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RESEARCH ARTICLE

Quartic and Quintic Hypersurfaces with Dense Rational Points

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Abstract

Let $X_4 \subset \mathbb{P}^{n+1}$ be a quartic hypersurface of dimension $n \geq 4$ over an infinite field k. We show that if either X_4 contains a linear subspace Λ of dimension $h \geq \max\{2, \dim(\Lambda \cap \mathrm{Sing}(X_4)) + 2\}$ or has double points along a linear subspace of dimension $h \geq 3$, a smooth k-rational point and is otherwise general, then X_4 is unirational over k. This improves previous results by A. Predonzan and J. Harris, B. Mazur and R. Pandharipande for quartics. We also provide a density result for the k-rational points of quartic 3-folds with a double plane over a number field, and several unirationality results for quintic hypersurfaces over a C_r field.

Contents

1. Introduction

An n-dimensional variety X over a field k is rational if it is birational to $\mathbb{P}^n_k$, while X is unirational if there is a dominant rational map $\mathbb{P}^n_k \dashrightarrow X$. If k is infinite and X is unirational, then the set $X(k)$ of the k-rational points of X is Zariski dense in X.

Since the first half of the twentieth century, the problem of establishing whether a degree d hypersurface $X_d \subset \mathbb{P}^{n+1}$ is rational or unirational has been central in birational projective geometry [Mor42], [Pre49], [Mor52], [IM71], [CG72], [Cil80], [Kol95], [Shi95], [HMP98], [HT00], [dF13], [BRS19], [RS19].

Quadric hypersurfaces with a smooth point are rational and as proven by J. Kollár, building on techniques of B. Segre [Seg43] and Y. I. Manin [Man86, Chapter 2, Section 12], cubic hypersurfaces with a smooth point are unirational [Kol02]. U. Morin proved that a general complex hypersurface $X_d \subset \mathbb{P}^{n+1}$ is unirational provided that n is large enough with respect to d [Mor42]. This result has then been reproved, in a different way, by C. Ciliberto [Cil80] and extended to complete intersections by A. Predonzan [Pre49], K. Paranjape, V. Srinivas [PS92] and L. Ramero [Ram90].

Furthermore, J. Harris, B. Mazur and R. Pandharipande proved that $X_d \subset \mathbb{P}^{n+1}$ is unirational if the codimension of its singular locus is sufficiently big with respect to n and d [HMP98].

Before stating our main results on the unirationality of quartics we briefly survey the state of the art. By the work of U. Morin, a general complex quartic $X_4 \subset \mathbb{P}^{n+1}$ with $n \geq 5$ is unirational [Mor36], [Mor52]. V. A. Iskovskikh and Y. I. Manin proved that the group of the birational automorphisms of a smooth quartic $X_4 \subset \mathbb{P}^4$ is finite so that X_4 is not rational [IM71].

Moreover, J. Harris and Y. Tschinkel showed that if $n \geq 3$ and k is a number field, then for some finite extension k' of k the set of k'-rational points of a smooth quartic $X_4 \subset \mathbb{P}^{n+1}$ is dense in the Zariski topology; in other terms, the k-rational points of X_4 are potentially dense [HT00]. Despite this great amount of effort, the unirationality of the general quartic $X_4 \subset \mathbb{P}^n$ for $n = 4, 5$ is still an open problem and only special families of quartic 3-folds, called quartics with separable asymptotics, are known to be unirational [Seg60]. For a nice survey on rationality and unirationality problems with a focus on their relation with the notion of rational connection, we refer to A. Verra's paper [Ver08]. We recall that a projective variety is rationally connected if any two of its points can be joined by a rational curve and refer to C. Araujo's paper [Ara05] for a survey on the subject.

In this paper, we address the unirationality of quartics $X_4 \subset \mathbb{P}^{n+1}$ containing a linear subspace whose dimension is larger than that of the singular locus of X_4 or containing a linear subspace with multiplicity two. Our main results in Theorems 3.15 and 3.22 can be summarized as follows:

Theorem 1.1. *Let $X_4 \subset \mathbb{P}^{n+1}$ be a quartic hypersurface and $\Lambda \subset \mathbb{P}^{n+1}$ an h-plane. Assume that either*

(i) *$n \geq 3, h \geq 2$, $\dim(\Lambda \cap \mathrm{Sing}(X_4)) \leq h - 2$, X_4 contains Λ and is not a cone over a smaller-dimensional quartic, or*
(ii) *$n \geq 4, h \geq 3$, X_4 has double points along Λ, a point $p \in X_4 \setminus \Lambda$ and is otherwise general,*

then X_4 is unirational.

All along the paper with the word 'general' we mean 'for a nonempty Zariski open subset of the parameter space of the objects we are considering'. We would like to stress that since all the proofs presented in the paper are constructive it is possible, given the equation cutting out the hypersurface, to establish whether or not it is general in the required sense.

Furthermore, for quartic 3-folds over a number field we prove, in Proposition 3.14, the following density result.

Theorem 1.2. *Let $X_4 \subset \mathbb{P}^4$ be a quartic hypersurface, over a number field k, having double points along a codimension two linear subspace $\Lambda \subset \mathbb{P}^4$, with a point $p \in X_4 \setminus \Lambda$ and otherwise general. The set $X_4(k)$ of the k-rational points of X_4 is Zariski dense in X_4.*

As Remark 3.17 shows, the assumption on the existence of a point $p \in X_4 \setminus \Lambda$ in the case of quartics singular along a linear subspace cannot be dropped. Under extra assumptions on the base field or on the existence of rational points in special subloci of X_4, Theorem 1.1 can be extended to smaller-dimensional quartics. For instance, by Proposition 3.19 a quartic surface $X_4 \subset \mathbb{P}^3$ with double points along a line $\Lambda \subset \mathbb{P}^3$, a point $p \in X_4 \setminus \Lambda$, a double point $q \in X_4 \setminus \Lambda$ with $q \neq p$ and otherwise general is unirational. Furthermore, by the second part of Theorem 3.15 a quartic $X_4 \subset \mathbb{P}^4$ over a C_r field, for which definition we refer to Remark 2.10, with $r = 0, 1$, having double points along a linear subspace Λ with $\dim(\Lambda) = 1, 2$, and otherwise general is unirational. Therefore, in order to complete Theorem 1.1 for all n and h we are left with the following open question.

Question 1.3. Let $X_4 \subset \mathbb{P}^{n+1}$ be a quartic hypersurface over a field k such that either

(i) $n = 3$, X_4 contains a line, or
(ii) $n = 3$, X_4 has double points along a linear subspace Λ with $\dim(\Lambda) = 1, 2$, a point $p \in X_4 \setminus \Lambda$, or
(iii) $n = 2$, X_4 has double points along a line Λ, a point $p \in X_4 \setminus \Lambda$,

and X_4 is otherwise general. Is then X_4 unirational?

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Note that a smooth quartic surface $X_4 \subset \mathbb{P}^3$ is $K3$, and hence, it cannot be unirational. As we said, (ii) has a positive answer when the base field is C_r with $r = 0, 1$, while (i) is open even over the complex numbers. Since any complex quartic 3-fold contains a line, (i) actually asks about the unirationality of a general quartic 3-fold and is probably one of the most interesting unirationality open problems. Since a quartic surface $X_4 \subset \mathbb{P}^3$ with a double line is birational to a conic bundle, (iii) is interesting only when the base field is not algebraically closed.

Note that by considering the generic fiber of the resolution of the linear projection from Λ as in the proof of Theorem 3.15 a positive answer to Question 1.3 would extend the first part of Theorem 1.1 to quartic hypersurfaces $X_4 \subset \mathbb{P}^{n+1}$ with $n \geq 3$ containing a line and the second part of Theorem 1.1 to quartic hypersurfaces $X_4 \subset \mathbb{P}^{n+1}$ with $n \geq 3$ having double points along either a line or a plane and an additional smooth point.

Remark 1.4. The main available results in the spirit of Theorem 1.1 can be found in [Pre49] and [HMP98]. By [Pre49, Theorem 1] a quartic $X_4 \subset \mathbb{P}^{n+1}$ containing an h-plane Λ is unirational provided that $\mathrm{Sing}(X_4) \cap \Lambda = \emptyset$ and $h \geq 4$. The same result has been proved in [HMP98, Corollary 3.7] for $h \geq 97$.

We would like to stress that both [Pre49] and [HMP98] as well as [Ram90] provide unirationality results for hypersurfaces of arbitrary degree and general unirationality bounds when the base field is algebraically closed.

In the case of quartics, Theorem 1.1 (i) improves [Pre49, Theorem 1] and [HMP98, Corollary 3.7] in two directions: on one side, it is enough to have that $h \geq 2$, on the other side, Λ is allowed to intersect the singular locus of X_4 as long as such intersection has codimension at least two in Λ.

In the last section, we investigate the unirationality of quintic hypersurfaces and divisors of bidegree $(3, 2)$ in products of projective spaces. As a by-product, we get new examples of unirational but not stably rational varieties.

A variety X is stably rational if $X \times \mathbb{P}^m$ is rational for some $m \geq 0$. Hence, a rational variety is stably rational, and a stably rational variety is unirational. The first examples of stably rational nonrational varieties had been given in [BCTSSD85], where the authors, using Châtelet surfaces, constructed a complex nonrational conic bundle T such that $T \times \mathbb{P}^3$ is rational.

In the last decade, important advances on stable rationality have been made, especially for hypersurfaces in projective spaces [Voi15], [CTP16], [Tot16], [HKT16], [AO18], [Sch18], [BvB18], [HPT18], [Sch19a], [Sch19b], [HPT19]. In [CTP16, Theorem 1.17], J. L. Colliot-Thélène and A. Pirutka proved that a very general smooth complex quartic 3-fold is not stably rational. In [Sch19b, Corollary 1.4], S. Schreieder gave the first examples of unirational nonstably rational smooth hypersurfaces. A. Auel, C. Böhning and A. Pirutka proved that a very general divisor of bidegree $(3, 2)$ in $\mathbb{P}^3 \times \mathbb{P}^2$, over the complex numbers, is not stably rational. By Theorem 4.7, we get that such a very general divisor is unirational but not stably rational.

Furthermore, thanks to our unirationality results for divisors of bidegree $(3, 2)$ we get new results on the unirationality of quintic hypersurfaces over C_r fields and number fields. The literature on the unirationality of quintics is much less rich than that on quartics. A general quintic hypersurface $X_5 \subset \mathbb{P}^{n+1}$, over an algebraically closed field, is unirational if $n \geq 17$ [Mor38]. Furthermore, a quintic $X_5 \subset \mathbb{P}^{n+1}$ containing a 3-plane and otherwise general is unirational if $n \geq 6$ [CMM08]. To the best of our knowledge, these are the only results on the unirationality of quintics.

Conventions on the base field, terminology and organization of the paper

All along the paper, the base field k will be of characteristic zero. Let X be a variety over k. When we say that X is rational or unirational, without specifying over which field, we will always mean that X is rational or unirational over k. Similarly, we will say that X has a point or contains a variety with certain properties meaning that X has a k-rational point or contains a variety defined over k with the required properties.

