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THE AMBROSETTI–PRODI PROBLEM FOR ELLIPTIC SYSTEMS WITH TRUDINGER–MOSER NONLINEARITIES

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Abstract In this work we study the following class of elliptic systems:

$$\begin{aligned} \Delta u &= a(x)u + b(x)v + H_u(x, u_+, v_+) + f_1(x) \quad \text{in } \Omega, \\ -\Delta v &= b(x)u + c(x)v + H_v(x, u_+, v_+) + f_2(x) \quad \text{in } \Omega, \\ & u = 0, \quad v = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, H is a C^1 function in $[0, +\infty) \times [0, +\infty)$ which is assumed to be in the critical growth range of Trudinger–Moser type and $f_1, f_2 \in L^r(\Omega)$, $r > 2$. Under suitable hypotheses on the functions $a, b, c \in C(\bar{\Omega})$ and using variational methods, we prove the existence of two solutions depending on f_1 and f_2.

Keywords: systems of elliptic equations; Ambrosetti–Prodi problem; Trudinger–Moser inequality

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1. Introduction

In this work we study the following elliptic system:

$$\left.\begin{aligned} -\Delta u &= a(x)u + b(x)v + H_u(x, u_+, v_+) + f_1(x) \quad \text{in } \Omega, \\ -\Delta v &= b(x)u + c(x)v + H_v(x, u_+, v_+) + f_2(x) \quad \text{in } \Omega, \\ & u = 0, \quad v = 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^2$, H is a C^1 function in $[0, +\infty) \times [0, +\infty)$ satisfying a Trudinger–Moser growth condition uniformly in $x \in \Omega$. We define $w_+ = \max\{w, 0\}$ and assume $f_1, f_2 \in L^r(\Omega)$, $r > 2$. By analysing the interaction between the matrix $A \in C(\bar{\Omega}, M_{2\times 2}(\mathbb{R}))$ given by

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix}$$

and the spectrum of $(-\Delta, H_0^1)$, we prove the existence of two solutions depending on the forcing terms f_1 and f_2.

This system is motivated by the famous paper by Ambrosetti and Prodi [**2**], which has been studied, explored and extended by an enormous variety of authors during the last 30 years. We refer the reader to [**9**, **18**] for a review. Bringing the discussion closer to our interests, we mention the work of de Figueiredo and Yang [**9**]; they considered the following problem:

$$\left.\begin{aligned} -\Delta u = \lambda u + g(x, u_+) + f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

where Ω is bounded and smooth in $\mathbb{R}^N$, $N \geqslant 7$, $g(x, u_+) = u_+^{2^*-1}$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent. They proved the existence of two solutions provided that f satisfies appropriate conditions regarding the sign of solutions to a related linear problem. Then, in an attempt to improve the restrictive condition on the dimension, Calanchi and Ruf [**5**] studied the same problem, providing an alternative approach that showed the existence of solutions for $N \geqslant 6$ and then, by adding a suitable subcritical perturbation, for the lower-dimensional cases $3 \leqslant N \leqslant 5$. Some of these results were extended to the system in [**19**]. Following these works, the natural problem concerning the bi-dimensional case, in which the critical growth of Sobolev type is replaced by a Trudinger–Moser growth condition, was investigated by Calanchi *et al*. [**6**].

Here we extend the results obtained in [**6**] for the scalar case, but we slightly change the arguments, weakening some hypotheses and strengthening others, in order to better explain some crucial results needed in both [**6**] and the present paper. Calanchi *et al*. [**6**] studied problem (1.2) in a bounded domain Ω in $\mathbb{R}^2$, where g is a function satisfying an unilateral critical Trudinger–Moser-type growth. They showed that for a given class of functions f and for $\lambda_k < \lambda < \lambda_{k+1}$, $k \geqslant 1$ (λ_i being the eigenvalues of $(-\Delta, H_0^1(\Omega))$), there exist two solutions for (1.2), one of which is negative. More precisely, this class of f is exactly the class for which the unique solution of the linear problem $-\Delta u = \lambda u + f(x)$ in Ω and $u = 0$ on $\partial\Omega$ is negative.

As is well known, one of the main difficulties in leading with this kind of growth condition in g is proving that the minimax level of the functional associated to this problem avoids levels of non-compactness. This is done in [**6**] by assuming an additional hypothesis, imposed to guarantee that this level lies below some critical constant. In order to reach the appropriate level, the techniques used therein require that the Moser functions z_n^r (defined below) must have support in a ball B_r such that $r > 0$ is chosen to be sufficiently small in many steps of the arguments.

The problem is that the minimax level and its estimates depend on r (since they depend on the Moser functions) and, as far as we are concerned, these estimates must occur uniformly in r (in order to prove that the weak limit of the Palais–Smale sequence of this level is non-trivial), but it is not clear whether such a uniform estimate can be proved in that case. Thus, in this paper we have considered different arguments in order to overcome this difficulty. Moreover, since Calanchi *et al*. assumed that $\lambda > \lambda_1$, the forcing term f was not a problem in the control of their minimax level: in such a case, f must be positive in some set of positive measure in Ω and so, by choosing a path in the linking geometry where the involved functions are supported in this set, they avoid possible loss of control. Being aware that f may be negative in all of Ω if λ is assumed to be below λ_1, they give a brief comment on how to proceed, but, as far as we know, that approach may not seem to work. We provide an alternative answer to that issue as well.

For the linear part of the problem, given by the continuous functions a, b and c, we analyse some generalizations of the scalar case, mainly inspired by [**7**] for non-constant a, b and c and by [**10**] for constant a, b and c. We also improve the hypothesis imposed in [**10**, **11**, **19**].

2. Hypothesis and main results

Let us rewrite (1.1) in its vector form:

$$\left.\begin{aligned} -\Delta U &= A(x)U + \nabla H(x, U_+) + F(x) \quad \text{in } \Omega, \\ U &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.1}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad U_+ = \begin{pmatrix} u_+ \\ v_+ \end{pmatrix}, \quad A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \quad \text{for all } x \in \Omega$$

and

$$F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \in L^r(\Omega) \times L^r(\Omega).$$

We also define

$$\nabla H(x, U_+) = \begin{pmatrix} H_u(x, U_+) \\ H_v(x, U_+) \end{pmatrix}.$$

Denoting by $\mu_1(x)$ and $\mu_2(x)$ the eigenvalues of $A(x)$ for each $x \in \Omega$, we suppose that the following hold.

(i) Either A is constant and

- (A_1) $\mu_1 \leqslant \mu_2 < \lambda_1$ or
- (A_2) there exists $k \geqslant 1$ such that $\lambda_k < \mu_1 \leqslant \mu_2 < \lambda_{k+1}$.

(ii) Or $b(x) \geqslant 0$ for all $x \in \Omega$ and $\max_{x\in\Omega} \max\{a(x), c(x)\} > 0$ and

- (A_3) $1 < \lambda_1^A$ or
- (A_4) there exists $k \geqslant 1$ such that $\lambda_k^A < 1 < \lambda_{k+1}^A$.

By λ_i^A we denote the eigenvalues associated to the linear problem

$$\left.\begin{aligned} -\Delta U &= \lambda A(x)U \quad && \text{in } \Omega, \\ U &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.2}$$

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Condition (ii) allows us to apply the theory of compact and self-adjoint operators (see [**8**] as well as some results obtained in [**7**]) to ensure the existence of an unbounded sequence of eigenvalues

$$0 < \lambda_1^A < \lambda_2^A \leqslant \lambda_3^A \leqslant \cdots$$

(and its corresponding eigenfunctions $\Phi_1^A, \Phi_2^A, \dots$) such that the first one is simple and is the only one admitting a strictly positive eigenfunction. Defining

$$\left.\begin{aligned} &E_0 = \{0\}, \\ &E_k = \operatorname{span}\{\Phi_1^A, \dots, \Phi_k^A\}, \end{aligned}\right\} \tag{2.3}$$

the following useful inequalities are consequences of this theory and will be used extensively in this work:

$$\left.\begin{aligned} \|V\|^2 &\leqslant \lambda_k^A \int_\Omega (A(x)V, V)_{\mathbb{R}^2} \quad \text{for all } V \in E_k, \\ \|V\|^2 &\geqslant \lambda_{k+1}^A \int_\Omega (A(x)V, V)_{\mathbb{R}^2} \quad \text{for all } V \in E_k^\perp. \end{aligned}\right\} \tag{2.4}$$

Here we denote by $(\cdot,\cdot)_{\mathbb{R}^2}$ the Euclidian inner product in $\mathbb{R}^2$ and by $|\cdot|$ its associated norm.

For the nonlinearity H we suppose that the following hold.

(H_0) $H \in C^1(\bar{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $H_u, H_v \geqslant 0$.

(H_1) ∇H is subcritical if

$$\lim_{|U|\to\infty} \frac{|\nabla H(x,U)|}{\exp(\alpha|U|^2)} = 0 \quad \text{for all } \alpha > 0$$

uniformly in $x \in \Omega$.

On the other hand, ∇H has a critical growth if there exists $\alpha_0 > 0$ such that

$$\lim_{|U|\to\infty} \frac{|\nabla H(x,U)|}{\exp(\alpha|U|^2)} = \begin{cases} 0 & \text{for all } \alpha > \alpha_0, \\ +\infty & \text{for all } \alpha < \alpha_0, \end{cases}$$

uniformly in $x \in \Omega$.

In order to establish a variational structure for (1.1) we should also consider the following assumptions.

(H_2) $H(x,0,0) = H_u(x,0,v) = H_v(x,u,0) = 0$ for all $u, v \geqslant 0$, $x \in \Omega$.

(H_3) $\displaystyle\lim_{|U|\to\infty} \frac{H(x,U) + |\nabla H(x,U)|}{(\nabla H(x,U), U)_{\mathbb{R}^2}} = 0$ uniformly in $x \in \Omega$.

(H_4) $|\nabla H(x,U)| = o(|U|)$ when $|U| \to 0$ uniformly in $x \in \Omega$.

Remark 2.1. By (H_2), H can be extended to the whole plane: letting $H(x,u,v) = H(x,u_+,v_+)$, we still have $H \in C^1(\Omega \times \mathbb{R}^2)$ satisfying (H_0)–(H_4). Therefore, H will always denote this extension.

Remark 2.2. We give some examples of functions H satisfying (H_0)–(H_4):

(i) $H(x,u,v) = \gamma(u) + \gamma(v)$, where

$$\gamma(s) = \int_0^s t_+ \exp(t^2 + t)\,\mathrm{d}t;$$

(ii) $H(x,u,v) = \gamma(u) + \gamma(v) + \gamma(u)\gamma(v)$.

