x) - (M_a f_j)(x)|^p \chi_{E_\alpha^c} \, dx \right\}^{1/p} \\ &:= \mathrm{I}_1 + \mathrm{I}_2 \,. \end{aligned}$$

Using the method in Case 2, we may get $\mathrm{I}_1 < \varepsilon/2$. And $\mathrm{I}_2 < \varepsilon/2$ can be obtained by applying the idea in Case 1.

Finally, let us show that the set $\mathcal{G}$ is a relative compact set in $L^{p,\lambda}(\mathbb{R}^n)$. Taking any sequence $\{f_j\}_{j=1}^\infty$in $\mathcal{G}$, by the relative compactness of $\{M_a f : f \in \mathcal{G}\}$ in $L^{p,\lambda}(\mathbb{R}^n)$, there exists a subsequence $\{M_a f_{j_k}\}_{k=1}^\infty$ of $\{M_a f_j : f_j\}$ that is convergent in $L^{p,\lambda}(\mathbb{R}^n)$. So, for any $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that for any $k > K$ and $m \in \mathbb{N}$, $\|M_a f_{j_k} - M_a f_{j+m}\|_{p,\lambda} < \varepsilon$. By the Minkowski inequality and (5.2), for any $r > 1$

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and $t \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\Big\{ \frac{1}{r^\lambda} \int_{B(t,r)} |f_j(x) - f_{j+m}(x)|^p \, dx \Big\}^p \\
&\quad \leq \Big\{ \frac{1}{r^\lambda} \int_{B(t,r)} |f_j(x) - M_a f_j(x)|^p \, dx \Big\}^p \\
&\qquad + \Big\{ \frac{1}{r^\lambda} \int_{B(t,r)} |M_a f_j(x) - M_a f_{j+m}(x)|^p \, dx \Big\}^p \\
&\qquad + \Big\{ \frac{1}{r^\lambda} \int_{B(t,r)} |M_a f_{j+m}(x) - f_{j+m}(x)|^p \, dx \Big\}^p \\
&\quad \leq \|M_a f_j - f_j\|_{L^{p,\lambda}} + \|M_a f_j - M_a f_{j+m}\|_{L^{p,\lambda}} + \|M_a f_{j+m} - f_{j+m}\|_{L^{p,\lambda}} \\
&\quad < 3\varepsilon.
\end{aligned}$$

This shows that the subsequence $\{f_j\}_{k=1}^{\infty}$ converges in $L^{p,\lambda}(\mathbb{R}^n)$, since $L^{p,\lambda}(\mathbb{R}^n)$ is a Banach space. Therefore, the set $\mathcal{G}$ is a relative compact set in $L^{p,\lambda}(\mathbb{R}^n)$, and we finish the proof of Theorem 1.12.

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