In Section 2, we will introduce the notation, prove some preliminary results about the relation between hypersurfaces in projective spaces and certain divisors in projective bundles and give an immediate

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generalization of a unirationality criterion due to F. Enriques. In Section 3, we will investigate the unirationality of quartic hypersurfaces and cubic complexes that are complete intersections of a quadric and a cubic. Finally, in Section 4 we will address the unirationality of quintics and divisors in products of projective spaces.

2. Hypersurfaces and divisors in projective bundles

Let $a_0, \dots, a_{h+1} \in \mathbb{Z}_{\geq 0}$ be nonnegative integers, and consider the simplicial toric variety $\mathcal{T}_{a_0,\dots,a_{h+1}}$ with Cox ring

$$\mathrm{Cox}(\mathcal{T}_{a_0,\dots,a_{h+1}}) \cong k[x_0, \dots, x_{n-h}, y_0, \dots, y_{h+1}]$$

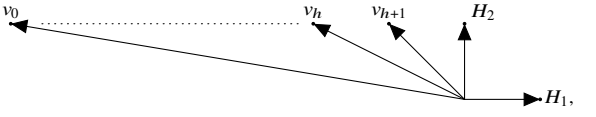
$\mathbb{Z}^2$-grading given, with respect to a fixed basis (H_1, H_2) of $\mathrm{Pic}(\mathcal{T}_{a_0,\dots,a_{h+1}})$, by the following matrix

$$\left(\begin{array}{cccccc} x_0 & \dots & x_{n-h} & y_0 & \dots & y_{h+1} \\ \hline 1 & \dots & 1 & -a_0 & \dots & -a_{h+1} \\ 0 & \dots & 0 & 1 & \dots & 1 \end{array}\right)$$

and irrelevant ideal $(x_0, \dots, x_{n-h}) \cap (y_0, \dots, y_{h+1})$. Then

$$\mathcal{T}_{a_0,\dots,a_{h+1}} \cong \mathbb{P}(\mathcal{E}_{a_0,\dots,a_{h+1}})$$

with $\mathcal{E}_{a_0,\dots,a_{h+1}} \cong \mathcal{O}_{\mathbb{P}^{n-h}}(a_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{n-h}}(a_{h+1})$. The secondary fan of $\mathcal{T}_{a_0,\dots,a_{h+1}}$ is as follows



where $H_1 = (1, 0)$ corresponds to the sections $x_0, \dots, x_{n-h}$, $H_2 = (0, 1)$, and $v_i = (-a_i, 1)$ corresponds to the section y_i for $i = 0, \dots, h+1$.

Definition 2.1. A divisor $D \subset \mathcal{T}_{a_0,\dots,a_{h+1}}$ of multidegree $(\delta_{d,0,\dots,0}, \dots, \delta_{0,\dots,0,d}; d)$ is a hypersurface given by an equation of the following form

$$D := \left\{ \sum_{0 \leq i_0 \leq \cdots \leq i_{h+1} \mid i_0 + \cdots + i_{h+1} = d} \sigma_{i_0,\dots,i_{h+1}}(x_0, \dots, x_{n-h}) y_0^{i_0} \dots y_{h+1}^{i_{h+1}} = 0 \right\} \subset \mathcal{T}_{a_0,\dots,a_{h+1}}, \tag{2.2}$$

where $\sigma_{i_0,\dots,i_{h+1}} \in k[x_0, \dots, x_{n-h}]_{\delta_{i_0,\dots,i_{h+1}}}$ and

$$\delta_{d,0,\dots,0} - da_0 = \delta_{d-1,1,0,\dots,0} - (d-1)a_0 - a_1 = \cdots = \delta_{0,\dots,0,d} - da_{h+1}. \tag{2.3}$$

Without loss of generality, we may assume that $a_0 \geq a_1 \geq \cdots \geq a_{h+1}$ so that (2.3) yields $\delta_{d,0,\dots,0} \geq \delta_{0,d,0,\dots,0} \geq \cdots \geq \delta_{0,\dots,0,d}$.

Lemma 2.4. *Let $X_d \subset \mathbb{P}^{n+1}$ be a hypersurface of degree d having multiplicity m along an h-plane Λ, and $\widetilde{X}_d$ the blowup of X_d along Λ with exceptional divisor $\widetilde{E} \subset \widetilde{X}_d$. Then $\widetilde{X}_d$ is isomorphic to a divisor of multidegree*

$$(d, d-1, \dots, d-1, \dots, d-j, \dots, d-j, \dots, m, \dots, m; d-m)$$

in $\mathcal{T}_{1,0,\dots,0}$, where $d-j$ is repeated $\binom{h+j}{j}$ times for $j = 0, \dots, d-m$. The exceptional divisor $\widetilde{E}$ is a divisor of bidegree $(m, d-m)$ in $\mathbb{P}^{n-h} \times \mathbb{P}^h$.

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Proof. We may assume that $\Lambda = \{z_0 = \cdots = z_{n-h} = 0\}$ and

$$X_d = \left\{ \sum_{m_0+\cdots+m_{n-h}=m} z_0^{m_0} \dots z_{n-h}^{m_{n-h}} A_{m_0,\dots,m_{n-h}}(z_0,\dots,z_{n+1}) = 0 \right\} \subset \mathbb{P}^{n+1}$$

with $A_{m_0,\dots,m_{n-h}} \in k[z_0,\dots,z_{n+1}]_{d-m}$. The blowup of $\mathbb{P}^{n+1}$ along Λ is the simplicial toric variety $\mathcal{T}$ with Cox ring

$$\mathrm{Cox}(\mathcal{T}) \cong k[x_0,\dots,x_{n-h},y_0,\dots,y_{h+1}]$$

$\mathbb{Z}^2$-grading given, with respect to a fixed basis (H_1,H_2) of $\mathrm{Pic}(\mathcal{T})$, by the following matrix:

$$\left(\begin{array}{ccccccc} x_0 & \dots & x_{n-h} & y_0 & y_1 & \dots & y_{h+1} \\ \hline 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ -1 & \dots & -1 & 1 & 0 & \dots & 0 \end{array}\right)$$

and irrelevant ideal $(x_0,\dots,x_{n-h}) \cap (y_0,\dots,y_{h+1})$. Substituting in the above matrix the first row with the sum of the rows and then swapping the rows an multiplying the top row by -1 we get to the following grading matrix

$$\left(\begin{array}{ccccccc} x_0 & \dots & x_{n-h} & y_0 & y_1 & \dots & y_{h+1} \\ \hline 1 & \dots & 1 & -1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 \end{array}\right) \tag{2.5}$$

and hence $\mathcal{T} = \mathcal{T}_{1,0,\dots,0}$. The blow-down morphism is given by

$$\begin{array}{rccc} \phi: & \mathcal{T}_{1,0,\dots,0} & \longrightarrow & \mathbb{P}^{n+1} \\ & (x_0,\dots,x_{n-h},y_0,\dots,y_{h+1}) & \mapsto & [x_0y_0 : \cdots : x_{n-h}y_0 : y_1 : \cdots : y_{h+1}] \end{array}$$

and the exceptional divisor is $E = \{y_0 = 0\}$. Hence, the strict transform of X_d is defined by

$$\widetilde{X}_d = \left\{ \sum_{m_0+\cdots+m_{n-h}=m} x_0^{m_0} \dots x_{n-h}^{m_{n-h}} A_{m_0,\dots,m_{n-h}}(x_0y_0,\dots,x_{n-h}y_0,y_1,\dots,y_{h+1}) = 0 \right\} \subset \mathcal{T}_{1,0,\dots,0}, \tag{2.6}$$

and the claim on the multidegree follows. Note that (2.5) yields that $E \cong \mathbb{P}^{n-h}_{(x_0,\dots,x_{n-h})} \times \mathbb{P}^{h}_{(y_1,\dots,y_{h+1})}$ and hence $\widetilde{E} = \widetilde{X}_d \cap E \subset \mathbb{P}^{n-h}_{(x_0,\dots,x_{n-h})} \times \mathbb{P}^{h}_{(y_1,\dots,y_{h+1})}$ is a divisor of bidegree $(m, d-m)$. □

The following is a straightforward generalization of a unirationality criterion for conic bundles due to F. Enriques [IP99, Proposition 10.1.1].

Proposition 2.7. *Let $f : X \to Y$ be a fibration over a unirational variety Y with X an irreducible variety. Assume that there exists a unirational subvariety $Z \subset X$ such that $f_{|Z} : Z \to Y$ is dominant, consider the fiber product*

$$\begin{array}{ccc} X_Z = X \times_Y Z & \longrightarrow & X \\ \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{f_{|Z}} & Y \end{array}$$

and denote by $X_{Z,\eta}$ the generic fiber of $\widetilde{f} : X_Z \to Z$. Assume that $X_{Z,\eta}$ is integral. Finally, assume that $X_{Z,\eta}$ is unirational over $k(Z)$ if and only if it has a $k(Z)$-rational point. Then X is unirational.

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Proof. Since $X_{Z,\eta}$ is integral after replacing Z by an open subset, we may assume that X_Z is irreducible.

Now, note that the dominant morphism $f_{|Z} : Z \to Y$ yields a rational section of $\widetilde{f} : X_Z \to Z$. So $X_{Z,\eta}$ has a $k(Z)$-rational point, and hence, it is unirational over $k(Z)$. Therefore, X_Z is unirational, and hence, X is unirational as well. □

We will apply Proposition 2.7 to fibrations in irreducible m-dimensional quadric or cubic hypersurfaces. In these cases, $X_{Z,\eta} \subset \mathbb{P}^{m+1}_{k(Z)}$ is an irreducible quadric or cubic hypersurface over $k(Z)$. So after replacing Z by an open subset, we may assume that X_Z is irreducible.

Remark 2.8. A quadric hypersurface over a field k with a smooth point is rational. Furthermore, by [Kol02, Theorem 1.2] a cubic hypersurface of dimension at least two with a point and which is not a cone is unirational.

Definition 2.9. Fix a real number $r \in \mathbb{R}_{\geq 0}$. A field k is C_r if and only if every homogeneous polynomial $f \in k[x_0, \ldots, n_n]_d$ of degree $d > 0$ in $n+1$ variables with $n+1 > d^r$ has a nontrivial zero in k^{n+1}.

Remark 2.10. (Lang's theorem) If k is a C_r field, $f_1, \ldots, f_s \in k[x_0, \ldots, n_n]_d$ are homogeneous polynomials of the same degree and $n+1 > sd^r$, then $f_1, \ldots, f_s$ have a nontrivial common zero in k^{n+1} [Poo17, Proposition 1.2.6]. Furthermore, if k is C_r, then $k(t)$ is C_{r+1} [Poo17, Theorem 1.2.7].

In the last section, we will need the following.