Remark 2.3. Assumption (H_3) is equivalent to the Ambrosetti–Rabinowitz condition for H and $|\nabla H|$ for all $\theta > 0$. This means that for each $\theta > 0$ there exists $s_\theta \geqslant 0$ such that

$$H(x,S) + |\nabla H(x,S)| \leqslant \frac{1}{\theta}(\nabla H(x,S), S)_{\mathbb{R}^2} \quad \text{for all } x \in \Omega \text{ and } |S| \geqslant s_\theta.$$

We also remark that in the scalar case there is no need to impose such a condition in $h = (H)'$, since it is obviously satisfied.

We seek solutions in $E = H_0^1(\Omega) \times H_0^1(\Omega)$, which is considered with its usual norm

$$\|(u,v)\|^2 = \|u\|^2_{H_0^1} + \|v\|^2_{H_0^1}.$$

More precisely, $U = (u,v) \in E$ is a (weak) solution of (2.1) if

$$\begin{aligned}\int_\Omega \nabla u \nabla \varphi + \int_\Omega \nabla v \nabla \psi - \int_\Omega (A(u,v), (\varphi,\psi))_{\mathbb{R}^2} \\ - \int_\Omega (\nabla H(x,u_+,v_+), (\varphi,\psi))_{\mathbb{R}^2} - \int_\Omega (F(x), (\varphi,\psi))_{\mathbb{R}^2} = 0 \quad \text{for all } (\varphi,\psi) \in E,\end{aligned}$$

where we define $\nabla u \nabla v = (\nabla u, \nabla v)_{\mathbb{R}^2}$ following the usual conventions.

We shall prove the existence of solutions for (2.1) in cases of both subcritical and critical growth. The results also differ according to the conditions on the matrix A, and we begin by showing that independently of H satisfying (H_0) there is always a region on $L^r(\Omega) \times L^r(\Omega)$ such that if $F = (f_1, f_2)$ belongs to it, then (2.1) has a negative solution, which will be denoted by $\Phi = (\varphi, \psi)$. This region is precisely determined by the subset of $L^r(\Omega) \times L^r(\Omega)$, where the unique solution of the linear problem

$$\left.\begin{aligned} -\Delta U &= A(x)U + F(x) && \text{in } \Omega, \\ U &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.5}$$

is negative (and therefore it is also a solution of (2.1)). As usual in Ambrosetti–Prodi problems, we can construct this region by a suitable parametrization related to the first eigenfunction: if (A_1) or (A_2) is satisfied, let

$$F(x) = F_T(x) = P(x) + Te_1(x), \tag{2.6}$$

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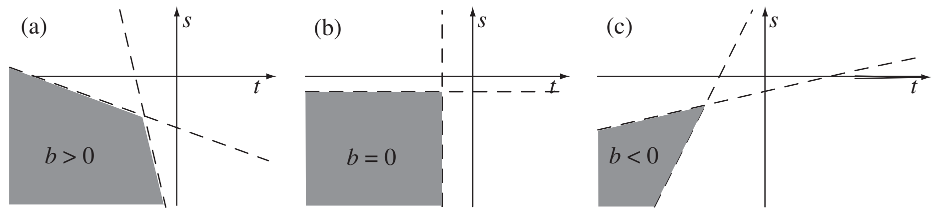

Figure 1. The regions determined in Theorem 2.4 under (A_1).

where $T = (s, t) \in \mathbb{R}^2$, e_1 is the first positive eigenfunction of $(-\Delta, H_0^1(\Omega))$ and $P = (p_1, p_2)$ is such that

$$\int_\Omega p_1(x)e_1(x) = \int_\Omega p_2(x)e_1(x) = 0.$$

If we are assuming (A_3) or (A_4), then the parametrization will occur in one dimension only, setting

$$F(x) = F_t(x) = P(x) + tA(x)\Phi_1^A(x), \tag{2.7}$$

where $t \in \mathbb{R}$ and Φ_1^A denotes the first positive eigenfunction of (2.2).

The first theorem gathers our results on the linear problem.

Theorem 2.4. *The following claims hold.*

(i) *If* (A_1) *is satisfied, then there exists an unbounded region* $\mathcal{R} \subset \mathbb{R}^2$ *(described below) such that (2.5) admits a negative solution* Φ_T *provided that* $F = F_T$ *with* $T = (s, t) \in \mathcal{R}$.

(ii) *If* (A_2) *is satisfied, then there exists an unbounded region* $\mathcal{S} \subset \mathbb{R}^2$ *(described below) such that (2.5) admits a negative solution* Φ_T *provided that* $F = F_T$ *with* $T = (s, t) \in \mathcal{S}$.

(iii) *If* (A_3) *is satisfied, then there exists* $C_1 \geqslant 0$ *such that (2.5) admits a negative solution* Φ_t *provided that* $F = F_t$ *with* $t \leqslant -C_1$.

(iv) *If* (A_4) *is satisfied, then there exists* $C_2 \geqslant 0$ *such that (2.5) admits a negative solution* Φ_t *provided that* $F = F_t$ *with* $t \geqslant C_2$.

The regions determined in Theorem 2.4 are delimited by a curve obtained by the intersection of two appropriate lines and depend on the sign of b. For (A_1), we have the following.

(i) If $b > 0$ (Figure 1 (a)),

$$\mathcal{R} = \{(\theta, \tau) \in \mathbb{R}^2;\ \tau < \alpha_1(\theta),\ \tau < \alpha_2(\theta)\},$$

where α_1 and α_2 are two intersecting lines with negative slopes.

(ii) If $b = 0$ (Figure 1 (b)),

$$\mathcal{R} = \{(\theta, \tau) \in \mathbb{R}^2;\ \theta < -C_2,\ \tau < -C_1\},$$

where $C_1, C_2 > 0$.

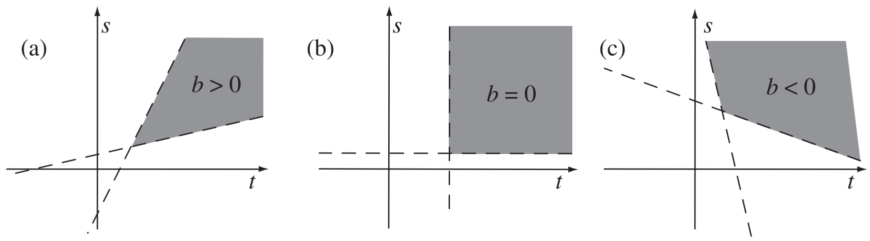

Figure 2. The regions determined in Theorem 2.4 under (A_2).

(iii) If $b < 0$ (Figure 1 (c)),

$$\mathcal{R} = \{(\theta,\tau) \in \mathbb{R}^2;\ \tau < \alpha_1(\theta),\ \tau > \alpha_2(\theta)\}.$$

Here α_1 and α_2 are two intersecting lines with positive slopes.

For (A_2) we have the following.

(i) If $b > 0$ (Figure 2 (a)),

$$\mathcal{S} = \{(\theta,\tau) \in \mathbb{R}^2;\ \tau > \beta_1(\theta),\ \tau < \beta_2(\theta)\},$$

where β_1 and β_2 are two intersecting lines with positive slopes.

(ii) If $b = 0$ (Figure 2 (b)),

$$\mathcal{S} = \{(\theta,\tau) \in \mathbb{R}^2;\ \theta > C_2,\ \tau > C_1\},$$

where $C_1, C_2 > 0$ (Figure 2 (c)).

(iii) If $b < 0$,

$$\mathcal{S} = \{(\theta,\tau) \in \mathbb{R}^2;\ \tau > \beta_1(\theta),\ \tau > \beta_2(\theta)\},$$

where β_1 and β_2 are two intersecting lines with negative slopes.

The precise definition of all these lines is given in the proof of Theorem 2.4 in the next section.

The main results of this work show that there exists a second solution to (2.1), provided that a negative one is already given. Theorems 2.4–2.6 give the multiplicity results of Ambrosetti–Prodi type for (2.1).

Theorem 2.5. *Let $F \in L^r(\Omega) \times L^r(\Omega)$ with $r > 2$ such that the solution $\Phi = (\phi,\psi)$ of (2.5) is negative. Suppose that* (H_0), (H_1) *(subcritical growth),* (H_2)–(H_4) *and one of hypotheses* (A_1)–(A_4) *hold. Then there exists a second solution for (2.1).*

We shall make use of variational methods and critical-point theorems such as the mountain pass or linking theorems. Since we are leading with subcritical growth, we can prove a Palais–Smale condition on the functional associated to the problem. So the arguments in the proof follow traditional methods and we shall give them briefly. The next theorem assumes critical growth of Trudinger–Moser type in H and more complex methods and proofs are used.

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Theorem 2.6. *Let $F \in L^r(\Omega) \times L^r(\Omega)$ with $r > 2$ such that the solution $\Phi = (\phi, \psi)$ of (2.5) is negative. Suppose that* (H_0), (H_1) *(critical growth),* (H_2)–(H_4) *and one of the hypotheses* (A_1)–(A_4) *hold. Moreover, consider the following assumption.*

(H_5) *For all $\gamma \geqslant 0$ there exists $c_\gamma \geqslant 0$ such that*

$$(\nabla H(x, S), S)_{\mathbb{R}^2} \geqslant \gamma h(x, u) \exp(\alpha_0 u^2)$$

for all $S = (s, t) \in \mathbb{R}^2$, $u \in \mathbb{R}$ and $x \in \Omega$; $s, t \geqslant u \geqslant c_\gamma$, where $h\colon \Omega \times \mathbb{R} \to \mathbb{R}^+$ is a Carathéodory function satisfying

$$\liminf_{u \to +\infty} \frac{\log(h(x, u))}{u} > 0,$$

uniformly in $x \in \Omega$.

Then there exists a second solution for (2.1).

In both Theorems 2.5 and 2.6, we suppose that F is such that (2.1) admits a negative solution Φ. A second solution is given by $V + \Phi$, where V is a non-trivial solution of the following problem:

$$\left.\begin{aligned} -\Delta V &= A(x)V + \nabla H(x, (V + \Phi)_+) \quad \text{in } \Omega, \\ V &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.8}$$

Therefore, our work consists of proving the existence of a non-trivial solution of (2.8) assuming the hypotheses in either Theorem 2.5 or Theorem 2.6.

3. The linear problem

This section is devoted to the proof of Theorem 2.4.

Let $\Phi_0 = (\phi_0, \psi_0)$ be the solution to the linear problem

$$\left.\begin{aligned} -\Delta U &= A(x)U + P(x) \quad \text{in } \Omega, \\ U &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.1}$$

where we assume one of the conditions (A_1)–(A_4) holds, with corresponding P given in either (2.6) or (2.7).

Proof of Theorem 2.4. For the moment, let us suppose (A_1) or (A_2) holds.

Notice that in either case, one must have $\det(\lambda_1 I - A) > 0$. Indeed, assuming (A_1), this inequality is proven by using that $\det(\mu_2 I - A) = 0$ and $a, c \leqslant \mu_2 < \lambda_1$. For condition (A_2), one must use that $\det(\mu_1 I - A) = 0$ and $a, c \geqslant \mu_1 > \lambda_1$.