Proposition 2.11. *Let $D \subset \mathcal{T}_{a_0,\ldots,a_{h+1}} \to \mathbb{P}^{n-h}$ be a divisor of multidegree $(\delta_{2,0,\ldots,0}, \ldots, \delta_{0,\ldots,0,2}, 2)$. Then D is birational to a hypersurface $X^n_{\delta_{2,0,\ldots,0}+2} \subset \mathbb{P}^{n+1}$ of degree $\delta_{2,0,\ldots,0}+2$ having multiplicity $\delta_{2,0,\ldots,0}$ along an h-plane Λ and multiplicity two along an $(n-h-1)$-plane Λ' such that $\Lambda \cap \Lambda' = \emptyset$.*

Proof. Write $D \subset \mathcal{T}_{a_0,\ldots,a_{h+1}}$ as in (2.2) and dehomogenize with respect to x_{n-h} and y_{h+1} to get an affine hypersurface

$$X = \{f(x_0, \ldots, x_{n-h-1}, y_0, \ldots, y_h) = 0\} \subset \mathbb{A}^{n+1}_{(x_0,\ldots,x_{n-h-1},y_0,\ldots,y_h)}.$$

Now, we introduce a new variable that we will keep denoting by x_{n-h} and homogenize f in order to get a polynomial $\overline{f}(x_0, \ldots, x_{n-h-1}, y_0, \ldots, y_h, x_{n-h})$ which is homogeneous of degree $\delta_{2,0,\ldots,0}+2$. Note that $\overline{f}$ has the following form

$$\overline{f} = \sum_{i_0+\cdots+i_{n-h}=\delta_{2,0,\ldots,0}} x_0^{i_0} \ldots x_{n-h}^{i_{n-h}} A_{i_0,\ldots,i_{n-h}}(y_0, \ldots, y_h, x_{n-h}) \tag{2.12}$$

with $A_{i_0,\ldots,i_{n-h}} \in k[y_0, \ldots, y_h, x_{n-h}]_2$ for all $0 \leq i_0 \leq \cdots \leq i_{n-h} \leq \delta_{2,0,\ldots,0}$. The hypersurface

$$X^n_{\delta_{2,0,\ldots,0}+2} = \{\overline{f}(x_0, \ldots, x_{n-h-1}, y_0, \ldots, y_h, x_{n-h}) = 0\} \subset \mathbb{P}^{n+1}$$

is birational to X, and hence, it is birational to D as well. To conclude, set

$$\Lambda = \{x_0 = \cdots = x_{n-h} = 0\},\ \Lambda' = \{y_0 = \cdots = y_h = x_{n-h} = 0\}$$

and note that (2.12) yields $\operatorname{mult}_\Lambda X^n_{\delta_{2,0,\ldots,0}+2} = \delta_{2,0,\ldots,0}$ and $\operatorname{mult}_{\Lambda'} X^n_{\delta_{2,0,\ldots,0}+2} = 2$. □

3. Cubic complexes and Quartics

In this section, we investigate the unirationality of quartic hypersurfaces containing linear subspaces and quadrics, and of complete intersections of quadric and cubic hypersurfaces. The following is an immediate consequence of [CMM07, Theorem 2.1].

Lemma 3.1. *Let $Y_{2,3} = Y_2 \cap Y_3 \subset \mathbb{P}^{n+2}$ with $n \geq 3$ be a smooth complete intersection, of a smooth quadric Y_2 and a cubic Y_3, defined over a field k with $\operatorname{char}(k) \neq 2, 3$. If Y_2 contains a 2-plane Π, then $Y_{2,3}$ is unirational.*

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Proof. For $n = 3$, the statement has been proven in [CMM07, Theorem 2.1]. Now, consider the incidence varieties

$$\begin{aligned} W_2 &= \{(y, H) \mid y \in H \cap Y_2\} \subset Y_2 \times \mathbb{G}(n-4, n-1); \\ W_3 &= \{(y, H) \mid y \in H \cap Y_3\} \subset Y_3 \times \mathbb{G}(n-4, n-1); \\ W_{2,3} &= \{(y, H) \mid y \in H \cap Y_{2,3}\} = W_2 \cap W_3 \subset Y_{2,3} \times \mathbb{G}(n-4, n-1), \end{aligned}$$

where $\mathbb{G}(n-4, n-1)$ is the Grassmannian of 5-planes in $\mathbb{P}^{n+2}$ containing Π. Let $W_{2,\eta}, W_{3,\eta}, W_{2,3,\eta}$ be the generic fibers of the second projection onto $\mathbb{G}(n-4, n-1)$ from $W_2, W_3, W_{2,3}$, respectively.

The 2-plane $\Pi \subset Y_2$ yields a 2-plane $\Pi' \subset W_{2,\eta}$ defined over $k(t_1, \ldots, t_{3(n-3)})$, and since Y_2 is smooth, we have that $W_{2,\eta}$ is smooth.

Hence, $W_{2,3,\eta} \subset \mathbb{P}^5_{k(t_1,\ldots,t_{3(n-3)})}$ is a smooth complete intersection over $k(t_1, \ldots, t_{3(n-3)})$ of a quadric and a cubic satisfying the hypotheses of [CMM07, Theorem 2.1], and so $W_{2,3,\eta}$ is unirational over the function field $k(t_1, \ldots, t_{3(n-3)})$. As in the proof of Proposition 2.7, after replacing $\mathbb{G}(n-4, n-1)$ by an open subset we may assume that $W_{2,3}$ is irreducible.

Therefore, $W_{2,3}$ is unirational over the base field k, and to conclude it is enough to observe that the first projection $W_{2,3} \to Y_{2,3}$ is dominant. □

Lemma 3.2. *Let $Q_0 \subset \mathbb{P}^{n+1}$ be a smooth $(n-1)$-dimensional quadric and $X_d \subset \mathbb{P}^{n+1}$ an irreducible hypersurface of degree d containing Q_0 with multiplicity one and otherwise general. Then there exists a complete intersection*

$$Y_{2,d-1} = Y_2 \cap Y_{d-1} \subset \mathbb{P}^{n+2}$$

of a quadric Y_2 and a hypersurface Y_{d-1} of degree $d-1$ such that:

(i) *$Y_{2,d-1}$ has a point $v \in Y_{2,d-1}$ of multiplicity $d-2$, and*
(ii) *the linear projection from v yields a birational map $\pi_v : Y_{2,d-1} \dashrightarrow X_d$.*

Furthermore, Y_2 is smooth and $\operatorname{mult}_v Y_{d-1} = d-2$.

Proof. We may write $Q_0 = \{z_1 = Q = 0\}$ with $Q \in k[z_0, \ldots, z_{n+1}]_2$. Then X_d is of the form

$$X_d = \{z_1 A + BQ = 0\}$$

with $A \in k[z_0, \ldots, z_{n+1}]_{d-1}$ and $B \in k[z_0, \ldots, z_{n+1}]_{d-2}$. Consider the quadric $Y_2 = \{z_1 u - Q = 0\}$ with $z_0, \ldots, z_{n+1}, u$ homogeneous coordinates on $\mathbb{P}^{n+2}$. Since Q_0 is smooth, we may assume that $Q = z_0^2 + \cdots + z_{n+1}^2$. Hence, Y_2 is a smooth as well.

Denote by CX_d, CQ_0 the cones, respectively, over X_d and Q_0 with vertex $v = [0 : \cdots : 0 : 1]$. Then $CQ_0 \subset Y_2$ and

$$Y_2 \cap CX_d = CQ_0 \cup \{z_1 u - Q = A + uB = 0\}.$$

Set $Y_{d-1} = \{A + uB = 0\}$ and $Y_{2,d-1} = \{z_1 u - Q = A + uB = 0\}$. Note that the tangent space of Y_2 at v is the hyperplane $\{z_1 = 0\}$, the tangent cone of Y_{d-1} at v is given by $\{B = 0\}$ and the tangent cone of $Y_{2,d-1}$ at v is cut out by $\{z_1 = B = 0\}$. Hence,

$$\operatorname{mult}_v Y_{d-1} = \operatorname{mult}_v Y_{2,d-1} = d-2.$$

Therefore, if $p \in Y_{2,d-1}$ is a general point the line $\langle v, p \rangle$ intersects $Y_{2,d-1}$ just in v with multiplicity $d-2$ and in p with multiplicity one. So the projection $\pi_v : Y_{2,d-1} \dashrightarrow \mathbb{P}^{n+1}$ is birational onto $\overline{\pi_v(Y_{2,d-1})}$ which must then be a hypersurface of degree $2(d-1) - (d-2) = d$. To conclude, it is enough to note that since $Y_{2,d-1}$ is not a cone of vertex v and $Y_{2,d-1} \subset CX_d$ we have $\overline{\pi_v(Y_{2,d-1})} = X_d$. □

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Lemma 3.3. *Consider the complete intersection*

$$Y_{2,d-1} = Y_2 \cap Y_{d-1} \subset \mathbb{P}^{n+2},$$

where $Y_2 = \{z_1 u - A_1 = 0\} \subset \mathbb{P}^{n+2}_{(z_0,\dots,z_{n-1},u)}$ *and*

$$Y_{d-1} = \{z_0^{d-2}u + z_0^{d-3}A_2 + z_0^{d-4}z_1A_3 + \cdots + z_1^{d-3}A_{d-1} = 0\} \subset \mathbb{P}^{n+2}_{(z_0,\dots,z_{n-1},u)}$$

with A_i *general homogeneous polynomial of degree two. Assume that the* $(n-2)$*-dimensional quadric* $\overline{Q} = \{z_0 = z_1 = A_1 = 0\}$ *has a point.*

Then there exists a variety W*, with a surjective morphism onto a rational surface* S*, such that if* W *is unirational, then there is a dominant rational map* $W \dashrightarrow Y_{2,d-1}$ *and hence* $Y_{2,d-1}$ *is unirational as well. Furthermore, the general fiber of* $W \to S$ *is a complete intersection of a quadric and a hypersurface of degree* $d-3$*.*

Proof. Set $v = [0 : \cdots : 0 : 1]$ and $H = \{z_0 = z_1 = 0\} \subset \mathbb{P}^{n+2}$. Then

$$\operatorname{mult}_v Y_{d-1} = d-2, \ \operatorname{mult}_H Y_{d-1} = d-3.$$

Take a general point $p \in H$. The lines through p that intersect Y_{d-1} at p with multiplicity at least $d-2$ are parametrized by a hypersurface W_{d-3} cut out in the $(n+1)$-dimensional projective space $\mathbb{P}(T_p\mathbb{P}^{n+2})$ of lines through p by a polynomial in $k[z_0, z_1]_{d-3}$.

Now, consider the cone $C\overline{Q}$ over the $(n-2)$-dimensional quadric $\overline{Q} = \{z_0 = z_1 = A_1 = 0\}$, and take a general point $p \in C\overline{Q}$. Note that $C\overline{Q} \subset Y_2$. The lines through p that are contained in Y_2 are parametrized by a quadric hypersurface $W_2 \subset \mathbb{P}(T_pY_2)$.

Let $\mathcal{F} = \mathcal{T}_{Y_2|C\overline{Q}}$ be the restriction of the tangent sheaf of Y_2 to $C\overline{Q}$. Summing up, there is a subvariety $W_{2,d-3} \subset \mathbb{P}(\mathcal{F})$ with a surjective morphism $\rho : W_{2,d-3} \to C\overline{Q}$ whose fiber over a general point of $p \in C\overline{Q}$ is a complete intersection of a quadric and a hypersurface of degree $d-3$ in $\mathbb{P}^n$. Hence, $\dim(W_{2,d-3}) = 2n-3$.

By hypothesis, $\overline{Q}$ has a point. Let $\overline{C}$ be a conic in $\overline{Q}$ through this point and S the cone over $\overline{C}$ with vertex v. Then $S \subset C\overline{Q}$ is a rational surface. Set $W = \rho^{-1}(S)$ and $W_s = \rho^{-1}(s)$ for $s \in S$. A general point $w \in W$ represents a pair (s, l_s), where $s \in S$ and l_s is a line through s which is contained in Y_2 and intersects Y_{d-1} with multiplicity $d-2$ at s. Since $\deg(Y_{d-1}) = d-1$ the line l_s intersects Y_{d-1} just at one more point $x_{(s,l_s)} \in Y_2 \cap Y_{d-1} = Y_{2,d-1}$ and we get a rational map

$$\begin{array}{rccc} \psi : & W & \dashrightarrow & Y_{2,d-1} \\ & (s, l_s) & \longmapsto & x_{(s,l_s)}. \end{array}$$

If W is unirational, in order to prove that ψ is dominant it is enough to prove that the induced map $\overline{\psi} : \overline{W} \dashrightarrow \overline{Y}_{2,d-1}$ between the algebraic closures is dominant. Take a general point $p \in \overline{Y}_{2,d-1}$, and assume that $x_{(s,l_s)} = p$. Then l_s lies in the tangent space of Y_2 at p which is given by $\{L = 0\}$. Such tangent space intersects $\overline{S}$ in a conic, and further intersecting with W_{d-3} we see that there are finitely may points $s \in \overline{S}$ such that $x_{(s,l_s)} = p$ for some $l_s \in W_s$. Furthermore, if $x_{(s,l_s)} = x_{(s,l'_s)}$, then $l_s = l'_s$. Hence, $\overline{\psi}$ is generically finite and since $\dim(W) = \dim(Y_{2,d-1})$ we conclude that $\overline{\psi}$ is dominant. □

Proposition 3.4. *Let* $X_d \subset \mathbb{P}^{n+1}$ *be a hypersurface of degree* d *having multiplicity* $d-2$ *along an* $(n-1)$*-plane* $\Lambda \subset \mathbb{P}^{n+1}$*. Assume that there is a quadric* Q_p *in the quadric fibration induced by the projection from* Λ *such that the quadric* $Q_p \cap \Lambda$ *is smooth and has a point and that* X_d *is otherwise general.*

Then there exist a rational surface S *and a variety* W *with a morphism onto* S *such that if* W *is unirational, then there is a dominant rational map* $W \dashrightarrow X_d$ *and hence* X_d *is unirational as well.*

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Proof. We may write $\Lambda = \{z_0 = z_1 = 0\}$ and

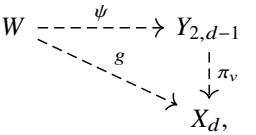
$$X_d = \{z_0^{d-2}A_1 + z_0^{d-3}z_1A_2 + \cdots + z_1^{d-2}A_{d-1} = 0\}$$

with $A_i \in k[z_0, \ldots, z_{n+1}]_2$. Note that X_d contains the smooth $(n-1)$-dimensional quadric $Q_p = \{z_1 = A_1 = 0\}$. Hence, by Lemma 3.2 there is an irreducible complete intersection $Y_{2,d-1} \subset \mathbb{P}^{n+2}$ which is birational to X_d.

Consider the quadric $Y_2 = \{z_1u - A_1 = 0\} \subset \mathbb{P}^{n+2}$ with homogeneous coordinates $z_0, \ldots, z_{n+1}, u$, and let CX_d be the cone over X_d as in the proof of Lemma 3.2. The intersection $Y_2 \cap CX_d$ has two components: the cone CQ_p over Q_p and the degree $d-1$ hypersurface

$$Y_{d-1} = \{z_0^{d-2}u + z_0^{d-3}A_2 + z_0^{d-4}z_1A_3 + \cdots + z_1^{d-3}A_{d-1} = 0\}.$$

Let $W \to S$ be the fibration constructed in Lemma 3.3 starting from the complete intersection $Y_{2,d-1} = Y_2 \cap Y_{d-1}$. If W is unirational, Lemma 3.3 yields a dominant rational map $\psi : W \dashrightarrow Y_{2,d-1}$. Finally, let $\pi_v : Y_{2,d-1} \dashrightarrow X_d$ be the dominant rational map in Lemma 3.2. By considering the composition



we get a dominant rational map $g : W \dashrightarrow X_d$, and hence X_d is unirational. □

Proposition 3.5. *Let $Y_{2,3} = Y_2 \cap Y_3 \subset \mathbb{P}^{n+2}$ be a complete intersection of a quadric and a cubic of the following form*

$$Y_{2,3} = \{uz_1 - A = z_0^2u + z_0B + z_1C = 0\}$$

with $A, B, C \in k[z_0, \ldots, z_{n+1}]_2$ general. If the quadric $\overline{Q} = \{z_0 = z_1 = A = 0\}$ is smooth and has a point, then $Y_{2,3}$ is unirational.

Proof. By Lemma 3.3 with $d = 4$, there exists a variety W with a morphism onto a rational surface S whose general fiber is a complete intersection of a quadric and a hyperplane. Hence, W has a structure of quadric bundle $W \to S$ over S with $(n-2)$-dimensional quadrics as fibers.

By the proof of Lemma 3.3, S is a cone over a conic and it is contained in $Y_2 = \{uz_1 - A = 0\}$. In particular, any line in S through its vertex is contained in Y_2. Moreover, S is contained in the intersection of Y_3 with its tangent cone at v. Hence, the lines in S through its vertex yield a section of the quadric bundle $W \to S$ and so W is rational. Finally, to conclude it is enough to apply Lemma 3.3. □

Corollary 3.6. *Let $X_4 \subset \mathbb{P}^{n+1}$ be a quartic hypersurface having multiplicity two along an $(n-1)$-plane $\Lambda \subset \mathbb{P}^{n+1}$ with $n \geq 3$. Assume that there is a quadric Q_p in the quadric fibration induced by the projection from Λ such that the quadric $Q_\Lambda = Q_p \cap \Lambda$ is smooth and has a point and that X_4 is otherwise general. Then X_4 is unirational.*

Proof. Up to a change of coordinates, we may assume that $\Lambda = \{z_0 = z_1 = 0\}$, X_4 is given by

$$X_4 = \{z_0^2A + z_0z_1B + z_1^2C = 0\} \subset \mathbb{P}^{n+1}$$

and $Q_\Lambda = \{z_1 = A = 0\}$. The intersection $\{z_1u - A = z_0^2A + z_0z_1B + z_1^2C = 0\} \subset \mathbb{P}^{n+2}_{(z_0,\ldots,z_{n+1},u)}$ has two components: $\{z_1 = A = 0\}$ and

$$Y_{2,3} = \{z_1u - A = uz_0^2 + z_0B + z_1^2C = 0\} \subset \mathbb{P}^{n+2}_{(z_0,\ldots,z_{n+1},u)}.$$

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By Lemma 3.2 and its proof, there exists a birational map $Y_{2,3} \dashrightarrow X_4$. To conclude, it is enough to note that $Y_{2,3}$ is a complete intersection of the form covered by Proposition 3.5. □

Corollary 3.7. *Let $X_4 \subset \mathbb{P}^{n+1}$, with $n \geq 5$, be a quartic hypersurface containing an $(n-1)$-dimensional smooth quadric Q_0 which contains a 2-plane and otherwise general. Then X_4 is unirational.*

Proof. By Lemma 3.2, X_4 is birational to a complete intersection $Y_{2,3} = Y_2 \cap Y_3$, where Y_2 is a smooth quadric. By the proof of Lemma 3.2, we have that since Q_0 contains a 2-plane Y_2 contains a 2-plane as well. Hence, Y_2 is a smooth quadric containing a 2-plane and we conclude by Lemma 3.1. □

Corollary 3.8. *Let $X_4 \subset \mathbb{P}^{n+1}$, with $n \geq 2$, be a quartic hypersurface having multiplicity two along an $(n-1)$-dimensional smooth quadric Q, with a point and otherwise general. Then X_4 is unirational.*

Proof. Slightly modifying the proof of Lemma 3.2, we see that in this case X_4 is birational to a complete intersection $Y_2 \cap Y_2' \subset \mathbb{P}^{n+2}$ of two quadrics, and hence, the claim follows from [CTSSD87, Proposition 2.3]. □

In the following, we will investigate the unirationality of quartic hypersurfaces by constructing explicit birational maps to divisors in products of projective spaces. In particular, we will get an improvement of Corollary 3.6 when $n \geq 4$.

Lemma 3.9. *Consider the hypersurface*

$$X_d = \{z_0^{d-2}(z_0L + Q) + z_0^{d-3}z_1A_1 + \cdots + z_0z_1^{d-3}A_{d-3} + z_1^{d-2}A_{d-2} = 0\} \subset \mathbb{P}^{n+1}_{(z_0,\ldots,z_{n+1})},$$

where $A_i = A_i^2 + z_0A_i^1 + z_0^2A_i^0$, and $L, A_i^1 \in k[z_1, \ldots, z_{n+1}]_1$, $Q, A_i^2 \in k[z_1, \ldots, z_{n+1}]_2$, $A_i^0 \in k$ for $i = 1, \ldots, d-2$. Then X_d is birational to the divisor

$$Y_{(d-1,2)} = \left\{ \sum_{i=0}^{d-1} x_0^{d-1-i}x_1^i B_i = 0 \right\} \subset \mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^n_{(w_1,\ldots,w_{n+1})}$$

of bidegree $(d-1, 2)$, where $B_0 = w_1L + A_1^0w_1^2$, $B_1 = Q + w_1A_1^1 + w_1^2A_2^0$, $B_j = A_{j-1}^2 + w_1A_j^1 + w_1^2A_{j+1}^0$ for $j = 2, \ldots, d-3$, $B_{d-2} = A_{d-3}^2 + w_1A_{d-2}^1$, $B_{d-1} = A_{d-2}^2$.