Consider a 2×1 matrix $\mu(T)$ such that $\mu(T)e_1$ solves

$$\begin{aligned} -\Delta U &= AU + Te_1(x) \quad \text{in } \Omega, \\ U &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

A direct calculation shows that

$$\mu(T) = \frac{1}{\det(\lambda_1 I - A)} \begin{pmatrix} (\lambda_1 - c)s + bt \\ bs + (\lambda_1 - a)t \end{pmatrix}.$$

Defining

$$\phi_T = \frac{(\lambda_1 - c)s + bt}{\det(\lambda_1 I - A)} e_1 + \phi_0$$

and

$$\psi_T = \frac{bs + (\lambda_1 - a)t}{\det(\lambda_1 I - A)} e_1 + \psi_0,$$

it is obvious that $\varPhi_T = (\phi_T, \psi_T)$ satisfies

$$\left.\begin{aligned} -\Delta \varPhi_T &= A\varPhi_T + F_T(x) \quad \text{in } \varOmega, \\ U &= 0 \quad \text{on } \partial\varOmega, \end{aligned}\right\} \tag{3.2}$$

where F_T is defined in (2.6).

We need to find parameters T such that $\varPhi_T$ is negative and therefore a solution of (2.1), with $F = F_T$.

We recall that $P \in L^r(\varOmega) \times L^r(\varOmega)$ with $r > 2$. Thus, elliptic regularity guarantees that $\varPhi_0 \in C^{1,\nu} \times C^{1,\nu}$ for some $0 < \nu < 1$. Then

$$\left\| \frac{\det(\lambda_1 I - A)}{(\lambda_1 - c)s + bt} \phi_T - e_1 \right\|_{C^1} = \left\| \frac{\det(\lambda_1 I - A)}{(\lambda_1 - c)s + bt} \phi_0 \right\|_{C^1}$$

and

$$\left\| \frac{\det(\lambda_1 I - A)}{bs + (\lambda_1 - a)t} \psi_T - e_1 \right\|_{C^1} = \left\| \frac{\det(\lambda_1 I - A)}{bs + (\lambda_1 - a)t} \psi_0 \right\|_{C^1}.$$

Let $\varepsilon > 0$ be such that if $\|\phi - e_1\|_{C^1} < \varepsilon$, then $\phi > 0$; since we want $\phi_T, \psi_T < 0$, we must have

$$\left.\begin{aligned} (c - \lambda_1)s - bt &> \varepsilon^{-1} \det(\lambda_1 I - A)\|\phi_0\|_{C^1}, \\ -bs + (a - \lambda_1)t &> \varepsilon^{-1} \det(\lambda_1 I - A)\|\psi_0\|_{C^1}. \end{aligned}\right\} \tag{3.3}$$

Assuming, for instance, (A_1), one has $\mu_1 \leqslant \mu_2 < \lambda_1$ and so $a, c < \lambda_1$. Therefore, (3.3) is satisfied in the following cases.

(a) If $b > 0$,

$$t < \left(\frac{c - \lambda_1}{b}\right)s - (\varepsilon b)^{-1} \det(\lambda_1 I - A)\|\phi_0\|_{C^1}, \tag{3.4}$$

$$t < \left(\frac{b}{a - \lambda_1}\right)s - [\varepsilon(\lambda_1 - a)]^{-1} \det(\lambda_1 I - A)\|\psi_0\|_{C^1}. \tag{3.5}$$

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(b) If $b = 0$,

$$s < -[\varepsilon(\lambda_1 - c)]^{-1} \det(\lambda_1 I - A)\|\phi_0\|_{C^1}, \tag{3.6}$$

$$t < -[\varepsilon(\lambda_1 - a)]^{-1} \det(\lambda_1 I - A)\|\psi_0\|_{C^1}. \tag{3.7}$$

(c) If $b < 0$,

$$t > \left(\frac{c - \lambda_1}{b}\right) s + (-\varepsilon b)^{-1} \det(\lambda_1 I - A)\|\phi_0\|_{C^1}, \tag{3.8}$$

$$t < \left(\frac{b}{\lambda_1 - a}\right) s - [\varepsilon(\lambda_1 - a)]^{-1} \det(\lambda_1 I - A)\|\psi_0\|_{C^1}. \tag{3.9}$$

Therefore, (3.4)–(3.9) determine exactly the region $\mathcal{R}$ in Theorem 2.4 (i), thus proving this claim.

The proof of Part (ii) of Theorem 2.4 is analogous. The only difference is that, assuming (A_2), we have $\mu_2 \geqslant \mu_1 > \lambda_1$ and then $a, c > \lambda_1$. Consequently, (3.4)–(3.9) are inverted and define the region $\mathcal{S}$.

If we are analysing conditions (A_3) or (A_4), we proceed in the following way: let again Φ_0 be the solution of (3.1) and consider the problem

$$\begin{aligned} -\Delta U &= A(x)U + tA(x)\Phi_1^A(x) \quad \text{in } \Omega, \\ U &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Notice that

$$\frac{t}{\lambda_1^A - 1}\Phi_1^A$$

is the solution of the above problem. We readily see that

$$\Phi_t = \Phi_0 + \frac{t}{\lambda_1^A - 1}\Phi_1^A$$

solves (2.1) (with $F = F_t$), provided it is negative. Let us show that this is possible for some values of t: notice that

$$\left\|\frac{\lambda_1^A - 1}{t}\Phi_t - \Phi_1^A\right\|_{C^1 \times C^1} = \left|\frac{\lambda_1^A - 1}{t}\right| \|\Phi_0\|_{C^1 \times C^1}.$$

But $\Phi_1^A = (\phi_1^A, \psi_1^A)$ is such that $\partial\phi_1^A/\partial\nu, \partial\psi_1^A/\partial\nu < 0$, where we denote by $\partial h/\partial\nu$ the outward normal derivative of h on $\partial\Omega$. Indeed, since $\phi_1^A, \psi_1^A > 0$ and $a \in L^\infty(\Omega)$, by letting $a(x) = \max\{a(x), 0\} + \min\{a(x), 0\} = a_+(x) + a_-(x)$ we have

$$-\Delta\phi_1^A = \lambda_1^A(a_+(x) + a_-(x))\phi_1^A + \lambda_1^A b(x)\psi_1^A \Rightarrow (-\Delta + K)\phi_1^A \geqslant 0,$$

where we have taken $0 \leqslant -a_-(x) \leqslant K$ for all $x \in \Omega$. Therefore, since $\phi_1^A > 0$ in Ω and $\phi_1^A = 0$ on $\partial\Omega$, Hopf's Lemma guarantees that $\partial\phi_1^A/\partial\nu < 0$ for all $x \in \partial\Omega$. The same procedure is carried out in order to prove that $\partial\psi_1^A/\partial\nu < 0$. Consequently, let $\varepsilon > 0$ be such that $\|\Phi - \Phi_1^A\|_{C^1\times C^1} < \varepsilon$ implies $\Phi > 0$. Then, we need to prove that

$$\frac{1-\lambda_1^A}{t} > 0 \quad \text{and} \quad \frac{1-\lambda_1^A}{t}\|\Phi_0\|_{C^1\times C^1} < \varepsilon.$$

So, in the case (A_3), since $1 < \lambda_1^A$, we get $t < -\varepsilon^{-1}\|\Phi_0\|_{C^1\times C^1}(\lambda_1^A - 1) \equiv C_1$, and if we assume ($\mathrm{A}_4$), since $1 > \lambda_1^A$, we must have $t > \varepsilon^{-1}\|\Phi_0\|_{C^1\times C^1}(1-\lambda_1^A) \equiv C_2$, as required. This completes the proof of Theorem 2.4. □

4. The subcritical case

This section is dedicated to the proof of Theorem 2.5.

4.1. Preliminaries

Since we shall use variational techniques, we require the well-known Trudinger–Moser inequality (see [**16**, **20**]):

$$\sup_{u\in B}\int_\Omega \exp(\beta u^2) < +\infty, \quad \beta \leqslant 4\pi, \quad \text{and} \quad \sup_{u\in B}\int_\Omega \exp(\beta u^2) = +\infty, \quad \beta > 4\pi, \tag{4.1}$$

where B denotes the unitary ball in $H_0^1(\Omega)$.

We shall make use of the classic mountain pass theorem of Ambrosetti–Rabinowitz [**3**] in case for (A_1) or (A_3), and the linking theorem of Rabinowitz [**17**] in the other cases.

For the sake of a better exposition let us define, for each $U = (u, v) \in E$,

$$\left.\begin{gathered} \tilde{U} := (U + \Phi)_+, \\ \tilde{u} = (u+\phi)_+, \qquad \tilde{v} = (v+\psi)_+. \end{gathered}\right\} \tag{4.2}$$

We seek non-trivial critical points for $J\colon E \to \mathbb{R}$ associated to (2.8) and given by

$$J(U) = \tfrac{1}{2}\|U\|^2 - \frac{1}{2}\int_\Omega (A(x)U, U)_{\mathbb{R}^2} - \int_\Omega H(x, \tilde{U}). \tag{4.3}$$

First of all, we see that $J \in C^1(E, \mathbb{R})$. For further details, we refer the reader to the results in [**12**].

The following lemma gives an estimate in H that will become useful in several steps of our arguments.

Lemma 4.1. *Suppose that* (H_0)–(H_4) *hold. Then, for all* $\varepsilon > 0$ *and* $\alpha > 0$ *(if* H *has critical growth,* $\alpha > \alpha_0$*) there exists* $K_\varepsilon > 0$ *such that*

$$H(x,u,v) \leqslant \varepsilon(u^2+v^2) + K_\varepsilon(u^3+v^3)\exp(\alpha(u^2+v^2)) \quad \textit{for all } x \in \Omega \textit{ and } (u,v) \geqslant (0,0).$$

Proof. By (H_4) we see that

$$H(x,u,v) - H(x,0,v) = \int_0^u H_u(x,s,v)\,\mathrm{d}s \leqslant \varepsilon\int_0^u |(s,v)|\,\mathrm{d}s \leqslant \varepsilon C(\tfrac{1}{2}u^2 + uv)$$

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and

$$H(x,0,v) - H(x,0,0) = \int_0^v H_v(x,0,t)\,\mathrm{d}t \leqslant \tfrac{1}{2}\varepsilon C v^2.$$

Therefore, since $H(x,0,0)=0$ we see that

$$H(x,u,v) \leqslant \varepsilon C(\tfrac{1}{2}u^2 + uv + \tfrac{1}{2}v^2) \leqslant \varepsilon C(u^2+v^2) = \varepsilon C|(u,v)|^2 \quad \text{if } |(u,v)| \leqslant c_\varepsilon.$$

By (H_0) and (H_1) there exists $C_1 \geqslant 1$ such that, if $|(u,v)| > C_1$, then

$$\begin{aligned}(\nabla H(x,u,v),(u,v))_{\mathbb{R}^2} &\leqslant (H_u(x,u,v) + H_v(x,u,v))(u+v)\\ &\leqslant \varepsilon \exp(\alpha|(u,v)|^2)(u+v)\\ &\leqslant \varepsilon C \exp(\alpha|(u,v)|^2)(u^3+v^3).\end{aligned}$$

Thus, let $C_2 > 0$ be such that $H(x,u,v) \leqslant (\nabla H(x,u,v),(u,v))_{\mathbb{R}^2}$ for all $|(u,v)| \geqslant C_2$ (here we use (H_3)). Then, if $|(u,v)| \geqslant \max\{C_1,C_2\}$, we get

$$H(x,u,v) \leqslant \varepsilon C \exp(\alpha|(u,v)|^2)(u^3+v^3),$$

which completes the proof. □

4.2. The Palais–Smale condition

The following result is easily proven for conditions (A_1) or (A_3) using arguments similar to the other cases, which are more delicate. We give the details below.

Lemma 4.2. *Under the hypotheses of Theorem 2.5, the functional (4.3) satisfies the Palais–Smale condition.*

Proof. Let $\{U_n\} = \{(u_n,v_n)\} \subset E$ be a Palais–Smale sequence. That means

$$\Big|\tfrac{1}{2}\|U_n\|^2 - \int_\Omega (A(x)U_n,U_n)_{\mathbb{R}^2} - \int_\Omega H(x,\tilde{U}_n)\Big| \leqslant C \tag{4.4}$$

and

$$\left|\int_\Omega \nabla U_n \nabla \Psi - \int_\Omega (A(x)U_n,\Psi)_{\mathbb{R}^2} - \int_\Omega (\nabla H(x,\tilde{U}_n),\Psi)_{\mathbb{R}^2}\right| \leqslant \epsilon_n\|\Psi\| \quad \text{for all } \Psi \in E. \tag{4.5}$$

By (H_3), let us take $c \geqslant 0$ such that

$$H(x,S) \leqslant \tfrac{1}{4}(\nabla H(x,S),S)_{\mathbb{R}^2} \quad \text{for all } x \in \Omega \text{ and } |S| \geqslant c.$$

Thus, using (4.4), (4.5), we have

$$\int_\Omega (\nabla H(x,\tilde{U}_n),\tilde{U}_n)_{\mathbb{R}^2} \leqslant C + \epsilon_n\|U_n\|. \tag{4.6}$$

However, for (A_2) or (A_4) we need to estimate the L^1 norm of $\nabla H(\cdot,\tilde{U}_n)$: by (H_3), there exist $C_1, C_2 > 0$ such that

$$|\nabla H(x,S)| \leqslant C_1 + C_2(\nabla H(x,S),S)_{\mathbb{R}^2} \quad \text{for all } S \in \mathbb{R}^2.$$

Then, we get

$$\int_\Omega |\nabla H(x,\tilde U_n)| \leqslant C + C\int_\Omega (\nabla H(x,\tilde U_n),\tilde U_n)_{\mathbb{R}^2},$$

and so

$$\int_\Omega |\nabla H(x,\tilde U_n)| \leqslant C + \epsilon_n\|U_n\|. \tag{4.7}$$

The situation is simpler if the interaction between the matrix $A(x)$ and the spectrum of $(-\Delta,\ H_0^1(\Omega))$ occurs only at the first eigenvalue. This is the case when (A_1) or (A_3) holds. In either case (using (2.4) when (A_3) holds), one can see that

$$\begin{aligned}
\epsilon_n\|U_n\| &\geqslant J'(U_n)U_n\\
&\geqslant C\|U_n\|^2 - \biggl(\int_\Omega H_u(x,\tilde U_n)u_n + \int_\Omega H_v(x,\tilde U_n)v_n\biggr)\\
&\geqslant C\|U_n\|^2 - \int_\Omega (\nabla H(x,\tilde U_n),\tilde U_n)_{\mathbb{R}^2}\\
&\geqslant C\|U_n\|^2 - (C+\epsilon_n\|U_n\|),
\end{aligned}$$

which implies that (U_n) is bounded.

The same holds for (A_2) or (A_4). It is now convenient to decompose E into appropriate subspaces. If (A_2) is valid, we consider $E = H_k \oplus H_k^\perp$, where $H_k = \operatorname{span}\{e_1,\dots,e_k\}$, and for (A_4) we take $E = E_k \oplus E_k^\perp$, with E_k defined in (2.3). For all $V \in E$ let us take $V = V^k + V^\perp$, where $V^k \in H_k$ and $V^\perp \in H_k^\perp$ if (A_2) holds or $V^k \in E_k$ and $V^\perp \in E_k^\perp$ for (A_4).

Using the variational inequalities of the eigenvalues (when (A_4) holds, they are given in (2.4)), in both cases we have

$$\begin{aligned}
-\epsilon_n\|U_n^k\| &\leqslant J'(U_n)U_n^k\\
&= \int_\Omega \nabla U_n\nabla U_n^k - \int_\Omega (A(x)U_n,U_n^k)_{\mathbb{R}^2} - \int_\Omega (\nabla H(x,\tilde U_n),U_n^k)_{\mathbb{R}^2}\\
&\leqslant -C\|U_n^k\|^2 - \int_\Omega (\nabla H(x,\tilde U_n),U_n^k)_{\mathbb{R}^2}.
\end{aligned}$$

Therefore,

$$C\|U_n^k\|^2 \leqslant \epsilon_n\|U_n^k\| - \int_\Omega (\nabla H(x,\tilde U_n),U_n^k)_{\mathbb{R}^2} \tag{4.8}$$

and, analogously,

$$C\|U_n^\perp\|^2 \leqslant \epsilon_n\|U_n^\perp\| + \int_\Omega (\nabla H(x,\tilde U_n),U_n^\perp)_{\mathbb{R}^2}. \tag{4.9}$$

Then, since $\dim(H_k)$ and $\dim(E_k)$ are finite, by (4.7), (4.8) we have

$$\begin{aligned}
C\|U_n^k\|^2 &\leqslant \epsilon_n\|U_n^k\| + \|U_n^k\|_\infty \int_\Omega |\nabla H(x,\tilde U_n)|\\
&\leqslant \epsilon_n\|U_n^k\| + C\|U_n\|(C+\epsilon_n\|U_n\|)\\
&\leqslant C + C\|U_n\| + C\epsilon_n\|U_n\|^2.
\end{aligned}$$

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On the other hand, from (4.6), (4.7) and (4.9) we can see that

$$\begin{aligned}C\|U_n^{\perp}\|^2 &\leqslant \epsilon_n\|U_n^{\perp}\| + \int_{\Omega}(\nabla H(x,\tilde{U}_n),U_n)_{\mathbb{R}^2} + \|U_n^k\|_{\infty}(C+\epsilon_n\|U_n\|)\\ &\leqslant \epsilon_n\|U_n^{\perp}\| + C + \epsilon_n\|U_n\| + \|\Phi\|_{\infty}\int_{\Omega}|\nabla H(x,\tilde{U}_n)| + C\|U_n\|(C+\epsilon_n\|U_n\|)\\ &\leqslant C + C\|U_n\| + C\epsilon_n\|U_n\|^2.\end{aligned}$$

By summing the latter two inequalities, we get

$$\|U_n\|^2 \leqslant C + C\|U_n\| + C\epsilon_n\|U_n\|^2, \tag{4.10}$$

proving the boundedness of the sequence (U_n) as desired.

Remark. Up to this point in this proof there is no difference between assuming subcritical or critical growth. We can therefore conclude that in the case of critical growth every Palais–Smale sequence is bounded.

To conclude, let $\{U_n\}$ be an appropriate subsequence such that $U_n \rightharpoonup U$ in E, $U_n \to U$ in $L^p(\Omega)\times L^p(\Omega)$ for all $p \geqslant 1$ and $U_n \to U$ almost everywhere in Ω for some $U \in E$. Notice that there is nothing else to prove in the case when $\|U_n\| \to 0$. Thus, one may suppose that $\|U_n\| \geqslant k > 0$ for n sufficiently large.

Claim 4.3. *We claim that*

$$\left.\begin{aligned}\int_{\Omega}(\nabla H(x,\tilde{U}_n),U)_{\mathbb{R}^2} &\to \int_{\Omega}(\nabla H(x,\tilde{U}),U)_{\mathbb{R}^2},\\ \int_{\Omega}(\nabla H(x,\tilde{U}_n),U_n)_{\mathbb{R}^2} &\to \int_{\Omega}(\nabla H(x,\tilde{U}),U)_{\mathbb{R}^2}.\end{aligned}\right\} \tag{4.11}$$

The proof of this claim is given later for completeness.

Taking $\Psi = U$ and $n\to\infty$ in (4.5) and using (4.11), we have

$$\|U\|^2 = \int_{\Omega}(A(x)U,U)_{\mathbb{R}^2} + \int_{\Omega}(\nabla H(x,\tilde{U}),U)_{\mathbb{R}^2}.$$

On the other hand, if $n\to\infty$ in (4.5) with $\Psi = U_n$,

$$\|U_n\|^2 \to \int_{\Omega}(A(x)U,U)_{\mathbb{R}^2} + \int_{\Omega}(\nabla H(x,\tilde{U}),U)_{\mathbb{R}^2},$$

again by (4.11). Consequently, $\|U_n\| \to \|U\|$ and so $U_n \to U$ in E. □

Proof of Claim 4.3. Let $K > 0$ be such that $\|U_n\| \leqslant K$ and take $\alpha > 0$ satisfying $K^2\alpha < 2\pi$. Since ∇H is subcritical, given $\epsilon > 0$, let M be sufficiently large that

$$|(\nabla H(x,\tilde{U}_n),U)_{\mathbb{R}^2}| \leqslant \epsilon\exp\left(K^2\alpha\left(\frac{|U_n|}{\|U_n\|}\right)^2\right)|U|$$

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if $|\tilde{U}_n| \geqslant M$. Choose also $\delta > 0$ such that

$$\int_A |(\nabla H(x,\tilde{U}),U)_{\mathbb{R}^2}| < \epsilon \quad \text{for all } A; \quad |A| < \delta. \tag{4.12}$$