Proof. Note that X_d passes through the point $p = [1 : 0 : \cdots : 0]$, and the rational map

$$\begin{array}{rcl} \varphi : \mathbb{P}^{n+1}_{(z_0,\ldots,z_{n+1})} & \dashrightarrow & \mathbb{P}^{n+1}_{(w_0,\ldots,w_{n+1})} \\ [z_0 : \cdots : z_{n+1}] & \mapsto & [z_0L : z_1^2 : z_1z_2 : \cdots : z_1z_{n+1}] \end{array}$$

is birational with birational inverse

$$\begin{array}{rcl} \varphi^{-1} : \mathbb{P}^{n+1}_{(w_0,\ldots,w_{n+1})} & \dashrightarrow & \mathbb{P}^{n+1}_{(z_0,\ldots,z_{n+1})} \\ [w_0 : \cdots : w_{n+1}] & \mapsto & [w_0w_1 : w_1L : w_2L : \cdots : w_{n+1}L], \end{array}$$

where $L = L(w_1, \ldots, w_{n+1})$. Note that φ^{-1} contracts the divisor $\{L = 0\}$ to the point p. The strict transform of X_d via φ^{-1} is given by

$$\begin{aligned} \widetilde{X}_d = \{&w_0^{d-2}(w_0w_1L + LQ) + w_0^{d-3}(w_0^2w_1^2A_1^0 + w_0w_1LA_1^1 + L^2A_1^2) + \ldots \\ &+ L^{d-3}(w_0^2w_1^2A_{d-2}^0 + w_0w_1LA_{d-2}^1 + L^2A_{d-2}^2) = 0\}, \end{aligned}$$

which we rewrite as

$$\begin{aligned} \widetilde{X}_d = \{&w_0^{d-1}(w_1L + w_1^2A_1^0) + w_0^{d-2}L(Q + w_1A_1^1 + w_1^2A_2^0) + \ldots \\ &+ w_0L^{d-2}(A_{d-3}^2 + w_1A_{d-2}^1) + L^{d-1}A_{d-2}^2 = 0\}. \end{aligned}$$

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Finally, substituting $w_0 = \frac{x_0}{x_1}L$ we get the equation cutting out the divisor $Y_{(d-1,2)} \subset \mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^n_{(w_1:\cdots:w_{n+1})}$ in the statement. □

Proposition 3.10. *Let $Y_{(3,2)}$ be a general divisor of the form*

$$Y_{(3,2)} = \left\{ \sum_{i=0}^{3} x_0^{3-i} x_1^i B_i = 0 \right\} \subset \mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^n_{(w_1,\ldots,w_{n+1})},$$

where $B_0 = w_1 L + A_1^0 w_1^2$, $B_1 = Q + w_1 A_1^1 + w_1^2 A_2^0$, $B_2 = A_1^2 + w_1 A_2^1$, $B_3 = A_2^2$, and $L, A_i^1 \in k[w_1, \ldots, w_{n+1}]_1$, $Q, A_i^2 \in k[w_1, \ldots, w_{n+1}]_2$, $A_i^0 \in k$ for $i = 1, 2$ are general. If $n \geq 4$, then $Y_{(3,2)}$ is unirational.

Proof. Consider the rational map

$$\begin{aligned} \eta : \mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^n_{(w_1,\ldots,w_{n+1})} &\dashrightarrow \mathbb{P}^2_{(u_1,u_2,u_3)} \\ ([x_0 : x_1], [w_1 : \cdots : w_{n+1}]) &\mapsto [\eta_1 : \eta_2 : \eta_3], \end{aligned} \tag{3.11}$$

where $\eta_i = B_i x_0^2 + B_{i+1} x_0 x_1 + \cdots + B_3 x_0^{i-1} x_1^{3-i}$ for $i = 1, 2, 3$. By [Ott15, Theorem 1.1 (ii)] the rational map

$$\begin{aligned} \mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^n_{(w_1,\ldots,w_{n+1})} &\dashrightarrow \mathbb{P}^2_{(u_1,u_2,u_3)} \times \mathbb{P}^n_{(w_1,\ldots,w_{n+1})} \\ ([x_0 : x_1], [w_1 : \cdots : w_{n+1}]) &\mapsto (\eta([x_0 : x_1], [w_1 : \cdots : w_{n+1}]), [w_1 : \cdots : w_{n+1}]) \end{aligned}$$

yields a small transformation $\eta^+ : Y_{(3,2)} \dashrightarrow Y^+_{(3,2)}$, where $Y^+_{(3,2)} \subset \mathbb{P}^2_{(u_1,u_2,u_3)} \times \mathbb{P}^n_{(w_1,\ldots,w_{n+1})}$ is cut out by the minors of order three of the following matrix:

$$M_{(u_1,u_2,u_3)} = \begin{pmatrix} 0 & u_1 & B_0 \\ -u_1 & u_2 & B_1 \\ -u_2 & u_3 & B_2 \\ -u_3 & 0 & B_3 \end{pmatrix}. \tag{3.12}$$

Consider the point $p = ([1 : 0], [0 : \cdots : 0 : 1]) \in Y_{(3,2)}$, its image

$$\begin{aligned} q &= ([B_1(0, \ldots, 0, 1), B_2(0, \ldots, 0, 1)], B_3(0, \ldots, 0, 1)]) \\ &= ([Q(0, \ldots, 0, 1), A_1^2(0, \ldots, 0, 1), A_2^2(0, \ldots, 0, 1)]) \end{aligned}$$

via η and set $\overline{u}_1 = Q(0, \ldots, 0, 1), \overline{u}_2 = A_1^2(0, \ldots, 0, 1), \overline{u}_3 = A_2^2(0, \ldots, 0, 1)$. Let $F_{\overline{u}}$ be the fiber of the first projection $\pi_1 : \mathbb{P}^2_{(u_1,u_2,u_3)} \times \mathbb{P}^n_{(w_1,\ldots,w_{n+1})} \to \mathbb{P}^2_{(u_1,u_2,u_3)}$ over $\overline{u} = [\overline{u}_1 : \overline{u}_2 : \overline{u}_3]$. Then

$$F_{\overline{u}} = \{\operatorname{rank}(M_{(\overline{u}_1,\overline{u}_2,\overline{u}_3)}) < 3\} \subset \mathbb{P}^n_{(w_1,\ldots,w_{n+1})}$$

is a complete intersection of two quadrics. Note that $q \in F_{\overline{u}}$ and since the $\overline{u}_i$ are general, $F_{\overline{u}}$ is smooth. Therefore, if $n \geq 4$ [CTSSD87, Proposition 2.3] yields the unirationality of $F_{\overline{u}}$. The strict transform of $F_{\overline{u}}$ via η^+ is given by

$$\widetilde{F}_{\overline{u}} = \left\{ \operatorname{rank} \begin{pmatrix} \overline{u}_1 & \overline{u}_2 & \overline{u}_3 \\ B_1 x_0^2 + B_2 x_0 x_1 + B_3 x_1^2 & B_2 x_0^2 + B_3 x_0 x_1 & B_3 x_0^2 \end{pmatrix} < 2 \right\} \subset \mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^n_{(w_1,\ldots,w_{n+1})}.$$

So $\widetilde{F}_{\overline{u}}$ is unirational and maps dominantly onto $\mathbb{P}^1_{(x_0,x_1)}$. Finally, to conclude it is enough to note that $Y_{(3,2)} \to \mathbb{P}^1_{(x_0,x_1)}$ is a fibration in quadric hypersurfaces and to apply Proposition 2.7 and Remark 2.8. □

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Proposition 3.13. *Let $X_4 \subset \mathbb{P}^{n+1}$ be a quartic hypersurface having double points along a codimension two linear subspace $\Lambda \subset \mathbb{P}^{n+1}$, with a point $p \in X_4 \setminus \Lambda$ and otherwise general. If $n \geq 4$, then X_4 is unirational.*

Proof. The equation of $X_4 \subset \mathbb{P}^{n+1}$ can be written as in Lemma 3.9 for $d = 4$. Hence, the claim follows from Lemma 3.9 and Proposition 3.10. □

For quartic 3-folds, we have the following density result.

Proposition 3.14. *Let $X_4 \subset \mathbb{P}^4$ be a quartic hypersurface, over a number field k, having double points along a codimension two linear subspace $\Lambda \subset \mathbb{P}^4$, with a point $p \in X_4 \setminus \Lambda$ and otherwise general. The set $X_4(k)$ of the k-rational points of X_4 is Zariski dense in X_4.*

Proof. By Lemma 3.9, it is enough to prove that a general divisor $Y_{3,2} \subset \mathbb{P}^1 \times \mathbb{P}^3$ as in Proposition 3.10 has dense k-points. Consider the 2-plane $H = \{x_1 = w_1 = 0\} \subset Y_{3,2}$ and the rational map

$$\begin{array}{rccc} \eta' : & Y_{(3,2)} & \dashrightarrow & \mathbb{P}^2_{(u_1,u_2,u_3)} \\ & ([x_0 : x_1], [w_1 : \cdots : w_4]) & \mapsto & [\eta_1 : \eta_2 : \eta_3] \end{array}$$

induced by (3.11). Note that η' maps H dominantly onto $\mathbb{P}^2_{(u_1,u_2,u_3)}$. Take a general point $p \in H$, its image $q = \eta'(p)$ and a general line $L = \{\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0\} \subset \mathbb{P}^2_{(u_1,\ldots,u_3)}$ through q. Set $S_{L,p} = \eta'^{-1}(L)$. Since $Y_{(3,2)}$ is general η' restricts to a morphism $\eta'_{|S_{L,p}} : S_{L,p} \to \mathbb{P}^1$. A straightforward computation shows that $S_{L,p}$ is smooth, and by the argument in the second part of the proof of Proposition 3.10, the general fiber of $\eta'_{|S_{L,p}} : S_{L,p} \to \mathbb{P}^1$ is a smooth curve of genus one.

Furthermore, $C = H \cap S_{L,p} = \{\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 = x_1 = w_1 = 0\}$. Hence, since $Y_{3,2}$ is general, C is a smooth conic with a point $p \in C$. Note that a general fiber of $\eta'_{|S_{L,p}} : S_{L,p} \to \mathbb{P}^1$ intersects a general fiber of the quadric bundle $Y_{3,2} \to \mathbb{P}^1$ in a zero-dimensional scheme of degree eight. So, $C \subset S_{L,p}$ is a rational 4-section of $\eta'_{|S_{L,p}}$.

Consider the fiber product $T = C \times_L S_{L,p}$. Since p, L and $Y_{3,2}$ are general, the ramification divisor of $\eta'_{|C} : C \to \mathbb{P}^1$ is disjoint from the singular fibers of $\eta'_{|S_{L,p}}$. Hence, T is smooth and the proof of [HT00, Theorem 8.1] goes through. So the set $S_{L,p}(k)$ of the k-rational points of $S_{L,p}$ is Zariski dense in $S_{L,p}$. Finally, letting the point p vary in H and the line L vary among the lines passing through $q = \eta'(p)$ we get the claim. □

The following is our main result on the unirationality of quadric hypersurfaces having double points along a linear subspace.

Theorem 3.15. *Let $X_4 \subset \mathbb{P}^{n+1}$ be a quartic hypersurface having double points along an h-plane $\Lambda \subset \mathbb{P}^{n+1}$, with a point $p \in X_4 \setminus \Lambda$ and otherwise general. If $n \geq 4$ and $h \geq 3$, then X_4 is unirational.*

Furthermore, if k is a C_r field, $n \geq 3$, and $s + 1 > 2^r$, where $s = \max\{n - h, h\}$ a general quartic hypersurface $X_4 \subset \mathbb{P}^{n+1}$ having double points along an h-plane is unirational.

Proof. The case $h = n - 1$ comes from Proposition 3.13. Let $\pi_H : X_4 \dashrightarrow \mathbb{P}^{n-h-1}$ be the projection from $H = \langle p, \Lambda \rangle$, $\widetilde{\pi}_H : \widetilde{X}_4 \to \mathbb{P}^{n-h-1}$, and $F_p \cong \mathbb{P}^{n-h-1}$ the fiber over p of the blowup $\widetilde{X}_4 \to X_4$ along $H \cap X_4$. Note that since X_4 is irreducible the blowup $\widetilde{X}_4$ is also irreducible.