Since

$$\int_\Omega |\tilde{U}_n| \leqslant C,$$

take M even larger, so that $A_n = \{x \in \Omega;\ |\tilde{U}_n(x)| \geqslant M\}$ is such that $|A_n| < \delta$. We have

$$\begin{aligned}\int_\Omega &|(\nabla H(x,\tilde{U}_n),U)_{\mathbb{R}^2} - (\nabla H(x,\tilde{U}),U)_{\mathbb{R}^2}| \\ &= \biggl(\int_{A_n} + \int_{|\tilde{U}_n|<M}\biggr)|(\nabla H(x,\tilde{U}_n),U)_{\mathbb{R}^2} - (\nabla H(x,\tilde{U}),U)_{\mathbb{R}^2}|\end{aligned}$$

and we notice that the second integral on the right tends to zero as $n \to \infty$ by the Dominated Convergence Theorem since

$$|(\nabla H(x,\tilde{U}_n),U)_{\mathbb{R}^2} - (\nabla H(x,\tilde{U}),U)_{\mathbb{R}^2}|\mathcal{X}_{|\tilde{U}_n|<M}(x) \to 0$$

and

$$|(\nabla H(x,\tilde{U}_n),U)_{\mathbb{R}^2} - (\nabla H(x,\tilde{U}),U)_{\mathbb{R}^2}|\mathcal{X}_{|\tilde{U}_n|<M}(x) \leqslant C,$$

for almost every $x \in \Omega$. Here $\mathcal{X}_A$ denotes the characteristic function of A. It remains to estimate the first integral on the right. We have that

$$\begin{aligned}\int_{A_n} &|(\nabla H(x,\tilde{U}_n),U)_{\mathbb{R}^2} - (\nabla H(x,\tilde{U}),U)_{\mathbb{R}^2}| \\ &\leqslant \int_{A_n} |(\nabla H(x,\tilde{U}_n),U)_{\mathbb{R}^2}| + \int_{A_n} |(\nabla H(x,\tilde{U}),U)_{\mathbb{R}^2}| \\ &= I_1 + I_2.\end{aligned}$$

But we can see that

$$\begin{aligned} I_1 &\leqslant \epsilon \int_\Omega \exp\biggl(K^2\alpha\biggl(\frac{|U_n|}{\|U_n\|}\biggr)^2\biggr)|U| \\ &\leqslant \epsilon\biggl(\int_\Omega \exp\biggl(2K^2\alpha\biggl(\frac{|U_n|}{\|U_n\|}\biggr)^2\biggr)\biggr)^{1/2}\|U\|_{L^2\times L^2} \\ &\leqslant \epsilon C.\end{aligned}$$

In this last estimate we used the Trudinger–Moser inequality, since we chose $\alpha > 0$ to be such that $2K^2\alpha < 4\pi$. Finally, we obtain $I_2 < \epsilon$, by using (4.12) and the definition of A_n. The proof of the second convergence in (4.11) follows the same arguments with obvious small modifications. $\square$

4.3. The geometric conditions and proof of Theorem 2.5

The case of the mountain pass geometry, which involves conditions (A_1) or (A_3), is easy to prove and can be concluded by making small modifications in parts of the proof of the case of the linking geometry. Therefore, we provide only this case.

Let us take the same decomposition of E given in the proof of the Palais–Smale condition (Lemma 4.2). The following proposition proves the geometric conditions needed in the linking theorem.

Proposition 4.4. *Suppose that* (A_2) *(or* (A_4)*),* (H_0), (H_1) *(subcritical) and* (H_2)–(H_4) *hold. Then,*

(i) *there exist* $\rho, \beta > 0$ *such that* $J(U) \geqslant \beta$ *if* $U \in \partial B_\rho \cap H_k^\perp$ *(or* $E_k^\perp$*),*

(ii) *there exist* $W \in H_k^\perp$ *(or* $E_k^\perp$*) and* $R > 0$ *such that* $R\|W\| > \rho$ *and if*

$$Q := \bar{B}_R \cap H_k \ (\text{or } E_k) \oplus \{sW : 0 \leqslant s \leqslant R\},$$

then $J(U) \leqslant 0$ *for all* $U \in \partial Q$.

Proof. (i) The variational characterization of the eigenvalues implies that

$$\frac{1}{2}\biggl(\|U\|^2 - \int_\Omega (A(x)U, U)_{\mathbb{R}^2}\biggr) \geqslant C\|U\|^2$$

for all $U \in H_k^\perp$ (or $E_k^\perp$). Therefore, Lemma 4.1, with $\alpha < \pi$, $\|U\| \leqslant 1$, shows that

$$\begin{aligned} J(U) &\geqslant \tfrac{1}{2}(C - C\varepsilon)\|U\|^2 - K_\varepsilon\biggl(\int_\Omega \exp(2\alpha(u^2+v^2))\biggr)^{1/2}\|U\|^3_{L^6\times L^6} \\ &\geqslant C\|U\|^2 - C\biggl(\int_\Omega \exp(4\alpha u^2)\biggr)^{1/4}\biggl(\int_\Omega \exp(4\alpha v^2)\biggr)^{1/4}\|U\|^3. \end{aligned}$$

Consequently, the Trudinger–Moser inequality gives us

$$J(U) \geqslant C\|U\|^2 - C\|U\|^3$$

for all $U \in H_k^\perp$ (or $E_k^\perp$) such that $\|U\| \leqslant 1$, proving item (i).

(ii) Let us choose W in the following way: fix $R_0 > \rho$ and take $W = (w_1, w_2)$ in $H_k^\perp$ (or $E_k^\perp$) such that

(a) we have

$$\|W\|^2 < \begin{cases} \dfrac{\mu_1}{\lambda_k} - 1 & \text{for } (A_2), \\[2ex] \dfrac{1}{\lambda_k^A} - 1 & \text{for } (A_4), \end{cases}$$

(b) there exists $\Gamma \subset \Omega$, $|\Gamma| > 0$, such that

$$w_1 \geqslant 2\biggl(K + \frac{\|\phi\|_\infty}{R_0}\biggr) \quad \text{and} \quad w_2 \geqslant 2\biggl(K + \frac{\|\psi\|_\infty}{R_0}\biggr) \quad \text{a.e. in } \Gamma,$$

with $K > 0$ satisfying $\|V\|_\infty \leqslant K\|V\|$ for all $V \in H_k$(or E_k).

This choice is possible because $H_k^\perp$ and $E_k^\perp$ possess unbounded functions.

As usual, let us split ∂Q: consider $\partial Q = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, where

$$\begin{aligned} \Sigma_1 &= \bar{B}_R \cap H_k \ (\text{or } E_k), \\ \Sigma_2 &= \{V + sW;\ \|V\| = R\}, \\ \Sigma_3 &= \{V + RW;\ \|V\| \leqslant R\}. \end{aligned}$$

If $U \in \Sigma_1$,

$$J(U) \leqslant \tfrac{1}{2}\|U\|^2 - \frac{1}{2}\int_\Omega (A(x)U, U)_{\mathbb{R}^2} \leqslant 0$$

independently of $R > 0$.

As for Σ_2, for (A_2) we get that

$$\begin{aligned} J(V + sW) &\leqslant \tfrac{1}{2}\|V\|^2 + \tfrac{1}{2}s^2\|W\|^2 - \frac{1}{2}\int_\Omega (AV, V)_{\mathbb{R}^2} \\ &\leqslant \tfrac{1}{2}R^2 + \tfrac{1}{2}R^2\|W\|^2 - \tfrac{1}{2}\mu_1\int_\Omega |V|^2 \\ &\leqslant \tfrac{1}{2}R^2\biggl(1 - \frac{\mu_1}{\lambda_k} + \|W\|^2\biggr) \\ &< 0, \end{aligned}$$

independently of $R > 0$.

If (A_4) is considered, by (2.4) we see that

$$\begin{aligned} J(V + sW) &\leqslant \tfrac{1}{2}\|V\|^2 + \tfrac{1}{2}s^2\|W\|^2 - \frac{1}{2}\int_\Omega (A(x)V, V)_{\mathbb{R}^2} \\ &\leqslant \tfrac{1}{2}R^2\biggl(1 - \frac{1}{\lambda_k^A} + \|W\|^2\biggr) < 0, \end{aligned}$$

also independently of $R > 0$.

To conclude, let $V + RW \in \Sigma_3$. By virtue of (H_3), it is possible to determine $\theta > 2$, $C_\theta > 0$ and $D_\theta \geqslant 0$ such that

$$H(u, v) \geqslant C_\theta(u^\theta + v^\theta) - D_\theta \quad \text{for all } u, v \geqslant 0. \tag{4.13}$$

We refer to [**13**] for the proof of this inequality. Consequently,

$$\begin{aligned} &J(V + RW) \\ &\quad\leqslant \tfrac{1}{2}R^2\|W\|^2 - D_\theta - R^\theta C_\theta\biggl(\int_\Omega \biggl[\biggl(w_1 + \frac{\phi + v_1}{R}\biggr)_+\biggr]^\theta - \int_\Omega \biggl[\biggl(w_2 + \frac{\psi + v_2}{R}\biggr)_+\biggr]^\theta\biggr), \end{aligned}$$

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and so we get

$$J(V+RW) \leqslant \tfrac{1}{2}R^2\|W\|^2 - D_\theta - R^\theta C_\theta(I_1(R)+I_2(R)). \tag{4.14}$$

$I_1(R)$ and $I_2(R)$ are the integrals that appear in the last estimate above. Consider $R \geqslant R_0$. So

$$\begin{aligned} I_1(R) &= \int_\Omega \left[\left(w_1 + \frac{\phi + v_1}{R}\right)_+\right]^\theta \\ &\geqslant \int_\Omega \left[\left(w_1 - \frac{\|\phi\|_\infty + \|v_1\|_\infty}{R}\right)_+\right]^\theta \\ &\geqslant \left(K + \frac{\|\phi\|_\infty}{R_0}\right)^\theta |\varGamma| \\ &=: \tau_1 > 0. \end{aligned}$$

Notice that τ_1 does not depend on $R \geqslant R_0$. Analogously, for I_2,

$$I_2(R) \geqslant \left(K + \frac{\|\psi\|_\infty}{R_0}\right)^\theta |\varGamma| =: \tau_2 > 0.$$

Therefore, coming back to (4.14), we obtain

$$J(V+RW) \leqslant \tfrac{1}{2}R^2\|W\|^2 - R^\theta C_\theta(\tau_1+\tau_2) - D_\theta.$$

Since $\theta > 2$, we have the desired result. □

Proof of Theorem 2.5. We have proved that J satisfies the geometric and compactness conditions required in the mountain pass theorem (for (A_1) and (A_3)) and in the linking theorem (assuming (A_2) or (A_4)). That means the existence of a non-trivial critical point for J and so a solution of (2.8). □

5. The critical case

This section is devoted to the proof of Theorem 2.6. So we shall assume the hypotheses of this theorem through this section.

One of the main problems involving critical growth of Trudinger–Moser type consists of proving that the minimax level determined by the geometric properties of the associated functional avoids the levels of non-compactness. In other words, we have to ensure that such a level lies below an appropriate constant, which, in our case, is given by $2\pi/\alpha_0$, where α_0 is as defined in the critical case of condition (H_1). This is where condition (H_5) plays its role: the presence of the function h 'brings down' suitable levels of J, given in (4.3). Such a condition has already been used, for instance, in [**1**].

5.1. Modified Moser functions

Consider the following well-known Moser sequence [**16**]: let $B_1(0)$ be the open ball of radius r, $r>0$, in $\mathbb{R}^2$. Thus, for each $m \in \mathbb{N}$, define

$$z_m^r(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log m)^{1/2} & \text{if } 0 \leqslant |x| \leqslant \dfrac{r}{m}, \\ \dfrac{\log(r/|x|)}{(\log m)^{1/2}} & \text{if } \dfrac{r}{m} \leqslant |x| \leqslant r, \\ 0 & \text{if } |x| \geqslant r. \end{cases}$$

We know that $z_m^r \in H_0^1(B_r(0))$, $\|z_m^r\| = 1$ and, for each $r>0$ fixed, $\|z_m^r\|_{L^2} = O(1/(\log m)^{1/2})$.

Let $r_0 \in \mathbb{R}$ be sufficiently small so that it is possible to choose $x_{r_0} \in \Omega$ satisfying $B_{4r_0}(x_{r_0}) \subset \Omega$, and

$$|e_1(x)|, |e_2(x)|, \dots |e_k(x)| \leqslant Cr_0 \quad (\text{for } (\mathrm{A}_2))$$

and

$$|\Phi_1^A(x)|, |\Phi_2^A(x)|, \dots, |\Phi_k^A(x)| \leqslant Cr_0 \quad (\text{for } (\mathrm{A}_4))$$

for all $x \in B_{4r_0}(x_{r_0})$ and some $C>0$. This is possible since e_i and Φ_i^A are Lipschitz continuous functions and vanish on $\partial\Omega$ (thus, x_0 may be chosen close to $\partial\Omega$). Notice that once r_0 and x_{r_0} have been chosen, then for each $r<r_0$ we can find x_r satisfying the above conditions. Notice also that we can always assume that the x_r are such that $\operatorname{dist}(x_r, \partial\Omega) \to 0$ when $r \to 0$.

For each $r \leqslant r_0$ define $Z_m^r \colon \Omega \to \mathbb{R}^2$ by

$$Z_m^r(x) = (z_m^r(x-x_r), z_m^r(x-x_r)). \tag{5.1}$$

Obviously, $\operatorname{supp}(Z_m^r) \subset \overline{B_r(x_r)} \subset B_{4r}(x_r) \subset \Omega$ and therefore $Z_m^r \in E = H_0^1(\Omega) \times H_0^1(\Omega)$ for all $r \leqslant r_0$ and $m \in \mathbb{N}$.

Our goal is to place the Moser sequence 'inside the ball', where the eigenfunctions are nearly zero. The next step consists of 'making holes' in these eigenfunctions in such a way that they vanish exactly where the Moser functions are supported. Consequently, we separate their supports, making many estimates easier to prove. This approach is inspired by the techniques developed in [**14**]. Evidently, this is only necessary under conditions (A_2) or (A_4), because of the interaction between $A(x)$ and higher-order eigenvalues.

Consider $r \leqslant r_0$ and x_r given above and $\zeta_r \in C^\infty(\mathbb{R}^N)$ such that $0 \leqslant \zeta_r \leqslant 1$, $|\nabla\zeta_r(x)| \leqslant 2/r$ and

$$\zeta_r(x) = \begin{cases} 0 & \text{if } x \in B_r(x_r), \\ 1 & \text{if } x \in \Omega \setminus B_{2r}(x_r). \end{cases}$$

Define

$$e_i^r = \zeta_r e_i,$$
$$\Phi_i^r = \zeta_r \Phi_i^A$$

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and consider the following finite-dimensional subspaces:

$$\begin{aligned} H_k^r &= \operatorname{span}\{(e_i^r, 0), (0, e_i^r);\ 1 \leqslant i \leqslant k\}, \\ E_k^r &= \operatorname{span}\{\Phi_i^r;\ 1 \leqslant i \leqslant k\}. \end{aligned} \tag{5.2}$$

The lemma below is needed in order to control the error caused by these truncated eigenfunctions. We omit its proof, since it can be proved exactly as in [**14**].

Lemma 5.1. *If $r \to 0$, then $e_i^r \to e_i$ in $H_0^1(\Omega)$ and $\Phi_i^r \to \Phi_i^A$ in E for all $1 \leqslant i \leqslant k$. Moreover, for each r small enough, we have the following.*

(i) *There exists $c_k > 0$ such that*

$$\|V\|^2 \leqslant (\lambda_k + c_k r^2) \int_\Omega |V|^2 \quad \text{for all } V \in H_k^r.$$

(ii) *There exists $c_k > 0$ such that*

$$\|V\|^2 \leqslant (\lambda_k^A + c_k r^2) \int_\Omega (A(x)V, V)_{\mathbb{R}^2} \quad \text{for all } V \in E_k^r.$$

The following lemma gives the boundedness of Palais–Smale sequences associated to (4.3). Its proof is exactly the same as that of Lemma 4.2, up to the point where we proved (4.10).

Lemma 5.2. *Under the conditions of Theorem 2.6, every Palais–Smale sequence of (4.3) is bounded.*

5.2. Geometric conditions

As in the subcritical case, the geometric hypothesis of the mountain pass theorem can be derived under assumption (A_1) or assumption (A_3). The proofs are small modifications of those for the linking geometry and are in fact simpler to handle. Therefore, we concentrate our efforts only on such a case: let us suppose (A_2) or (A_4) holds. As usual, we must choose an appropriate decomposition for E. Observe that, for r sufficiently small, one can split the space as follows: $E = H_k^r \oplus H_k^\perp$ and $E = E_k^r \oplus E_k^\perp$, where H_k^r and E_k^r are defined in (5.2).

Now, fix r small enough in order to ensure that

$$\left(\frac{\mu_1}{\lambda_k + c_k r^2} - 1\right), \left(\frac{1}{\lambda_k^A + c_k r^2} - 1\right) > 0. \tag{5.3}$$

and $\delta > 0$ such that

$$\delta^2 < \begin{cases} \dfrac{\mu_1}{\lambda_k + c_k r^2} - 1 & \text{for } (\mathrm{A}_2), \\ \dfrac{1}{\lambda_k^A + c_k r^2} - 1 & \text{for } (\mathrm{A}_4). \end{cases} \tag{5.4}$$

We have the following proposition.

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Proposition 5.3. *Suppose that* (A_2) (*or* (A_4)), (H_0), (H_1) (*critical growth*) *and* (H_2)–(H_4) *hold. Then*

(i) *there exist* $\rho, \beta > 0$ *such that* $J(U) \geqslant \beta$ *if* $U \in \partial B_\rho \cap H_k^\perp$ (*or* $E_k^\perp$),

(ii) *for each* m *large enough, there exists* $R = R(m) > 0$ *such that if*

$$Q := \bar{B}_R \cap H_k^r \text{ (or } E_k^r) \oplus \{s\delta Z_m^r : 0 \leqslant s \leqslant R\},$$

then $J(U) \leqslant 0$ *for all* $U \in \partial Q$; *moreover,* $R = R(m) \to \infty$ *when* $m \to \infty$.

Proof. The proof of (i) is exactly the same as that of Proposition 4.4.

Let us prove (ii): first, notice that the way we defined H_k^r and E_k^r, given in (5.2), implies that

$$J(V + s\delta Z_m^r) = J(V) + J(s\delta Z_m^r) \quad \text{for all } V \in H_k^r \text{ or } V \in E_k^r, \tag{5.5}$$

since the support of the modified eigenfunctions in H_k^r or E_k^r is disjoint from the support of Z_m^r.

Therefore, let us estimate $J(V)$ for all $V \in H_k^r$ or E_k^r.

Let $V \in H_k^r$ and notice that by Lemma 5.1 we obtain

$$\begin{aligned} J(V) &\leqslant \tfrac{1}{2}\|V\|^2 - \tfrac{1}{2}\mu_1 \int_\Omega |V|^2 \\ &\leqslant \frac{1}{2}\left(1 - \frac{\mu_1}{\lambda_k + c_k r^2}\right)\|V\|^2. \end{aligned}$$

Analogously, if $V \in E_k^r$, Lemma 5.1 shows that

$$\begin{aligned} J(V) &\leqslant \tfrac{1}{2}\|V\|^2 - \frac{1}{2}\int_\Omega (A(x)V, V)_{\mathbb{R}^2} \\ &\leqslant \frac{1}{2}\left(1 - \frac{1}{\lambda_k^A + c_k r^2}\right)\|V\|^2. \end{aligned}$$

Since $\mu_1 < \lambda_k$ and $1 < \lambda_k^A$, we can take $C_1 > 0$ such that

$$C_1 < \left(\frac{\mu_1}{\lambda_k + c_k r^2} - 1\right) \quad \text{or} \quad C_1 < \left(\frac{1}{\lambda_k^A + c_k r^2} - 1\right)$$

for all r small enough. Therefore,

$$J(V) \leqslant -C_1\|V\|^2 \quad \text{for all } V \in H_k^r \text{ or } E_k^r. \tag{5.6}$$

As in Proposition 4.4, we split ∂Q as follows: let $\partial Q = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, where

$$\begin{aligned} \Sigma_1 &= \bar{B}_R \cap H_k^r \text{ (or } E_k^r), \\ \Sigma_2 &= \{V + s\delta Z_m^r;\ \|V\| = R\}, \\ \Sigma_3 &= \{V + R\delta Z_m^r;\ \|V\| \leqslant R\}. \end{aligned}$$

If $U \in \Sigma_1$, from (5.6) we see that

$$J(U) \leqslant 0 \quad \text{independently of } R > 0.$$

For Σ_2, if (A_2) is assumed, the choices of r and δ in (5.3) and (5.4) give us

$$\begin{aligned} J(V + s\delta Z_m^r) &= J(V) + J(s\delta Z_m^r) \\ &\leqslant \tfrac{1}{2} R^2 \left(1 - \frac{\mu_1}{\lambda_k + c_k r^2} + \delta^2\right) \\ &< 0, \end{aligned}$$

independently of $R > 0$.

For (A_4), using (2.4), we see that

$$J(V + s\delta Z_m^r) \leqslant \tfrac{1}{2} R^2 \left(1 - \frac{1}{\lambda_k^A + c_k r^2} + \delta^2\right) < 0,$$

again by (5.3) and (5.4) and independently of $R > 0$.