The generic fiber $\widetilde{X}_{4,\eta}$ of $\widetilde{\pi}_H : \widetilde{X}_4 \to \mathbb{P}^{n-h-1}$ is a quartic hypersurface $\widetilde{X}_{4,\eta} \subset \mathbb{P}^{h+2}_{k(t_1,\ldots,t_{n-h-1})}$ with double points along an h-plane and with a $k(t_1, \ldots, t_{n-h-1})$-rational point induced by F_p.

Therefore, Proposition 3.13 yields that $\widetilde{X}_{4,\eta}$ is unirational over $k(t_1, \ldots, t_{n-h-1})$ and since $\widetilde{X}_4$ is irreducible $\widetilde{X}_4$ is unirational.

Now, assume k to be C_r. The exceptional divisor $\widetilde{E}$ in Lemma 2.4 is a divisor of bidegree $(2, 2)$ in $\mathbb{P}^{n-h} \times \mathbb{P}^h$, and since X_4 is general $\widetilde{E}$ maps dominantly onto $\mathbb{P}^{n-h}$. Furthermore, since both the fibrations on $\widetilde{E}$ have quadric hypersurfaces as fibers and $s + 1 > 2^r$ by Remark 2.10 the divisor $\widetilde{E}$ has a point. To conclude, it is enough to apply [Mas22, Theorem 1.8] and Proposition 2.7. □

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Remark 3.16. In the second part of Theorem 3.15, the assumption on the base field is needed in order to ensure the existence of a point in the exceptional divisor $\widetilde{E} \subset \mathbb{P}^{n-h} \times \mathbb{P}^h$. Indeed, over non C_r fields there are divisors of bidegree $(2,2)$ in $\mathbb{P}^{n-h} \times \mathbb{P}^h$ without points [Mas22, Remark 4.15].

Remark 3.17. In Theorem 3.15, the assumption on the existence of a point in $X_4 \setminus \Lambda$ is necessary as the following argument shows. Take, for instance, $n = 3$, $h = 1$ and $k = \mathbb{R}$. By Lemma 2.4, X_4 is birational to a conic bundle $D \subset \mathcal{T}_{1,0,0}$ of multidegree $(4,3,3,2,2,2)$. We may write D as the zero set of a polynomial

$$f = a_0 x_0^4 + a_1 x_0^3 x_1 + a_2 x_0^2 x_1^2 + a_3 x_0 x_1^3 + a_4 x_1^4,$$

where $a_i \in \mathbb{R}[x_1, x_2, y_0, y_1, y_2]$. Let V be the $\mathbb{R}$-vector space parametrizing these conic bundles and consider the map

$$\begin{array}{rccc} ev: & V \times \mathbb{R}^5 & \longrightarrow & \mathbb{R}^5 \\ & (f, (\overline{x}_1, \overline{x}_2, \overline{y}_0, \overline{y}_1, \overline{y}_2)) & \mapsto & P_{f,\overline{x}_1,\overline{x}_2,\overline{y}_0,\overline{y}_1,\overline{y}_2}(x_0), \end{array}$$

where $P_{f,\overline{x}_1,\overline{x}_2,\overline{y}_0,\overline{y}_1,\overline{y}_2}(x_0) = f(x_0, \overline{x}_1, \overline{x}_2, \overline{y}_0, \overline{y}_1, \overline{y}_2) \in \mathbb{R}[x_0]$. The polynomials $P_{f,\overline{x}_1,\overline{x}_2,\overline{y}_0,\overline{y}_1,\overline{y}_2}(x_0)$ with a real root form a semialgebraic subset $R_{rr} \subset \mathbb{R}^5$ of maximal dimension; see, for instance, [BPR06, Section 4.2.3]. Hence, $ev^{-1}(R_{rr}) \subset V \times \mathbb{R}^5$ is also a semialgebraic subset of maximal dimension and then the Tarski–Seidenberg principle [BCR98, Section 2.2] yields that $Z_{rp} = \pi_1(ev^{-1}(R_{rr})) \subset V$, where $\pi_1 : V \times \mathbb{R}^5 \to V$ is the projection, is a semialgebraic subset of maximal dimension. Note that

$$Z_{rp} = \{D \subset \mathcal{T}_{1,0,0} \text{ of mutidegree } (4,3,3,2,2,2) \text{ having a rational point}\}.$$

The complementary set $Z_{nrp} = Z_{rp}^c$ is nonempty, take, for instance,

$$D = \{(x_0^2 + x_1^2 + x_2^2)^2 y_0^2 + (x_0^2 + x_1^2 + 2x_2^2) y_1^2 + (x_0^2 + x_1^2 + 3x_2^2) y_2^2 = 0\} \subset \mathcal{T}_{1,0,0}.$$

Hence, $Z_{nrp} \subset V$ is also a semialgebraic subset of maximal dimension. By Lemma 2.4, the quartics $X_4 \subset \mathbb{P}^4$ corresponding to the conic bundles in Z_{nrp} do not have a point in $X_4 \setminus \Lambda$, and hence, they cannot be unirational.

Lemma 3.18. *Consider the hypersurface*

$$X_d = \{z_0^{d-2} Q + z_0^{d-3} z_1 A_1 + \cdots + z_0 z_1^{d-3} A_{d-3} + z_1^{d-2} A_{d-2} = 0\} \subset \mathbb{P}^{n+1}_{(z_0,\ldots,z_{n+1})},$$

where $A_i = A_i^2 + z_0 A_i^1 + z_0^2 A_i^0$, *and* $A_i^1 \in k[z_1, \ldots, z_{n+1}]_1$, $Q, A_i^2 \in k[z_1, \ldots, z_{n+1}]_2$, $A_i^0 \in k$ *for* $i = 1, \ldots, d-2$. *Then* X_d *is birational to the divisor*

$$Y_{(d-2,2)} = \left\{ x_0^{d-2} Q + \sum_{i=1}^{d-2} x_0^{d-2-i} x_1^i A_i = 0 \right\} \subset \mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^n_{(z_1,\ldots,z_{n+1})}$$

of bidegree $(d-2, 2)$.

Proof. It is enough to substitute $z_0 = \frac{x_0}{x_1} z_1$ in the equation of X_d and to clear the denominators of the resulting polynomial. □

For quartic hypersurfaces with a double point, we get a version of Theorem 3.15 with a slightly less restrictive condition on n.

Proposition 3.19. *Let* $X_d \subset \mathbb{P}^{n+1}$ *be a hypersurface of degree* d *having multiplicity* $d-2$ *along a codimension two linear subspace* $\Lambda \subset \mathbb{P}^{n+1}$, *with a point* $p \in X_d \setminus \Lambda$, *a double point* $q \in X_d \setminus \Lambda$ *and otherwise general. If either* $d = 4$ *and* $n \geq 2$ *or* $d = 5$ *and* $n \geq 4$, *then* X_d *is unirational.*

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Proof. We may assume that $\Lambda = \{z_0 = z_1 = 0\}$ and $q = [1 : 0 : \cdots : 0]$ so that the equation of $X_d \subset \mathbb{P}^{n+1}$ can be written as in Lemma 3.18. First, consider the case $d = 4$. By Lemma 3.18 X_4 is birational to a divisor $Y_{(2,2)} \subset \mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^n_{(z_1,\ldots,z_{n+1})}$ of bidegree $(2,2)$. The point $p \in X_d \setminus \Lambda$ yields a point $p' \in Y_{(2,2)}$. Let H be a general 2-plane in $\mathbb{P}^n_{(z_1,\ldots,z_{n+1})}$ through the projection of p' and set $\Gamma = \mathbb{P}^1_{(x_0,x_1)} \times H$. Then $W = Y_{(2,2)} \cap \Gamma$ is a divisor of bidegree $(2,2)$ in $\mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^2$ which by [KM17, Corollary 8] is unirational. Since W maps dominantly onto $\mathbb{P}^1_{(x_0,x_1)}$, to conclude it is enough to apply Proposition 2.7.

Now, let $d = 5$. Then by Lemma 3.18 X_5 is birational to a divisor $Y_{(3,2)} \subset \mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^n_{(z_1,\ldots,z_{n+1})}$ of bidegree $(3,2)$. This divisor is not of the same form of those considered in Proposition 3.10. However, the argument in the proof of Proposition 3.10 goes through smoothly and shows $Y_{(3,2)}$ is unirational. □

Remark 3.20. The argument in the proof of Proposition 3.10 works indeed for a general divisor of bidegree $(3,2)$ as shown in [Mas22, Proposition 3.2] which works for a divisor of bidegree $(3,2)$ in $\mathbb{P}^1 \times \mathbb{P}^n$ with a point and otherwise general. So [Mas22, Proposition 3.2] cannot be applied directly to the divisors $Y_{(3,2)}$ in Proposition 3.10 which are special. In the proof of Proposition 3.10, it is shown that the argument in the proof of [Mas22, Proposition 3.2] works for a general $Y_{(3,2)}$, in other words that a general divisor of the form $Y_{(3,2)}$ is general enough, among the divisor of bidegree $(3,2)$ in $\mathbb{P}^1 \times \mathbb{P}^n$, for [Mas22, Proposition 3.2] to apply.

Lemma 3.21. *Let $D \subset \mathbb{P}^{n-h} \times \mathbb{P}^h$ be an irreducible divisor of bidegree $(1,a)$. Then D is rational.*

Proof. If $n - h = 1$, then the projection $\pi_{2|D} : D \to \mathbb{P}^h$ is birational. Assume $n - h \geq 2$. The generic fiber of $\pi_{2|D} : D \to \mathbb{P}^h$ is an $(n-h-1)$-dimensional projective space over the function field of $\mathbb{P}^h$. So we can find an open subset $U \subset \mathbb{P}^h$ such that $\pi^{-1}_{2|D}(U)$ is birational to $U \times \mathbb{P}^{n-h-1}$. Finally, since D is irreducible we conclude that D is rational. □

The following is our main result on the unirationality of quartic hypersurface containing a linear subspace.

Theorem 3.22. *Let $X_4 \subset \mathbb{P}^{n+1}$ be an irreducible quartic hypersurface containing an h-plane $\Lambda \subset \mathbb{P}^{n+1}$. Assume that $\Lambda \cap \mathrm{Sing}(X_4)$ has dimension at most $h-2$ and that X_4 is not a cone over a quartic of smaller dimension. If $n \geq 3$ and $h \geq 2$, then X_4 is unirational.*

Proof. Consider the blowup $\widetilde{X}_4$ along Λ in Lemma 2.4. Under our hypotheses, X_4 cannot be singular at the general point of Λ. So the exceptional divisor $\widetilde{E} \subset \mathbb{P}^{n-h} \times \mathbb{P}^h$ has bidegree $(1,3)$, and Lemma 3.21 yields that $\widetilde{E}$ is unirational. The projection $\pi : \widetilde{X}_4 \to \mathbb{P}^{n-h}$ is a fibration in h-dimensional cubic hypersurfaces. Therefore, thanks to Proposition 2.7 and Remark 2.8 to conclude it is enough to show that $\pi_{|\widetilde{E}} : \widetilde{E} \to \mathbb{P}^{n-h}$ is dominant.