To conclude, let $V + R\delta Z_m^r \in \Sigma_3$. Making use of (4.13),

$$\begin{aligned} &J(V + R\delta Z_m^r) \\ &\quad = J(V) + J(R\delta Z_m^r) \\ &\quad \leqslant -C_1 \|V\|^2 + \tfrac{1}{2} R^2 \delta^2 - \int_\Omega H((R\delta z_m^r(\cdot - x_r) + \phi)_+, (R\delta z_m^r(\cdot - x_r) + \psi)_+) \\ &\quad \leqslant \tfrac{1}{2}\delta^2 R^2 - R^\theta C_\theta \left(\int_{B_{r/m}(0)} \left[\left(\delta z_m^r - \frac{\|\phi\|_\infty}{R}\right)_+\right]^\theta + \left[\left(\delta z_m^r - \frac{\|\psi\|_\infty}{R}\right)_+\right]^\theta \right) - D_\theta. \end{aligned}$$

Let us choose m sufficiently large that

$$\delta z_m^r(x) = \frac{\delta}{\sqrt{2\pi}} \log^{1/2} m \geqslant \max\left\{\frac{2\|\phi\|_\infty}{R_0}, \frac{2\|\psi\|_\infty}{R_0}\right\} \quad \text{for all } x \in B_{r/m}(0), \tag{5.7}$$

and, consequently,

$$J(V + R\delta Z_m^r) \leqslant \tfrac{1}{2}\delta^2 R^2 - R^\theta C_\theta \left(\int_{B_{r/m}(0)} \left(\frac{\|\phi\|_\infty}{R_0}\right)_+^\theta + \left(\frac{\|\psi\|_\infty}{R_0}\right)_+^\theta \right) - D_\theta.$$

Thus, we obtain

$$J(V + R\delta Z_m^r) \leqslant \tfrac{1}{2}\delta^2 R^2 - C_m R^\theta - D_\theta,$$

where

$$C_m = \left[\left(\frac{\|\phi\|_\infty}{R_0}\right)_+^\theta + \left(\frac{\|\psi\|_\infty}{R_0}\right)_+^\theta\right] \pi \left(\frac{r}{m}\right)^2 \to 0 \quad \text{if } m \to \infty.$$

Since $\theta > 2$, this completes the proof. □

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For the mountain pass case, which involves (A_1) or (A_3), we define the minimax level of J by

$$\tilde{c} = \tilde{c}(m) = \inf_{\upsilon \in \Upsilon} \sup_{W \in \upsilon(E)} J(W) \tag{5.8}$$

where

$$\Upsilon = \{\upsilon \in C(E, E)\colon \upsilon(0) = 0 \text{ and } \upsilon(R_m Z_m^r) = R_m Z_m^r\},$$

R_m being such that $J(R_m Z_m^r) \leqslant 0$.

It is well known that such a minimax level has a Palais–Smale sequence. That means that there exists $(V_n) \subset E$ such that $J(V_n) \to \tilde{c}$ and $J'(V_n) \to 0$. We refer the reader to [**15**] for the proof of this claim.

For the linking geometry case, we define

$$\hat{c} = \hat{c}(m) = \inf_{\gamma \in \Gamma} \max_{W \in \gamma(Q)} J(W), \tag{5.9}$$

where $\Gamma = \{h \in C(Q, E);\ h(u) = u \text{ if } u \in \partial Q\}$. Once again, the existence of a Palais–Smale sequence in this level is proven in [**15**].

5.3. Control of the minimax levels

In this subsection we prove that the minimax levels given in (5.8) and (5.9) stay below $2\pi/\alpha_0$ for m sufficiently large. The additional assumption (H_5) is used here.

Proposition 5.4. *Let $\tilde{c}(m)$ be given as in (5.8). Then there exists m large enough such that*

$$\tilde{c}(m) < \frac{2\pi}{\alpha_0}.$$

Proof. We begin by fixing some constants that we shall use in this proof. We can assume, without loss of generality, that

$$\limsup_{u \to +\infty} \frac{\log(h(x, u))}{u} < \infty.$$

Thus, from (H_5) we have the existence of $0 < \varepsilon_0 < C_0 < \infty$ such that

$$\varepsilon_0 \leqslant \frac{\log(h(x, u))}{u} \leqslant C_0 \tag{5.10}$$

for all u large enough.

Let us also choose and fix r and the corresponding $x_r \in \Omega$ close enough to $\partial\Omega$, such that, denoting by $\|\Phi\|_{\infty,r}$ the $L^\infty(B_r(x_r)) \times L^\infty(B_r(x_r))$ norm of Φ, we have

$$\|\Phi\|_{\infty,r} \leqslant \frac{\varepsilon_0}{2\alpha_0}. \tag{5.11}$$

Finally, consider γ such that

$$\gamma > \frac{4}{\alpha_0 r^2} \exp\left(\frac{(C_0)^2}{4\alpha_0}\right). \tag{5.12}$$

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The proof is by contradiction: assume that for all m we have $\tilde{c}(m) \geqslant 2\pi/\alpha_0$. By definition,

$$\tilde{c}(m) \leqslant \max_{t \geqslant 0} J(tZ_m^r).$$

But notice that for $t \geqslant R_m$ we have $J(tZ_m^r) \leqslant 0$ and therefore for each m there exists $t_m > 0$ such that

$$J(t_m Z_m^r) = \max_{t \geqslant 0} J(tZ_m^r). \tag{5.13}$$

Then

$$J(t_m Z_m^r) \geqslant \frac{2\pi}{\alpha_0} \quad \text{for all } m \in \mathbb{N}$$

and, consequently,

$$t_m^2 \geqslant \frac{4\pi}{\alpha_0} \quad \text{for all } m \in \mathbb{N}. \tag{5.14}$$

Let us prove that $t_m^2 \to 4\pi/\alpha_0$: from (5.13) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(J(tZ_m^r))\bigg|_{t=t_m} = 0.$$

So

$$t_m \|Z_m^r\|^2 - t_m \int_\Omega (A(x)Z_m^r, Z_m^r)_{\mathbb{R}^2} - \int_\Omega (\nabla H(x, (t_m Z_m^r + \Phi)_+), Z_m^r)_{\mathbb{R}^2} = 0.$$

Multiplying this last equation by t_m and noticing that $\|Z_m^r\|^2 = 1$, $\Phi < 0$ and $H_u, H_v \geqslant 0$, we have

$$\begin{aligned} t_m^2 &\geqslant \int_{B_{r/m}(x_r)} (\nabla H(x, (t_m Z_m^r + \Phi)_+), t_m Z_m^r)_{\mathbb{R}^2} \\ &\geqslant \int_{B_{r/m}(x_r)} (\nabla H(x, (t_m Z_m^r + \Phi)_+), (t_m Z_m^r + \Phi)_+)_{\mathbb{R}^2}. \end{aligned}$$

Since $t_m \geqslant 2\pi/\sqrt{\alpha_0} > 0$, we can take m so large that $(t_m(\sqrt{2\pi})^{-1}\log^{1/2} m - \|\Phi\|_{\infty,r}) \geqslant c_\gamma$, where c_γ is given in (H$_5$). Then

$$(t_m Z_m^r + \Phi)_+ = (t_m Z_m^r + \Phi) = (t_m(\sqrt{2\pi})^{-1}\log^{1/2} m + \phi, t_m(\sqrt{2\pi})^{-1}\log^{1/2} m + \psi)$$

in $B_{r/m}(x_r)$ and so (H$_5$) implies that

$$t_m^2 \geqslant \gamma \int_{B_{r/m}(x_r)} h\biggl(x, \frac{t_m}{\sqrt{2\pi}}\log^{1/2} m - \|\Phi\|_{\infty,r}\biggr) \exp\biggl(\alpha_0\biggl(\frac{t_m}{\sqrt{2\pi}}\log^{1/2} m - \|\Phi\|_{\infty,r}\biggr)^2\biggr).$$

For convenience, let us rewrite e^x as $\exp(x)$ in what follows. We have

$$t_m^2 \geqslant \gamma \int_{B_{r/m}(x_r)} \exp\biggl[-\biggl(\frac{\log[h(x,(t_m/\sqrt{2\pi})\log^{1/2} m - \|\Phi\|_{\infty,r})]}{2\sqrt{\alpha}_0((t_m/\sqrt{2\pi})\log^{1/2} m - \|\Phi\|_{\infty,r})}\biggr)^2 + \alpha_0((t_m/\sqrt{2\pi})\log^{1/2} m - \|\Phi\|_{\infty,r} + \frac{\log[h(x,(t_m/\sqrt{2\pi})\log^{1/2} m - \|\Phi\|_{\infty,r})]}{2\alpha_0((t_m/\sqrt{2\pi})\log^{1/2} m - \|\Phi\|_{\infty,r})}\biggr)^2\biggr].$$

But if m is large, (5.10) shows that

$$-\biggl(\frac{\log[h(x,(t_m/\sqrt{2\pi})\log^{1/2} m - \|\Phi\|_{\infty,r})]}{2\sqrt{\alpha}_0((t_m/\sqrt{2\pi})\log^{1/2} m - \|\Phi\|_{\infty,r})}\biggr)^2 \geqslant -\frac{(C_0)^2}{4\alpha_0}$$

and

$$\frac{\log[h(x,(t_m/\sqrt{2\pi})\log^{1/2} m - \|\Phi\|_{\infty,r})]}{2\alpha_0((t_m/\sqrt{2\pi})\log^{1/2} m - \|\Phi\|_{\infty,r})} \geqslant \frac{\varepsilon_0}{2\alpha_0},$$

from which we obtain

$$t_m^2 \geqslant \gamma\pi\frac{r^2}{m^2}\exp\biggl(-\frac{(C_0)^2}{4\alpha_0}\biggr)\exp\biggl(\alpha_0\biggl(\frac{t_m}{\sqrt{2\pi}}\log^{1/2} m - \|\Phi\|_{\infty,r} + \frac{\varepsilon_0}{2\alpha_0}\biggr)^2\biggr),$$

and by the choice of r and x_r in (5.11) we see that

$$\begin{aligned} t_m^2 &\geqslant \gamma\pi\frac{r^2}{m^2}\exp\biggl(-\frac{(C_0)^2}{4\alpha_0}\biggr)\exp\biggl(\alpha_0\frac{t_m^2}{2\pi}\log m\biggr) \\ &= \exp\biggl(-\frac{(C_0)^2}{4\alpha_0}\biggr)\gamma\pi r^2\exp\biggl(\biggl(\alpha_0\frac{t_m^2}{2\pi} - 2\biggr)\log m\biggr). \end{aligned}$$

Therefore, (t_m) is a bounded sequence. Because of (5.14), we have $t_m^2 \to 4\pi/\alpha_0$. Letting $m \to \infty$ in the inequality above, one gets

$$\gamma \leqslant \frac{4}{\alpha_0 r^2}\exp\biggl(\frac{(C_0)^2}{4\alpha_0}\biggr),$$

which is contrary to the choice of γ in (5.12). This contradiction follows from the assumption $\tilde{c}(m) \geqslant 2\pi/\alpha_0$ for all $m \in \mathbb{N}$, which concludes the proof. □

Analogously to the cases for (A_2) or (A_4), we have the following.