Since X_4 is not a cone over a quartic of smaller dimension by the expression of $\widetilde{X}_4$ in (2.6), we see that $\pi_{|\widetilde{E}} : \widetilde{E} \to \mathbb{P}^{n-h}$ is not dominant if and only if the equation of X_d is of the following form

$$X_d = \left\{ \sum_{i=0}^{n-h} z_i (C_i(z_0, \ldots, z_{n-h}) + c_i P) = 0 \right\} \subset \mathbb{P}^{n+1},$$

where $C_i \in k[z_0, \ldots, z_{n-h}]_{d-1}$, $c_i \in k$ and $P \in k[z_0, \ldots, z_{n+1}]_{d-1}$. In particular,

$$Z = \{z_0 = \cdots = z_{n-h} = P = 0\} \subset \Lambda \cap \mathrm{Sing}(X_d)$$

and since $\dim(Z) \geq h - 1$ we get a contradiction. □

Corollary 3.23. *Assume that $h \geq 2$, $n > 2h - 1$, and let $X_4 \subset \mathbb{P}^{n+1}$ be a general quartic hypersurface containing an h-plane. Then X_4 is unirational.*

Proof. Note that $n > 2h - 1$ yields that a general quartic containing a fixed h-plane is smooth. The claim follows from Theorem 3.22. □

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4. Divisors in products of projective spaces and Quintics

We now study the unirationality of quintic hypersurfaces that are singular along linear subspaces and of divisors of bidegree $(3,2)$ in products of projective spaces. We begin with a series of preliminary results that we will need later on.

Lemma 4.1. *Let $\mathcal{C} \to W$ be a fibration in m-dimensional cubic hypersurfaces with $m \geq 2$, $\mathcal{C}$ irreducible and W a rational variety over a C_r field k. If the general fiber of $\mathcal{C} \to W$ is an irreducible cubic with no triple point and*

$$m > 3^{r+\dim(W)} - 2,$$

then $\mathcal{C}$ is unirational.

Proof. Let $\mathcal{C}'$ be the generic fiber of $\mathcal{C} \to W$. Then $\mathcal{C}'$ is a cubic hypersurface over the function field $F = k(t_1, \dots, t_{\dim(W)})$. Since $m + 2 > 3^{r+\dim(W)}$, Remark 2.10 yields that $\mathcal{C}'$ has a point. Hence, by [Kol02, Theorem 1] $\mathcal{C}'$ is unirational over F and so since $\mathcal{C}$ is irreducible $\mathcal{C}$ is unirational. □

Lemma 4.2. *Let $Q^{N-1} \subset \mathbb{P}^N$ be a smooth $(N-1)$-dimensional quadric hypersurface over a C_r field. If*

$$N - 2s + 1 > 2^r$$

for some $0 \leq s \leq \lfloor \frac{N-1}{2} \rfloor$, then through any point of Q^{N-1} there is an s-plane contained in Q^{N-1}.

Proof. Since $N-1 > 2^r - 2$ by Remark 2.10 Q^{N-1} has a point. Hence, Q^{N-1} is rational and in particular the set of its rational points is dense. Take a point $x_0 \in Q^{N-1}$.

The tangent space $T_{x_0}Q^{N-1}$ cuts out on Q^{N-1} a cone with vertex x_0 over an $(N-3)$-dimensional quadric Q^{N-3}. Since $N - 3 > 2^r - 2$ Remark 2.10 yields the existence of a point $x_1 \in Q^{N-3}$. The line $\langle x_0, x_1 \rangle$ is therefore contained in Q^{N-1}.

Now, $T_{x_1}Q^{N-3} \cap Q^{N-3}$ is a cone with vertex x_1 over an $(N-5)$-dimensional quadric Q^{N-5} which again by Remark 2.10 has a point $x_2 \in Q^{N-5}$. The 2-plane $\langle x_0, x_1, x_2 \rangle$ is contained in Q^{N-1}.

Proceeding recursively, we have that $T_{x_{s-1}}Q^{N-2(s-1)-1} \cap Q^{N-2(s-1)-1}$ is a cone with vertex x_{s-1} over an $(N-2s-1)$-dimensional quadric Q^{N-2s-1}. Since $N - 2s + 1 > 2^r$, the quadric Q^{N-2s-1} has a point x_s and the s-plane $\langle x_0, \dots, x_s \rangle$ is then contained in Q^{N-1}. □

Lemma 4.3. *Let $X = Q_1 \cap \cdots \cap Q_c \subset \mathbb{P}^N$ be a smooth complete intersection of quadrics. Assume that X contains a $(c-1)$-plane Λ. Then X is rational.*

Proof. Note that since X is smooth Q_i must be smooth along X. Let H be a general c-plane containing Λ. Then H intersects Q_i along $\Lambda \cup H_i$ where H_i is a $(c-1)$-plane. Hence, we get c linear subspaces of dimension $c-1$ of $H \cong \mathbb{P}^c$.

Since H is general, these $(c-1)$-planes intersect in a point $x_H = H_1 \cap \cdots \cap H_c \in X$. To conclude, it is enough to parametrize X with the $\mathbb{P}^{N-c}$ of c-planes containing Λ. □

Proposition 4.4. *Let $X = Q_1 \cap \cdots \cap Q_c \subset \mathbb{P}^N$ be a smooth complete intersection of quadrics over a C_r field. If*

$$N - s(c+1) + 1 > 2^r c,$$

then through any point of X there is an s-plane contained in X.

Proof. For $s = 0$, the claim follows from Remark 2.10. We proceed by induction on s. Since

$$N - (s-1)(c+1) + 1 > N - s(c+1) + 1 > 2^r c$$

through any point of X, there is an $(s-1)$-plane. Fix one of these $(s-1)$-planes, and denote it by $\Lambda^{s-1} \subset X$. The s-planes in $\mathbb{P}^N$ containing Λ^{s-1} are parametrized by $\mathbb{P}^{N-s}$.

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Arguing as in the proof of Lemma 4.2, we see that requiring such an s-plane to be contained in Q_i yields s linear equations given by the tangent spaces of Q_i at s general points of Λ^{s-1} plus a quadratic equation induced by Q_i itself.

Hence, the points of $\mathbb{P}^{N-s}$ corresponding to s-planes contained in X are parametrized by a subvariety cut out by sc linear equations and c quadratic equations that is an intersection Y of c quadrics in $\mathbb{P}^{N-s-sc}$. Since $\dim(Y) = N - s(c+1) - c > 2^r c - 1 - c \geq -1$ and $N - s(c+1) + 1 > 2^r c$, Remark 2.10 yields that Y has a point and so there is an s-plane through Λ^{s-1} contained in X. □

Corollary 4.5. *Let $X = Q_1 \cap \cdots \cap Q_c \subset \mathbb{P}^N$ be a smooth complete intersection of quadrics over a C_r field. If*

$$N - c^2 + 2 > 2^r c,$$

then X is rational.

Proof. By Proposition 4.4 with $s = c - 1$, the complete intersection X contains a $(c-1)$-plane, and hence, the claim follows from Lemma 4.3. □

We are now ready to prove our first result on unirationality of quintic hypersurfaces.

Proposition 4.6. *Let $X_5^n = \{\overline{f} = 0\} \subset \mathbb{P}^{n+1}$ be a quintic hypersurface of the form*

$$\overline{f} = \sum_{i_0+\cdots+i_{n-h}=3} x_0^{i_0} \dots x_{n-h}^{i_{n-h}} A_{i_0,\dots,i_{n-h}}(y_0, \dots, y_h, x_{n-h})$$

with $A_{i_0,\dots,i_{n-h}}$ quadratic polynomials. If either

(i) *$h \geq 4$ and the complete intersection*

$$W' = \{A_{3,\dots,0} = A_{2,1,0,\dots,0} = x_0 = \cdots = x_{n-h} = 0\} \subset \mathbb{P}^h_{(y_0,\dots,y_h)}$$

contains a line, or

(ii) *$h \geq 5$ and the quadric*

$$\widetilde{Q} = \{x_0 = \cdots = x_{n-h} = A_{3,\dots,0} = 0\} \subset \mathbb{P}^h_{(y_0,\dots,y_h)}$$

contains a 2-plane,

and X_5^n is otherwise general, then X_5^n is unirational.

Proof. Note that X_5^n is a hypersurface of the form (2.12) in Proposition 2.11. Hence, X_5^n has multiplicity three along $\Lambda = \{x_0 = \cdots = x_{n-h} = 0\}$, and multiplicity two along $\Lambda' = \{y_0 = \cdots = y_h = x_{n-h} = 0\}$.

First, consider (i). Fix a point $p \in \Lambda'$, say $p = [1 : 0 : \cdots : 0]$. The lines through p intersecting X_5^n at p with multiplicity at least four are parametrized by the variety

$$Y = \{A_{3,\dots,0} = x_1 A_{2,1,0,\dots,0} + \cdots + x_{n-h} A_{2,0,\dots,0,1} = 0\} \subset \mathbb{P}^n_{(x_1,\dots,x_{n-h},y_0,\dots,y_h)}.$$

Assume Y to be unirational. Associating to a general point $y \in Y$, representing a line l_y, the fifth intersection point of l_y and X_5^n we get a rational map

$$\psi : Y \dashrightarrow X_5^n.$$

Set $Y_\psi = \overline{\psi(Y)} \subset X_5^n$, and let $\pi : X_5^n \dashrightarrow \mathbb{P}^{n-h}$ be the restriction to X_5^n of the projection from Λ. We want to prove that $\pi_{|Y_\psi} : Y_\psi \dashrightarrow \mathbb{P}^{n-h}$ is dominant. Indeed, if so Y_ψ would be a unirational variety transverse to the quadric fibration induced by the projection from Λ and dominating $\mathbb{P}^{n-h}$, and Proposition 2.7 would imply that X_5^n is unirational.

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Since Y_ψ is unirational it is enough to prove that the induced map $\overline{\pi}_{|Y_\psi} : \overline{Y}_\psi \dashrightarrow \mathbb{P}^{n-h}$ between the algebraic closures is dominant. Consider the complete intersection

$$Z = Y \cap X_5^n = \{A_{3,\dots,0} = x_1 A_{2,1,0,\dots,0} + \cdots + x_{n-h} A_{2,0,\dots,0,1} = \overline{f} = 0\} \subset \mathbb{P}^{n+1}_{(x_0,\dots,x_{n-h},y_0,\dots,y_h)}.$$

A general point of $z \in Z \subset X_5^n$ represents a line l_z intersecting X_5^n with multiplicity four at p and multiplicity one at z. Hence, $Z \subset Y_\psi$.

Fix a general point $q \in \mathbb{P}^{n-h}$, and consider

$$Z_q = \overline{\pi}^{-1}_{|Y_\psi}(q) \cap Z.$$

Note that since $q \in \mathbb{P}^{n-h}$ is defined by $n-h$ equations, $Z \subset \mathbb{P}^{n+1}_{(x_0,\dots,x_{n-h},y_0,\dots,y_h)}$ is cut out by three equations and by hypothesis $n+1-(n-h)-3 = h-2 > 0$. So Z_q is nonempty. Then $\overline{\pi}_{|W_\psi}$ is dominant when restricted to $\overline{Z}$, and hence, it is dominant.

Now, we prove that Y is unirational. In $\mathbb{P}^n_{(x_1,\dots,x_{n-h},y_0,\dots,y_h)}$, fix the point $p' = [1 : 0 : \cdots : 0]$. The quadric

$$\{x_1 = \cdots = x_{n-h} = A_{2,1,0,\dots,0}\} \subset \mathbb{P}^h_{(y_0,\dots,y_h)}$$

parametrizes lines that are contained in

$$Y_3 = \{x_1 A_{2,1,0,\dots,0} + \cdots + x_{n-h} A_{2,0,\dots,0,1}\}.$$

Indeed, the tangent cone of Y_3 at p' is defined by $\{A_{2,1,0,\dots,0} = 0\}$ and $\{x_1 = \cdots = x_{n-h} = 0\} \subset Y_3$. So these lines intersect Y_3 with multiplicity three at p' and at another point in the linear space $\{x_1 = \cdots = x_{n-h} = 0\}$.

Set $Y_2 = \{A_{3,\dots,0} = 0\}$, and let W be the cone over W' with vertex p'. Since p' is in the vertex of Y_2, we have that

$$W \subset Y = Y_2 \cap Y_3.$$

Furthermore, since W' contains a line by Lemma 4.3 it is rational, and hence, W is rational as well.

As in the proof of Proposition 3.4, we construct a quadric bundle $\widetilde{\mathcal{Q}} \to W$ with $(n-4)$-dimensional fibers whose general point (w, l_w) represents a point $w \in W$ and a line l_w which is contained in Y_2 and intersects Y_3 with multiplicity two at w.

Associating to (w, l_w) the third point of intersection of l_w and Y_3, we get a rational map $W \dashrightarrow Y$ and arguing as in the proof of Proposition 3.4 we see that such rational map is dominant.

Now, we consider (ii). Note that

$$\widetilde{Q}' = \{x_1 = \cdots = x_{n-h} = A_{3,\dots,0} = 0\} \subset X_5^n \subset \mathbb{P}^{n+1}$$

is an h-dimensional quadric cone over $\widetilde{Q}$ with vertex $[1 : 0 : \cdots : 0]$. Since $\widetilde{Q}$ contains a 2-plane $\widetilde{Q}'$ contains a 3-plane $H \subset X_5^n$. If $x \in H$ is a general point the lines through x intersecting X_5^n with multiplicity four at x are parametrized by a complete intersection of a quadric and a cubic in $\mathbb{P}^{n-1}$. Hence, we get a fibration $\mathcal{Y}_{2,3} \to H$ whose fiber over a general $x \in H$ is a complete intersection $Y_{2,3,x}$ of a quadric $Y_{2,x} \subset \mathbb{P}^{n-1}$ and a cubic $Y_{3,x} \subset \mathbb{P}^{n-1}$. Note that since X_5^n is general among the quintics satisfying (ii) for $x \in H$ both $Y_{2,x}$ and $Y_{2,3,x}$ are smooth.

The generic fiber of $\mathcal{Y}_{2,3} \to H$ is then a complete intersection $\mathcal{Y}_{2,3,k(H)} = \mathcal{Y}_{2,k(H)} \cap \mathcal{Y}_{3,k(H)}$ satisfying the hypotheses of Lemma 3.1. Indeed, by considering the 2-plane parametrizing lines in H through a general $x \in H$ we get a 2-plane over $k(H)$ contained in $\mathcal{Y}_{2,k(H)}$.

Therefore, by Lemma 3.1 $\mathcal{Y}_{2,3,k(H)}$ is unirational over $k(H)$. As usual after replacing H by an open subset, we may assume that $\mathcal{Y}_{2,3}$ is irreducible and so we get that $\mathcal{Y}_{2,3}$ is unirational. Now, a general point

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of $\mathcal{Y}_{2,3}$ represents a pair (x, l_x) with $x \in H$ general and l_x a line intersecting X_5^n with multiplicity four at x. As usual, associating to $(x, l_x) \in \mathcal{Y}_{2,3}$ the fifth point of intersection of l_x with X_5^n we get a rational map

$$\phi : \mathcal{Y}_{2,3} \dashrightarrow X_5^n.$$

Note that $\dim(\mathcal{Y}_{2,3}) = 3 + (n-3) = n$. Arguing as in the last part of the proof of Proposition 3.4, we see that ϕ is generically finite and therefore it is dominant. □

Theorem 4.7. *Let $D \subset \mathbb{P}^{n-h} \times \mathbb{P}^{h+1}$, with $n - h \geq 2$, be a general divisor of bidegree $(3,2)$ over a C_r field. If either*

(i) $n \geq 4$, $2h \geq n+3$ *and* $h > 2^{r+1} + 2$, *or*
(ii) $n \geq 6$, $h \geq 5$ *and* $h > 2^r + 3$, *or*
(iii) $n - h - 1 > 3^{r+1} - 2$,

then D is unirational.

Proof. By Proposition 2.11, D is birational to a quintic hypersurface $X_5^n \subset \mathbb{P}^{n+1}$ of the form considered in Proposition 4.6.

Consider (i). Since, $h > 2^{r+1} + 2$ by Proposition 4.4 the complete intersection W' in Proposition 4.6 contains a line, and hence, Proposition 4.6 yields that X_5^n is unirational.

For (ii), note that since $h > 2^r + 3$ Lemma 4.2 implies that the quadric $\widetilde{Q}$ in Proposition 4.6 contains a 2-plane.

Now, consider (iii). The projection $\pi_2 : D \to \mathbb{P}^{h+1}$ endows D with a structure of fibration in $(n-h-1)$-dimensional cubic hypersurfaces. Take a general line $L \subset \mathbb{P}^{h+1}$, and set $\mathcal{C}_L = \pi_2^{-1}(L)$. Then $\mathcal{C}_L$ is a fibration in $(n-h-1)$-dimensional cubic hypersurfaces over $\mathbb{P}^1$ and since $n - h - 1 > 3^{r+1} - 2$ Lemma 4.1 yields that $\mathcal{C}_L$ is unirational. Finally, to conclude it is enough to note that $\pi_{|\mathcal{C}_L} : \mathcal{C}_L \to \mathbb{P}^{n-h}$ is dominant and to apply Proposition 2.7. □

Remark 4.8. Take for instance $r = 0$ that is k is algebraically closed. Remark 2.10 gives the rationality of a $D \subset \mathbb{P}^{n-h} \times \mathbb{P}^{h+1}$ as in Theorem 4.7 for $h > 2^{n-h} - 2$ while Theorem 4.7 gives the unirationality of D for $h > 4$ as long as $n \geq 4$ and $h \geq n - h + 3$. For example, take $h = 10$. Then Remark 2.10 yields that D is rational for $n \leq 13$ while Theorem 4.7 gives the unirationality of D for $n \leq 17$.

Furthermore, there are cases covered by (iii) but not by (i). For instance, by (iii) we get that a general $D \subset \mathbb{P}^{n-1} \times \mathbb{P}^2$ of bidegree $(3,2)$ and dimension at least four is unirational.

For $r \geq 1$, (i) generally performs better that (ii). For instance, the case $h = 7, n = 11$ is covered by (i) but not by (ii).

Remark 4.9. In particular, Theorem 4.7 (ii) together with [ABP18, Theorem A] yields that a very general, meaning outside of a countable union of closed subsets of the corresponding parameter space, divisor of bidegree $(3,2)$ in $\mathbb{P}^3 \times \mathbb{P}^2$, over an algebraically closed field of characteristic zero, is unirational but not stably rational.

We end this section with our main results on the unirationality of quintic hypersurfaces which are singular along a linear subspace.

Theorem 4.10. *Let $X_5 \subset \mathbb{P}^{n+1}$ be a quintic hypersurface over a C_r field having multiplicity three along an h-plane and otherwise general. Assume that $n - h \geq 2$. If either*

(i) $n \geq 5$, $2h \geq n+4$ *and* $h > 2^{r+1} + 3$, *or*
(ii) $n \geq 7$, $h \geq 6$ *and* $h > 2^r + 4$, *or*
(iii) $n - h - 1 > 3^{r+1} - 2$,

then X_5 is unirational.

Similarly, if $X_5 \subset \mathbb{P}^{n+1}$ has multiplicity two along an h-plane with $h \geq 2$ and is otherwise general, and either

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(i) $n \geq 5$, $2h \leq n-4$ *and* $n-h > 2^{r+1}+3$, *or*
(ii) $n \geq 7$, $n-h-1 \geq 5$ *and* $n-h-1 > 2^r+3$, *or*
(iii) $h > 3^{r+1}-1$,

then X_5 is unirational.

Proof. By Lemma 2.4, the exceptional divisor $\widetilde{E} \subset \widetilde{X}_5$ is a divisor of bidegree $(3,2)$ in $\mathbb{P}^{n-h} \times \mathbb{P}^h$. Since X_5 is general, $\widetilde{E}$ maps dominantly onto $\mathbb{P}^{n-h}$, and hence, to conclude we apply Proposition 2.7 and Theorem 4.7. For the second statement, it is enough to argue as in the previous case on a divisor of bidegree $(2,3)$ in $\mathbb{P}^{n-h} \times \mathbb{P}^h$. □

Proposition 4.11. *Let $X_5 \subset \mathbb{P}^{n+1}$ be a quintic hypersurface over a field k having multiplicity three along an $(n-1)$-plane and otherwise general. If either*

(i) $n \geq 5$ *and k is either a number field or a real closed field, or*
(ii) *k is C_r, $n > 4$ and $n > 2^r$,*

then X_5 is unirational.

Proof. The exceptional divisor $\widetilde{E} \subset \widetilde{X}_5$ is a general divisor of bidegree $(3,2)$ in $\mathbb{P}^1 \times \mathbb{P}^{n-1}$. By [Mas22, Corollary 4.13, Lemma 4.18] and Remark 2.10, under our hypotheses $\widetilde{E}$ has a point and hence [Mas22, Theorem 1.8] yields that $\widetilde{E}$ is unirational. To conclude, it is enough to argue as in the proof of Theorem 4.10 applying Propositions 2.7. □

Remark 4.12. When k is a real closed field, the unirationality of a quintic hypersurface $X_5 \subset \mathbb{P}^{n+1}$ as in Proposition 4.11 follows from [Kol99, Corollary 1.8].

Proposition 4.13. *Let $X_d \subset \mathbb{P}^{n+1}$ be a hypersurface of degree d over a field k having multiplicity $d-2$ along an h-plane and otherwise general. Assume that $(h+1)(d-2)$ is odd. If either*

(i) $d \leq \frac{5h+3}{h+1}$, *or*
(ii) $d \leq \frac{6h+1}{h+1}$, $h \leq 4$, *k is C_r and $h+1 > 2^{r+n-h-1}$,*

then X_5 is unirational.

Proof. In this case, the exceptional divisor $\widetilde{E} \subset \widetilde{X}_d$ is a general divisor of bidegree $(d-2,2)$ in $\mathbb{P}^{n-h} \times \mathbb{P}^h$. Note that the discriminant of the quadric bundle $\widetilde{E} \to \mathbb{P}^{n-h}$ has degree $(h+1)(d-2)$. Hence, the claim follows from [Mas22, Theorem 1.7] and Propositions 2.7. □

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