Proposition 5.5. *Let $\hat{c}(m)$ be given as in (5.9). Then there exists m large enough that*

$$\hat{c}(m) < \frac{2\pi}{\alpha_0}.$$

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Proof. The proof is almost the same as that of Proposition 5.4, since we are able to separate the supports of functions in H_k^r and E_k^r from the Moser functions Z_m^r. Indeed, by definition of $\hat{c}(m)$ one gets

$$\begin{aligned}\hat{c}(m) &\leqslant \max\{J(V+tZ_m^r);\ V\in H_k^r\ (\text{or } E_k^r)\cap B_{R(m)},\ t\geqslant 0\}\\ &= \max\{J(V)+J(tZ_m^r);\ V\in H_k^r\ (\text{or } E_k^r)\cap B_{R(m)},\ t\geqslant 0\}\\ &\leqslant \max\{J(V);\ V\in H_k^r\ (\text{or } E_k^r)\cap B_{R(m)}\}+\max\{J(tZ_m^r);\ t\geqslant 0\}.\end{aligned}$$

But we have already seen in (5.6) that $J(V)\leqslant 0$ for all $V\in H_k^r$ (or E_k^r) and therefore

$$\hat{c}(m)\leqslant \max\{J(tZ_m^r);\ t\geqslant 0\}.$$

The rest of the proof is exactly the same as that of Proposition 5.4. □

5.4. Proof of Theorem 2.6

From now on we shall make no distinction between the mountain pass minimax level and the linking minimax level. This means that we define $c(m)=\tilde{c}(m)$ for (A_1) or (A_3), and $c(m)=\hat{c}(m)$ for (A_2) or (A_4).

Let us take m such that $c(m)<2\pi/\alpha_0$. Let (U_n), $U_n=(u_n,v_n)$ be a Palais–Smale sequence of this level $c(m)$. Since (U_n) is bounded, consider a suitable subsequence still denoted by (U_n) and $U\in H_0^1(\Omega)$ such that $U_n\rightharpoonup U$ weakly in $E=H_0^1(\Omega)\times H_0^1(\Omega)$, $U_n\to U$ in $L^p(\Omega)\times L^p(\Omega)$ for all $p\geqslant 1$ and almost everywhere in Ω. The following lemma is an auxiliary convergence result needed later.

Lemma 5.6. *If $U_n\rightharpoonup U$ weakly in E and (U_n) satisfies (4.6), then, if $n\to\infty$, the following convergences hold.*

(i) $\nabla H(\cdot,\tilde{U}_n)\to\nabla H(\cdot,\tilde{U})$ *in* $L^1(\Omega)\times L^1(\Omega)$.

(ii) $H(\cdot,\tilde{U}_n)\to H(\cdot,\tilde{U})$ *in* $L^1(\Omega)$.

Proof. We can suppose that $\tilde{U}_n(x)\to\tilde{U}(x)$ for almost every $x\in\Omega$. Since (U_n) is bounded, from (4.6) we see that

$$\int_\Omega(\nabla H(x,\tilde{U}_n),\tilde{U}_n)_{\mathbb{R}^2}\leqslant\int_\Omega(\nabla H(x,\tilde{U}_n),U_n)_{\mathbb{R}^2}\leqslant C_0. \tag{5.15}$$

For (i) we must prove that $H_u(\cdot,\tilde{U}_n)\to H_u(\cdot,\tilde{U})$ and $H_v(\cdot,\tilde{U}_n)\to H_v(\cdot,\tilde{U})$, both in $L^1(\Omega)$. Let us prove only the expression involving H_u, since the other is exactly the same. This proof is similar to that of Claim 4.3, but we give it here for the sake of completeness. Indeed, by $H_u(\cdot,\tilde{U})\in L^1(\Omega)$, we have that for all $\epsilon>0$ there exists $\delta>0$ such that

$$\int_A H_u(x,\tilde{U}(x))<\epsilon\quad\text{for all } A;\quad |A|<\delta. \tag{5.16}$$

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Since

$$\int_\Omega |\tilde{U}_n(x)| \leqslant C \quad \text{for all } n,$$

let us take M_0 such that the sets $A_n \equiv \{x \in \Omega;\ |\tilde{U}_n| \geqslant M_0\}$ satisfy

$$|A_n| < \delta. \tag{5.17}$$

By (H_3), given $\epsilon > 0$ there exists $c_\epsilon > 0$ such that

$$H_u(x,S) \leqslant \frac{\epsilon}{C_0}(\nabla H(x,S), S)_{\mathbb{R}^2} \quad \text{for all } |S| \geqslant c_\epsilon, \tag{5.18}$$

where C_0 is given by (5.15).

So take $M = \max\{M_0, c_\epsilon\}$. Observe that

$$\begin{aligned}\int_\Omega |H_u(x,\tilde{U}_n(x)) - H_u(x,\tilde{U}(x))| &\leqslant \int_{|\tilde{U}_n|<M} |H_u(x,\tilde{U}_n(x)) - H_u(x,\tilde{U}(x))| \\ &\quad + \int_{|\tilde{U}_n|\geqslant M} H_u(x,\tilde{U}_n(x)) + \int_{|\tilde{U}_n|\geqslant M} H_u(x,\tilde{U}(x)) \\ &\equiv I_1 + I_2 + I_3.\end{aligned}$$

Using (5.15), (5.18) and the M chosen above, we obtain

$$I_2 \leqslant \frac{\epsilon}{C_0}\int_\Omega (\nabla H(x,\tilde{U}_n), \tilde{U}_n)_{\mathbb{R}^2} \leqslant \epsilon.$$

By (5.16) and (5.17) we also have

$$I_3 \leqslant \int_{A_n} H_u(x,\tilde{U}(x)) \leqslant \epsilon.$$

Therefore, it remains to prove that $I_1 \to 0$ if $n \to \infty$. Since H_u is continuous, we have that

$$|H_u(x,\tilde{U}_n(x)) - H_u(x,\tilde{U}(x))|\mathcal{X}_{|\tilde{U}_n|<M}(x) \to 0$$

for almost every $x \in \Omega$. Here $\mathcal{X}_A$ denotes the characteristic function of A. Since

$$|H_u(x,\tilde{U}_n(x)) - H_u(x,\tilde{U}(x))|\mathcal{X}_{|\tilde{U}_n|<M}(x) \leqslant C,$$

we prove part (i) by the Dominated Convergence Theorem.

Let us prove (ii): we follow a similar scheme to the proof of [**4**, Lemma A.1]. Since H is continuous, we get $H(x,\tilde{U}_n(x)) \to H(x,\tilde{U}(x))$. As $|\Omega| < \infty$, Egorov's Theorem ensures that this pointwise convergence is also a convergence in measure. Consequently, Vitali's Theorem implies that $H(\cdot,\tilde{U}_n) \to H(\cdot,\tilde{U})$ in L^1 occurs when $H(\cdot,\tilde{U}_n)$ is uniformly integrable in n. That means that we need to prove that for each $\epsilon > 0$ there exists $K_\epsilon \geqslant 0$ such that

$$\int_{H(x,\tilde{U}_n)\geqslant K_\epsilon} H(x,\tilde{U}_n) < \epsilon.$$

Indeed, by (H_3) and given $\epsilon > 0$, let us take c_ϵ such that

$$H(x,S) \leqslant \frac{\epsilon}{C_0}(\nabla H(x,S),S)_{\mathbb{R}^2} \quad \text{for all } x \in \Omega \quad \text{and } |S| \geqslant c_\epsilon,$$

where C_0 is given in (5.15). Moreover, we can take $K_\epsilon > 0$ such that

$$H(x,\tilde{U}_n(x)) \geqslant K_\epsilon \Rightarrow |\tilde{U}_n(x)| \geqslant c_\epsilon.$$

Therefore,

$$\int_{H(x,\tilde{U}_n)\geqslant K_\epsilon} H(x,\tilde{U}_n) \leqslant \int_{|\tilde{U}_n|\geqslant c_\epsilon} H(x,\tilde{U}_n) \leqslant \frac{\epsilon}{C_0}\int_\Omega (\nabla H(x,\tilde{U}_n),\tilde{U}_n)_{\mathbb{R}^2} < \epsilon,$$

as required. □

Proof of Theorem 2.6. Notice that U is a solution to (2.8): letting $V \in C_c^\infty(\Omega) \times C_c^\infty(\Omega)$ one gets

$$0 \leftarrow J'(U_n)V = \int_\Omega \nabla U_n \nabla V - \int_\Omega (A(x)U_n,V)_{\mathbb{R}^2} - \int_\Omega (\nabla H(x,\tilde{U}_n,)V)_{\mathbb{R}^2}.$$

But then, since

$$\int_\Omega \nabla U_n \nabla V \to \int_\Omega \nabla U \nabla V,$$
$$\int_\Omega (A(x)U_n,V)_{\mathbb{R}^2} \to \int_\Omega (A(x)U,V)_{\mathbb{R}^2}$$

and

$$\int_\Omega (\nabla H(x,\tilde{U}_n),V)_{\mathbb{R}^2} \to \int_\Omega (\nabla H(x,\tilde{U}),V)_{\mathbb{R}^2}$$

(the latter by Lemma 5.6), we have $J'(U)V = 0$ for all $V \in C_c^\infty(\Omega) \times C_c^\infty(\Omega)$. However, we still have to prove that $U \neq 0$.

Let us suppose, by contradiction, that $U = 0$. Then

$$\|U_n\|_{L^2(\Omega)\times L^2(\Omega)} \to 0 \quad \text{and} \quad \int_\Omega H(x,\tilde{U}_n) \to \int_\Omega H(x,(\Phi)_+) = 0$$

(again by Lemma 5.6). Therefore,

$$c(m) = \lim_{n\to\infty} J(U_n) = \tfrac{1}{2}\lim_{n\to\infty} \|U_n\|^2.$$

So, since $c(m) < 2\pi/\alpha_0$, we can pick $\delta > 0$ such that

$$\|U_n\|^2 \leqslant \frac{4\pi}{\alpha_0} - \delta \tag{5.19}$$

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for all n large enough. Let us consider $\epsilon > 0$ and $p > 1$ such that

$$p(\alpha_0 + \epsilon)\left(\frac{4\pi}{\alpha_0} - \delta\right) \leqslant 4\pi.$$

By (H_1) we can take $C > 0$ sufficiently large such that

$$H_u(x, s, t) + H_v(x, s, t) \leqslant \exp((\alpha_0 + \epsilon)(s^2 + t^2)) + C \quad \text{for all } s, t \geqslant 0, \quad x \in \Omega.$$

Then we see that

$$\begin{aligned} \|U_n\|^2 &\leqslant o(1) + C\biggl(\int_\Omega \exp\left(p(\alpha_0 + \epsilon)\left(\frac{4\pi}{\alpha_0} - \delta\right)\left(\frac{|U_n|}{\|U_n\|}\right)^2\right) + C\biggr)^{1/p} \|U_n\|_{L^{p'}(\Omega)\times L^{p'}(\Omega)} \\ &\leqslant o(1) + C\biggl(\int_\Omega \exp\left(4\pi\left(\frac{|U_n|}{\|U_n\|}\right)^2\right) + C\biggr)^{1/p} \|U_n\|_{L^{p'}(\Omega)\times L^{p'}(\Omega)}. \end{aligned}$$

However,

$$\|U_n\|_{L^{p'}(\Omega)\times L^{p'}(\Omega)} \to 0.$$

Since, by the Trudinger–Moser inequality, the last integral is bounded, we have $\|U_n\| \to 0$. Therefore, $U_n \to 0$ in E and so $J(U_n) \to 0 = c(m)$. This contradicts the definition of the minimax level, concluding the proof. □

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